Localisation of Dirac eigenmodes in finite-temperature lattice gauge theory

Aula Leonardi

Eötvös Loránd University (ELTE)
Budapest

Workshop on Gauge Topology, Flux Tubes and Holographic Models

ECT*, Villazzano (TN)
27 May 2022

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Matteo Giordano

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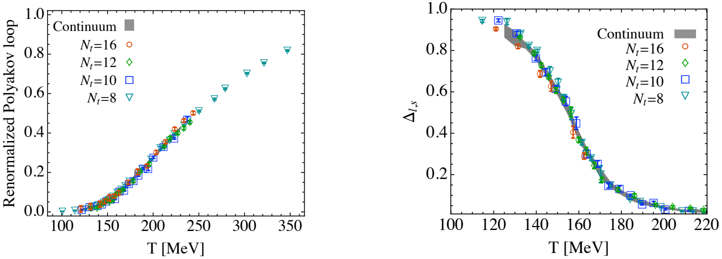
STRONG 2020

QCD at finite temperature

QCD: crossover at $T \simeq 145 - 165$ MeV, (approximate) deconfinement and chiral symmetry restoration, relation between the two still not fully clear



Renormalised Polyakov loop and chiral condensate
Figures from [BW collaboration (2010)]

QCD-like gauge theories with genuine phase transitions:

- deconfinement improves chiral symmetry properties if single transition (e.g., pure gauge theory, $N_f = 3$ staggered fermions on coarse lattices)
- $T_{\rm dec} < T_\chi$ if two transitions are present (e.g., adjoint fermions)

Deconfinement and χSB from spontaneous SB

Deconfinement and χSB from SSB in opposite quark-mass limits

Quark mass $m \to \infty$ (pure gauge theory)

- Exact $\mathbb{Z}_3$ centre symmetry
- Spontaneously broken above $T_c \approx 290\,\mathrm{MeV}$ [Boyd *et al.* (1996)]

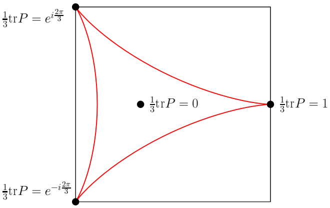
Quark free energy from Polyakov loop

$$\langle \mathrm{tr}\, P \rangle \propto e^{-F_q/T}$$

$T < T_c$: $\langle \mathrm{tr}\, P \rangle = 0 \Rightarrow F_q = \infty$

$T > T_c$: $\langle \mathrm{tr}\, P \rangle \neq 0 \Rightarrow F_q < \infty$

Deconfinement = PL ordering



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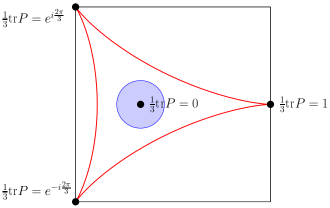

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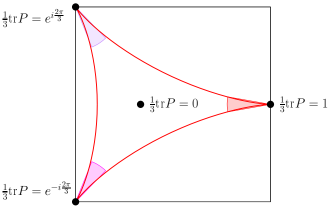


Deconfinement and χSB from spontaneous SB

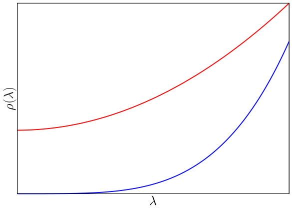
Quark mass $m \to 0$

- Exact $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry
- Spontaneously broken below $T_c(N_f = 2) \approx 132\,\mathrm{MeV}$ [Ding *et al.* (2019)]

$$|\langle\bar{\psi}\psi\rangle| = \int_0^\infty d\lambda\, \frac{2m\rho(\lambda)}{\lambda^2 + m^2} \underset{m\to 0}{\to} \pi\rho(0^+)$$

$$\rho(\lambda) = \lim_{V\to\infty} \frac{T}{V}\Big\langle \sum_n \delta(\lambda - \lambda_n)\Big\rangle$$

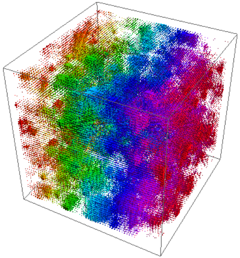
χSB = accumulation of Dirac modes at $\lambda = 0$



Different symmetries, approximate @$m_{\rm phys}$: how do they affect each other?

Localisation of Dirac eigenmodes

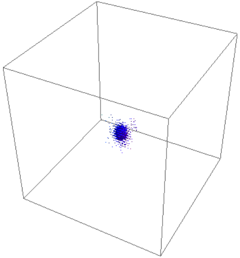
Low Dirac modes become localised at the QCD transition



Delocalised mode

- extends throughout system
- $\|\psi(x)\|^2 \sim 1/L^\alpha$ with $0 < \alpha \leq d$

cond-mat: delocalised if $\alpha = d$, critical if $0 < \alpha < d$



Localised mode

- confined in finite region
- $\|\psi(x)\|^2 \sim 1/L^0$ inside, negligible outside

$$\int_0^\beta dt \int d^3x \, \|\psi_n(t,\vec{x})\|^2 = 1$$

$$\|\psi_n(t,\vec{x})\|^2 \equiv \sum_{c,\eta} |\psi_{n\,c,\eta}(t,\vec{x})|^2$$

Figures from [Ujfalusi *et al.* (2015)]

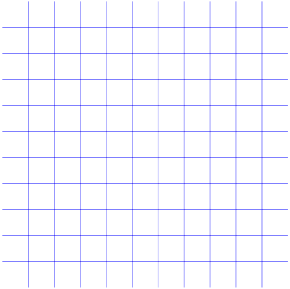
Localisation and Anderson transitions

Anderson model for "dirty" conductors

[Anderson (1958)]

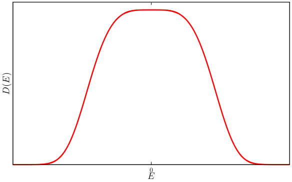
$$H_{\vec{x},\vec{y}} \quad = \qquad \sum_{\mu=1}^{3}(\delta_{\vec{x}+\hat{\mu},\vec{y}} + \delta_{\vec{x}-\hat{\mu},\vec{y}})$$

Random potential $|\varepsilon_{\vec{x}}| \leq \frac{W}{2}$ (orthogonal AM)



Average over realisations of disorder

- $W=0$ (no disorder): delocalised modes
- $W \neq 0$: localised beyond mobility edge $E_c(W)$, $E_c \to 0$ as W increases
- All states localised for $W > W_c$: metal-insulator transition
- Second-order (Anderson) transition at E_c with divergent correlation length $\xi \sim |E - E_c|^{-\nu}$



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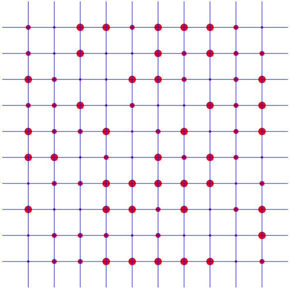

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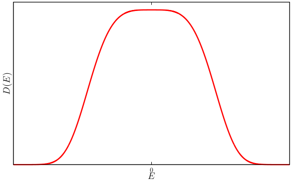

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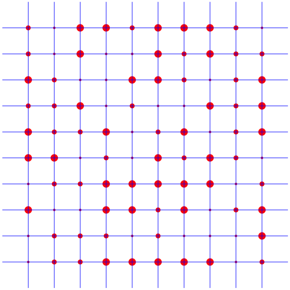

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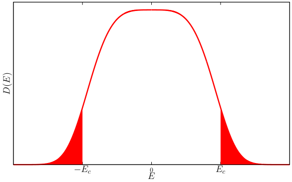

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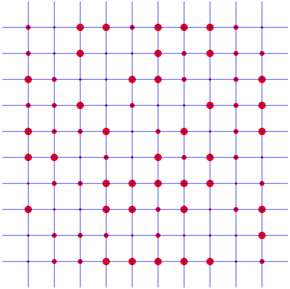

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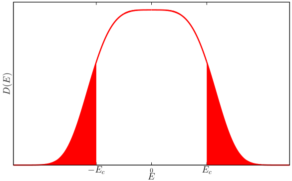

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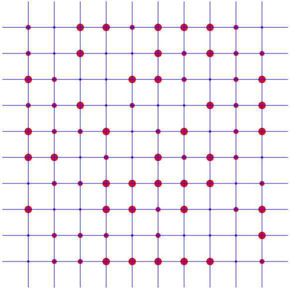

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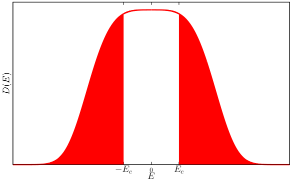

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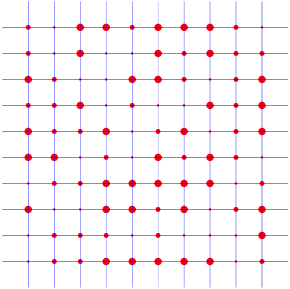

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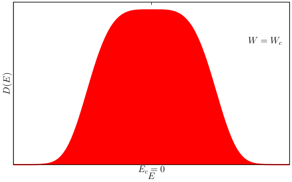

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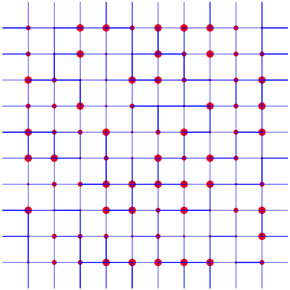

Random potential $|\varepsilon_{\vec{x}}| \leq \frac{W}{2}$ (orthogonal AM)
$+$ random phases $\phi_{\vec{y},\vec{x}} = -\phi_{\vec{x},\vec{y}}$ (unitary AM)

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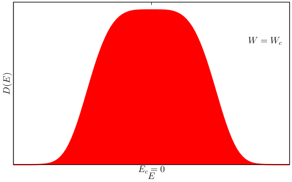
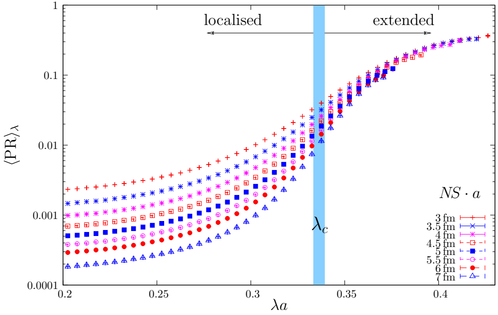


Localisation of Dirac eigenmodes in QCD

Participation ratio ≈ fraction of system occupied by a mode



Data for $T \simeq 2.6 T_c$
from [MG *et al.* (2014)]

$$\mathrm{IPR}_n = \int_0^{\frac{1}{T}} dt \int d^d x \, \|\psi_n(t, \vec{x})\|^4 \quad \sim L^{-\alpha}$$

$$\langle \mathrm{PR} \rangle_\lambda = \frac{T}{L^d} \left\langle \sum_n \delta(\lambda - \lambda_n) \mathrm{IPR}_n^{-1} \right\rangle \quad \sim L^{-(d-\alpha)}$$

Localisation and spectral statistics

Localisation of eigenmodes reflects on statistical properties of eigenvalues

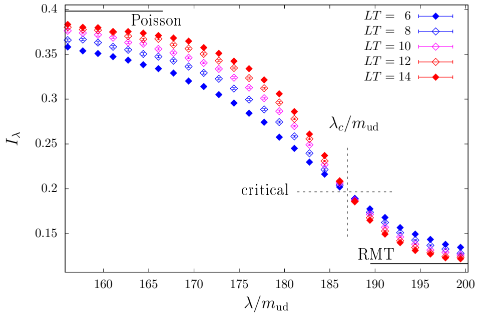
- delocalised modes easily mixed by fluctuations $\to$ RMT- type statistics
- localised modes fluctuate independently $\to$ Poisson statistics

Universal expectations for unfolded level spacings $s_n = \frac{\lambda_{n+1} - \lambda_n}{\langle \lambda_{n+1} - \lambda_n \rangle_\lambda}$

- local spacing distribution $p_\lambda(s)$ changes with L

$$I_\lambda = \int_0^{s_0} ds \, p_\lambda(s)$$

- $p_\lambda \to p_{\mathrm{Poisson}}$ (localised) or p_{RMT} (delocalised) as $L \to \infty$
- λ_c = scale-invariant point, critical statistics p_{crit}



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Localisation of Dirac eigenmodes in QCD - mobility edge

Mobility edge extrapolates to 0 in the crossover region [Kovács, Pittler (2012)]
No localised modes in the confined/chirally broken phase at low T

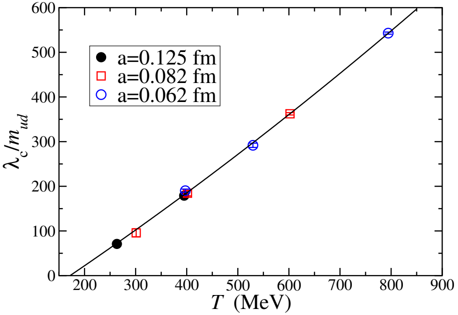


Figure from [Kovács, Pittler (2012)]

Numerical evidence from lattice
[review: MG, Kovács (2021)]

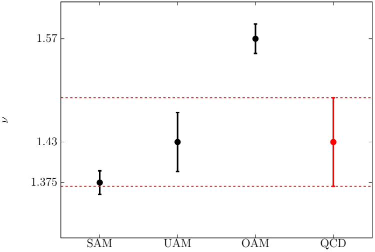
- Various fermion discretisations
  - staggered [García-García, Osborn (2007), Kovács, Pittler (2012), MG *et al.* (2014)]
  - overlap [Kovács (2010), Dick *et al.* (2015)]
  - domain wall [Cossu, Hashimoto (2016)]
  - twisted mass [Holicki *et al.* (2018)]
- Survive continuum limit
  $\Longrightarrow$ not a lattice artefact

Ratio $\lambda_c/m_{\rm ud}$ expected to be RG-invariant [Kovács, Pittler (2012)]
Proof in preparation [MG (202ish)]

Dirac operator as a disordered Hamiltonian

disordered Hamiltonian	$\longleftrightarrow$	Dirac operator
random hopping, potential	$\longleftrightarrow$	gauge fields
ensemble average $\langle\langle \ldots \rangle\rangle$	$\longleftrightarrow$	path integral $\langle \ldots \rangle$

- Same critical features as 3D UAM...
  - localisation length critical exponent [MG *et al.* (2014)]
  - multifractal exponents [Ujfalusi *et al.* (2015)]
  - critical statistics [Nishigaki *et al.* (2014)]
- ...but why localisation near $\lambda = 0$ above T_c?



Data from [MG *et al.* (2014), Slevin, Ohtsuki (1997), (1999), Asada *et al.*, (2005)]

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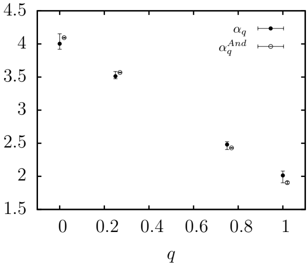


Figure from [Ujfalusi *et al.* (2015)]

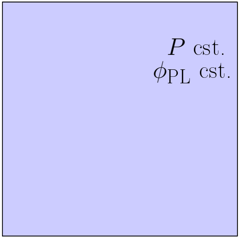
Localisation and deconfinement - sea/islands picture

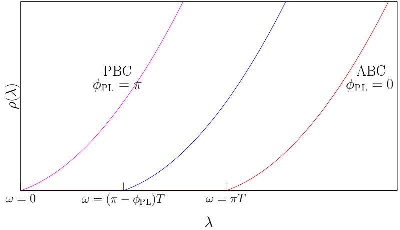
Qualitative explanation of localisation: “sea/islands” picture

[Bruckmann *et al.* (2011); MG *et al.* (2015, 2016, 2017)]

Ordered phase ≈ configs. fluctuate around $\vec{A} = 0$, $P(\vec{x}) = \mathrm{diag}(e^{i\phi_a})$

Gapped spectrum $|\lambda| \geq \omega = (\pi - \phi_{\mathrm{PL}})T$ with $\phi_{\mathrm{PL}} = \max_{a, \phi_a \in (-\pi, \pi]} |\phi_a|$





- “Sea” of $\phi_{\mathrm{PL}} = 0$ selected by fermions because of largest spectral gap
- “Islands” with $|\phi(\vec{x})| < \phi_{\mathrm{PL}}$ can support localised modes in the gap

Prediction: localisation in the deconfined phase of a generic gauge theory

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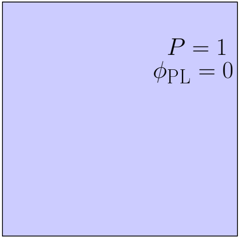

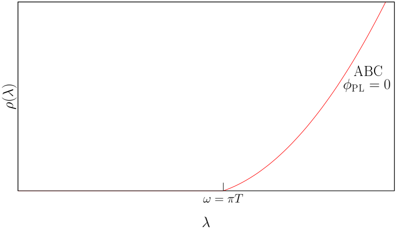

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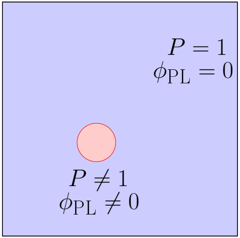

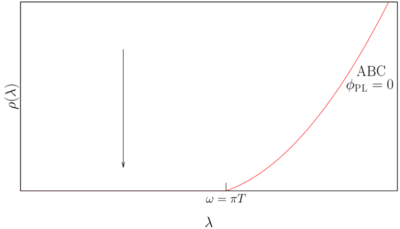

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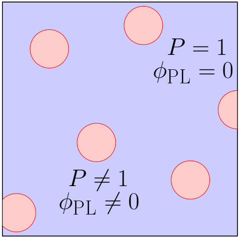
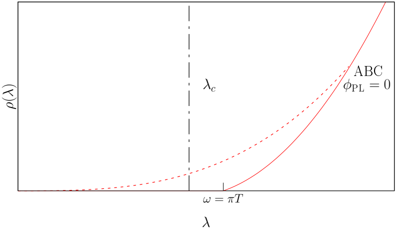
Ordered phase ≈ configs. fluctuate around $\vec{A} = 0$, $P(\vec{x}) = \mathrm{diag}(e^{i\phi_a})$

Gapped spectrum $|\lambda| \geq \omega = (\pi - \phi_{\mathrm{PL}})T$ with $\phi_{\mathrm{PL}} = \max_{a, \phi_a \in (-\pi,\pi]} |\phi_a|$



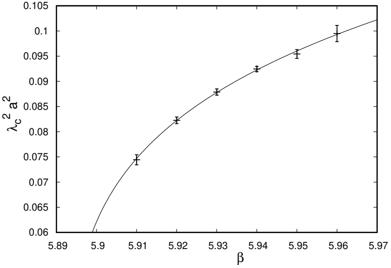


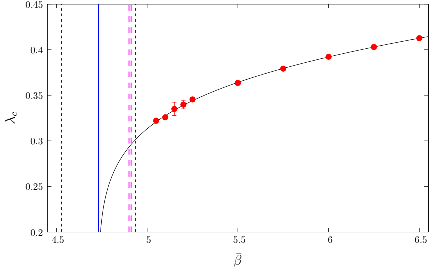
- "Sea" of $\phi_{\mathrm{PL}} = 0$ selected by fermions because of largest spectral gap
- "Islands" with $|\phi(\vec{x})| < \phi_{\mathrm{PL}}$ can support localised modes in the gap

Prediction: localisation in the deconfined phase of a generic gauge theory

Localisation and deconfinement - pure gauge theory

Localisation observed in the deconfined phase in many pure gauge systems

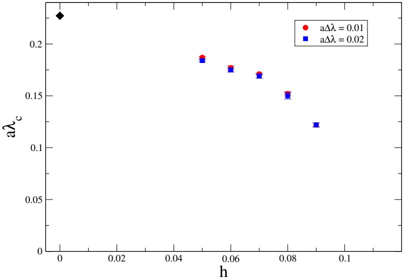




- SU(3) in 3+1D [Kovács, Vig (2018, 2020)] and 2+1D [MG (2019)]
  - first order (3+1D)/second order (2+1D) thermal phase transition
  - λ_c vanishes at T_c (actually jumping in 3+1D [Kovács (2021)])
  - Anderson transition is second order (3+1D)/BKT (2+1D)

Localisation and deconfinement - pure gauge theory

Localisation observed in the deconfined phase in many pure gauge systems



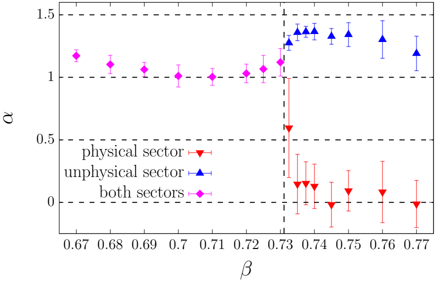
- SU(3) + trace deformation [Bonati *et al.* (2021)]

$$S_{\mathrm{td}} = S_{\mathrm{SU(3)}} + h \sum_{\vec{x}} |\mathrm{tr}\, P(\vec{x})|^2$$

  - first order "reconfining" transition at fixed $T > T_c$ as deformation parameter h increased
  - λ_c vanishes at $h_c \simeq 0.1$

Localisation and deconfinement - pure gauge theory

Localisation observed in the deconfined phase in many pure gauge systems



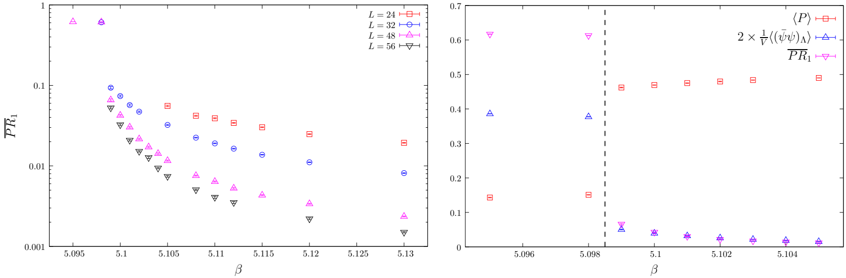
- $\mathbb{Z}_2$ in 2+1D [Baranka, MG (2021)]
  - simplest gauge theory with deconfinement transition
  - lowest modes "critical" in the confined phase, at β_c become

    $\rightarrow$ localised in the "physical" sector $\langle P \rangle > 0$

    $\rightarrow$ more delocalised in the "unphysical" sector $\langle P \rangle < 0$

Localisation and deconfinement - fermions

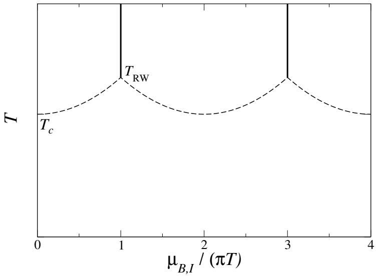
Localisation in systems with fermions displaying genuine phase transition



- SU(3) + $N_f = 3$ unimproved rooted staggered fermions on coarse ($N_t = 4$) lattices [MG *et al.* (2017)]
  - first-order, (partially) deconfining and chirally restoring transition on the lattice [De Forcrand, Philipsen (2003)], does not survive continuum limit
  - localised modes appear at the transition

Localisation and deconfinement - fermions

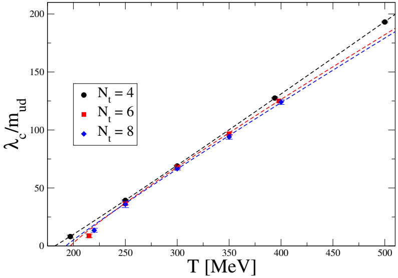
Localisation in systems with fermions displaying genuine phase transition

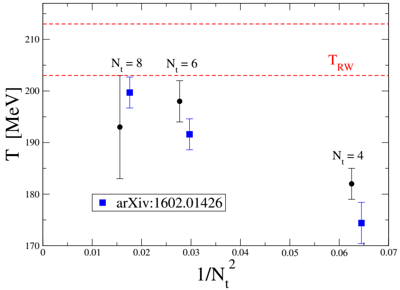


- SU(3) + imaginary baryochemical potential $\hat{\mu}_{B,I}$ [Cardinali *et al.* (2022)]
  - periodic partition function $Z(\hat{\mu}_{B,I} \pm 2\pi) = Z(\hat{\mu}_{B,I})$ [Roberge, Weiss (1986)]
  - first-order lines above second-order Ising point at the Roberge-Weiss points $T_{\rm RW} = 208(5)\,{\rm MeV}$, $\hat{\mu}_{B,I} = \pi \bmod 2\pi$ [Bonati *et al.* (2016)]
  - λ_c vanishes at $T_{\rm RW}$

Localisation and deconfinement - fermions

Localisation in systems with fermions displaying genuine phase transition





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  - λ_c vanishes at $T_{\rm RW}$

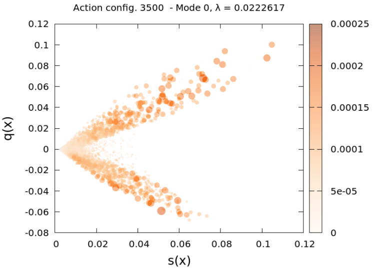
Topology and localisation

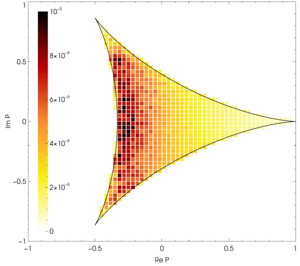
Relation with topological objects not fully understood, very partial picture

Instantons:

- disordered medium scenario of QCD: localised modes from mixing of localised zero modes sitting on calorons (see [García-García, Osborn (2007)])
- localised near-zero topological modes [Dick *et al.* (2015)]
- low modes like (anti)selfdual sites and Polyakov loop phases $\phi \sim (\pi, \pi, 0)$: L-type monopoles? [Cossu, Hashimoto (2016)]

see also talk by S. Sharma





Figures from [Cossu, Hashimoto (2016)]

Topology and localisation

. . . but:

- not all localised modes are topological – at most 50% in pure gauge SU(3) [Kovács, Vig (2018)]

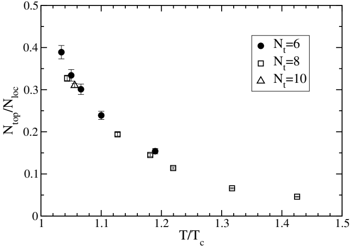


Figure from [Kovács, Vig (2018)]

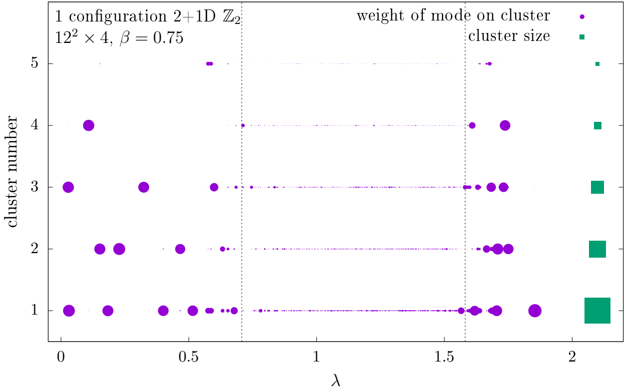
- topology and P-loop fluctuations anticorrelated [Larsen *et al.* (2022)], not really alternative to sea/islands picture
- localisation also in models without instantons

Near-zero modes: see talks by T.G. Kovács, A. Alexandru, and I. Horváth

Topology and localisation

Thermal monopoles: ???

Centre vortices: strong correlation with localised modes. . . in 2+1D $\mathbb{Z}_2$ pure gauge theory – not much else in the theory [work in progress]



Localisation and Goldstone modes

At $m = 0$ chiral symmetry broken if density of near-zero modes is finite...

$$\Sigma = -\langle \bar{\psi}\psi \rangle = \pi\rho(0^+) \neq 0 \Longrightarrow \text{massless pions}$$

...but if near-zero modes are localised up to $\lambda_c(m)$ [MG (2020)]

$$\boxed{\text{massless pions} \quad \Longleftrightarrow \quad \pi\rho(0^+)\,(1-\xi) \neq 0\,, \quad \xi = \lim_{m\to 0} \tfrac{2}{\pi} \arctan \tfrac{\lambda_c}{m}}$$

localisation alternative to Goldstone's theorem
[McKane, Stone (1981), Golterman, Shamir (2003)]
see also [Greensite *et al.* (2005,2006)]

Coordinate-space Ward identity:

$$\partial_\mu \langle \mathcal{A}^a_\mu(x) \mathcal{P}^b(0) \rangle = \delta^{(4)}(x)\delta^{ab}\Sigma + 2m\langle \mathcal{P}^a(x)\mathcal{P}^b(0)\rangle$$

$\mathcal{A}^a_\mu = \bar{\psi}\gamma_5\gamma_\mu t^a \psi$: non-singlet axial-vector currents
$\mathcal{P}^a = \bar{\psi}\gamma_5 t^a \psi$: non-singlet pseudoscalar densities

Localisation and Goldstone modes

Explicit breaking:

$$2m\langle\mathcal{P}^a(x)\mathcal{P}^a(0)\rangle \sim \int d\lambda\, \frac{2m}{\lambda^2+m^2}\Big\langle \sum_n \delta(\lambda-\lambda_n)\mathcal{O}_n(x)\mathcal{O}_n(0)\Big\rangle$$

$$\mathcal{O}_n(x) = (\psi_n(x), \gamma^5\psi_n(x))$$

$$\Big|\Big\langle \sum_n \delta(\lambda-\lambda_n)\mathcal{O}_n(x)\mathcal{O}_n(0)\Big\rangle\Big| \le \frac{T}{V}\Big\langle \sum_n \delta(\lambda-\lambda_n)\mathrm{IPR}_n\Big\rangle \sim \rho(\lambda)V^{-\alpha(\lambda)}$$

Nonvanishing if $\alpha = 0$, $\frac{2m}{\lambda^2+m^2} \to \pi\delta(\lambda) \Longrightarrow$ "anomalous remnant" in the chiral limit if nonzero density of localised near-zero modes is present

Momentum-space Ward identity:

$$-ik_\mu\mathcal{G}_{AP}(k) = \Sigma + 2m\mathcal{G}_{PP}(k) \underset{\substack{m\to 0\\ k\to 0}}{=} \pi\rho(0^+) - \pi\rho_{\mathrm{loc}}(0^+)\xi$$

Massless pole disappears if $\xi = 1$ — e.g., $\lambda_c(m{=}0)$ finite
$\Longrightarrow$ pions "melt" without chiral restoration?

Summary and outlook

Low Dirac modes in gauge theories localised in high-T deconfined/chirally restored phase, delocalised in low-T confined/chirally broken phase

- Strong connection with deconfinement: localised low modes appear at the transition when genuine
- Polyakov loop ordering & fluctuations responsible for localisation (possibly connected to topology)
- In the chiral limit it can affect (even kill) Goldstone excitations

Open issues:

- Physical meaning of localisation still unclear
- Something is still missing in connecting localisation and deconfinement
  - systems without centre symmetry?
  - systems with separate deconfining and chiral transitions?
- Is low-mode localisation how deconfinement improves chiral symmetry properties?

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