Topological Confinement in Skyrme Holography

ECT Workshop “Gauge Topology, Flux Tubes and Holographic Models: The Intricate Dynamics of QCD in Vacuum and Extreme Environments”

May 26th, 2022

[Cartwright,Harms,Kaminski,Thomale; Class.Quant.Grav (2022]
[Cartwright,Harms,Kaminski; JHEP (2021)]

[Photo credit: Marco Kröner]

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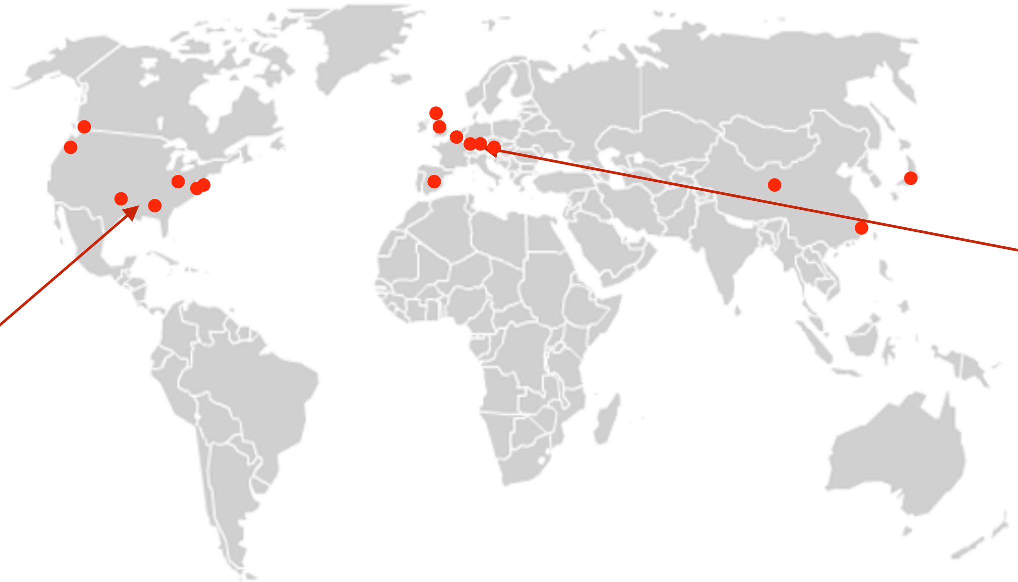

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Understand Effect of Gauge Topology on Deconfinement in Toy Model

Outline

1. **Holographic Deconfinement**
2. Einstein-Skyrme Model
3. Topological Confinement
4. Discussion

similar in spirit to talk by Maximilian Attems

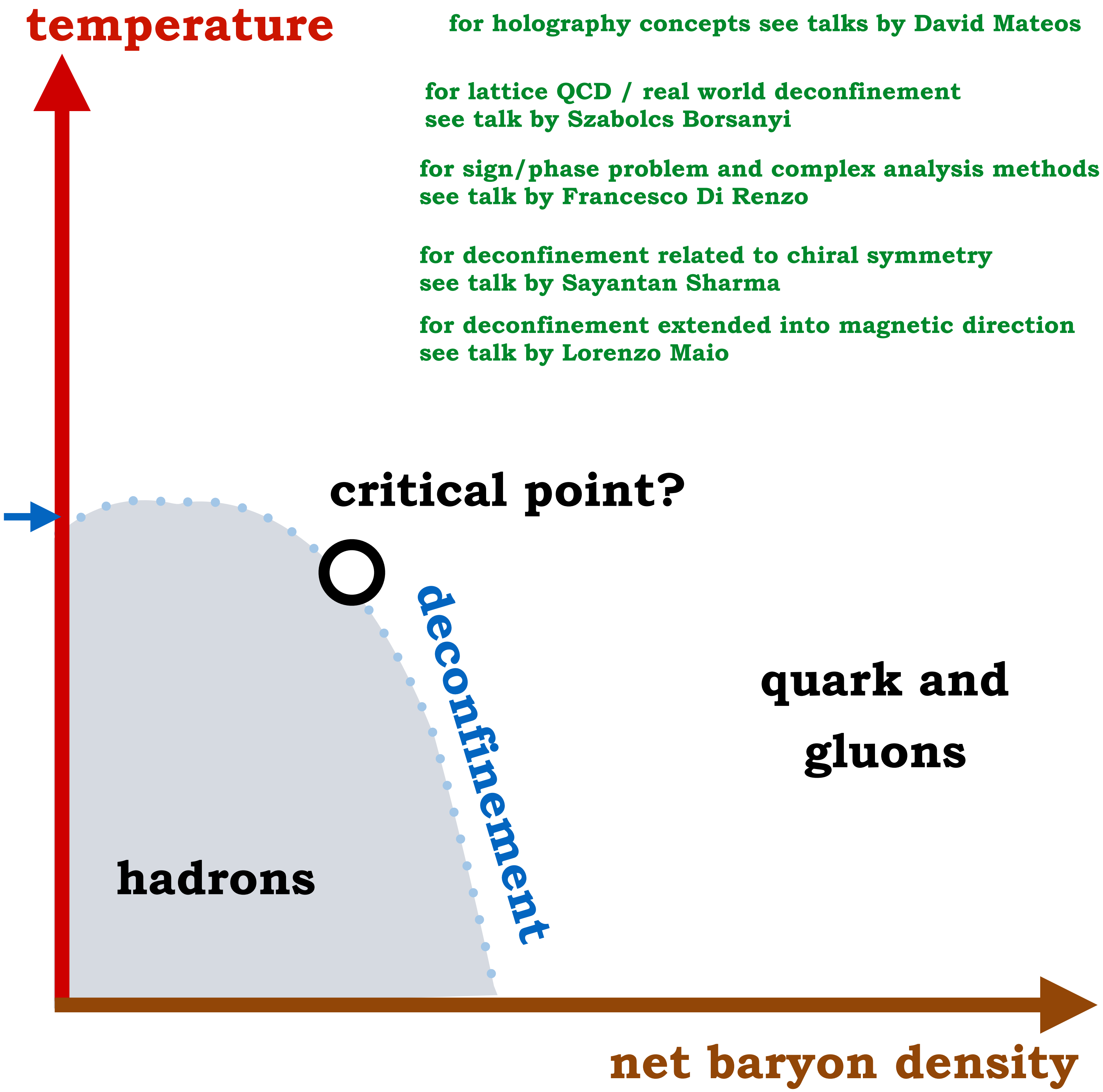
for holography concepts see talks by David Mateos

for lattice QCD / real world deconfinement
see talk by Szabolcs Borsanyi

for sign/phase problem and complex analysis methods
see talk by Francesco Di Renzo

for deconfinement related to chiral symmetry
see talk by Sayantan Sharma

for deconfinement extended into magnetic direction
see talk by Lorenzo Maio



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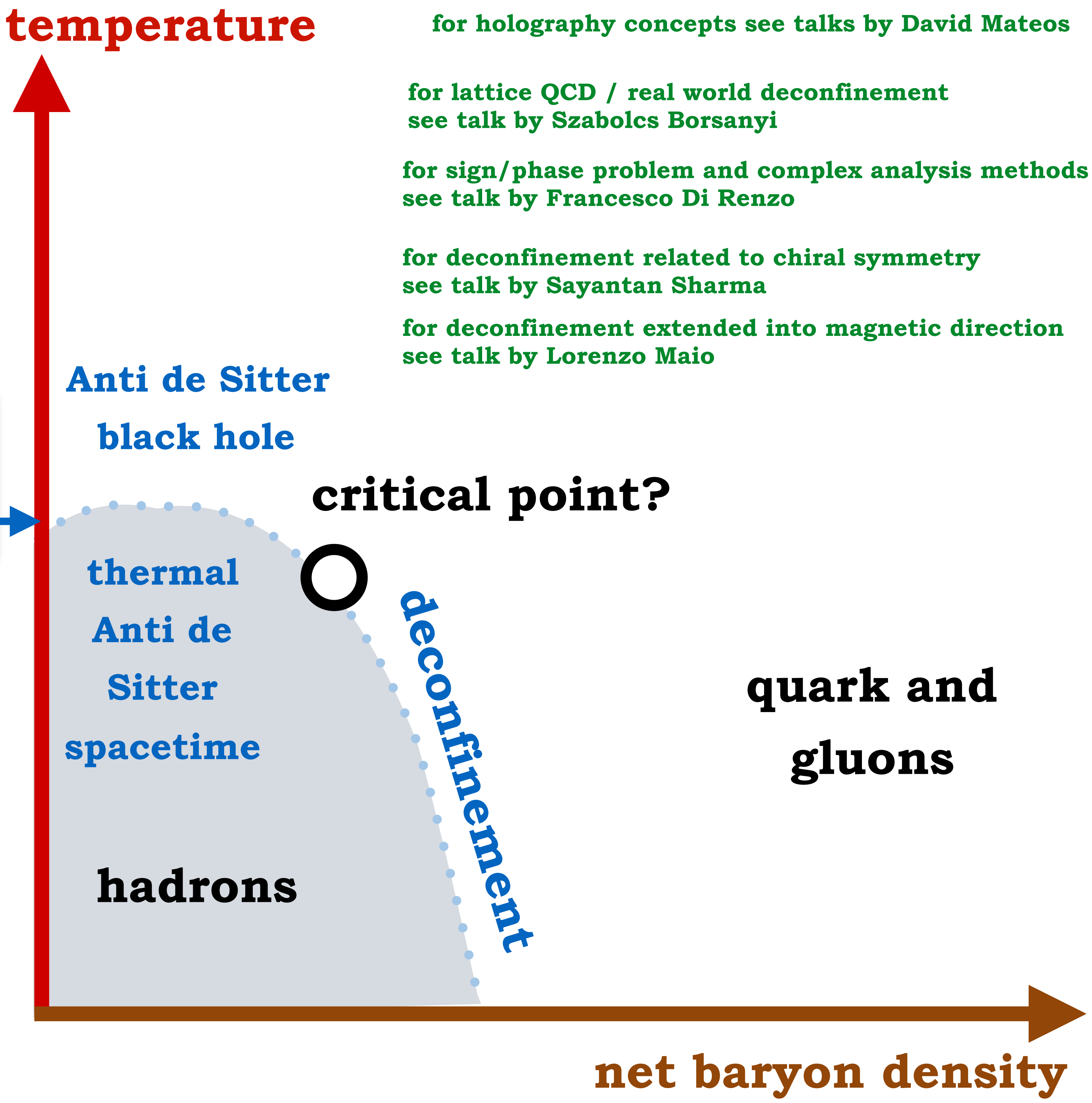

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Summary

- add Skyrme matter
- Skyrme matter = gauge fields
- novel black hole solutions with non-trivial gauge field topology
- gauge field topology affects Hawking-Page transition

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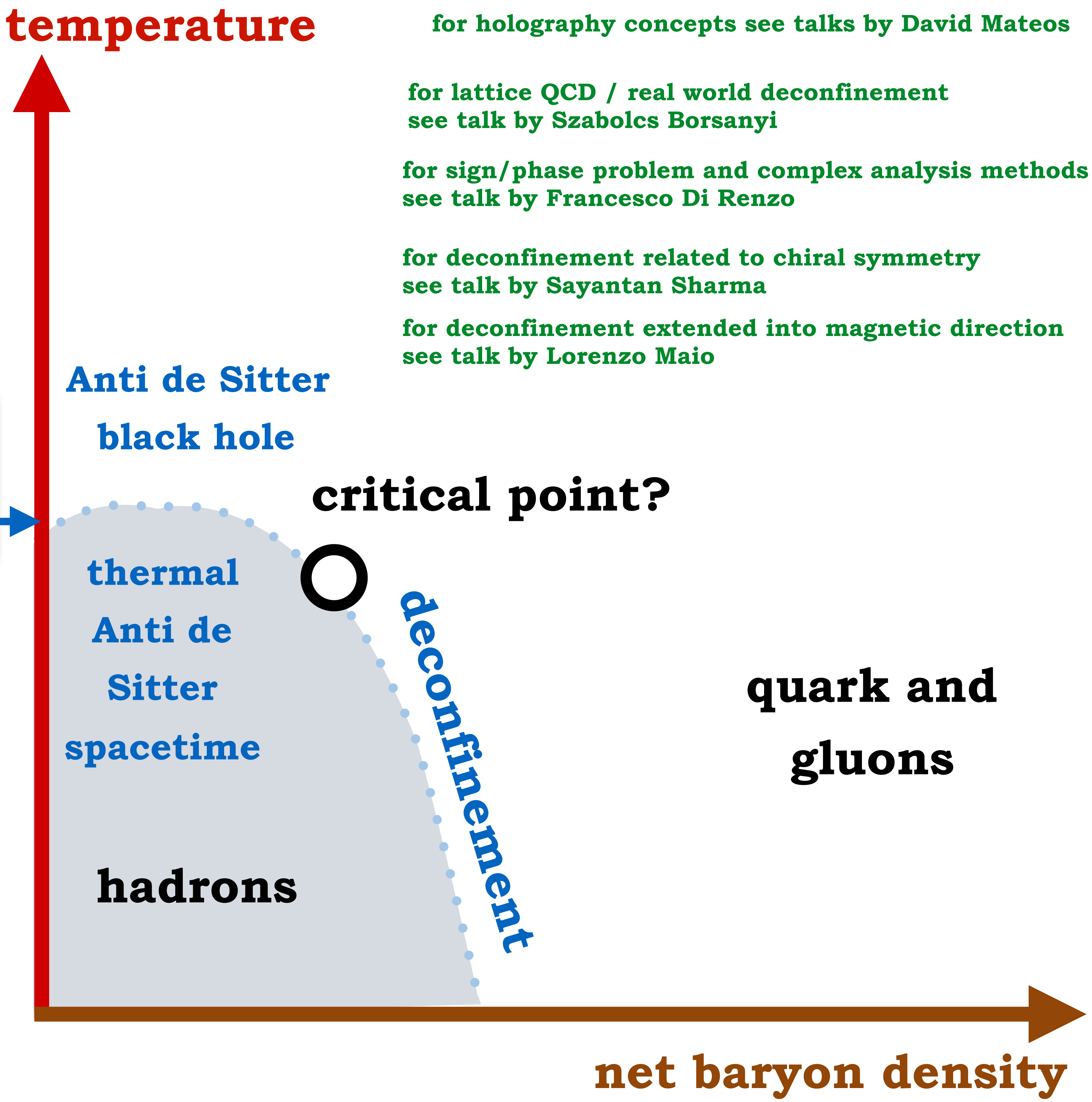

➡holographic deconfinement is affected by gauge field topology; analytic!

1. Holographic Deconfinement: Hawking-Page (HP)

[Hawking,Page; (1983)]

Hawking-Page (HP) phase transition (boundary topology $S^3 \times S^1$)

- between thermal AdS$_5$

$$ds^2_{thermal} = \left(1 + r^2/L^2\right) d\tau^2 + \frac{dr^2}{1 + r^2/L^2} + r^2 d\Omega_3^2$$

with temperature given by $\tau = \tau + 1/T_{thermal}$,

Action: Einstein-Hilbert

$$S_{EH} = \frac{1}{\kappa} \int d^5x \sqrt{g}\,(R - 2\Lambda)$$

- and AdS$_5$ black hole

$$ds^2_{black\,hole} = \underbrace{\left(1 + r^2/L^2 - m/r^2\right)}_{f(r)} d\tau^2 + \frac{dr^2}{1 + r^2/L^2 - m/r^2} + r^2 d\Omega_3^2$$

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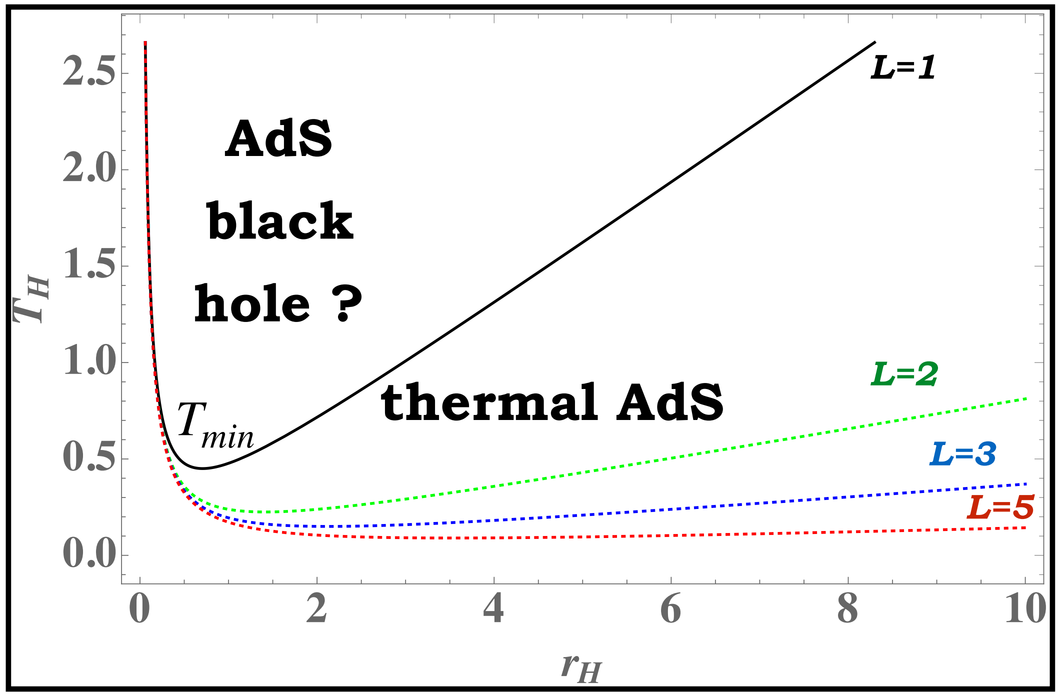
near the horizon

$$ds^2_{black\, hole} \approx f'(r_h)(r - r_h) d\tau^2 + \frac{dr^2}{f'(r_h)(r - r_h)} + r_h^2 d\Omega_3^2$$

with Hawking temperature

$$T_H = \frac{|f'(r_h)|}{4\pi} = \frac{4r_h^2 + 2L^2}{4\pi L^2 r_h} \quad \text{with minimum value } T_{min} = \frac{\sqrt{2}}{\pi L} \text{ at } r_h = \frac{L}{\sqrt{2}}$$



L is “size” of AdS

- comparison of on-shell actions evaluated on each solution

$$S_{thermal} - S_{black\, hole} \propto L^2 - r_h^2 = 0$$

yields transition temperature:

$$\boxed{T_{HP} = \frac{3}{2\pi L}}$$

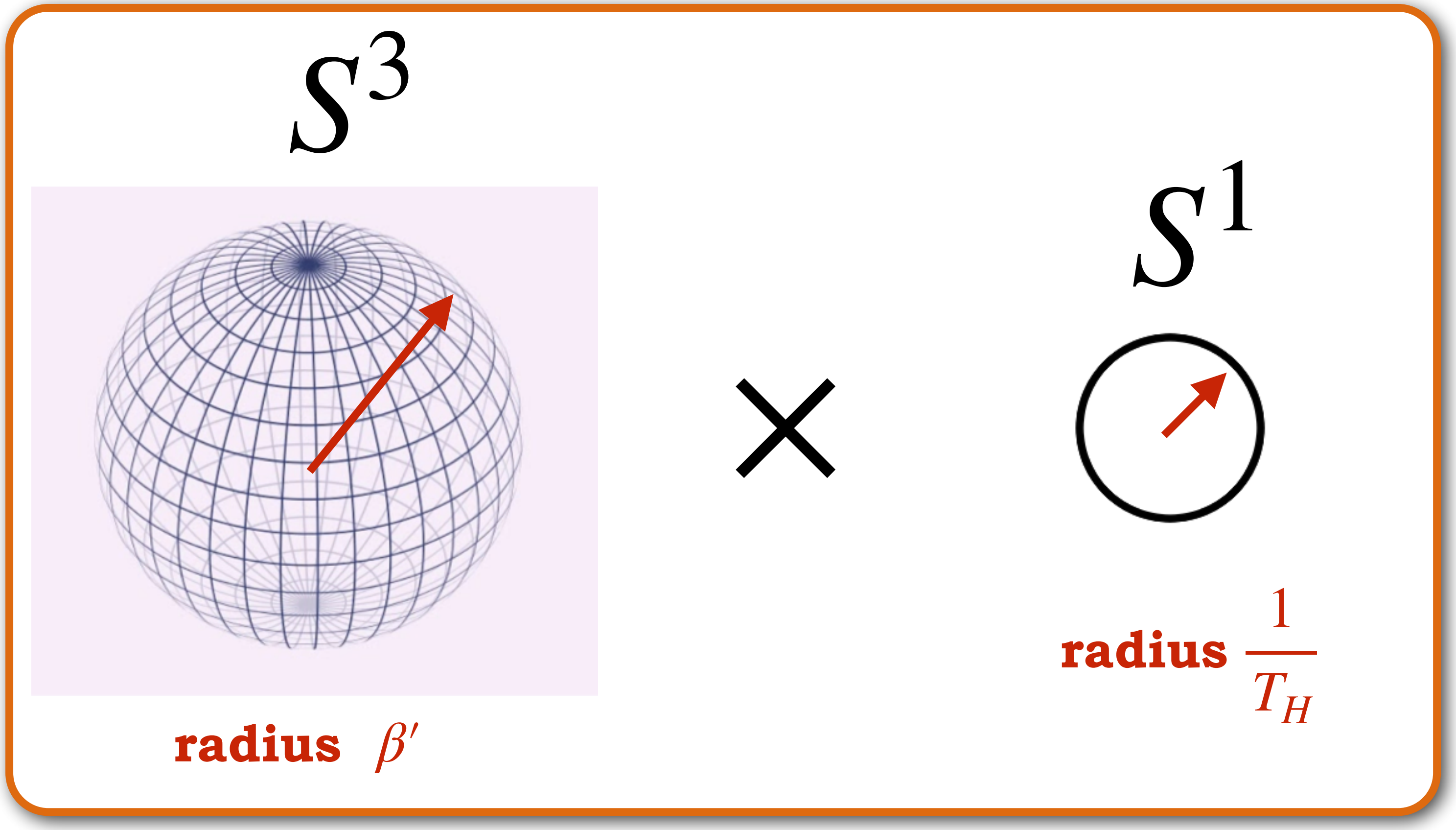
1. Holographic Deconfinement: Witten's Interpretation

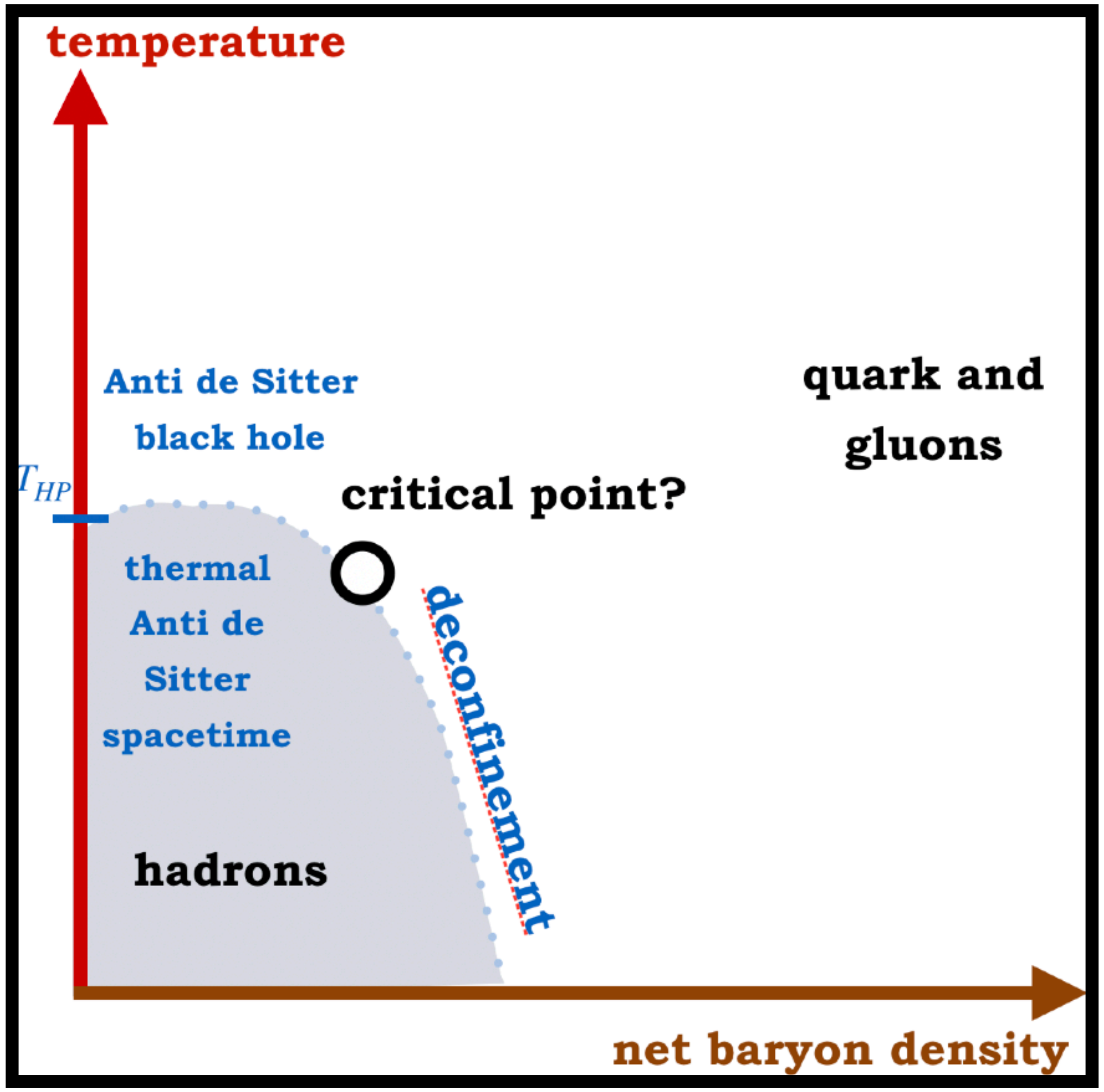
- **strongly coupled** thermal **Conformal** Field Theory (CFT) at **large number of colors** N_c *have* ***phase transition*** ***dual to HP transition***

[Witten; (1998)]

- these are ***deconfinement*** transitions (Wilson loops calculated) *[Witten; (1998)]*

- specific example: N=4 Super-Yang-Mills (SYM) is holographically dual to Einstein-Hilbert action with negative cosmological constant (AdS5) *[Maldacena; (1997)]*





holography concepts mentioned
see talks by David Mateos, Matti Jarvinen, Maximilian Attems

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2. **Einstein-Skyrme Model**
3. Topological Confinement
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2. Einstein-Skyrme Model

Action:

$$S = \int d^5x\sqrt{-g}\left(\frac{R-2\Lambda}{\kappa}+\mathcal{L}_m\right)$$

Cosmological constant: $\Lambda = -6/L^2$

$\kappa = 16\,\pi\,G$

gravitational coupling

Skyrme matter Lagrangian:

$$\boxed{\mathcal{L}_m = \frac{f_\pi}{16\pi}K_\mu K^\mu + \frac{1}{32\,\tilde{e}^2}\mathrm{Tr}\left([K_\mu, K_\nu]\,[K^\mu, K^\nu]\right)}$$

kinetic Skyrme coupling ***Skyrme coupling***

SU(2)-valued Lorentz-4-vector:
(**pure gauge** field)

$$K_\mu = U^{-1}\partial_\mu U \quad \in SU(2)$$

Skyrme field U

Dimensionless Skyrme coupling: $e = \dfrac{\tilde{e}}{8\pi G}$

2. Einstein-Skyrme: Equivalence to Merons

Einstein-Skyrme theory has the same action (and equations of motion) as a particular Einstein-Yang-Mills theory.

Meron action:

$$S_{\text{meron}} = \int d^5x\sqrt{-g}\left(\frac{1}{16\pi G}(R-2\Lambda) + \underbrace{\frac{1}{16\pi\gamma^2}}_{\textit{Yang-Mills coupling}}\text{Tr}[F_{\mu\nu}F^{\mu\nu} - \underbrace{2m^2}_{\textit{Proca mass}} A^\mu A_\nu]\right)$$

Identifications:

$$A_\mu = \lambda K_\mu\,, \quad F_{\mu\nu} = \lambda(\lambda-1)[K_\mu, K_\nu]\,, \quad \frac{m^2\lambda^2}{\pi\gamma^2} = \frac{f_\pi^2}{2}\,, \quad \frac{\lambda^2(\lambda-1)^2}{\pi\gamma^2} = \frac{1}{2\tilde{e}^2}$$

and choose $|\lambda| = \frac{1}{2}$ (merons are **half-pure-gauge**) *[Manton,Sutcliffe (2004)]*

lead to Skyrme action, recall: $\mathcal{L}_m = \frac{f_\pi}{16\pi}K_\mu K^\mu + \frac{1}{32\,\tilde{e}^2}\text{Tr}\left([K_\mu, K_\nu]\,[K^\mu, K^\nu]\right)$

➡ allows interpretation as consistent truncation of type IIB Supergravity on asymptotically AdS5 dual to *N*=4 SYM on $\mathbb{R}^1 \times S^3$ with an SU(2) subgroup of the R-symmetry

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3. Novel Topological Skyrme AdS5 Black Hole

[Cartwright,Harms,Kaminski; JHEP (2021)]

Coordinates: $\psi \in (0,\pi),\ \theta \in (0,\pi),\ \phi \in (0,2\pi)$

Pauli matrices: $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Gauge configuration:

$$U = e^{in\chi v^i \tau^i}$$

$$v = (\cos(\theta), \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi))$$

Metric:

$$ds^2 = \frac{1}{A(r)}\mathrm{d}r^2 - A(r)\mathrm{d}t^2 + r^2\left(h_1(\psi)\mathrm{d}\psi^2 + h_2(\psi)(\mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\phi^2)\right)$$

$$K_\mu = U^{-1}\partial_\mu U$$

with $h_1(\psi) = n^2, \quad h_2(\psi) = \sin(n\psi), \quad A(r) = \underbrace{\frac{1}{e^2r^2} + \frac{\log(r)}{e^2r^2}}_{\text{Skyrmion contributions}} + \frac{r^2}{L^2} - \underbrace{\frac{m_t}{r^2}}_{\text{mass parameter}} + 1\,.$

3. Interpretation within Dual Field Theory

[Cartwright,Harms,Kaminski; JHEP (2021)]

SU(2)-current vanishes in field theory:

$$\langle J_i \rangle = \lim_{r\to\infty} r^2 A_i = \frac{-1}{2} A_i^{(2),a} \tau_a \, , \quad = 0$$

$$A_i^{ext} = \lim_{r\to\infty} A_i = \frac{-1}{2} A_i^{(0),a} \tau_a \, .$$

Two trace contributions:

$$\bar{\kappa} \, \langle T^i_{\ i} \rangle = \frac{3L^2}{4e^2} - \frac{3L^4}{4}$$

external field (explicit breaking)

conformal anomaly of N=4 SYM on $\mathbb{R} \times S^3$

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Explicit field configuration of SU(2)-gauge field:

[Manton,Sutcliffe (2004)]

$$A_\psi^{(0),1} = -n\cos(\theta)\,,$$
$$A_\psi^{(0),2} = -n\sin(\theta)\cos(\phi)\,,$$
$$A_\psi^{(0),3} = -n\sin(\theta)\sin(\phi)\,.$$

$$A_\phi^{(0),1} = \sin^2(\theta)\sin^2(n\psi)\,,$$
$$A_\phi^{(0),2} = \sin(\theta)\sin(n\psi)\left(\sin(\phi)\cos(n\psi) - \cos(\theta)\cos(\phi)\sin(n\psi)\right)\,,$$
$$A_\phi^{(0),3} = -\sin(\theta)\sin(n\psi)\left(\cos(\theta)\sin(\phi)\sin(n\psi) + \cos(\phi)\cos(n\psi)\right)$$

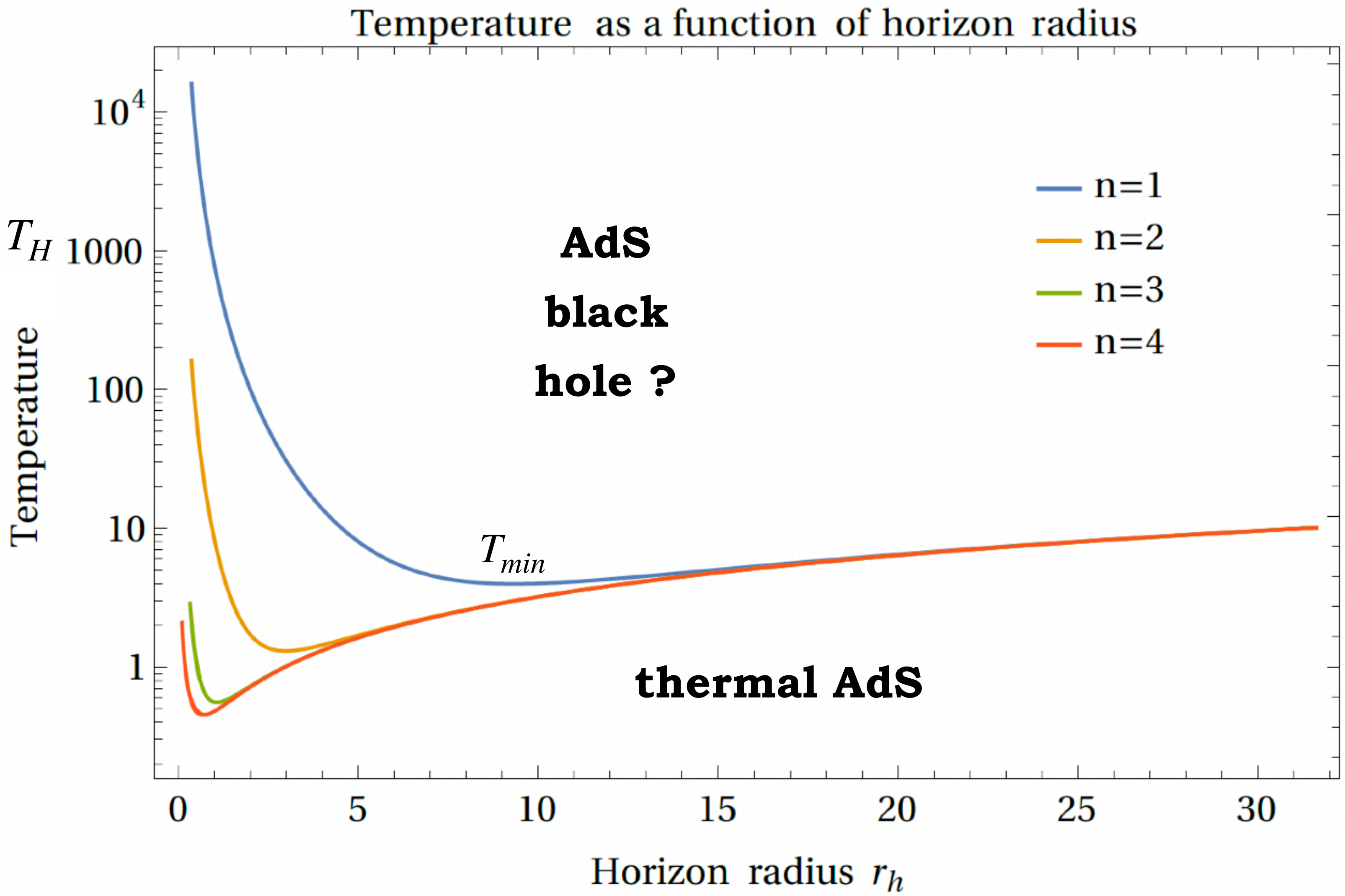
$$A_\theta^{(0),1} = \sin(n\psi)\sin(\theta)\cos(n\psi)\,,$$
$$A_\theta^{(0),2} = \sin(n\psi)\left(\cos(\theta)\cos(\phi)\cos(n\psi) + \sin(\phi)\sin(n\psi)\right)\,,$$
$$A_\theta^{(0),3} = \sin(n\psi)\left(\cos(\phi)\sin(n\psi) - \cos(\theta)\sin(\phi)\cos(n\psi)\right)\,,$$

Gauge fields map $A_j^a \tau_a(x) : S^3 \to S^3$

➡gauge field configuration has non-trivial topology with Chern number *n, counting how often coordinate sphere wraps SU(2) sphere*

3. Thermodynamics of Topological Skyrme AdS5 Black Hole

[Cartwright,Harms,Kaminski,Thomale; Class.Quant.Grav (2022)]
[Cartwright,Harms,Kaminski; JHEP (2021)]



Hawking temperature "quantized":

$$T_H = \frac{|A'(r_h)|}{4\pi}$$

Entropy is "quantized":

$$S = \frac{2\pi^2 n}{4\,G_5} r_h^3\,, \quad n \in \mathbb{N}$$

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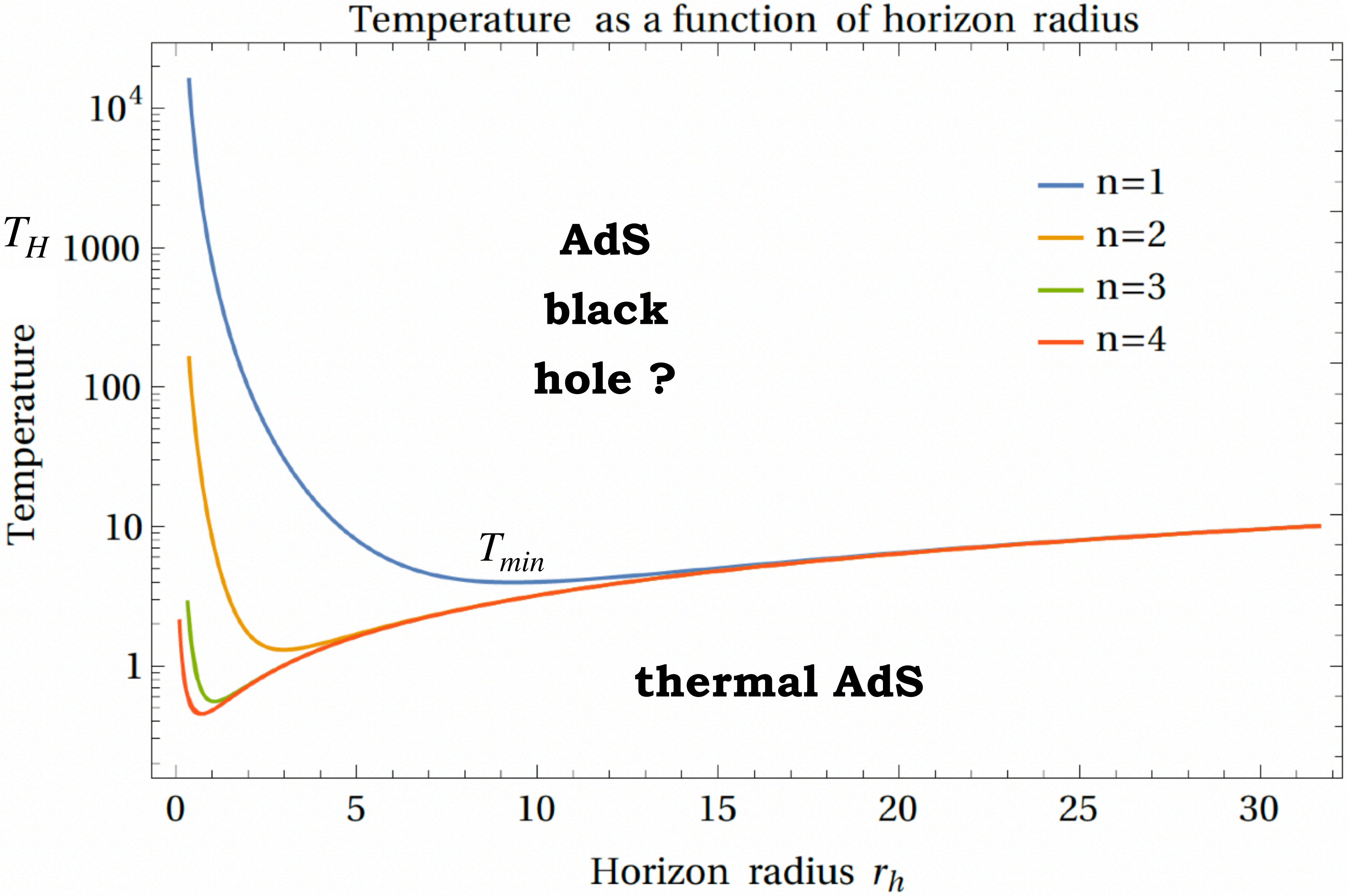

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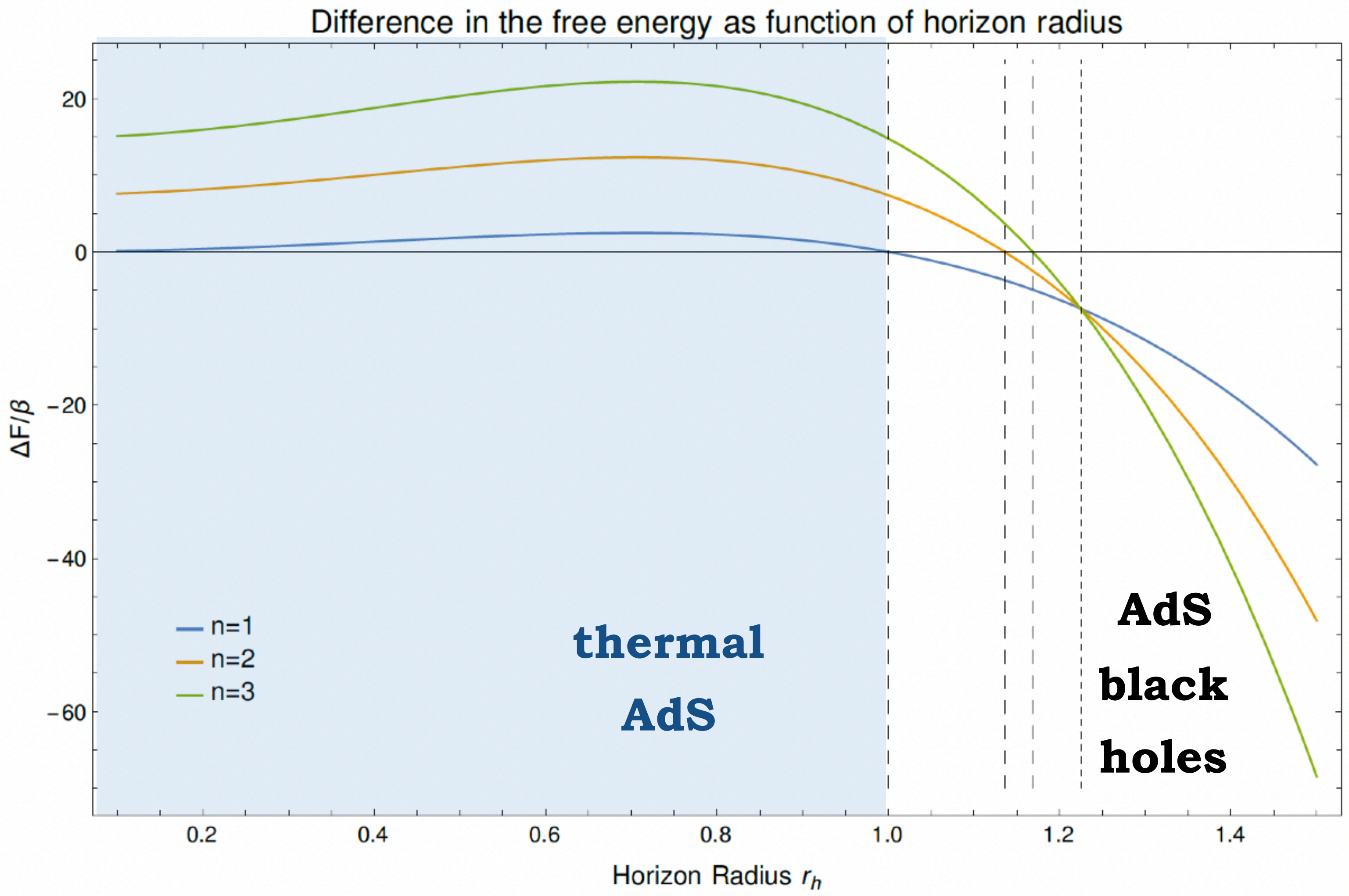
Free energy proportional to renormalized on-shell action evaluated on the solution: $F = T\,S_{ren}$

$$S_{ren} = \lim_{\epsilon\to 0}\left(\int \mathrm{d}^4x \left(\int_{r_h}^{1/\epsilon} \mathrm{d}r\sqrt{-g}\left(\frac{(R-2\Lambda)}{16\pi G_5} + \frac{\mathrm{Tr}(F^2)}{16\pi\gamma^2}\right)\right) + S_{ct}\right)$$

$$S_{ct} = \frac{1}{8\pi G_5}\int \mathrm{d}^4x\sqrt{\gamma}\left(K - \frac{1}{2L}\left(2(1-d) - \frac{L^2}{d-2}R(\gamma)\right)\right) + \frac{L}{16\pi\gamma^2}\log(\epsilon)\int \mathrm{d}^4x\sqrt{\gamma_0}\,\mathrm{Tr}(F_0^2)$$

3. Topology Effects on Hawking-Page Transition

[Cartwright,Harms,Kaminski; JHEP (2021)]



$$\boxed{e = 10^6}$$

Small Skyrmion effects:

- larger transition temperature for larger n
- several n at same horizon radius

Free energy difference:

$$\Delta F = F_{\text{BH}} - F_{\text{thermal}} = \frac{\pi^2 \beta n r_h^2}{\kappa}\left(-\frac{\tilde{e}^2 L^4 \left(\frac{3}{n} - 3\right) + 4\tilde{e}^2 r_h^4 + 4L^2}{4\tilde{e}^2 L^2 r_h^2} + \frac{3\kappa \log(r_h)}{\tilde{e}^2 r_h^2} + 1\right)$$

Original Hawking-Page transition: $\lim_{e\to\infty} \frac{\kappa}{\pi^2 \beta r_h^2} \Delta F = \left(1 - \frac{r_h^2}{L^2}\right)$

➡ if $\Delta F < 0$, then black hole phase is preferred

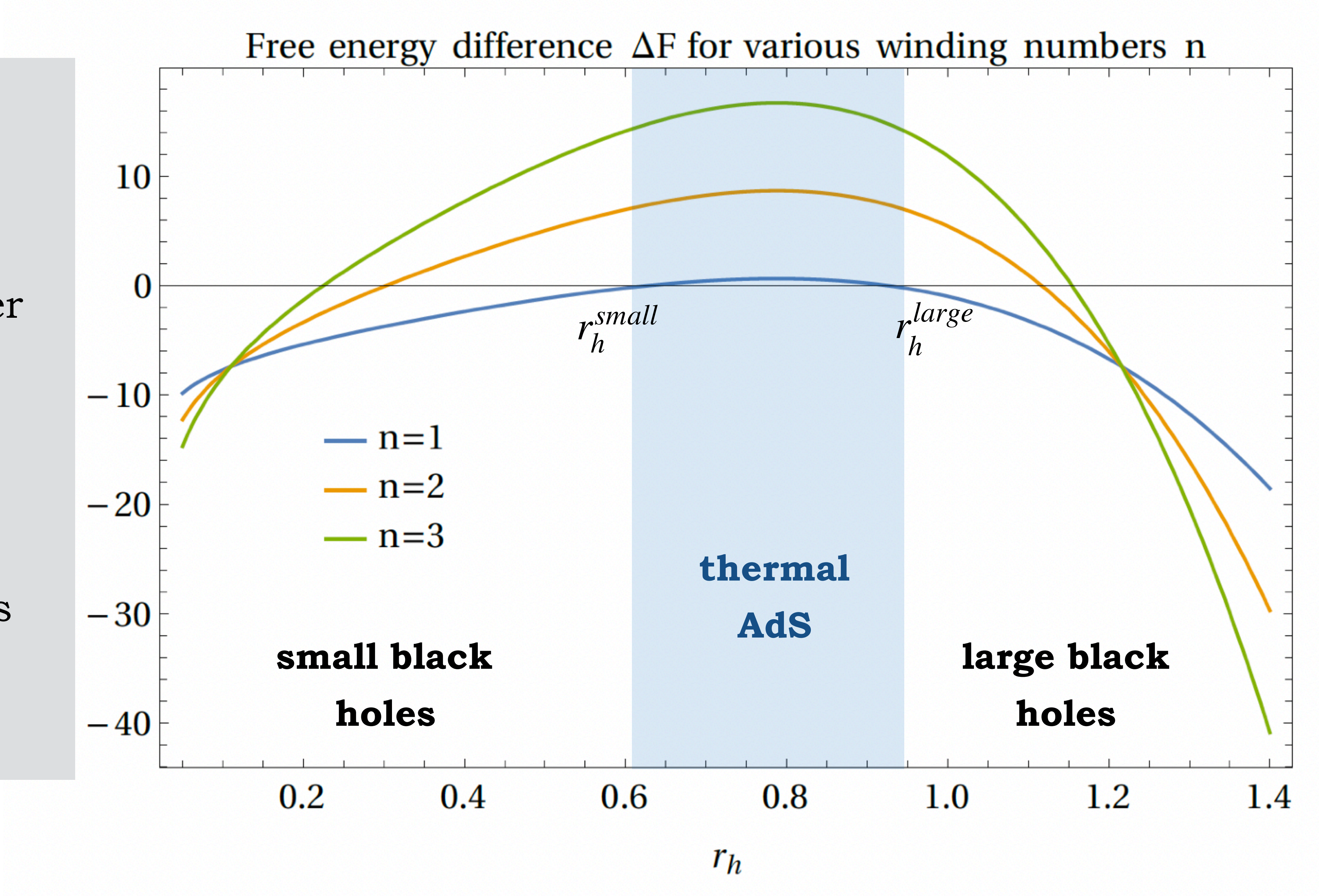
3. Topology Effects on Hawking-Page Transition

[Cartwright,Harms,Kaminski; JHEP (2021)]
[Cartwright,Harms,Kaminski,Thomale; Class.Quant.Grav (2022)]

$$\boxed{e = \sqrt{10}}$$

Large Skyrmion effects:

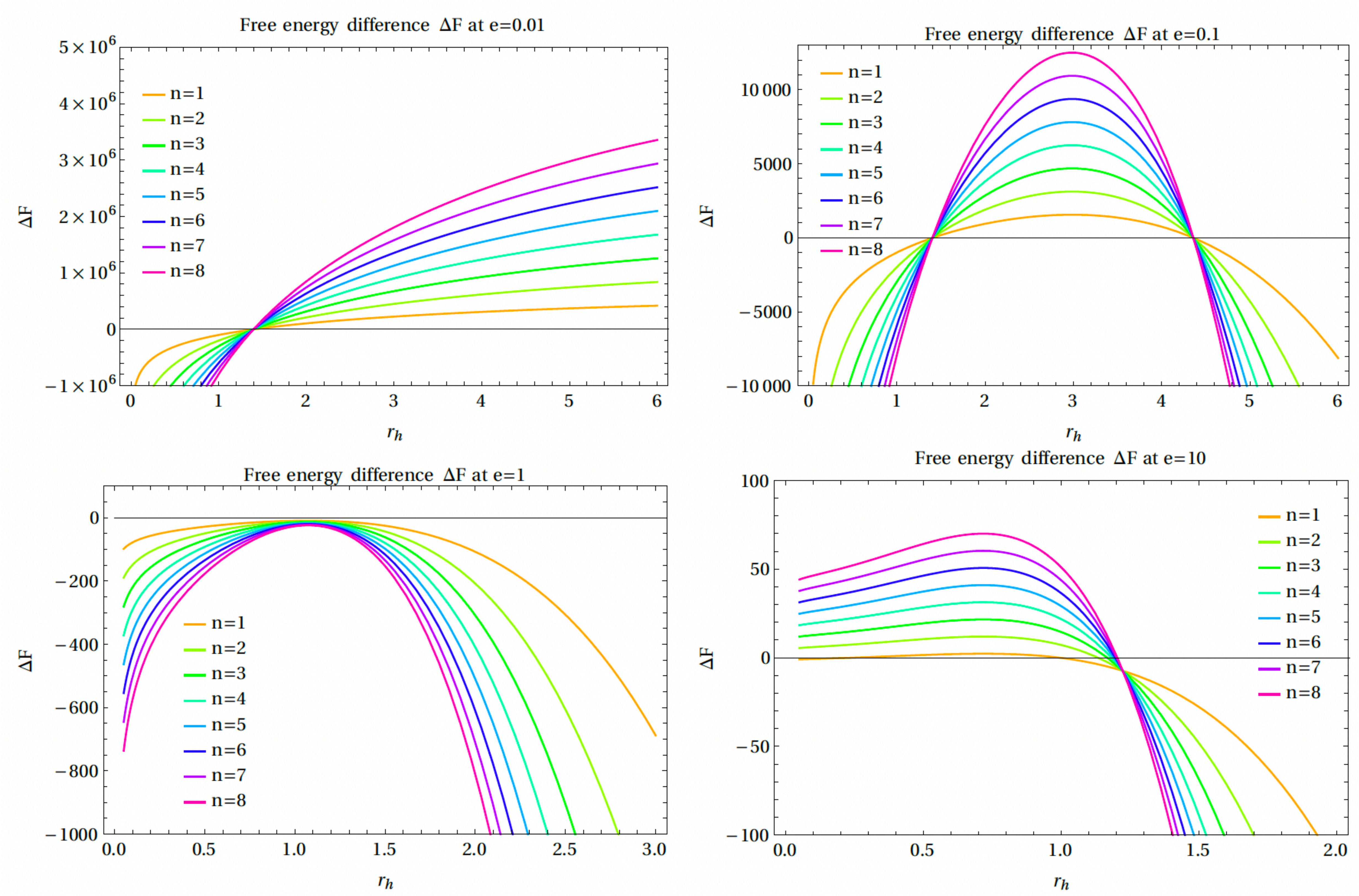
- two transitions for larger or small black holes (r_h^{large} or r_h^{small})
- transition radii pushed towards extreme values for larger n



3. Topology Effects on Hawking-Page Transition

[Cartwright,Harms,Kaminski; JHEP (2021)]
[Cartwright,Harms,Kaminski,Thomale; Class.Quant.Grav (2022)]

There are many effects:



Free energy difference ΔF at e=0.01

Free energy difference ΔF at e=0.1

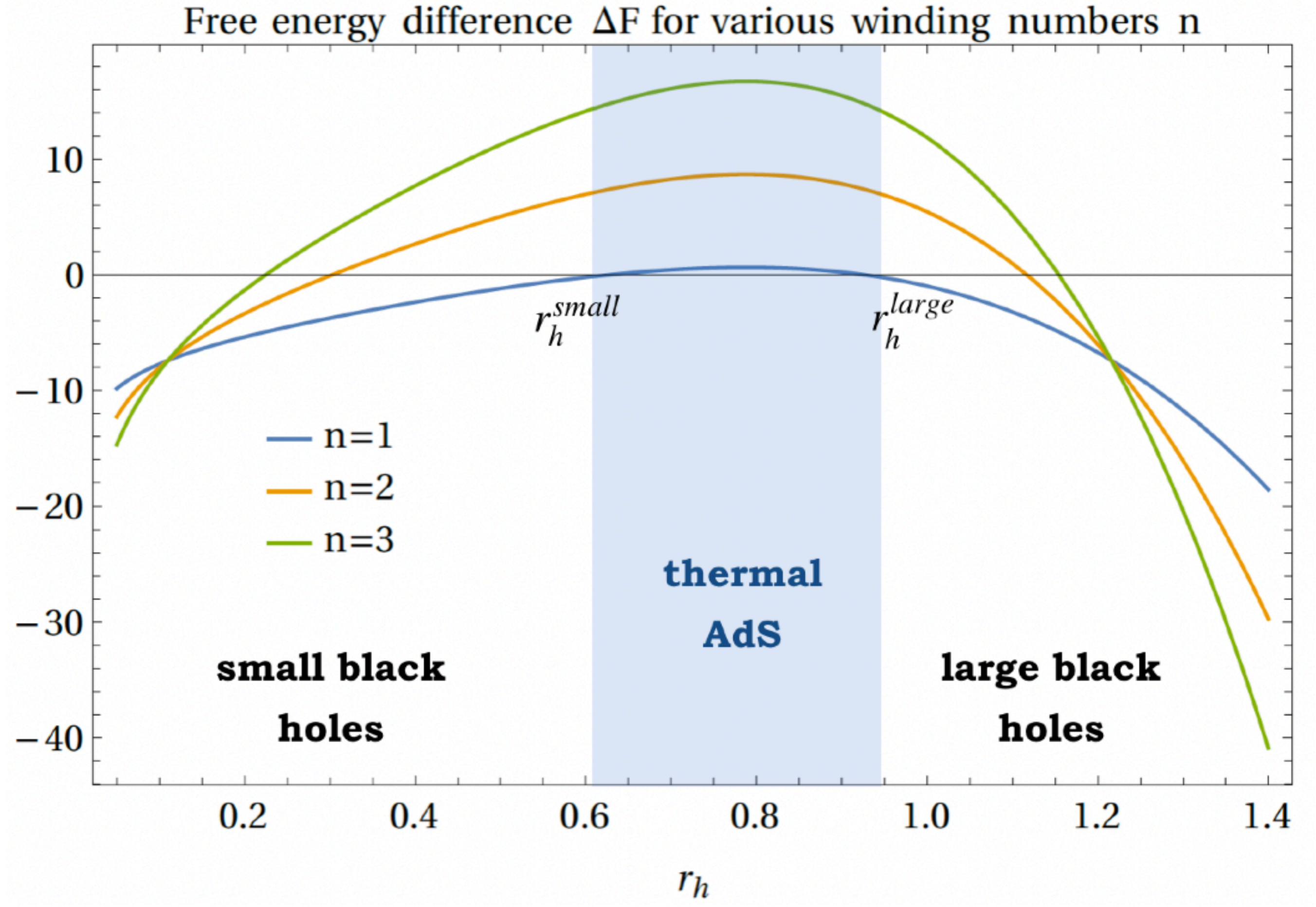
Free energy difference ΔF at e=1

Free energy difference ΔF at e=10

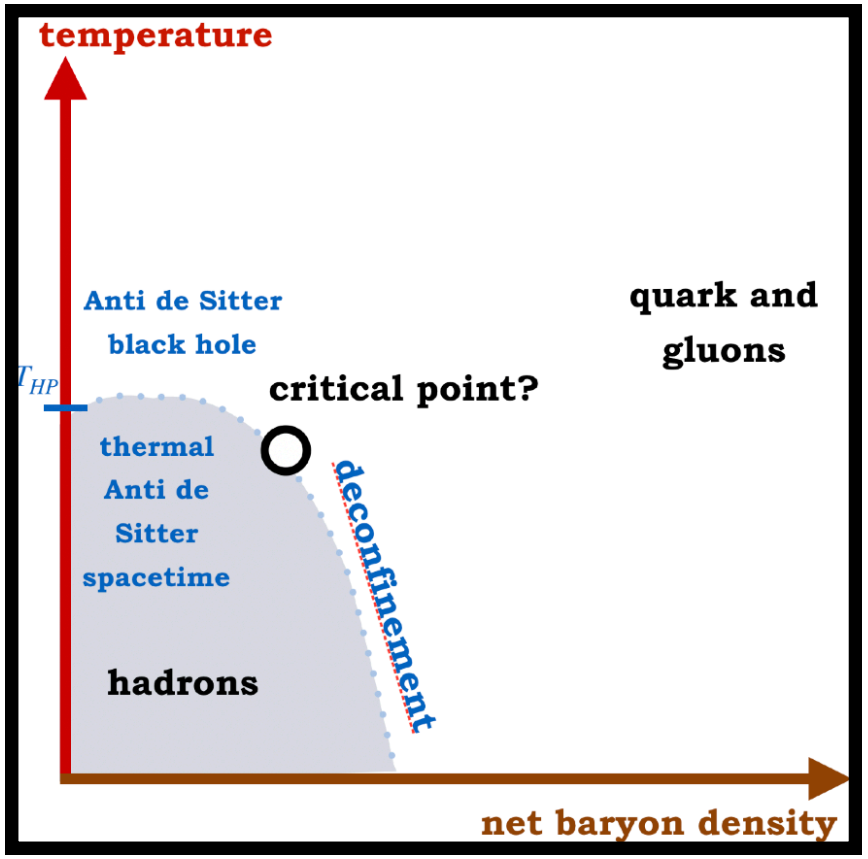
Discussion

Summary

- added Skyrme matter to HP transition
- Skyrme matter = gauge fields
- novel black hole solutions with non-trivial gauge field topology



Deconfinement transition = **Hawking-Page (HP) transition**



Topological effects on deconfinement

➡**holographic deconfinement is affected qualitatively by gauge field topology**

➡**2*n* transition temperatures (shifted)**

➡**“quantized” entropy and energy**

➡**applications to condensed matter?**

Thank you!

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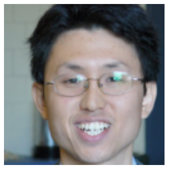

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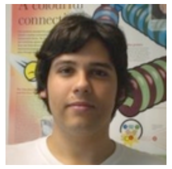

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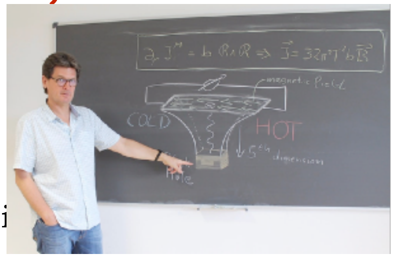

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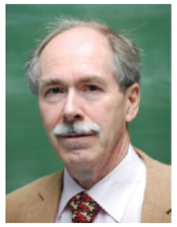

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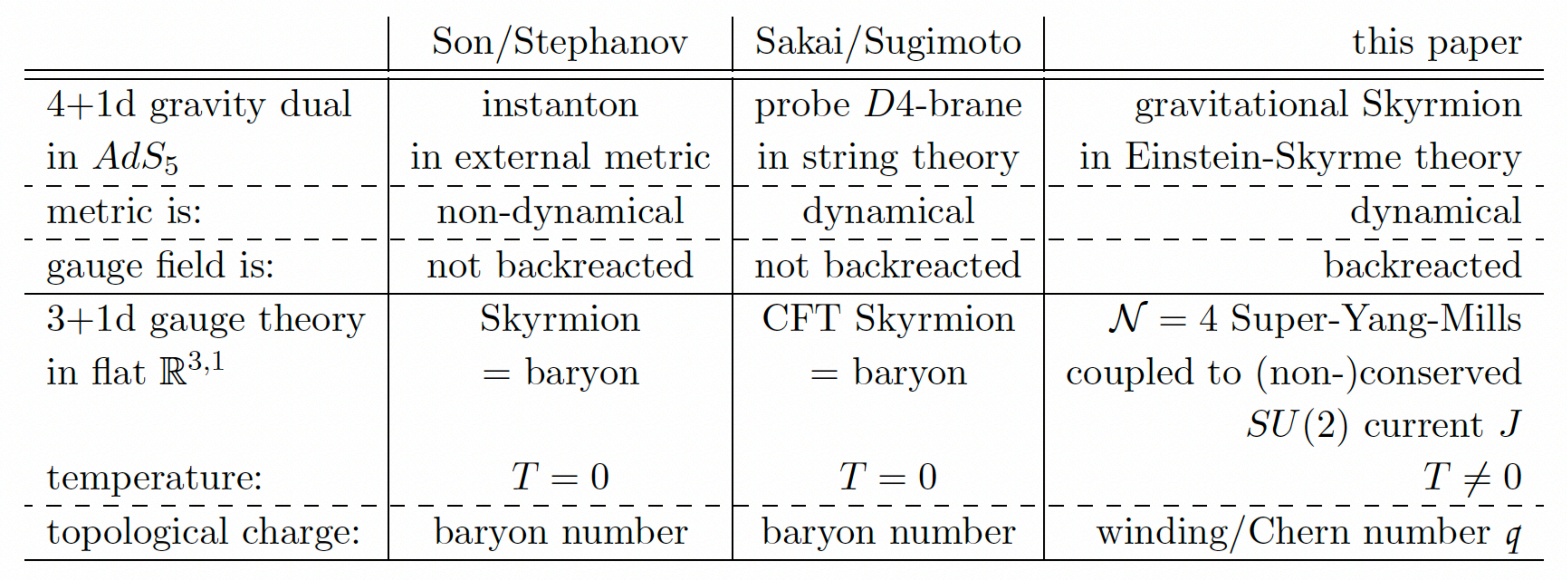
Gerard 't Hooft
(Nobel Laureate)

APPENDIX

3. Comparison to other Skyrmion dualities



	Son/Stephanov	Sakai/Sugimoto	this paper
4+1d gravity dual in AdS_5	instanton in external metric	probe $D4$-brane in string theory	gravitational Skyrmion in Einstein-Skyrme theory
metric is:	non-dynamical	dynamical	dynamical
gauge field is:	not backreacted	not backreacted	backreacted
3+1d gauge theory in flat $\mathbb{R}^{3,1}$	Skyrmion $=$ baryon	CFT Skyrmion $=$ baryon	$\mathcal{N}=4$ Super-Yang-Mills coupled to (non-)conserved $SU(2)$ current J
temperature:	$T=0$	$T=0$	$T\neq 0$
topological charge:	baryon number	baryon number	winding/Chern number q

for confinement in light-front holography
see talks by
Stanley Brodsky
and Yang Li

Details of Topological Skyrme AdS5 Black Hole

Energy-momentum tensor:

$$\bar{\kappa}\,\langle T^{00}\rangle = \frac{3}{16}L^4\left(\frac{4\log(\Lambda)}{e^2}+L^2+4\,m_t\right),$$

$$\bar{\kappa}\,\langle T^{\psi\psi}\rangle = \frac{L^2\left(e^2\left(4\,m_t-3L^2\right)+4\log(\Lambda)+4\right)}{16e^2n^2},$$

$$\bar{\kappa}\,\langle T^{\theta\theta}\rangle = \frac{L^2\csc^2(n\psi)\left(e^2\left(4\,m_t-3L^2\right)+4\log(\Lambda)+4\right)}{16e^2},$$

$$\bar{\kappa}\,\langle T^{\phi\phi}\rangle = \frac{L^2\csc^2(\theta)\csc^2(n\psi)\left(e^2\left(4\,m_t-3L^2\right)+4\log(\Lambda)+4\right)}{16e^2}.$$

Topological charge of gauge field configuration:

$$q=\frac{2}{3\pi^2}\int_0^{\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\psi\int_0^{2\pi}\mathrm{d}\phi\frac{3}{4}n\sin(\theta)\sin^2(n\psi)=n\in\mathbb{N}.$$

Second Chern number:

$$c_2 \;=\; \int C_2$$

$$C_2 \;=\; \tfrac{1}{8\pi^2}\left(\mathrm{tr}(F\wedge F)-\mathrm{tr}F\wedge\mathrm{tr}F\right)$$

$$q=\frac{{g_{\mathrm{YM}}}^2}{8\pi^2}\int Dd^4x=\frac{1}{16\pi^2}\oint_{S^3}\hat{n}_iJ^id^3x$$

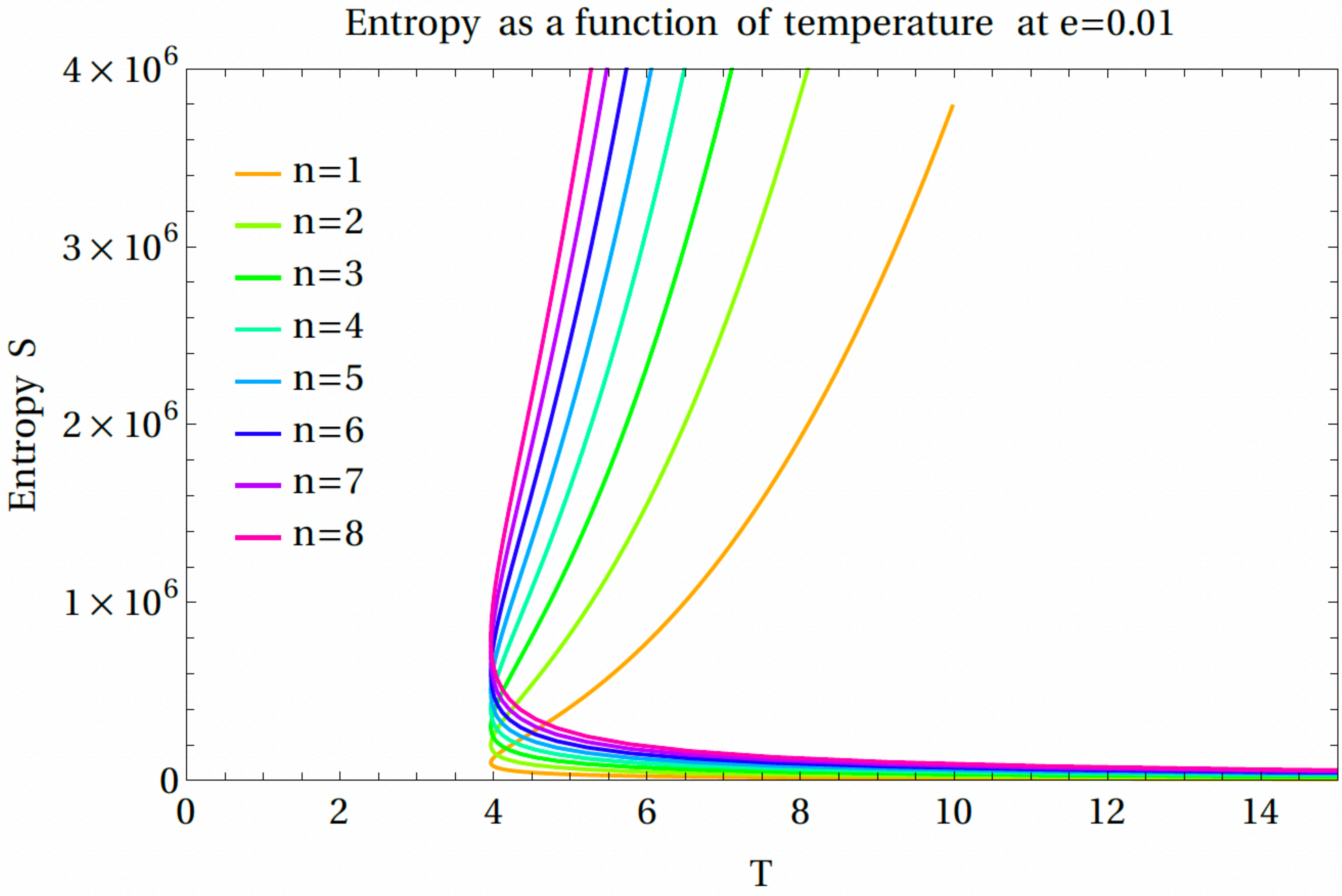
$$D=\frac{1}{4}F^a_{ij}\tilde{F}^a_{ij}=\frac{1}{2{g_{\mathrm{YM}}}^2}\partial_iJ^i$$

$$J_i={g_{\mathrm{YM}}}^2\epsilon_i{}^{jmn}\left[A^a_j\partial_mA^a_n+\frac{g_{\mathrm{YM}}}{3}\epsilon_{abc}A^a_jA^b_mA^c_n\right]$$

$$A_i\;\to\;A_i'\;=\omega A_i\omega^{-1}\;-\;(i/g_{\mathrm{YM}})(\partial_i\omega)\omega^{-1}$$

$$J_i\to J_i'=J_i+2ig_{\mathrm{YM}}\epsilon_i{}^{jmn}\mathrm{tr}\{\partial_j(\partial_m\omega A_n\omega^{-1})\}-\frac{2}{3}\epsilon_i{}^{jmn}\mathrm{tr}\{\partial_j\omega\omega^{-1}\partial_m\omega\omega^{-1}\partial_n\omega\omega^{-1}\}$$

Details of Topological Skyrme AdS5 Black Hole



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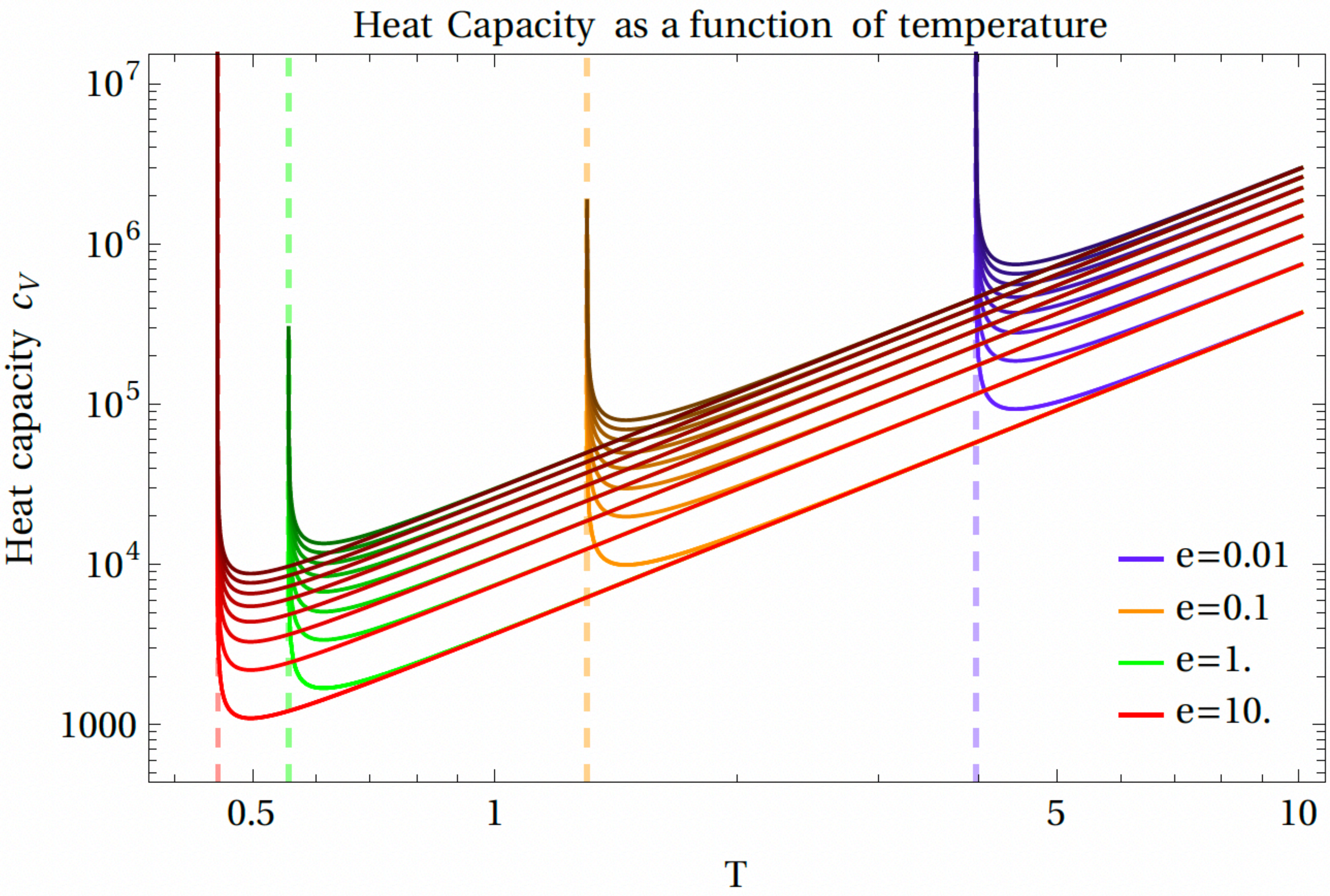


Stability condition:

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V \geq 0$$

$$L^2\left(2e^2r_h^2+3\right)-4e^2r_h^4<0$$

Charge susceptibility:

$$\chi_\rho = \left(\frac{\partial \mu}{\partial \rho}\right)_V \geq 0$$

Conserved charge via Komar integral:

$$M = -\int_\Sigma \sqrt{\det(g_{(0)})}\,\xi_i^t u_j \left\langle T^{ij}\right\rangle$$

$$M = \frac{3\pi L^2 n}{32G_5} + \frac{3\pi\, m_t\, n}{8G_5} + \frac{3\pi n}{8e^2G_5}\log\left(\Lambda L\right)$$

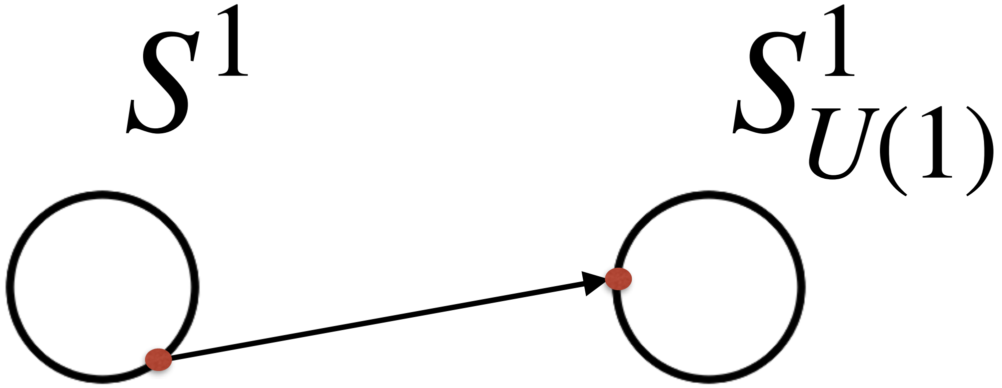
$$\begin{aligned} c_V &= \left(\frac{\partial M}{\partial m_t}\right)\left(\frac{\partial m_t}{\partial r_h}\right)\left(\frac{\partial r_h}{\partial T}\right) \\ &= \frac{3\pi^2 n r_h^3\left(L^2\left(2e^2r_h^2+1\right)+4e^2r_h^4\right)}{2G_5\left(L^2\left(-2e^2r_h^2+3\right)+4e^2r_h^4\right)} \end{aligned}$$

Topological Charge: U(1) Example

$$q_{circle} = \frac{1}{2\pi}\oint d\Omega_i J^i = -\frac{i}{2\pi}\oint d\Omega_i \epsilon^{ij}(\partial_j\omega)\omega^{-1} = \frac{n}{2\pi}\oint d\Omega_i \epsilon^{ij}\partial_j\theta\,,$$

$$J^i = -i\epsilon^{ij}(\partial_j\omega)\,\omega^{-1}$$

$$\omega = e^{in\theta},\ \theta \in [0, 2\pi].$$



$$\hat{n}_i = (\cos\psi, \sin\psi)$$

$$x_i = (x, y) = r(\cos\psi, \sin\psi),\ \psi \in [0, 2\pi],\ \text{and}\ d\Omega_i = \hat{n}_i d\psi$$

$$q_{circle} = \frac{1}{2\pi} n \int_0^{2\pi} d\theta = n$$

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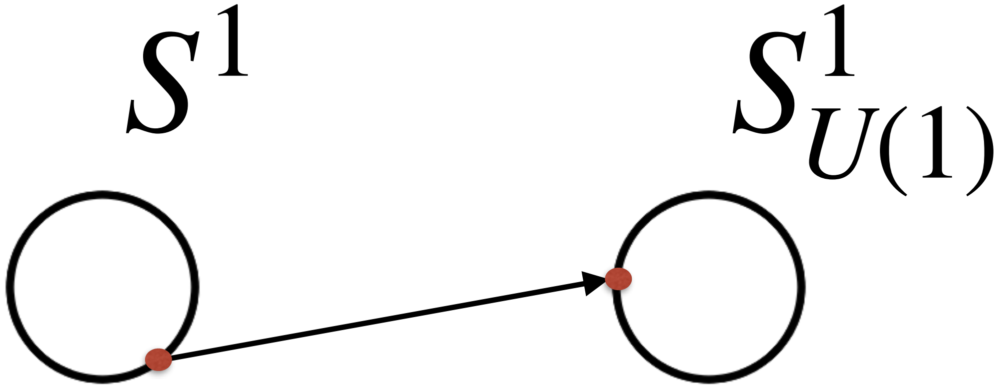

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