Characteristic momentum of Hydro+
&
a bound on the sound velocity enhancement near the QCD critical point

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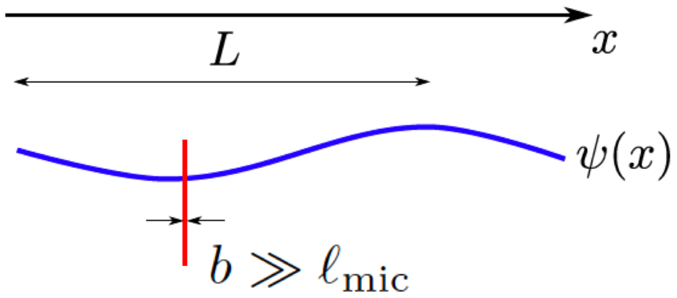
outline

- Correlation function of hydro fluctuations
- Why Hydro+? (Critical slowing down)
- Single-mode Hydro+
- Hydro+ near the QCD critical point
- Impact of critical slowing down on the speed of sound
- Range of applicability of Hydro+
- Concluding remarks

1. Hydrodynamics

- Hydro variables are macroscopically averaged values of conserved densities $\hat{\psi}$: $\psi = \langle \hat{\psi} \rangle$

- The averaging is one over a region of size b :
  the so-called hydrodynamic cell



[An, Basar, Stephanov, Yee 1902.09517]

- The resulting coarse-grained variables are used to describe the evolution of inhomogeneities at $L \gg b$

- These Hydro variables obey deterministic hydro equations :

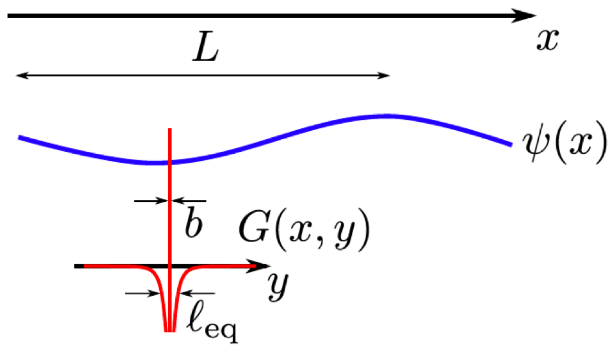
$$\partial_t \psi_A(t, \boldsymbol{q}) = -M_{AB} \psi_B(t, \boldsymbol{q}) \qquad (q \sim L^{-1})$$

2. Correlation function of hydro variables

In addition to hydro variables, the two-point function $\langle\phi_A(t,\boldsymbol{x}_1)\phi_B(t,\boldsymbol{x}_2)\rangle$ must be considered

where $\phi \equiv \hat{\psi} - \psi$

It is convenient to define $\langle\phi_A(t,\boldsymbol{x}+\boldsymbol{y}/2)\phi_B(t,\boldsymbol{x}-\boldsymbol{y}/2)\rangle \equiv G_{AB}(x,\boldsymbol{y})$



- The dependence of $G(x,\boldsymbol{y})$ on x and $\boldsymbol{y}$ is like this:

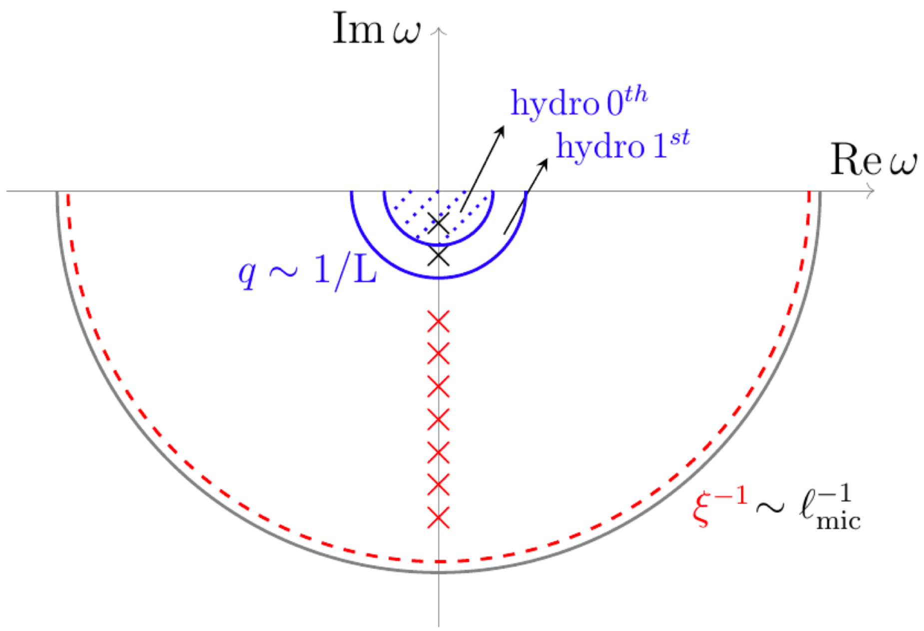
[An, Basar, Stephanov, Yee 1902.09517]

- The separation of scales $q \ll Q \ll b^{-1} \ll \ell_{\rm mic}^{-1}$ suggests to work with local modes

$$G_{\boldsymbol{Q}}(x) = \int_{\boldsymbol{Q}} G(x,\boldsymbol{y})e^{-i\boldsymbol{Q}\cdot\boldsymbol{y}}$$

3. In and out of equilibrium $G_{\boldsymbol{Q}}$ modes

- In a static background, $G_{\boldsymbol{Q}}(x)$ does not depend on $\boldsymbol{x}$.
- It is easy to show that $G_{\boldsymbol{Q}}(t)$ obeys: $\partial_t G_{\boldsymbol{Q}}(t) = -\Gamma_{\boldsymbol{Q}}\left(G_{\boldsymbol{Q}}(t) - \bar{G}_{\boldsymbol{Q}}\right),$ $\qquad \Gamma_{\boldsymbol{Q}} = \gamma\, \boldsymbol{Q}^2$

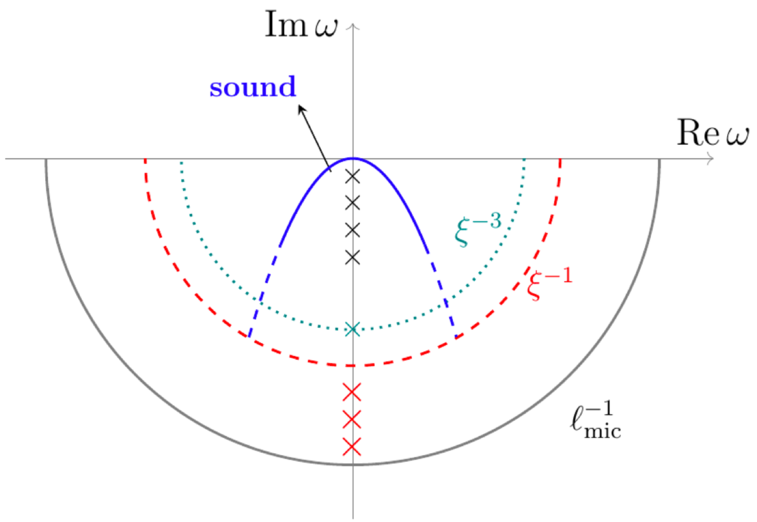


$q \ll Q \lesssim \xi^{-1}$: equilibrated $G_{\boldsymbol{Q}}$ modes

$Q \ll \xi^{-1}$: out of equilibrium $G_{\boldsymbol{Q}}$ modes (hydro)

4. Break down of hydro near the critical point

Near a critical point, $\xi \gg 1/T$, so additional hierarchy of scales emerges



- Break down of hydro at $\omega_{\rm sound} \sim \xi^{-1}$? ? ?

  $\times$: $Q \ll \xi^{-1} \rightarrow \Gamma_Q \sim Q^2$ (hydro)

  $\times$: $Q \sim \xi^{-1} \rightarrow \Gamma_Q \sim \xi^{-3}$

- Hydro breaks down at $\omega_{\rm sound} \sim \xi^{-3}$

- Away from the critical point $\xi \ll b \ll L$, $G_{\boldsymbol{Q}}(x)$ modes with $Q \sim \xi^{-1}$ are fast-decaying.
- Near the critical point $b \ll \xi$ and the relaxation time of these $G_{\boldsymbol{Q}}$ modes diverges:

critical slowing down [Berdnikov, Rajagopal 992274]

5. Hydro+: extending hydro beyond $Q \sim \xi^{-3}$

In the simplest setup, the so-called **"single-mode hydro+"** : [Stephanov, Yin 1712.10305]

there is only one single slow mode with decay rate $\sim \xi^{-3}$: ϕ

Hydro+ equations = hydro eqs. + relaxation equation of the slow mode ϕ

$$
\begin{aligned}
D\epsilon &= -w_{(+)}\theta - \partial_{(\mu}u_{\nu)}\,\Pi^{\mu\nu}, \\
Dn &= -n\theta - \partial \cdot \Delta J, \\
w_{(+)}Du^{\nu} &= -\partial_{\perp}^{\nu}p - \delta_{\perp\lambda}^{\nu}\,\partial_{\mu}\Pi^{\mu\lambda}, \\
D\phi &= -\gamma_{\pi}\pi - A_{\phi}\theta + \cdots\,,
\end{aligned}
$$

(+) indicates that thermo functions are now functions of ϵ, n and ϕ

$$ds_{(+)} = \beta_{(+)}\,d\varepsilon - \alpha_{(+)}\,dn - \pi d\,\phi$$

6. Linearized Hydro+: enhancement in c_s^2

- **The spectral curve is** $$F(\omega, q^2) = \omega^2 - q^2 \left(c_s^2 + \frac{\omega}{\omega + i\Gamma_\pi} \frac{\beta p_\pi^2}{\phi_\pi w} \right) = 0$$

We see that due to the critical slowing down: $c_s^2 \quad \to \quad c_s^2 + \Delta c_s^2$

where: $$\Delta c_s^2 = \frac{\omega^2}{\omega^2 + \Gamma_\pi^2} \frac{\beta p_\pi^2}{\phi_\pi w}$$

$$\boxed{\Delta c_s^2(\infty) = \frac{\beta p_\pi^2}{\phi_\pi w}}$$

- **In terms of dimensionless quantities** $\mathfrak{w} = \frac{\omega}{\Gamma_\pi}, \quad \mathfrak{q} = \frac{c_s\, q}{\Gamma_\pi}, \quad \alpha = \frac{\Delta c_s^2(\infty)}{c_s^2}$ **:**

[N.A., Kaminski 2112.14747]

$$F(\mathfrak{w}, \mathfrak{q}^2) = \mathfrak{w}^2 - \mathfrak{q}^2 \frac{i + \mathfrak{w} + \alpha\, \mathfrak{w}}{i + \mathfrak{w}} = 0$$

The only parameter which should be determined is **alpha**

7. Spectrum of linear excitations

The spectral curve is a polynomial of order 3.

So there are 3 modes:

$$\mathfrak{w}_1(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{7/3}(-1+3(1+\alpha)\mathfrak{q}^2)}{3\,\mathcal{D}(\mathfrak{q})} - \frac{2^{2/3}}{3}\mathcal{D}(\mathfrak{q})\right),$$
$$\mathfrak{w}_2(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{4/3}(-i+\sqrt{3})(-1+3(1+\alpha)\mathfrak{q}^2)}{\mathcal{D}(\mathfrak{q})} - 2^{2/3}(i+\sqrt{3})\mathcal{D}(\mathfrak{q})\right)$$
$$\mathfrak{w}_3(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{4/3}(-i-\sqrt{3})(-1+3(1+\alpha)\mathfrak{q}^2)}{\mathcal{D}(\mathfrak{q})} - 2^{2/3}(1+i\sqrt{3})\mathcal{D}(\mathfrak{q})\right)$$

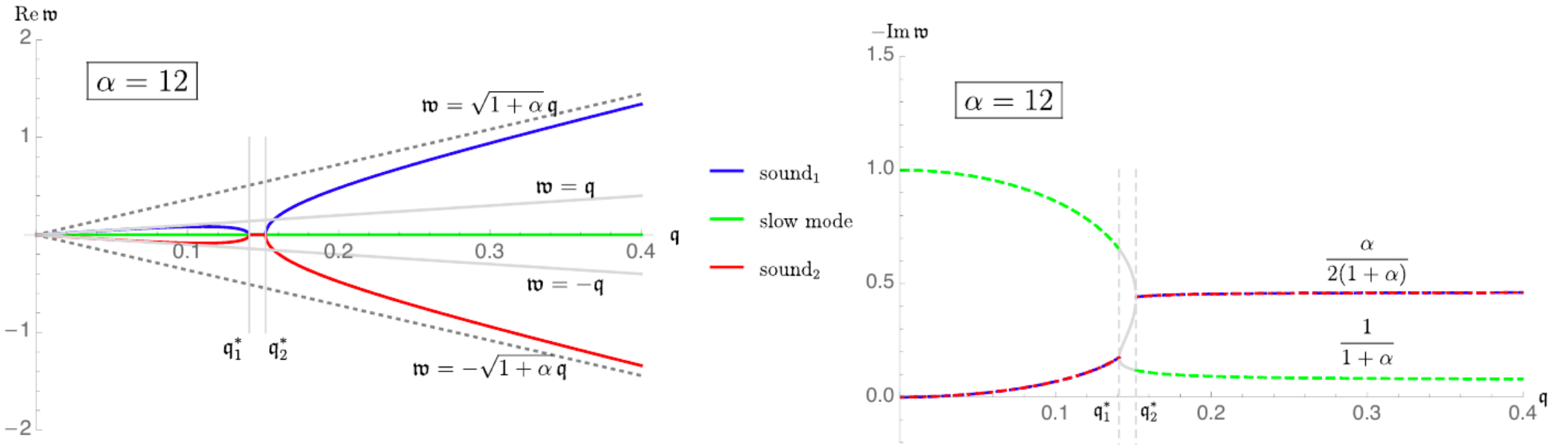
Interestingly:

$$\mathcal{D}(\mathfrak{q}) = \left(2i + 9i(2-\alpha)\mathfrak{q}^2 + 3\sqrt{3}\sqrt{-4 - 4\mathfrak{q}^4(1+\alpha^3) + \mathfrak{q}^2(-8+20\alpha+\alpha^2)}\right)^{1/3}$$

The square root has four branch points:

$$(\mathfrak{q}_1^*)^2 = \frac{\alpha^2 + 20\alpha - 8 + \sqrt{\alpha-8}\,(\alpha^{3/2} - 8\alpha^{1/2})}{8(1+\alpha)^3}$$
$$(\mathfrak{q}_2^*)^2 = \frac{\alpha^2 + 20\alpha - 8 - \sqrt{\alpha-8}\,(\alpha^{3/2} - 8\alpha^{1/2})}{8(1+\alpha)^3}$$

8. Example: Spectrum at $\alpha = 12$



[N.A., Kaminski 2112.14747]

- At $q < q_1$, the slowest modes are the two sound modes
- At $q > q_2$, the slow mode is the slowest mode

We refer to $\mathfrak{q}_c = \min\{|\mathfrak{q}_1^*|, |\mathfrak{q}_2^*|\}$ as the **characteristic momentum of Hydro+,**

beyond which, the standard hydrodynamics breaks down

9. Hydro+ near the QCD critical point

- The slowest mode is the $G_{\boldsymbol{Q}}(x)$ corresponding to **order parameter field.**

Since this mode is the slow mode of Hydro+,

we call it $\phi_{\boldsymbol{Q}}(t, \boldsymbol{x})$: $D\phi_{\boldsymbol{Q}} = -\Gamma_{\boldsymbol{Q}} \left(\phi_{\boldsymbol{Q}} - \bar{\phi}_{\boldsymbol{Q}}\right)$ with

$$\bar{\phi}_{\boldsymbol{Q}} \approx \frac{c_M\,\xi^2}{1+(Q\xi)^2}$$

[Rajagopal, Ridgway, Weller, Yin 1908.08539]

$$\Gamma_{\boldsymbol{Q}} = \frac{2D_0\xi_0}{\xi^3} K(Q\xi)$$

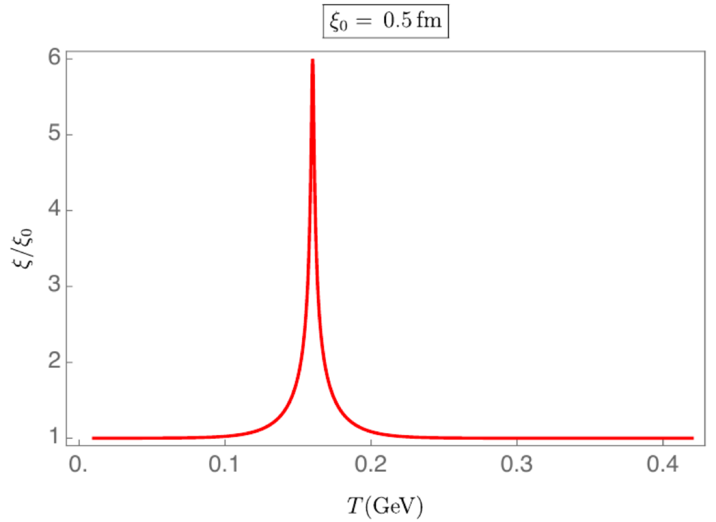
[Kawasaki 1970]

- In our calculations we choose to work with: $D_0 = 0.1,\, 0.5\,\mathrm{fm}$

- We also parametrize ξ, as [Rajagopal, Ridgway, Weller, Yin 1908.08539]

$$\left(\frac{\xi}{\xi_0}\right)^{-2} = \sqrt{\tanh^2\left(\frac{T-T_c}{\Delta T}\right)\left(1-\left(\frac{\xi_{\max}}{\xi_0}\right)^{-4}\right)+\left(\frac{\xi_{\max}}{\xi_0}\right)^{-4}}$$

$$\Delta T = 0.2T_c$$



10. Question that we want to address:

Similar to the single-mode Hydro+, here, the presence of slow mode leads to

1. enhancement in the sound velocity **(our main goal)**
2. enhancement in the value of bulk viscosity [Martinez, Schafer, Skokov 1906.11306]

We find

$$\Delta c_s^2(\omega) \approx \frac{c_s^4}{2s} \int \frac{d^3\boldsymbol{Q}}{(2\pi)^3} \left[f_2(Q\xi)\right]^2 \left(\frac{\xi}{\xi_0}\right)^4 \left(T\frac{\partial}{\partial T}\left(\frac{\xi}{\xi_0}\right)^{-2}\right)^2 \frac{\omega^2}{\omega^2+\Gamma_{\boldsymbol{Q}}^2}$$

To calculate Δc_s^2, we need to know

1) The equation of state near the CP. [Rajagopal, Ridgway, Weller, Yin 1908.08539]
2) The range of integration. Our idea is that

the characteristic momentum of single-mode Hydro+

can be used

to constrain the momentum of the slow mode contributing to the above integral.

11. Relation with single-mode Hydro+

The contribution of any mode is given by $\alpha_{\boldsymbol{Q}}(\omega \gg \Gamma_{\boldsymbol{Q}}) = \frac{\Delta c^2_{s,\boldsymbol{Q}}(\infty)}{c_s^2} \approx \frac{c_s^2}{2s} \frac{Q^2 \, \Delta Q}{2\pi^2} [f_2(Q\xi)]^2 \left(\frac{\xi}{\xi_0}\right)^4 \left(T \frac{\partial}{\partial T}\left(\frac{\xi}{\xi_0}\right)^{-2}\right)^2$

- This $\alpha_{\boldsymbol{Q}}$ is the Q-dependent version of the $\alpha = \frac{\Delta c_s^2(\infty)}{c_s^2}$ in the single-mode Hydro+.

  As if we treat any $\phi_{\boldsymbol{Q}}$ mode as a single-mode in a distinct Hydro+ .

- Similarly, we define a Q-dependent characteristic momentum $q_c \equiv q_c(Q)$

- Then the above picture can be applied to any point (μ, T) near the critical point (in the phase space).

  Therefore we find : $q_c \equiv q_c(Q, \mu, T)$.

- We limit our study to small μ region; then we have $q_c \equiv q_c(Q, T)$

Now let us see how the existence of $q_c \equiv q_c(Q, T)$ limits the range of momentum →

12. Constraint on the momentum of slow mode

We argue that at any temperature T near the critical point,

Only $\phi_{\boldsymbol{Q}}$ modes with $q_c(Q,T) \ll Q$ contribute to the sound velocity enhancement.

[N.A., Kaminski 2112.14747]

Why?

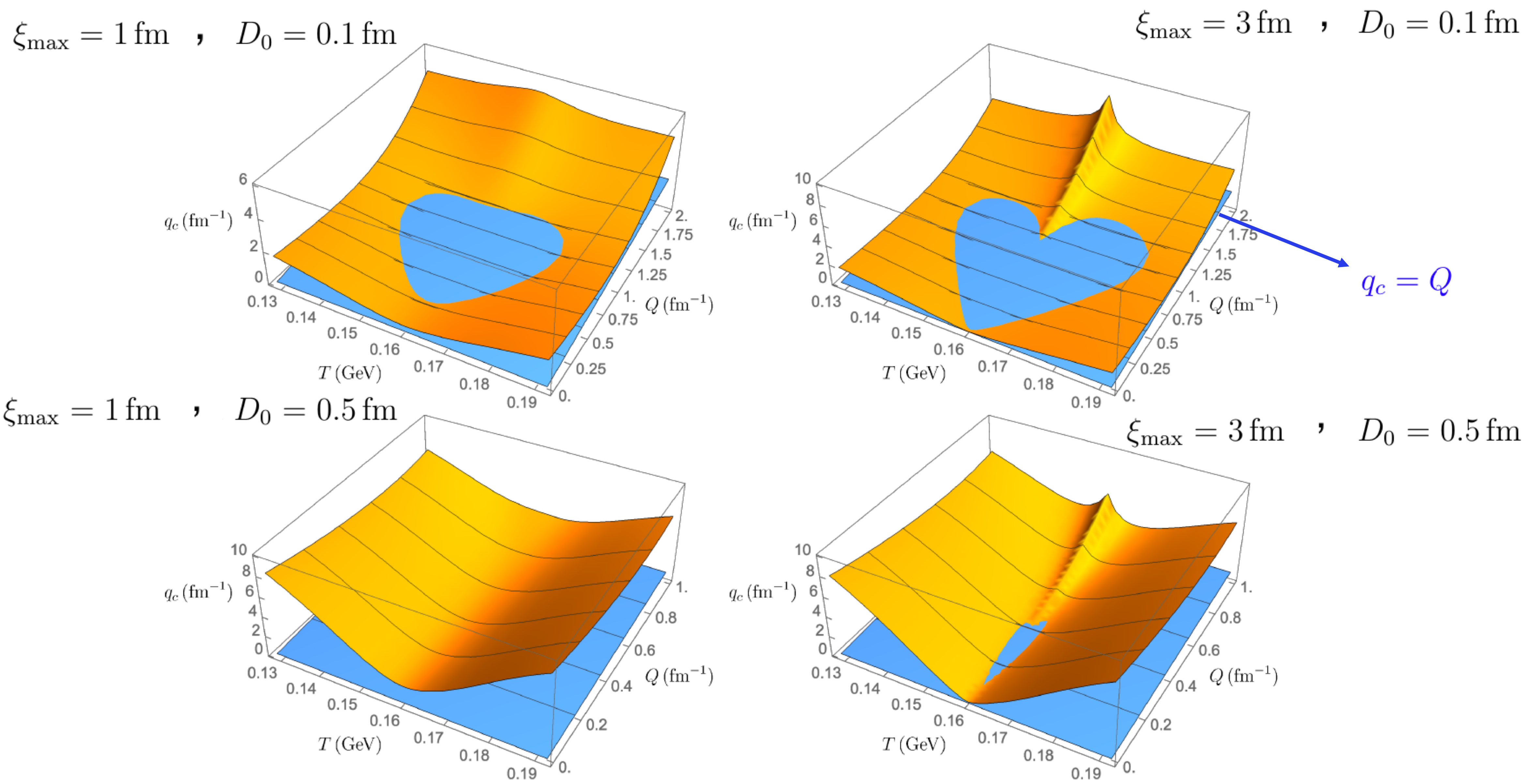
- Let us consider modes whose momentum belongs to the complement range: $Q \lesssim q_c(Q,T)$
- On the other hand, for the partially equilibrated states: $q \ll Q$

⇒ we arrive at $q \ll q_c$ which is range of conventional hydro

with no need to consider the critical slowing down ! ! !

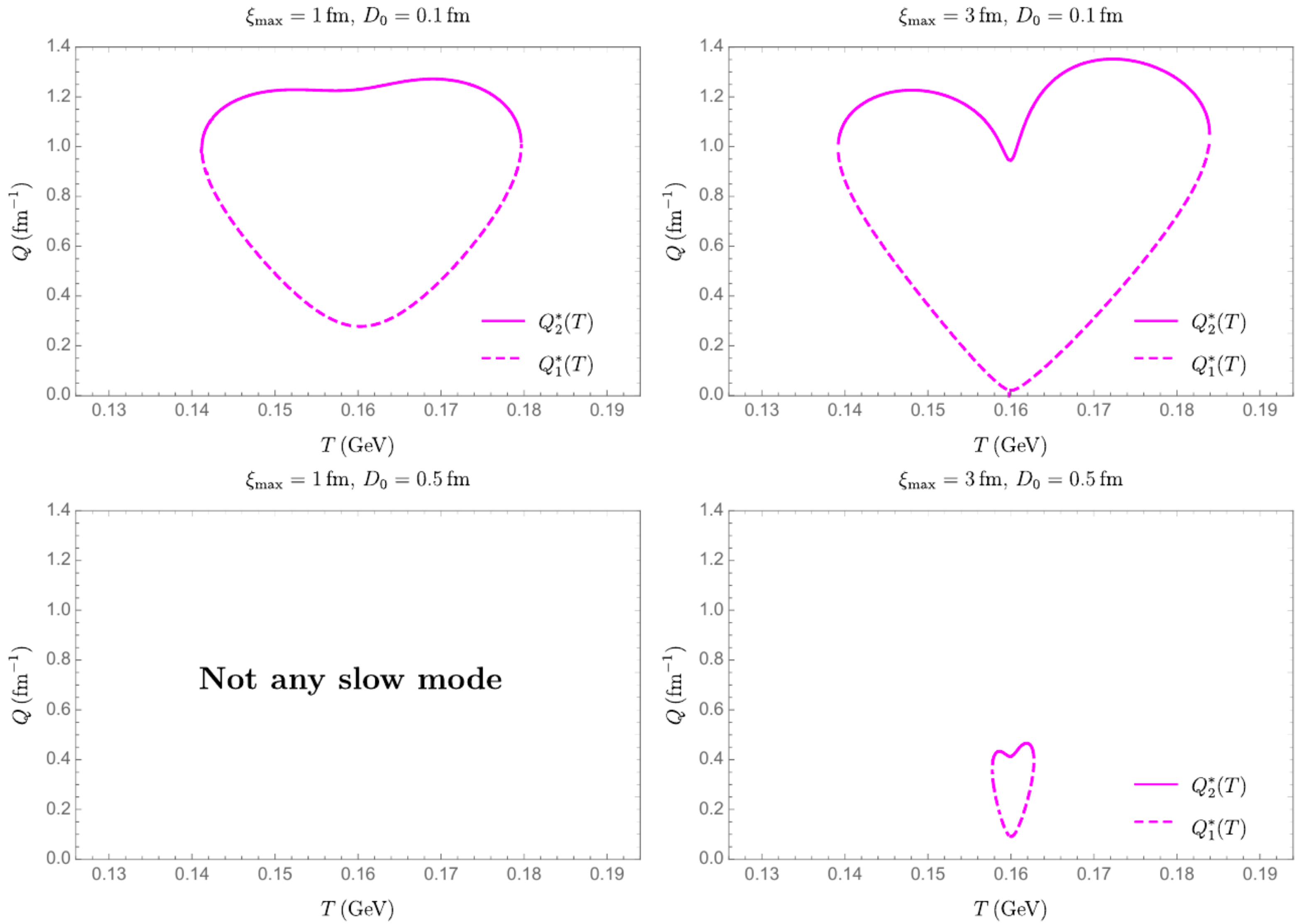
Thus we should find $q_c(Q,T)$ and determine when the inequality $q_c(Q,T) \ll Q$ holds.

13. Characteristic momentum near the critical point



14. Contributing modes

$$\Delta c_s^2(\omega) \approx \frac{c_s^4}{2s} \int \frac{d^3\boldsymbol{Q}}{(2\pi)^3} [f_2(Q\xi)]^2 \left(\frac{\xi}{\xi_0}\right)^4 \left(T\frac{\partial}{\partial T}\left(\frac{\xi}{\xi_0}\right)^{-2}\right)^2 \frac{\omega^2}{\omega^2 + \Gamma_{\boldsymbol{Q}}^2}$$

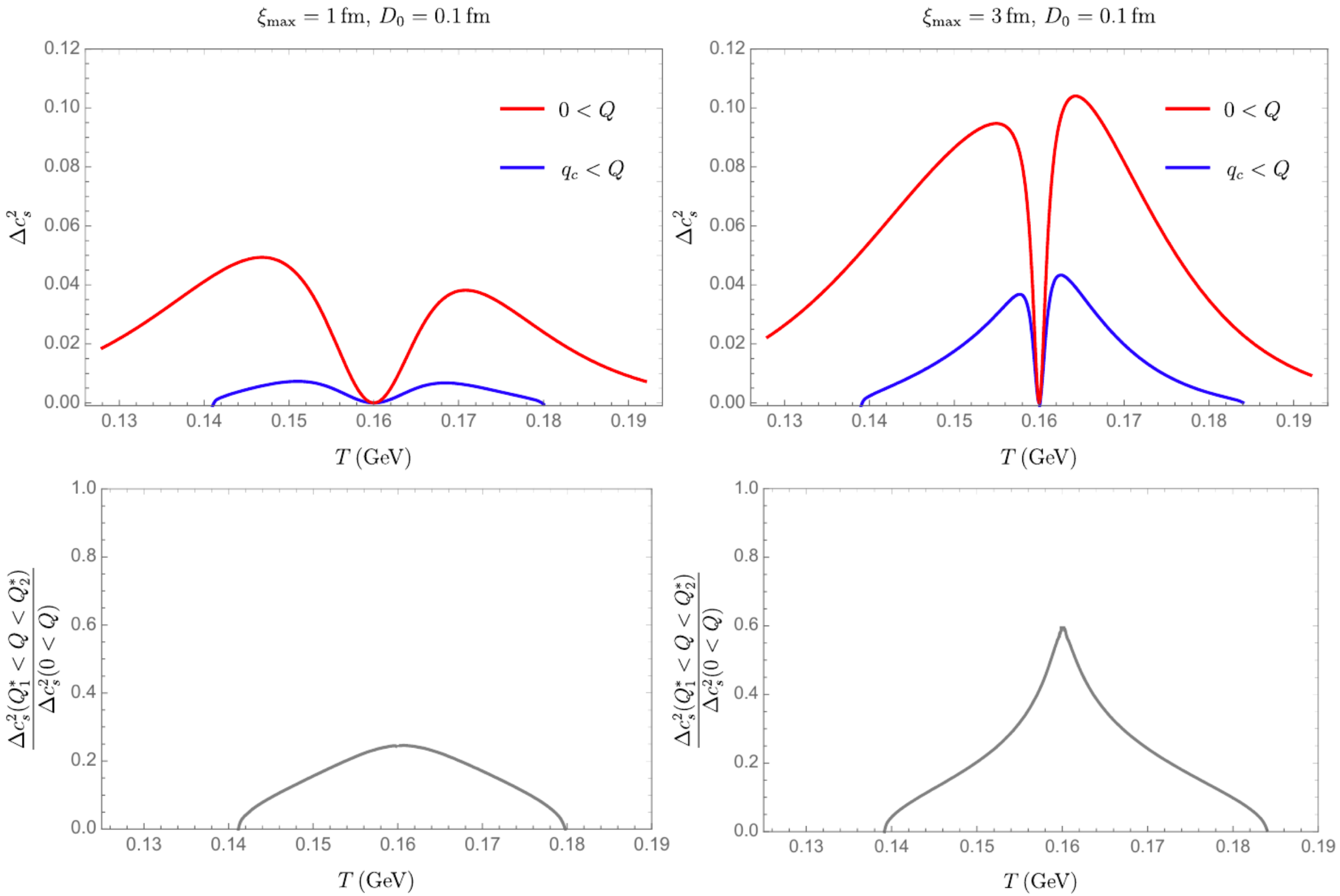


At any T, only modes

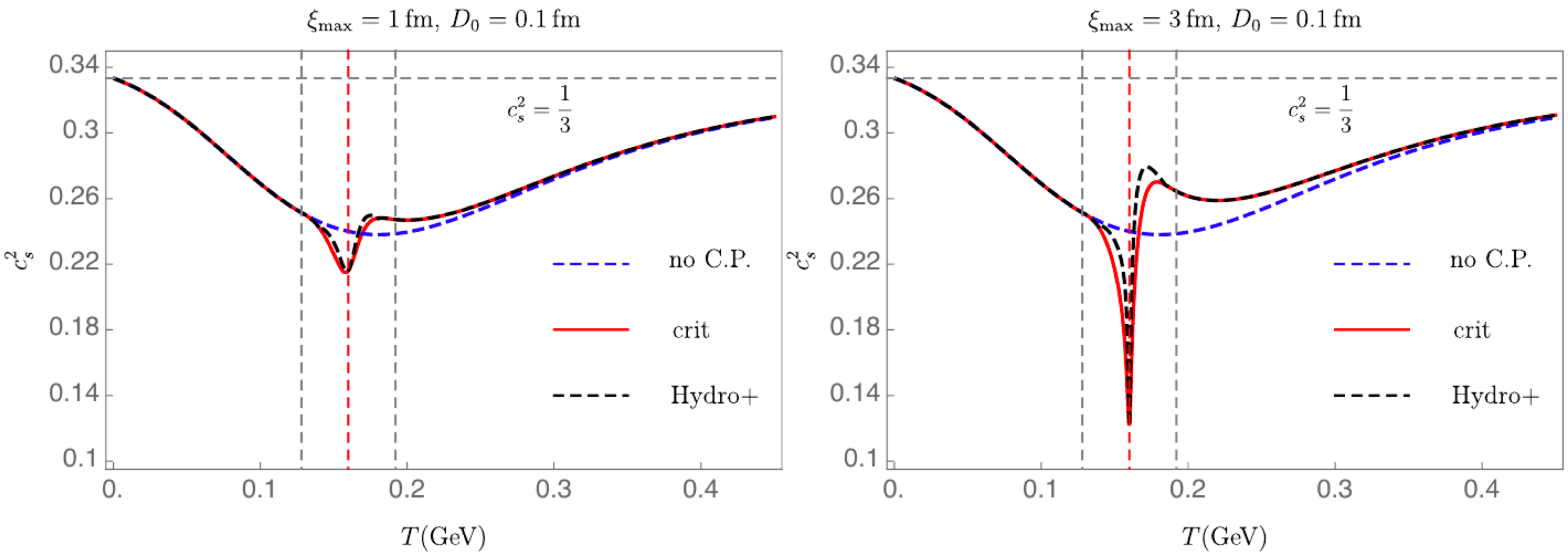
with $Q_1^*(T) \ll Q \ll Q_2^*(T)$

contribute to Δc_s^2.

15. Bound on Δc_s^2



16. c_s^2 near the critical point



- By increasing the value of $\xi_{\max}$, the enhancement of the speed of sound also increases.
- The enhancement of the speed of sound in any case is small, which is similar to the case of the bulk viscosity enhancement being small in [Martinez, Schafer, Skokov 1906.11306].

17. Conclusion and outlook

- Our study reveals the range of applicability of Hydro+ in the linear regime.

- To improve the results, it is needed to include the nonlinear effects. It would be important to calculate the sound velocity enhancement in the full nonlinear analysis of [An, Basar, Stephanov, Yee 1912.13456] and compare it with our results.

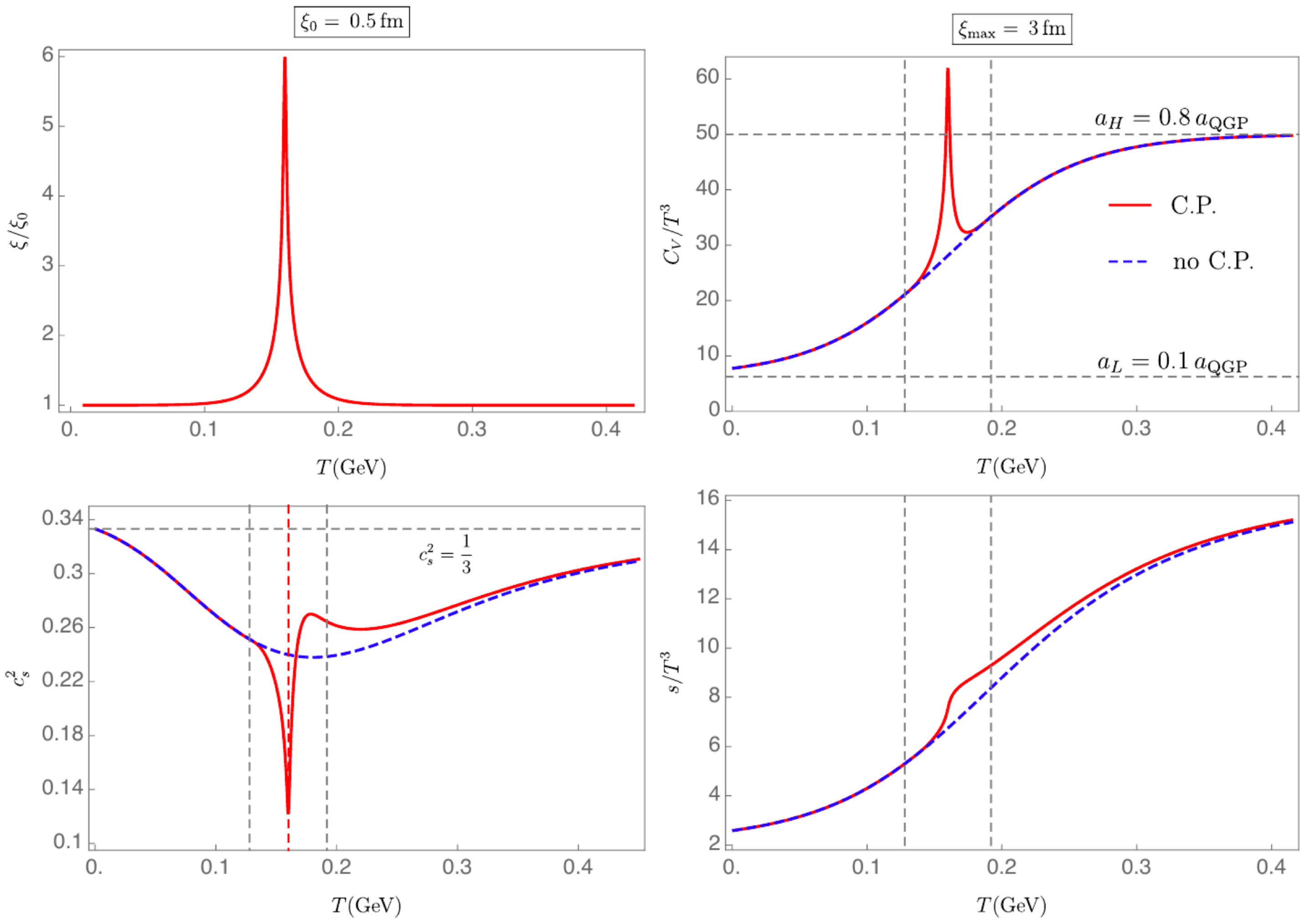
- Then one important place to follow the issues is in the freezeout analysis. One should investigate how the characteristic momentum of Hydro+ limit the modes contributing to the proton multiplicity correlator. [Pradeep, Rajagopal, Stephanov, Yin 2204.00639]

- It is also possible to include the hydro nonlinear effects analytically; for instance one should see whether some physical constraint like the existence of characteristic momentum in Hydro+ limits the momentum of modes in [An, Basar, Stephanov, Yee 1912.13456].

Thank you for your attention

1. EoS

$$\frac{c_V^{\text{no C.P.}}}{T^3} = \left[\left(\frac{a_H + a_L}{2}\right) + \left(\frac{a_H - a_L}{2}\right)\tanh\left(\frac{T - T_{\text{C.O.}}}{\Delta T_{\text{C.O.}}}\right)\right]$$



$$T_{\text{C.O.}} = T_c \quad \text{and} \quad \Delta T_{\text{C.O.}} = 0.6\,T_c$$

$$a_L = 0.1\,a_{\text{QGP}}, \quad a_H = 0.8\,a_{\text{QGP}}$$

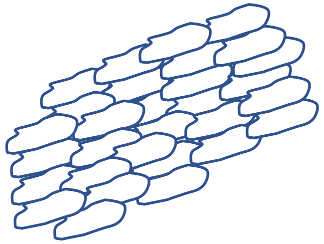
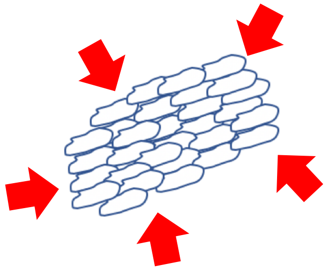
$$a_{\text{QGP}} = \frac{4\pi^2(N_c^2 - 1) + 21\pi^2 N_f}{15}$$

$$s(T) = \int_0^T dT' \frac{c_V(T')}{T'},$$

$$c_s^2 = \frac{s}{c_V}.$$

2. Hydro response

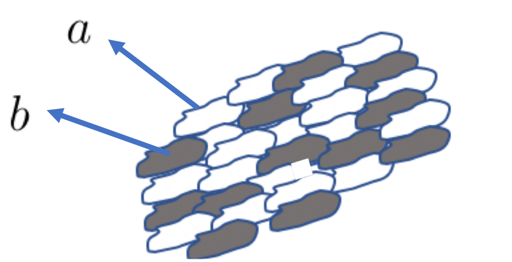
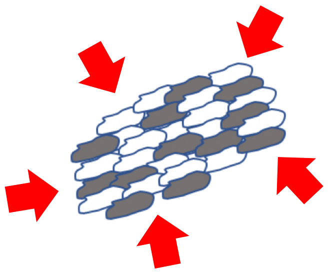
Compression of a pure fluid:

thermal equilibrium	out of equilibrium	equilibrium restoration
 $P_1,\ T_1,\ V_1$	 $V_2 < V_1$	**immediately equilibrated** $P_2,\ T_2,\ V_2$

- The process is irreversible $\longrightarrow$ the entropy is increased
- This dissipation is determined by the **bulk viscosity**

3. When Hydro response is not enough

Compression of a mixed fluid of “two substances a and b in chemical equilibrium”:

thermal equilibrium	out of equilibrium	equilibrium restoration
 $P_1,\ T_1,\ V_1,\ C_1^a,\ C_1^b$		immediately reaches $P_2,\ T_2,\ V_2$ but chemical equilibrium has not been reached ↓ if the reaction between a and b is not rapid equilibrium restoration occurs slowly

The chemical reaction is irreversible $\longrightarrow$ we expect a **larger bulk viscosity**

4. The conjectured QCD critical point and Hydro+

Compression of a pure fluid: