Quantum simulation of field theories with ultracold atoms

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Outline

- Ultracold atoms as quantum simulators for field theories
- Symmetry-locked superfluid phases
- Link models

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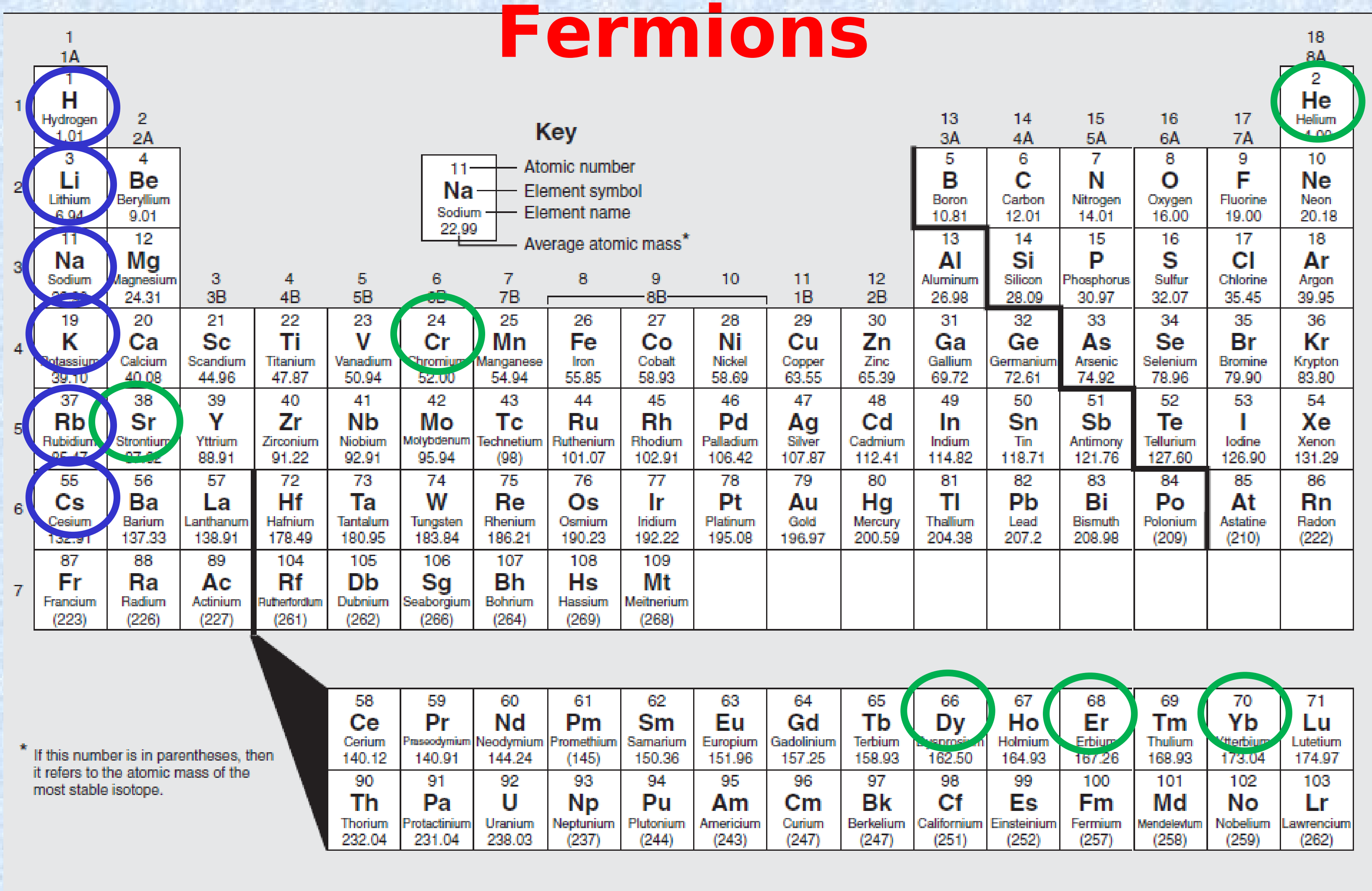
- **Ultracold atoms as quantum simulators for field theories**
- **Symmetry-locked superfluid phases**
- **Link models**

The ultracold family: Bosons & Fermions



Typical values:

$$\begin{cases} numbers:\ 10^3 - 10^6\ atoms \\ temperatures: 10 - 100\, nanoKelvin \\ sizes:\ 1 - 50\, micrometers \end{cases}$$

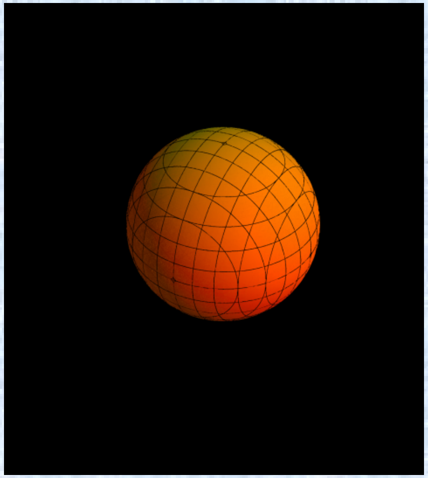
Controlling the system...

- Bosons and/or fermions
- Geometry (1D / 2D)
- Long-range interactions
- Add disorder
- Time-dependence (and to a certain extent space-dependence) of the parameters of the Hamiltonian
- Simulate a magnetic field through a rotation or with optical tools
- Explicit tuning of the interactions via Feshbach resonances
- Optical lattices (i.e., periodic potentials and minima of the potential located on a lattice)

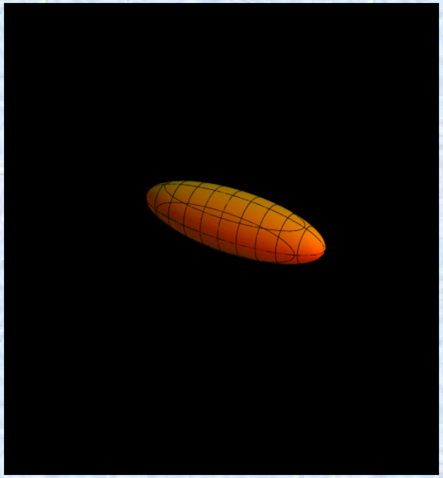
Experimental setup (I)

Magnetic harmonic potential: $V(x,y,z) = \frac{1}{2} m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$

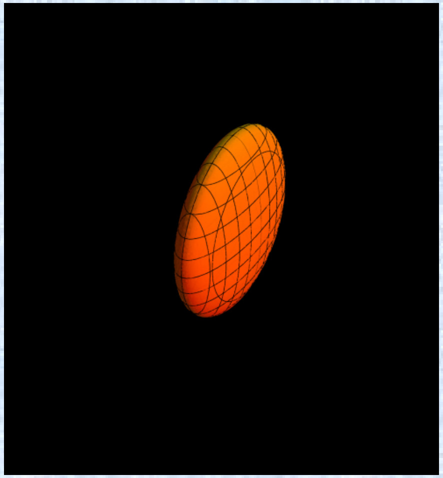
$\omega_x = \omega_y = \omega_z$



$\omega_x << \omega_y = \omega_z$



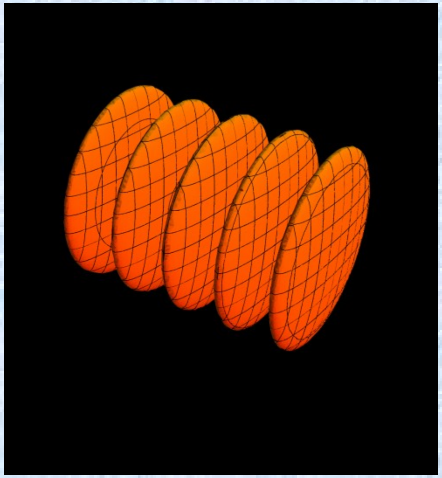
$\omega_x >> \omega_y = \omega_z$



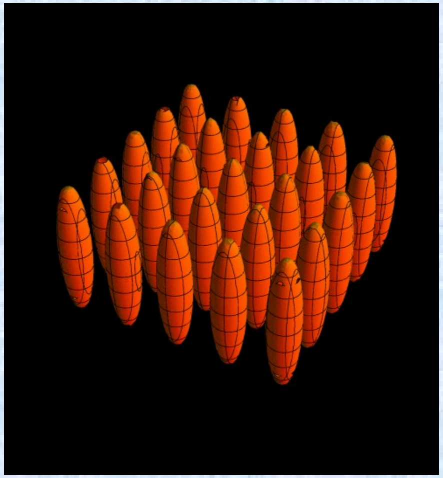
Experimental setup (II)

Using optical lattices:

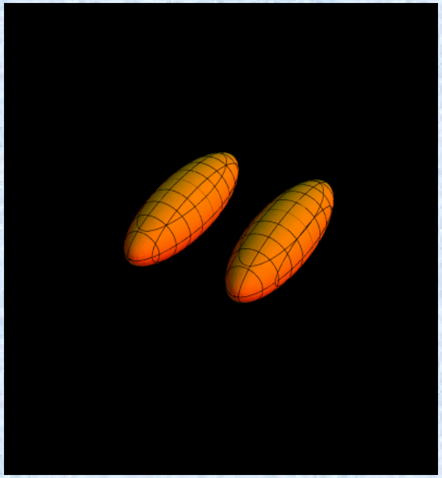
$$V = V_0 \cos^2(kx)$$

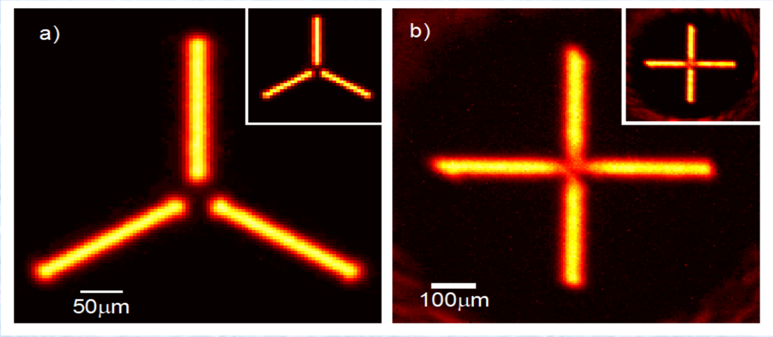


$$V = V_0 \left[\cos^2(kx) + \cos^2(ky)\right]$$



Other configurations (like ladders, coupled cigars or stars) are as well possible:





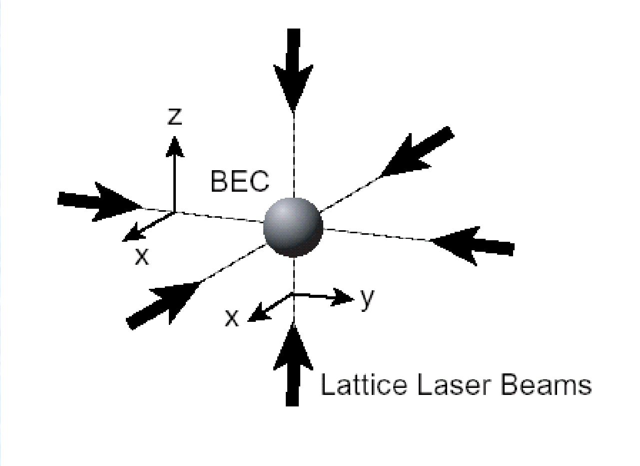
Quantum Simulations

Realization by purpose of a model Hamiltonian (or an effective model) of interest in an experimental setup with highly tunable parameters

Ultracold atoms as quantum simulators of:

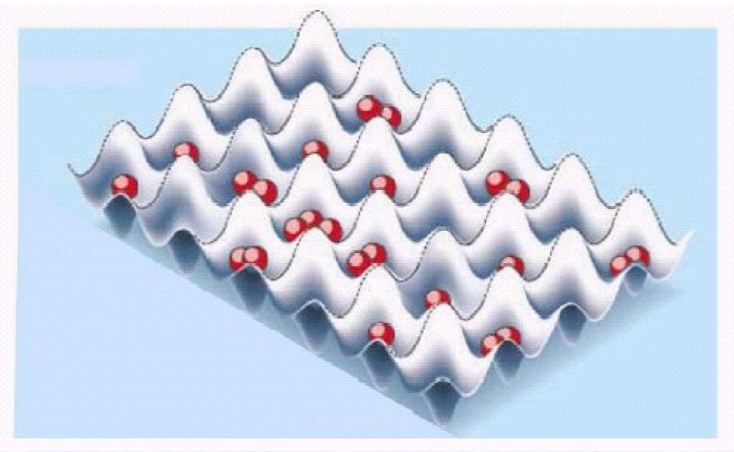
- **Strongly interacting lattice systems (e.g., Fermi and/or Bose Hubbard-like models)**
- **Quantum magnetism**
- **Fermionic superfluids with Cooper pairs**
- **Low-dimensional systems**
- **Quantum Hall physics**
- **Topological states of matter**
- **Field theories**
- **…**

Ultracold bosons in an optical lattice



$$V_{opt}(x) = V_0 \sin^2(kx)$$

e.g., a 1D lattice



It is possible to control:

- barrier height

- interaction term

- the shape of the network

- the dimensionality (1D, 2D, ...)

- the tunneling among planes or among tubes (in order to have a layered structure)

...

Effective Hamiltonian for ultracold bosons in optical lattices (I)

In second quantization, the full quantum many-body Hamiltonian is

$$\hat{H} = \int d\vec{r} \left(\hat{\psi}^{+}(\vec{r}) \left[\frac{-\hbar^2}{2\mathrm{m}} \nabla^2 + V_{opt}(\vec{r}) \right] \hat{\psi}(\vec{r}) + g_0 \hat{\psi}^{+}(\vec{r}) \hat{\psi}^{+}(\vec{r}) \hat{\psi}(\vec{r}) \hat{\psi}(\vec{r}) \right)$$

A very good description of (equilibrium and dynamical) low-energy properties – valid for large values of lattice height - is obtained using the Ansatz

$$\hat{\psi}(\vec{r}) = \sum_i \hat{b}_i \Phi_i(\vec{r})$$

tight-binding Ansatz
[D. Jaksch et al., PRL (1998)]

Wannier functions
(to be determined)

One gets...

Effective Hamiltonian for ultracold bosons in optical lattices (II)

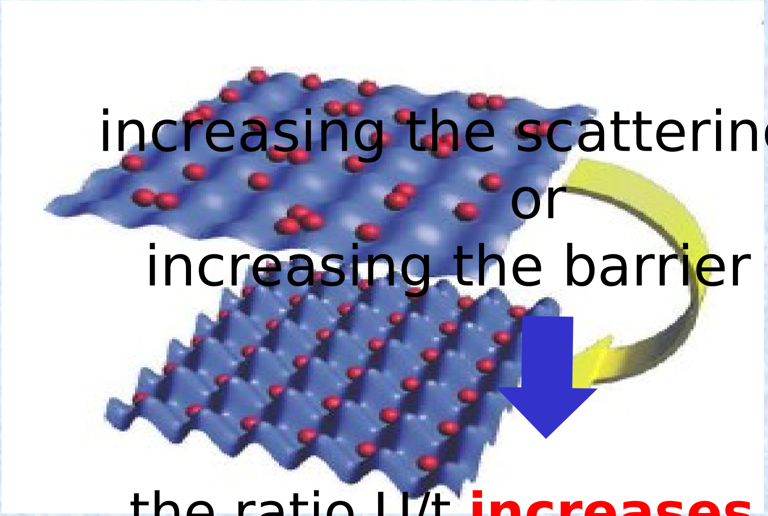
$$\hat{H} = -t \sum_{\langle i,j\rangle} \left(\hat{b}_i^{+}\,\hat{b}_j + h.c.\right) + \frac{U}{2}\sum_i \hat{n}_i\left(\hat{n}_i - 1\right)$$

$$\hat{n}_i \equiv \hat{b}_i^{+}\,\hat{b}_i \qquad N_T \text{ number of particles on } N \text{ sites} \qquad \text{filling } f = \frac{N_T}{N}$$

<u>Bose-Hubbard Hamiltonian</u>

t/U>>1 → **<u>Superfluid</u>**
dynamics described by the discrete nonlinear Schroedinger equation

t/U<<1 → **<u>Mott insulator</u>** quantum fluctuations dominate



increasing the scattering length
or
increasing the barrier height

the ratio U/t **<u>increases</u>**

Effective Hamiltonian for ultracold fermions in optical lattices

Similarly, for a dilute single-species Fermi gas the effective Hamiltonian is

$$\hat{H} = -t \sum_{<i,j>} \left(\hat{c}_i^+ \hat{c}_j + h.c. \right) \equiv -t \sum_{i,j} A_{ij} \hat{c}_i^+ \hat{c}_j$$

$$\left(f \leq 1 \right)$$

TIGHT-BINDING HAMILTONIAN

Notice that informations about the geometry and the Wannier functions are into the matrix A and the coefficients t, U:

$$t = -\int d\vec{r}\, \Phi_i^*\left(\vec{r}\right) \left[\frac{-\hbar^2}{2\mathrm{m}} \nabla^2 + V_{opt}\left(\vec{r}\right) \right] \Phi_j\left(\vec{r}\right)$$

$$U = g_0 \int d\vec{r} \left| \Phi_i\left(\vec{r}\right) \right|^4$$

Quantum simulation of graphene properties (I)

↓

Implementable putting ultracold fermions in lattices having Dirac points

e.g., in 2D, using the honeycomb lattice itself: using three optical lattices

$$V(x,y) = \sum_{j=1,2,3} V_j \sin^2[k_L(x\cos\theta_j + y\sin\theta_j) + \pi/2]$$

$$\theta_1 = \pi/3, \theta_2 = 2\pi/3, \theta_3 = 0$$

↓

Tight-binding model on the honeycomb (alias, graphene)

The 3D case

Not a straigthforward generalization of the 2D case: indeed, having 2D honeycomb coupled along the z-direction in general destroys the Dirac cones.

More formally:

$$\hat{H} = -t \sum_{i,j} A_{ij} \hat{c}_i^{+} \hat{c}_j$$

adjacency matrix of the graph [cfr. N. Biggs, Algebraic Graph Theory]

$$-t \sum_{j} A_{ij} \phi_\alpha(j) = \epsilon_\alpha \phi_\alpha(i) \qquad \hat{d}_\alpha = \sum_{\vec{j}} \phi_\alpha(j) \hat{c}_{\vec{j}}$$

$$\hat{H} = \sum_\alpha \epsilon_\alpha \hat{d}_\alpha^{+} \hat{d}_\alpha$$

The requests are that:

i) the single particle spectrum has Dirac points (and cones) and the graph has spectral dimension 3

ii) that the the adjacency matrix has nearest-neighbour couplings

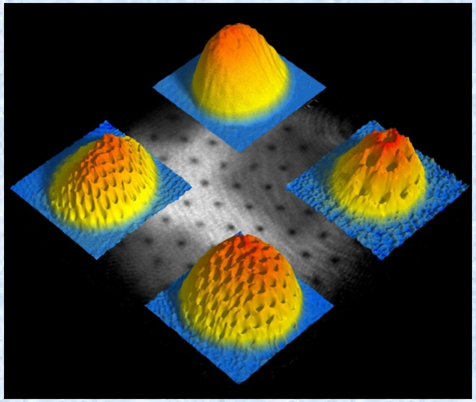
iii) not too many lasers are needed...

Although symmetries have been studied *[A.A. Abrikosov and S.D. Beneslavskii, JETP (1970) – J.L. Manes, PRB (2012)]*, **not easy to satisfy in practice i)-ii)-iii)...**

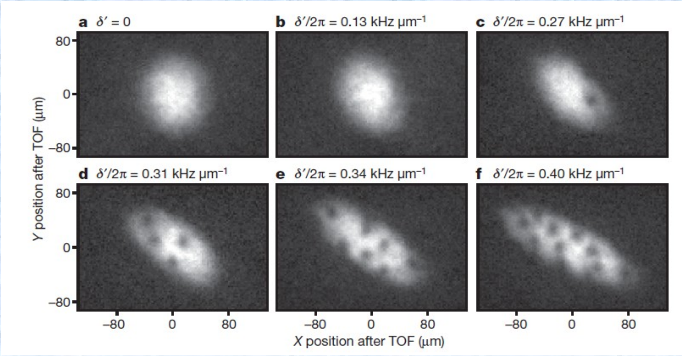
A possible solution: use a synthetic magnetic field

- **Using rotating traps**

[see the review N. Cooper, Adv. Phys. (2008)]



- **With spatially dependent optical couplings between internal states of the atoms** **[Y.-J. Lin et al., Nature (2009) – J. Dalibard et al, RMP (2011)]**

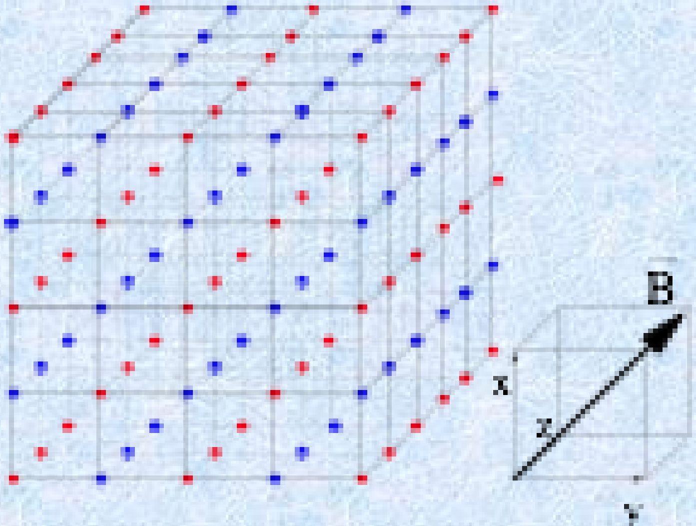


For our purposes: single-species Fermi gas in a π-flux magnetic field (at half filling)

$$\hat{H} = -t \sum_{<i,j>} \hat{c}_i^{+} e^{-i a_{ij}} \hat{c}_j + h.c.$$

$$a_{ij} = \int_i^j \vec{A} \cdot d\vec{l} \qquad \vec{B} \equiv rot\,\vec{A} = \pi\left(1,1,1\right)$$

(we can also assume different hoppings t_x, t_y and t_z along the three directions x, y and z)



Single-particle spectrum and Dirac cones (I)

Using the Hasegawa's gauge:

$$\vec{A} = \pi\left(0, x-y, y-x\right)$$

[Y. Hasegawa, J. Ph. Soc. Jap. (1990)]

one gets

$$E_{\vec{k}} = \pm 2\sqrt{t_x^2\cos^2 k_x + t_y^2\cos^2 k_y + t_z^2\cos^2 k_z}$$

with k belonging to the first (magnetic) Brillouin zone.

[L. Lepori et al, Europhys. Lett. (2010), PRB (2016);
M. Burrello et al, JPA (2017); J. Pinto Barros, M. Burrello, and A. Trombettoni (2020)]

Single-particle spectrum and Dirac cones (II)

- For t_z =0 the results for the 2D case with π-flux are retrieved [I. Affleck and J.B. Marston, PRB (1988)] **are retrieved.**

- Excitations around the two inequivalent Dirac points <u>obey the 3D Dirac equation</u>.

- In the limit of vanishing t_z one retrieves the 2D Dirac equation.

- A mass term can be added using a Bragg pulse.

- The Dirac points does not depend on t_x, t_y and t_z.

- With an attractive interaction U one has a semimetal-superfluid transition at a <u>finite</u> value of U

- Extendable to many components

Applications

- With a spatial control of the synthetic magnetic field one can an e.m. field.

- With a *<u>dynamical</u>* gauge field one can then have a simulation of the 3+1 QED.

- One may also think to have the fermions living in one dimension and the gauge field in an higher dimension: this has been studied in the context of graphene, giving rise to pseudo-QED *[E.C. Marino, Nucl. Phys. B, 1993]***. In one dimension one would then have the pseudo-Schwinger model.**

Outline

- Ultracold atoms as quantum simulators for relativistic field theories

- **Symmetry-locked superfluid phases**

- Topological Kondo model in junctions of Tonks-Girardeau gases

Small Dictionary (I)

Abelian transformation: only one parameter/generator

$$U = e^{i\phi}$$

Non Abelian symmetry : more than one parameter/generator, not commuting each others

$$U = e^{i\alpha_j T^j}$$

$$[T_i, T_k] \neq 0$$

Small Dictionary (II)

Global transformation :
parameters NOT space-time dependent

$$U = e^{ia_j T^j}$$

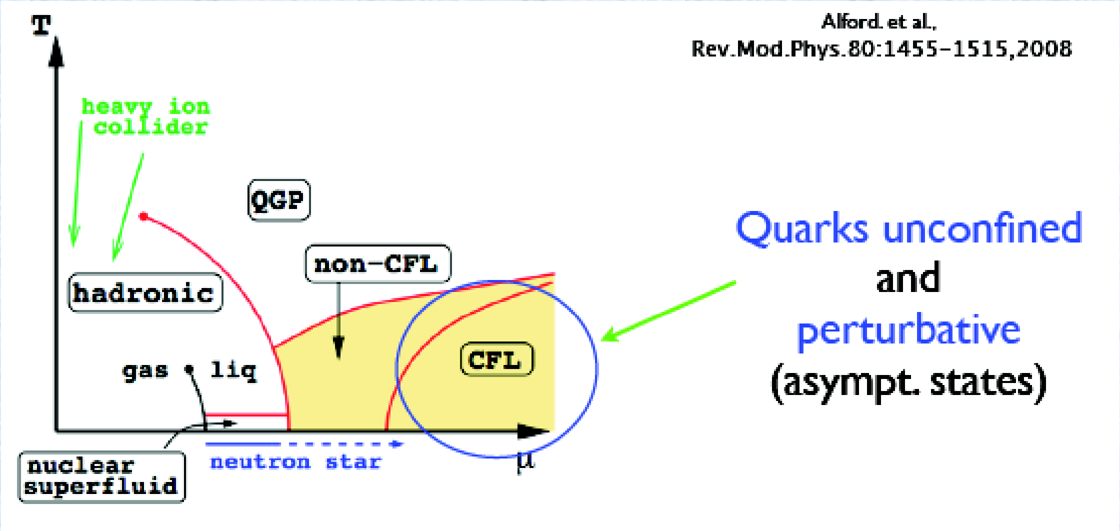
Local (gauge) trasnformation :
parameters space-time dependent

$$U(x^\mu) = e^{ia_j(x^\mu)T^j}$$

Motivations

→ **physics of interacting fermionic mixtures**

→ **QCD inspired problem(s)**



The proposed quantum simulatio

→ 4 fermionic species: e.g., 4 species of Yb or a mixture ^{171}Yb-^{173}Yb

→ 2 doublets $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

→ $U(2)_c \times U(2)_f$ (global) invariant Hamiltonian

→ $$\hat{H}_{int} = -U_c \sum_{i,c} (c^\dagger_{i;\,c} c_{i;\,c})^2 - U_f \sum_{i,f} (c^\dagger_{i;\,f} c_{i;\,f})^2 + \\ - U_{cf} \sum_{i,c,f} c^\dagger_{i;\,c} c_{i;\,c} c^\dagger_{i;\,f} c_{i;\,f}$$

$$\boxed{\begin{matrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \equiv \begin{pmatrix} c_g \\ c_r \end{pmatrix} \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv \begin{pmatrix} c_u \\ c_d \end{pmatrix} \end{matrix}}$$

Order parameters

$$\langle c_{i;\,u} c_{i;\,d} \rangle \equiv \Delta_f\,,\ \langle c_{i;\,r} c_{i;\,g} \rangle \equiv \Delta_c,\ \langle c_{i;\,c} c_{i;\,f} \rangle \equiv \mathbf{\Delta}_{cf}$$

a 2x2 matrix

Order parameters:

$$2|\Delta_0|^2 = (|\Delta_c|^2 + |\Delta_f|^2)$$

$$\Delta_+^2 = \mathrm{Tr}\left(\mathbf{\Delta}_{cf}^\dagger \mathbf{\Delta}_{cf}\right),\ \Delta_-^2 = 2\det\mathbf{\Delta}_{cf}$$

We consider $U_c = U_f \equiv U$

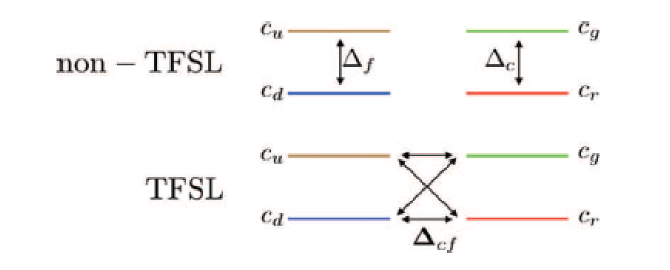
Results (I)

$$2|\Delta_0|^2 = (|\Delta_c|^2 + |\Delta_f|^2)$$

$$\Delta_+^2 = \mathrm{Tr}\left(\mathbf{\Delta}_{cf}^\dagger \mathbf{\Delta}_{cf}\right), \ \Delta_-^2 = 2\det \mathbf{\Delta}_{cf}$$

We find a “two-flavors” symmetry-locked phase (TFSL) for U_{cf}>U:

The minimization of F with respect to $\Delta_\pm$ and Δ_0 gives $|\Delta_+| = |\Delta_-|$ and $|\Delta_c| = |\Delta_f|$. We find that for $U_{cf} \neq U$ the gap equations are not consistent if both Δ_+ and Δ_0 are non-zero both $T = 0$ and finite temperature and two phases are found as follows (see fig. 1): i) Non-TFSL phase: for $U_{cf} < U$ it is $\Delta_+ = 0$ and $\Delta_0 \neq 0$; ii) TFSL phase: for $U_{cf} > U$ it is $\Delta_0 = 0$ and $\Delta_+ \neq 0$.



[L. Lepori, A. Trombettoni, and W. Vinci, Europhys. Lett. (2015); J. Pinto Barros, L. Lepori, and A. Trombettoni PRA (2017); J. Pinto Barros, M. Burrello, and A. Trombettoni (2021)]

Results (II)

Non-TFSL superfluid phase abelian:

$$U(2)_c \times U(2)_f \to SU(2)_c \times SU(2)_f$$

“Two-Flavours locking” phase :

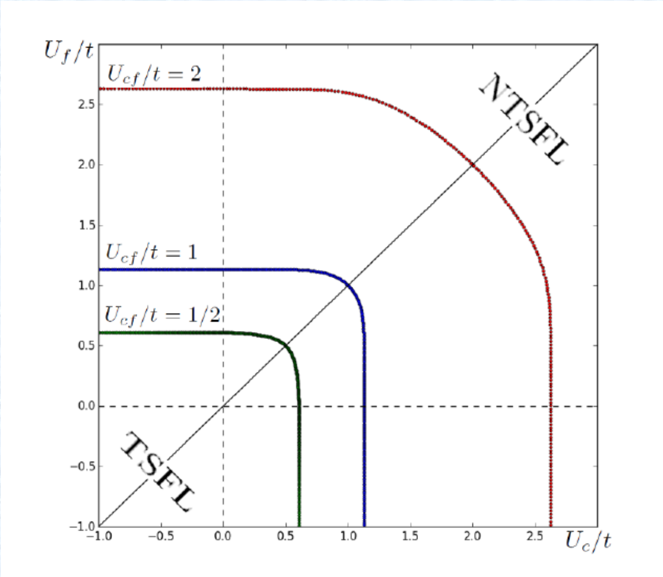
Spont. Symm. Break. $U(2)_c \times U(2)_f \to U(2)_{c+f}$

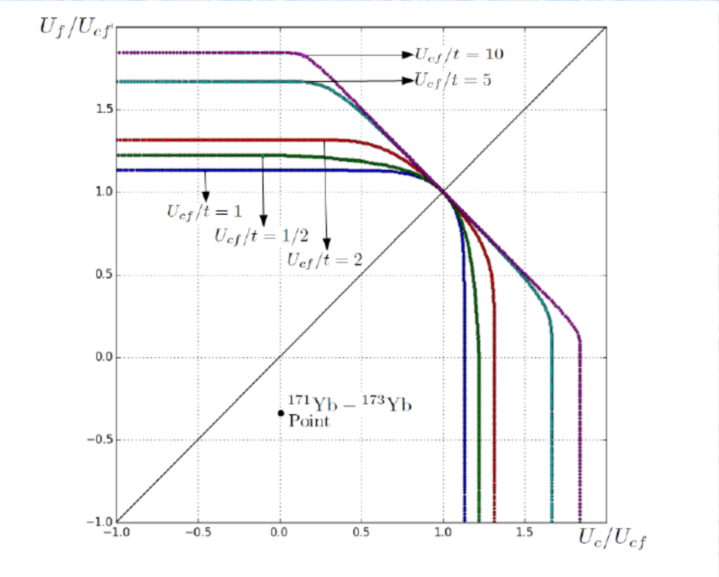
$$\Downarrow$$

vortices in the CFL phase have fractional flux

Results (III)

But it does survive when interactions are different? Especially if $U_f<0$...

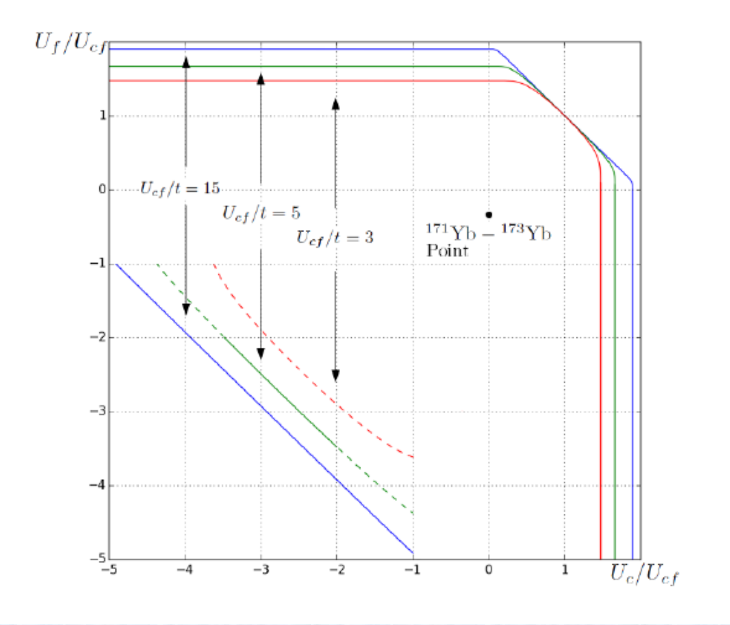




$$a_{171-173} = -578a_0$$
$$a_{173-173} = +200a_0$$
$$a_{171-171} = -3a_0$$

Results (IV)

The previous are mean-field results → combining with a strong-coupling computation



To have a “true” color-flavor locking: even without putting interactions, use dynamical gauge fields

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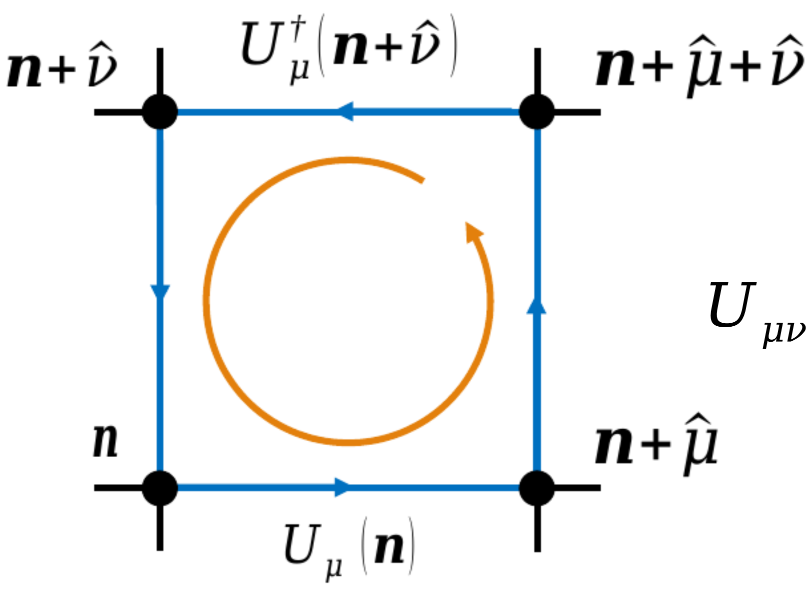
- Ultracold atoms as quantum simulators for field theories

- Symmetry-locked superfluid phases

- Link models → how to generate plaquette terms?

Basic quantities on the lattice



[Wilson, PRD (1974)]

$$U_\mu(\boldsymbol{n}) = e^{iea A_\mu(\boldsymbol{n})}$$

LINKS

$$U_{\mu\nu}(\boldsymbol{n}) = U_\mu(\boldsymbol{n})\, U_\nu(\boldsymbol{n}+\hat{\mu})\, U^\dagger_\mu(\boldsymbol{n}+\hat{\nu})\, U^\dagger_\nu(\boldsymbol{n})$$

$$U_{\mu\nu}(\boldsymbol{n}) = e^{iea^2 F_{\mu\nu}(\boldsymbol{n})}$$

PLAQUETTES

Link models

ELECTRIC FIELD

$$[U_\mu(\mathbf{n}), E_\nu(\mathbf{n}')] = -\delta_{\mu,\nu}\delta_{\mathbf{n},\mathbf{n}'}U_\mu(\mathbf{n})$$

KOGUT-SUSSKIND HAMILTONIAN

$$H_g = \frac{e^2}{2}\sum_{\mathbf{n},\mu} E_\mu^2(\mathbf{n}) - \frac{1}{4a^2e^2}\sum_P (U_{\mu\nu} + U_{\mu\nu}^\dagger)$$

ROKHSAR-KIVELSON HAMILTONIAN

$$H_{RK} = H_g + \lambda \sum_P (U_{\mu\nu} + U_{\mu\nu}^\dagger)^2$$

Quantum link models → replace the Wilson operators by discrete quantum degrees of freedom, still living on the links of the lattice (quantum links)

Link models (II)

Bosonic quantum link models

$$U_\mu(\mathbf{n}) = S_\mu^+(\mathbf{n}), \qquad U_\mu^\dagger(\mathbf{n}) = S_\mu^-(\mathbf{n}), \qquad E_\mu(\mathbf{n}) = S_\mu^z(\mathbf{n})$$

Fermionic quantum link models: in terms of fermionic states and occupation numbers, we denote the two states of the local Hilbert space with $|0\rangle$ and $|1\rangle$

$$|1\rangle = c_\mu^\dagger(\mathbf{n})|0\rangle \qquad U_{\mu\nu}(\mathbf{n}) = c_\mu(\mathbf{n})c_\nu(\mathbf{n}+\hat{\nu})c_\mu^\dagger(\mathbf{n}+\hat{\nu})c_\nu^\dagger(\mathbf{n})$$

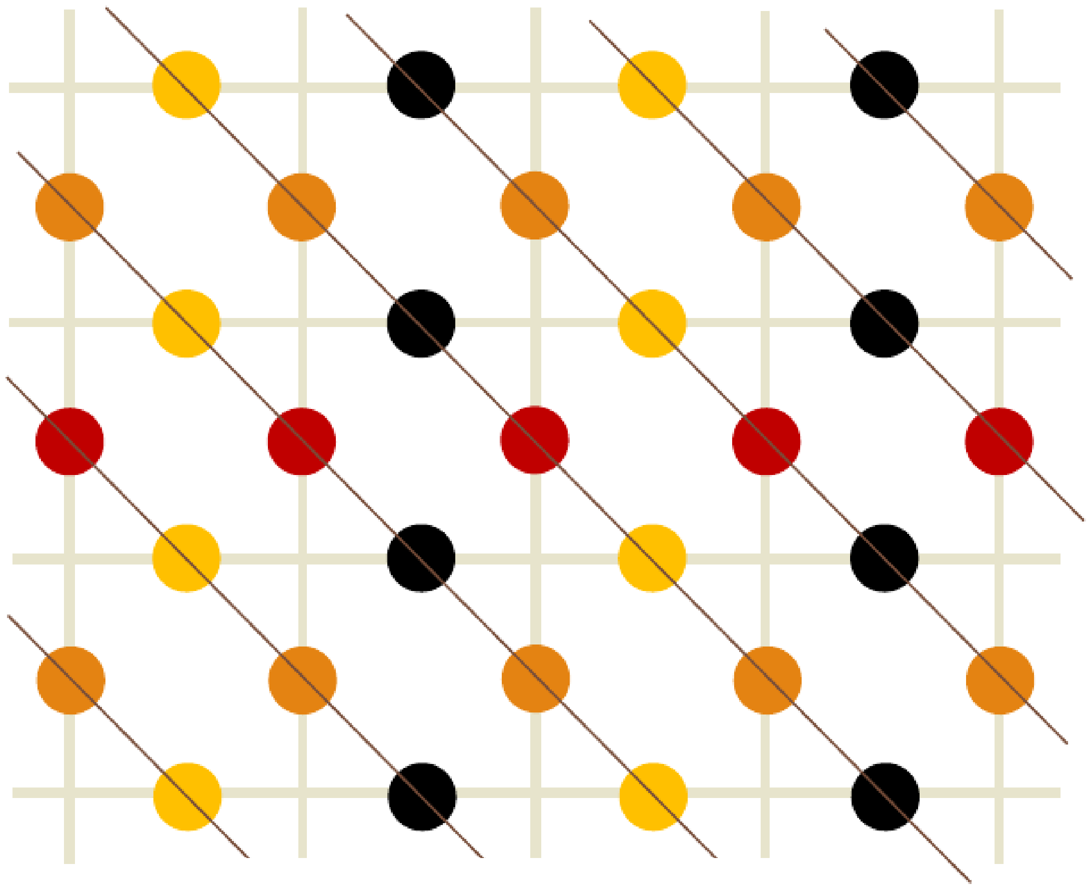
Proposals for having plaquette terms

→ 4 correlated hoppings + angular momentum conservation
[Zohar, Cirac, and Reznik, PRA (2013)]

→ Dual formulation: single hopping + conditional operations on the

nearest-neigbours [A. Celi et al, PRX (2020)]

A proposal using a spin dependent-optical lattice



$$H_0 = -h \sum_{x,m} \epsilon_{xm} b^{\dagger}_{xm} b_{xm}$$

Derivation of the plaquette term in perturbation theory

$$H_1 = H_{\text{hop}} + H_{\text{int}} \equiv -t \sum_{\langle i,j \rangle_d, m} (b^\dagger_{im} b_{jm} + \text{h.c.}) + \frac{1}{2} \sum_{\langle i,j \rangle, m, m'} V^{i,j}_{mm'} b^\dagger_{xm} b^\dagger_{ym'} b_{xm'} b_{ym}$$

(hard-core condition assumed)

At the third order of the perturbation theory for large h one finds:

$$H^{(\text{eff})} = \frac{t^2}{h} \sum_{\langle i,j \rangle_d, m, m'} n_{im} n_{jm'} - \frac{1}{h} \sum_{\langle i,j \rangle, m, m'} (V^{i,j}_{mm'})^2 n_{im} n_{jm'} + \frac{t^2}{h} \sum_{\substack{i,i',j,j' \in \square \\ m,m'}} V^{(i,i')}_{mm'} b^\dagger_{j'm} b^\dagger_{jm'} b_{i'm'} b_{im}$$

[P. Fontana, J. Pinto-Barros, M. Burrello, and A. Trombettoni, to be submitted]

Connections with link models

$$U_{im} = b_{im}^{\dagger}, \qquad U_{im}^{\dagger} = b_{im}$$

$$E_{im} \equiv n_{im} - \frac{1}{2}$$

$$H^{(\mathrm{eff})} = \sum_{\langle x,y \rangle_d, m, m'} \lambda_1^{(mm')} E_{xm} E_{ym'} - \sum_{\langle x,y \rangle, m, m'} \lambda_2^{(mm')} E_{xm} E_{ym} - J \sum_{\square} (U_{\square} + U_{\square}^{\dagger})$$

Conclusions

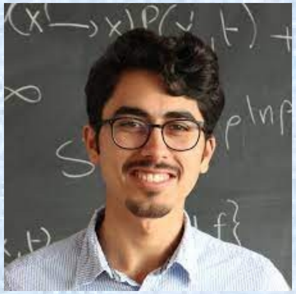
Ultracold atoms as quantum simulators for field theories models

Ultracold fermions and gauge fields are a tool to emulate mechanisms such as the color-flavor locking

Quantum simulation of link models with plaquette terms is demanding → we discussed a proposal involves spinor dipolar gases

Acknoledgements

P. Fontana
(SISSA → Barcelona)



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Thank you!