The non-equilibrium hydrodynamic attractor

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Primary references: MS, J. Noronha, G. Denicol, 1709.06644
D. Almaalol and MS, 1801.10173

Outline/Motivation

- Today, I would like to discuss the **non-equilibrium dynamical attractor** in different settings. [see e.g. Heller and Spalinski, Phys. Rev. Lett. 115 (7), 072501 (2015)]
- First, I will try to present things pseudo-pedagogically using both Israel-Stewart type theories (DNMR and MIS) and anisotropic hydrodynamics (aHydro) within RTA.
- Second, I will present the aHydro dynamics and attractor using a LO massless scalar collisional kernel.

Not just “trivial” systems…

Generalized aHydro formalism

In aHydro, one starts from kinetic theory and assumes that the distribution function is of the form

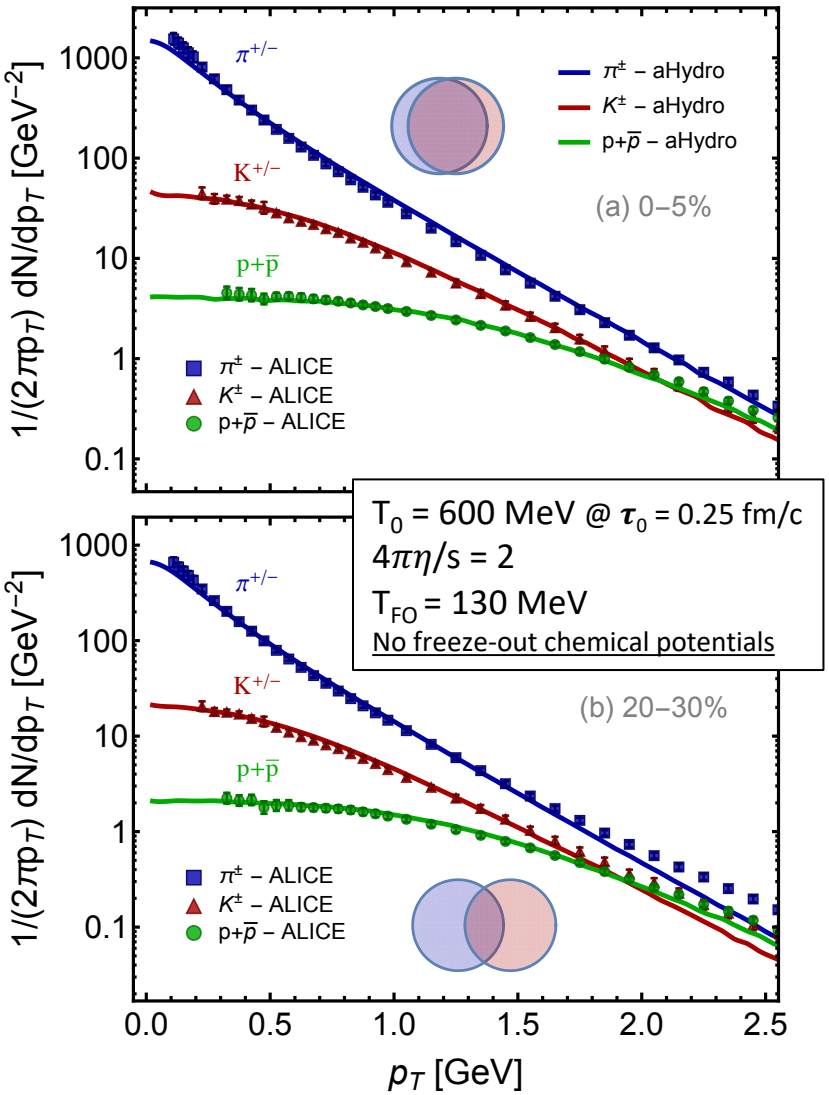
$$f(x,p) = f_{\rm eq}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu}(x) p^\nu}}{\lambda(x)}, \frac{\mu(x)}{\lambda(x)}\right) + \delta\tilde{f}(x,p)$$

$$\Xi^{\mu\nu} = \underbrace{u^\mu u^\nu}_{\text{LRF four velocity}} + \underbrace{\xi^{\mu\nu}}_{\text{Traceless symmetric anisotropy tensor}} - \underbrace{\Delta^{\mu\nu}}_{\text{Transverse projector}} \underbrace{\Phi}_{\text{“Bulk”}}$$

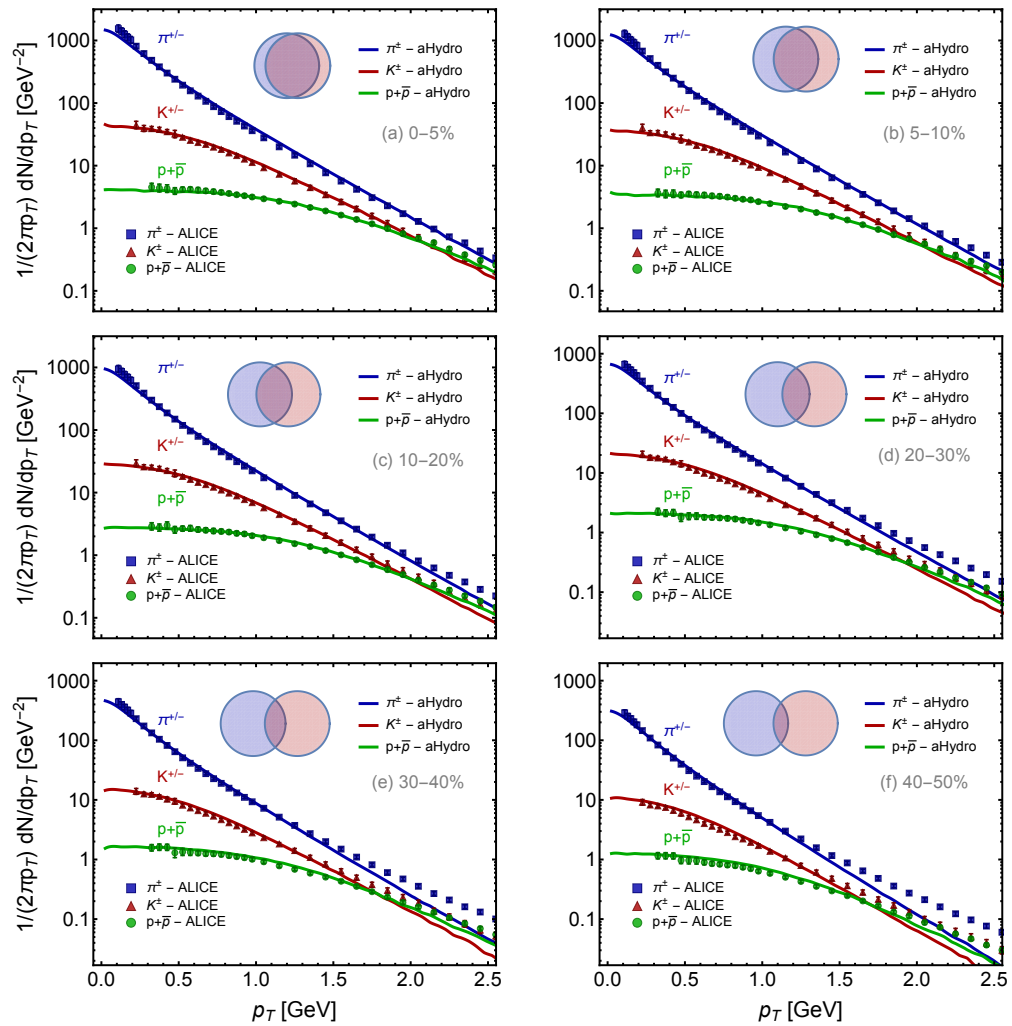
$$u^\mu u_\mu = 1$$
$$\xi^\mu{}_\mu = 0$$
$$\Delta^\mu{}_\mu = 3$$
$$u_\mu \xi^{\mu\nu} = u_\mu \Delta^{\mu\nu} = 0$$

Identified particle spectra

Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191



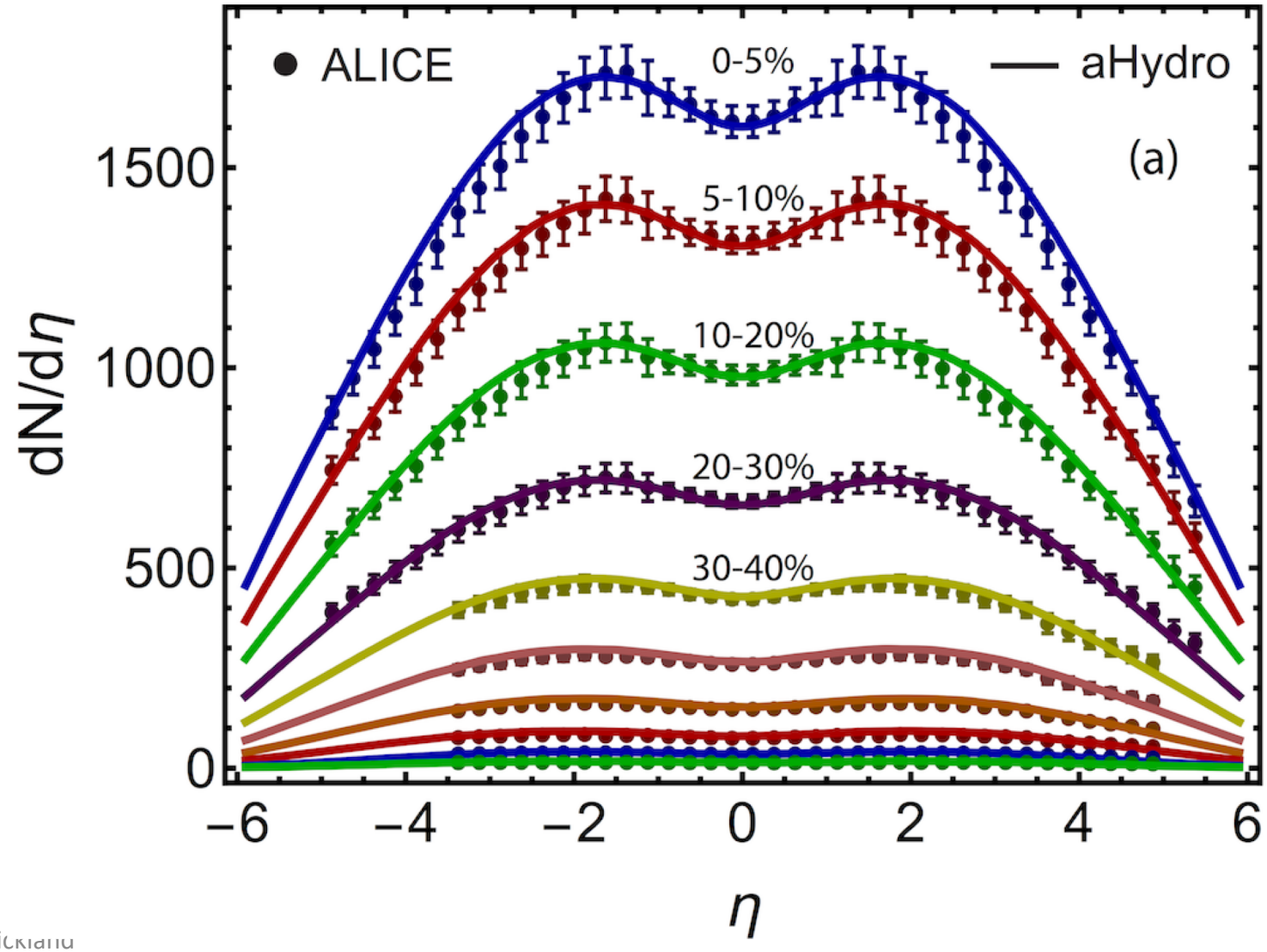
T_0 = 600 MeV @ τ_0 = 0.25 fm/c
$4\pi\eta/s = 2$
T_{FO} = 130 MeV
No freeze-out chemical potentials



Data are from the ALICE collaboration data for **Pb-Pb collisions @ 2.76 TeV/nucleon**

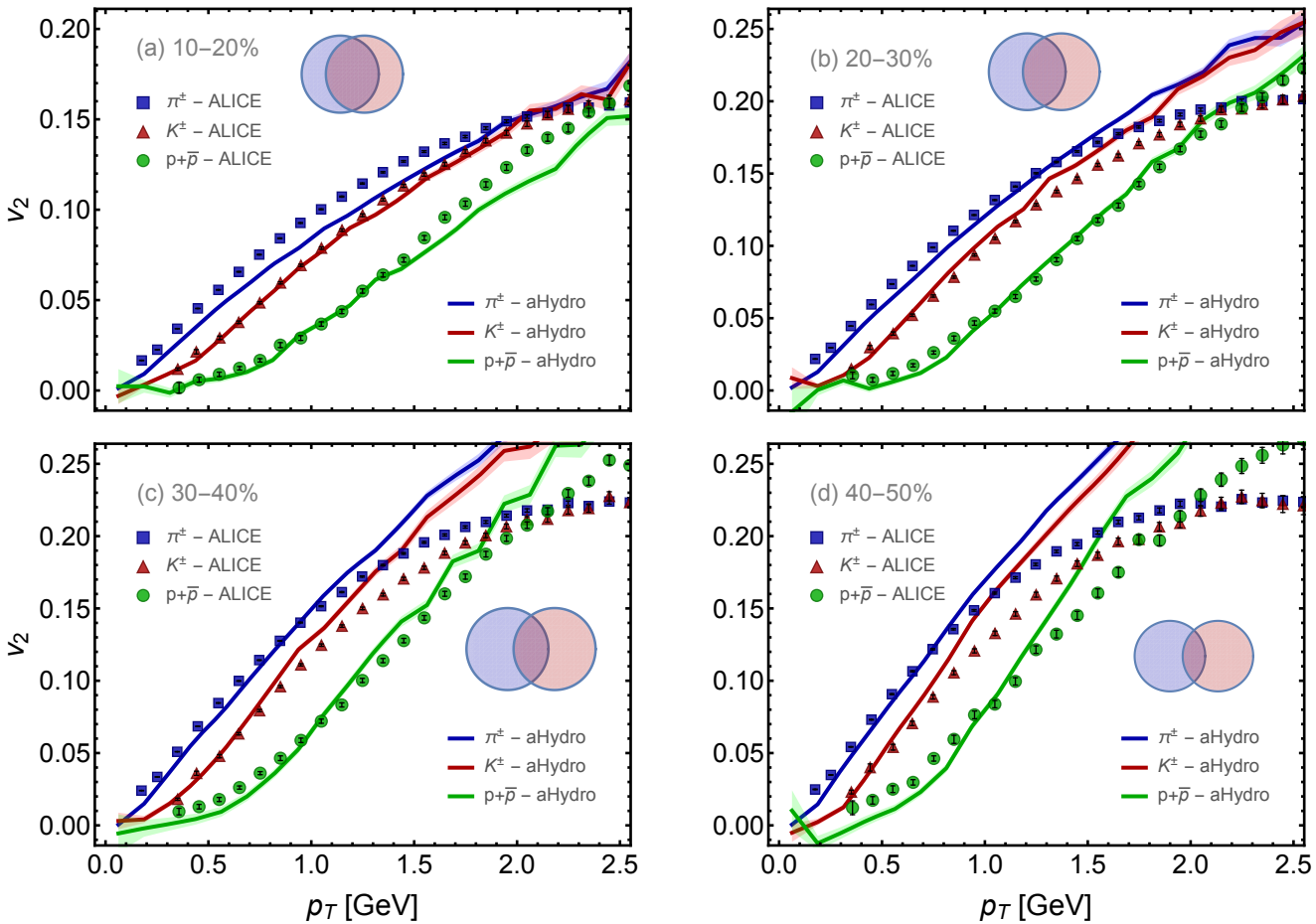
Charged particle multiplicity

Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191



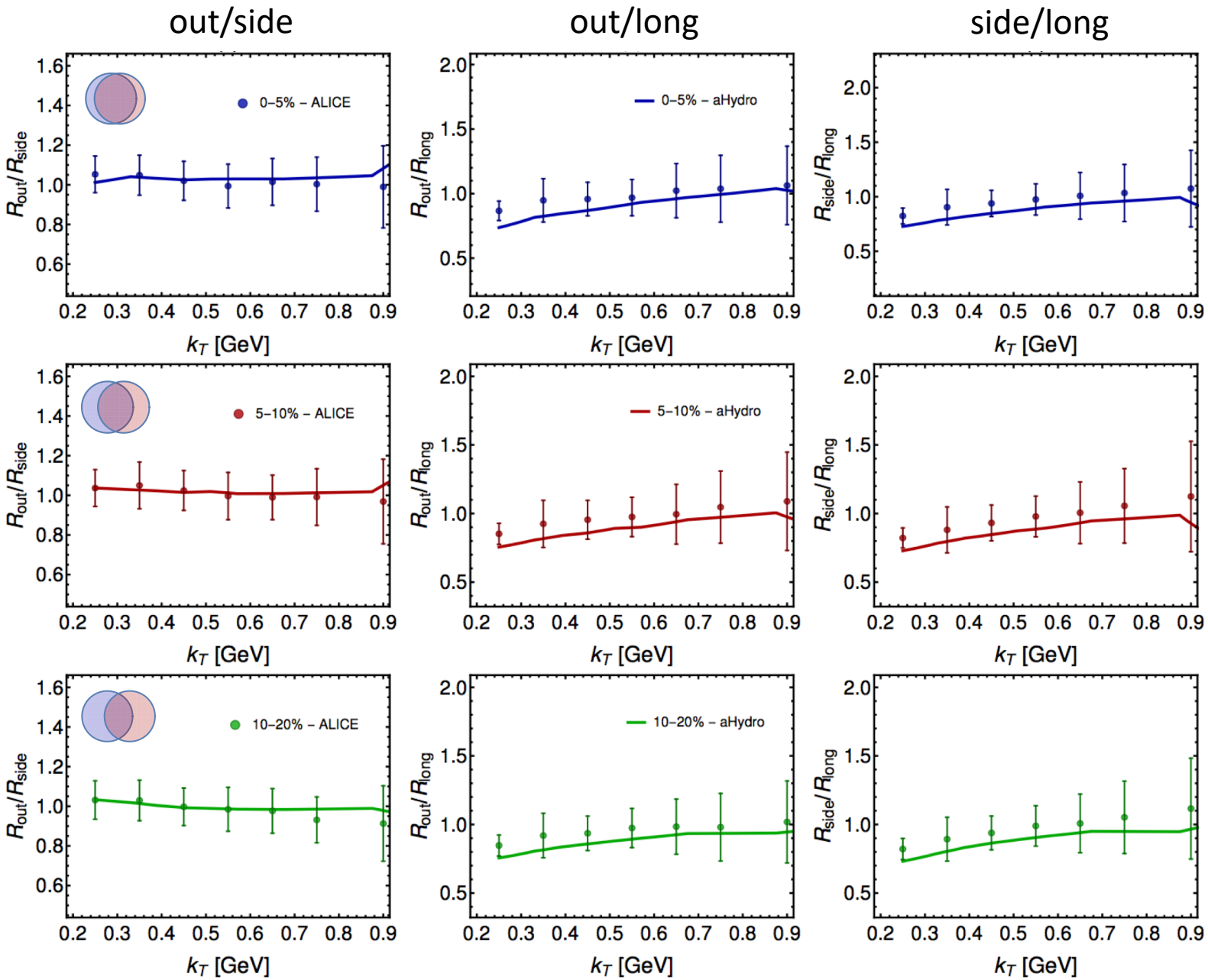
Elliptic flow

- Quite good description of identified particle elliptic flow as well
- Central collisions → need to include fluctuating init. Conditions!



Alqahtani, Nopoush, Ryblewski, MS, 1703.05808 (PRL); 1705.10191

HBT Radii Ratios

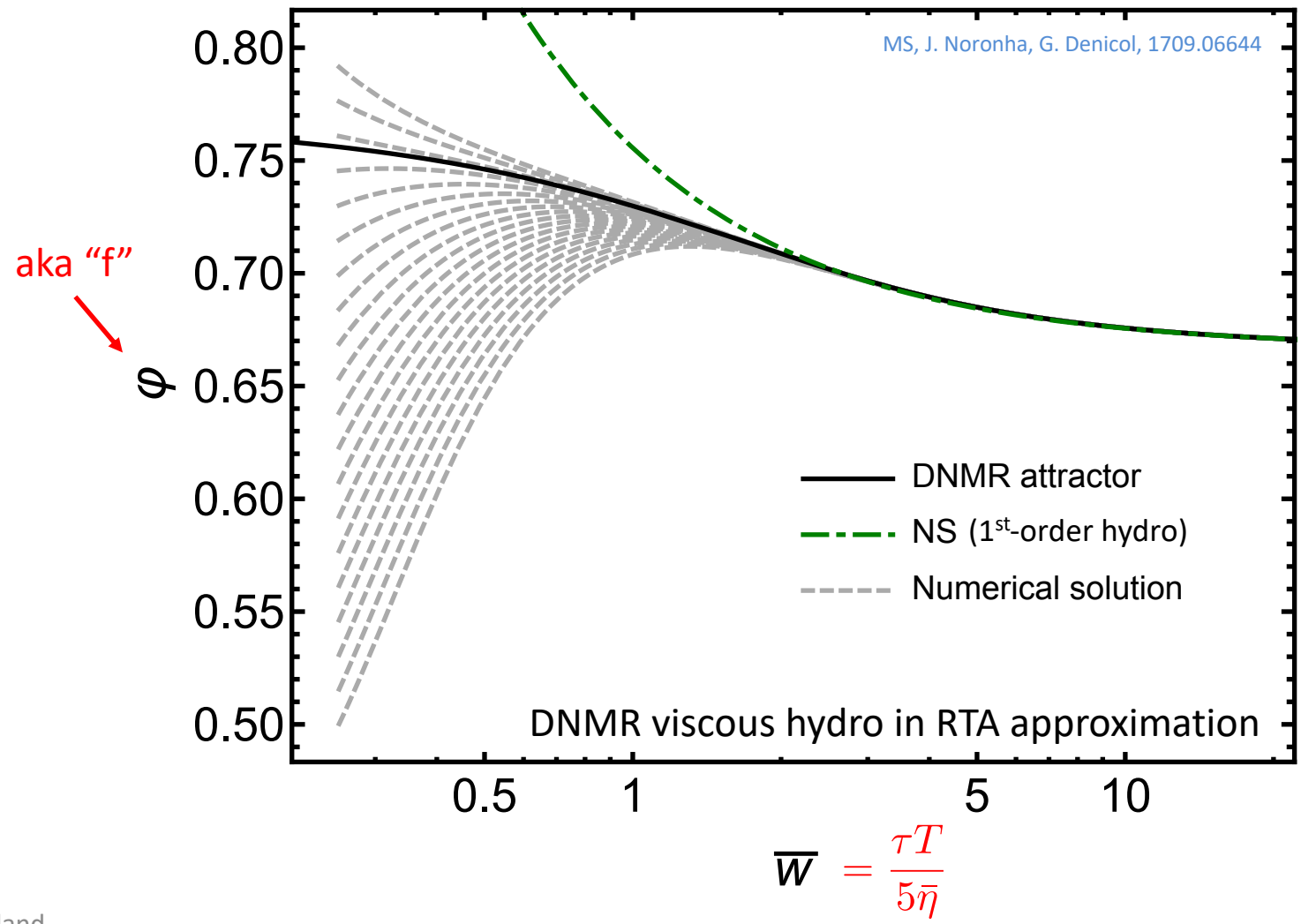


Part 1

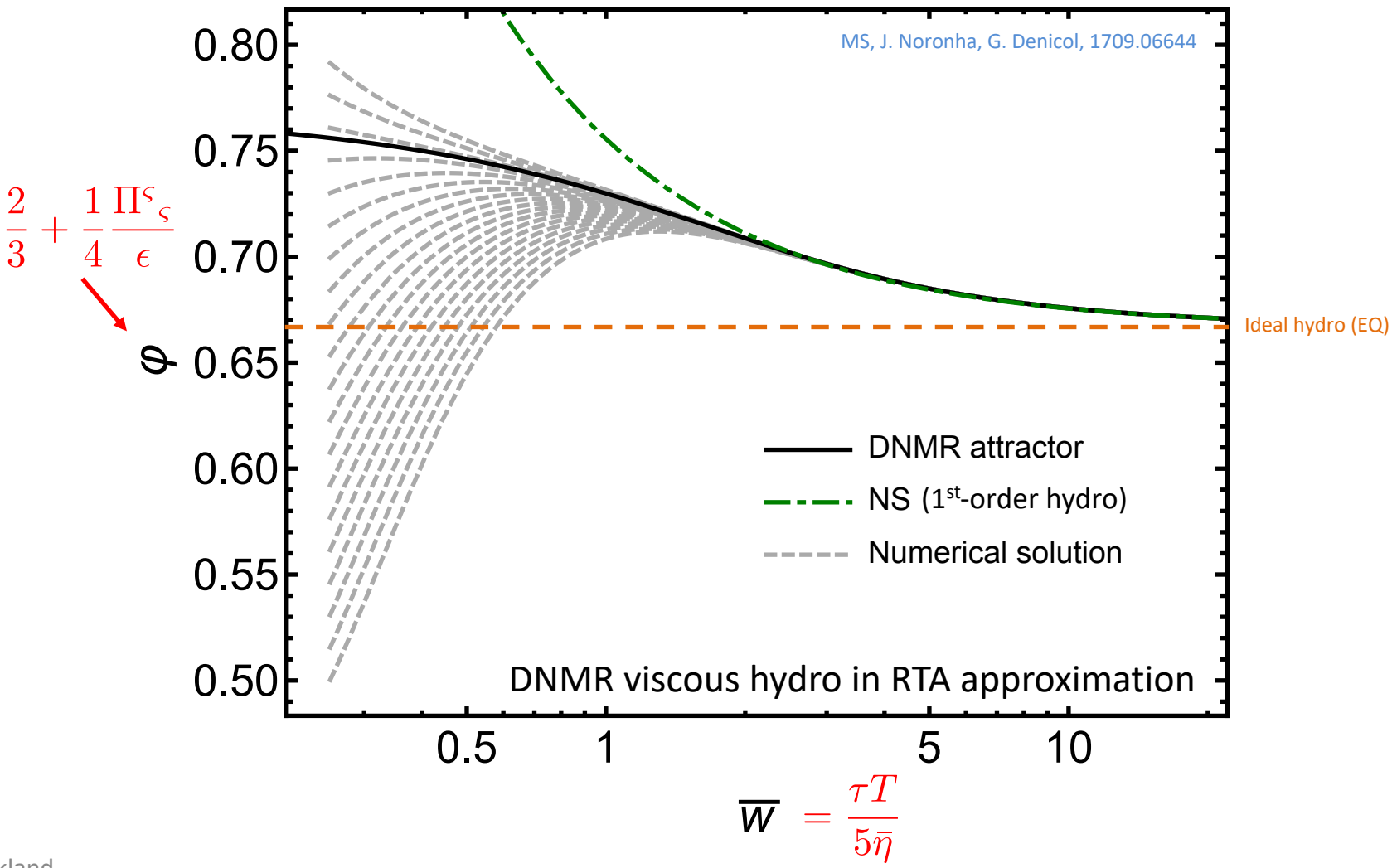
The DNMR and aHydro conformal 0+1d non-equilibrium attractor

MS, J. Noronha, G. Denicol, 1709.06644

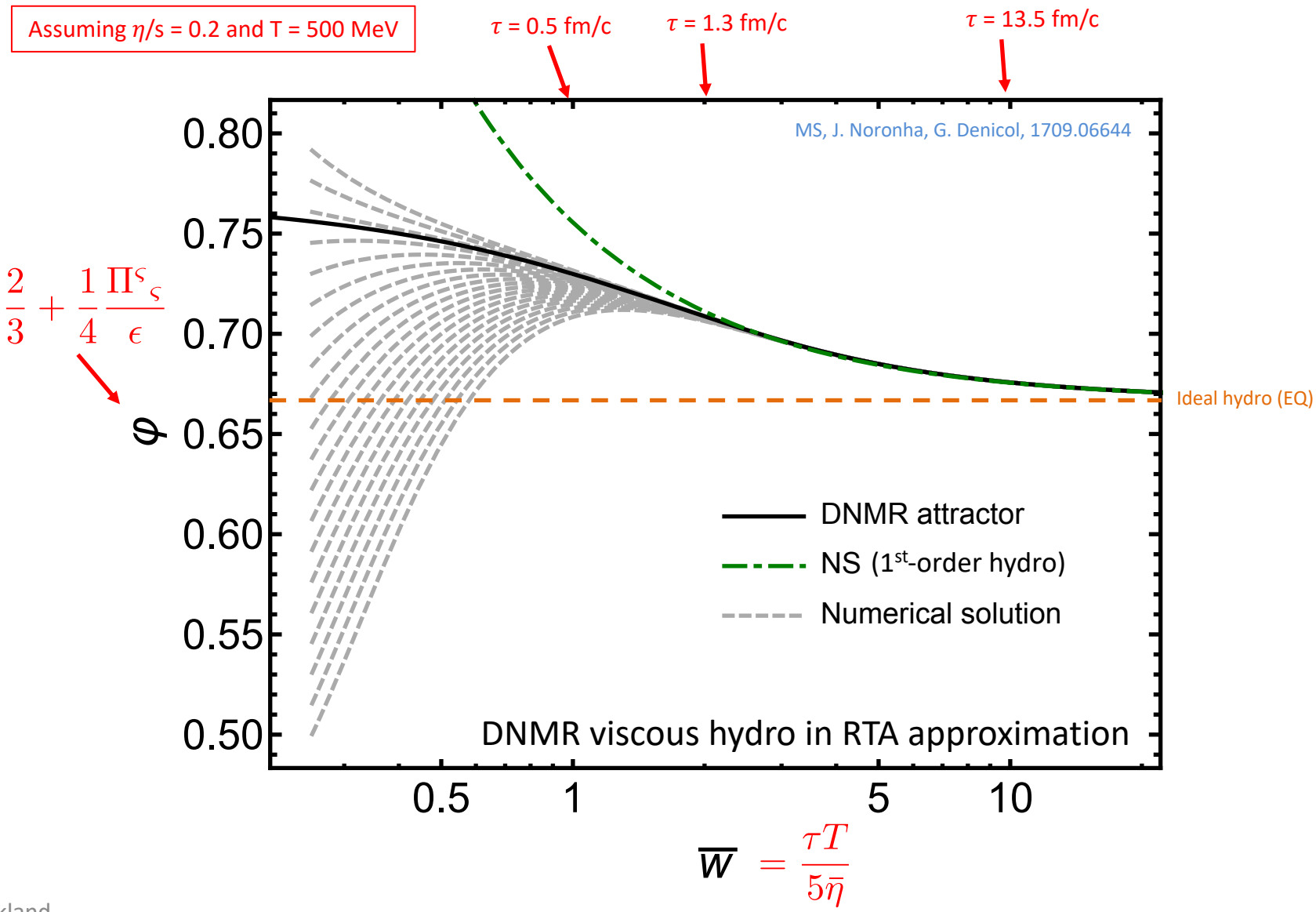
The attractor concept – 0+1d



The attractor concept – 0+1d



The attractor concept – 0+1d



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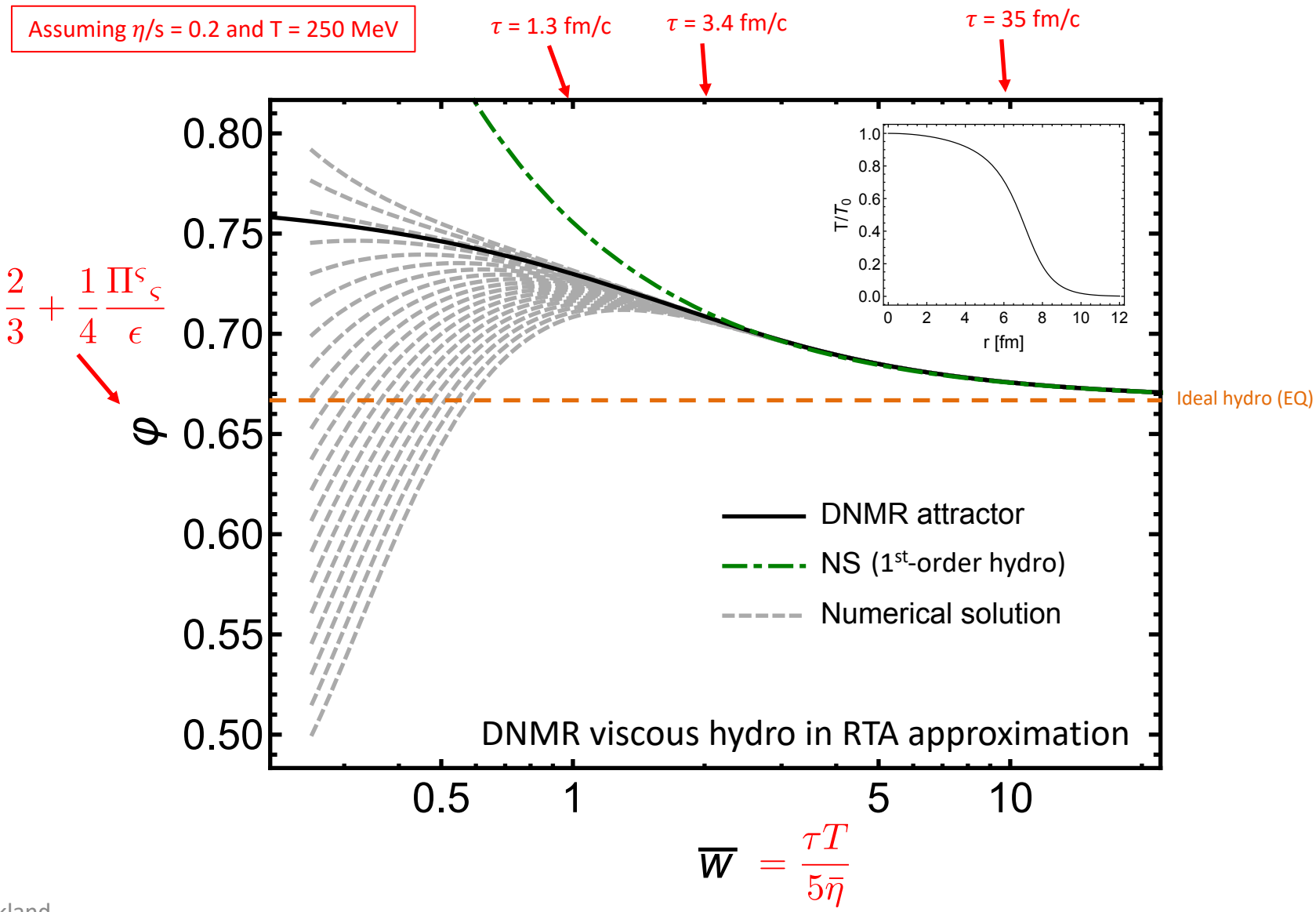
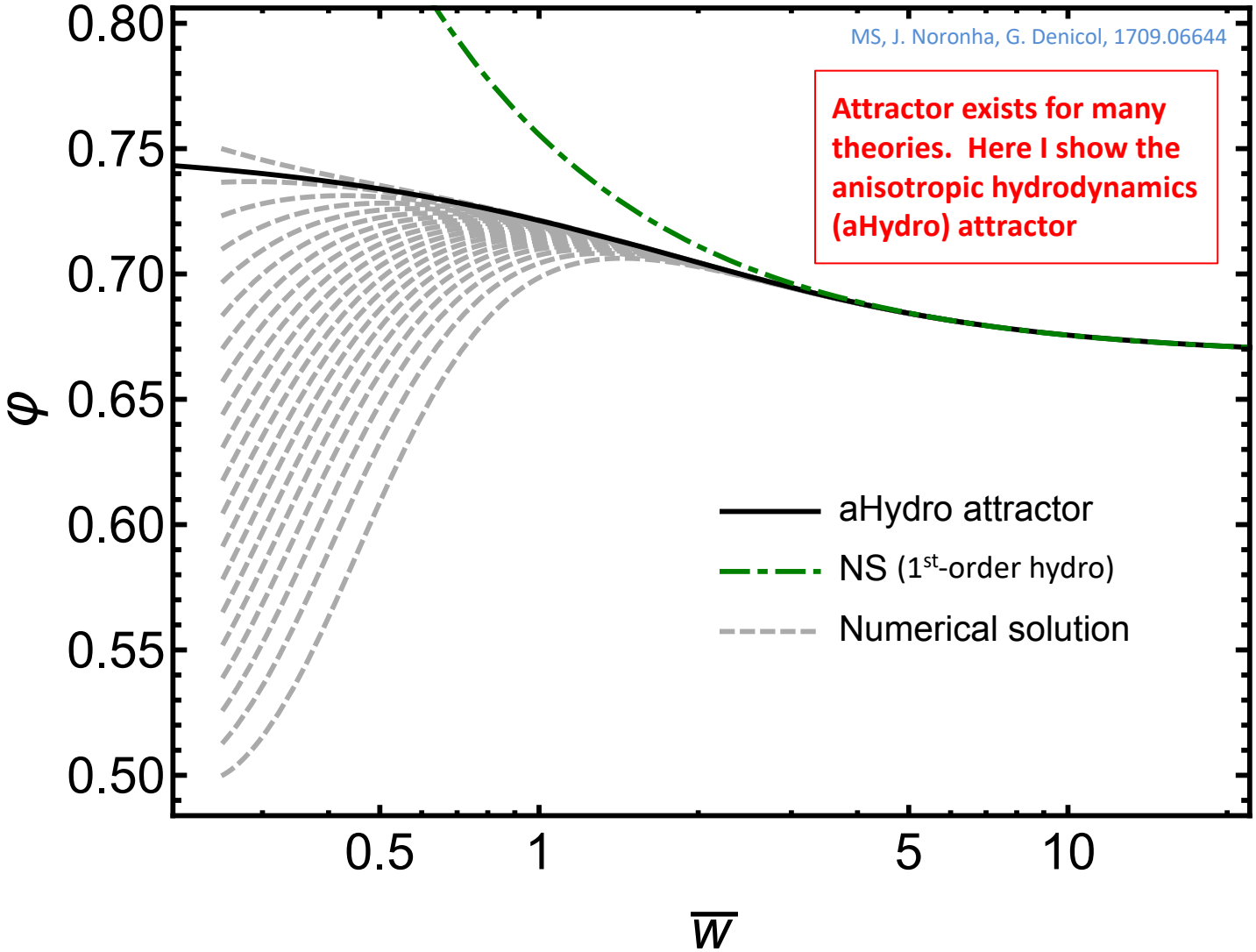
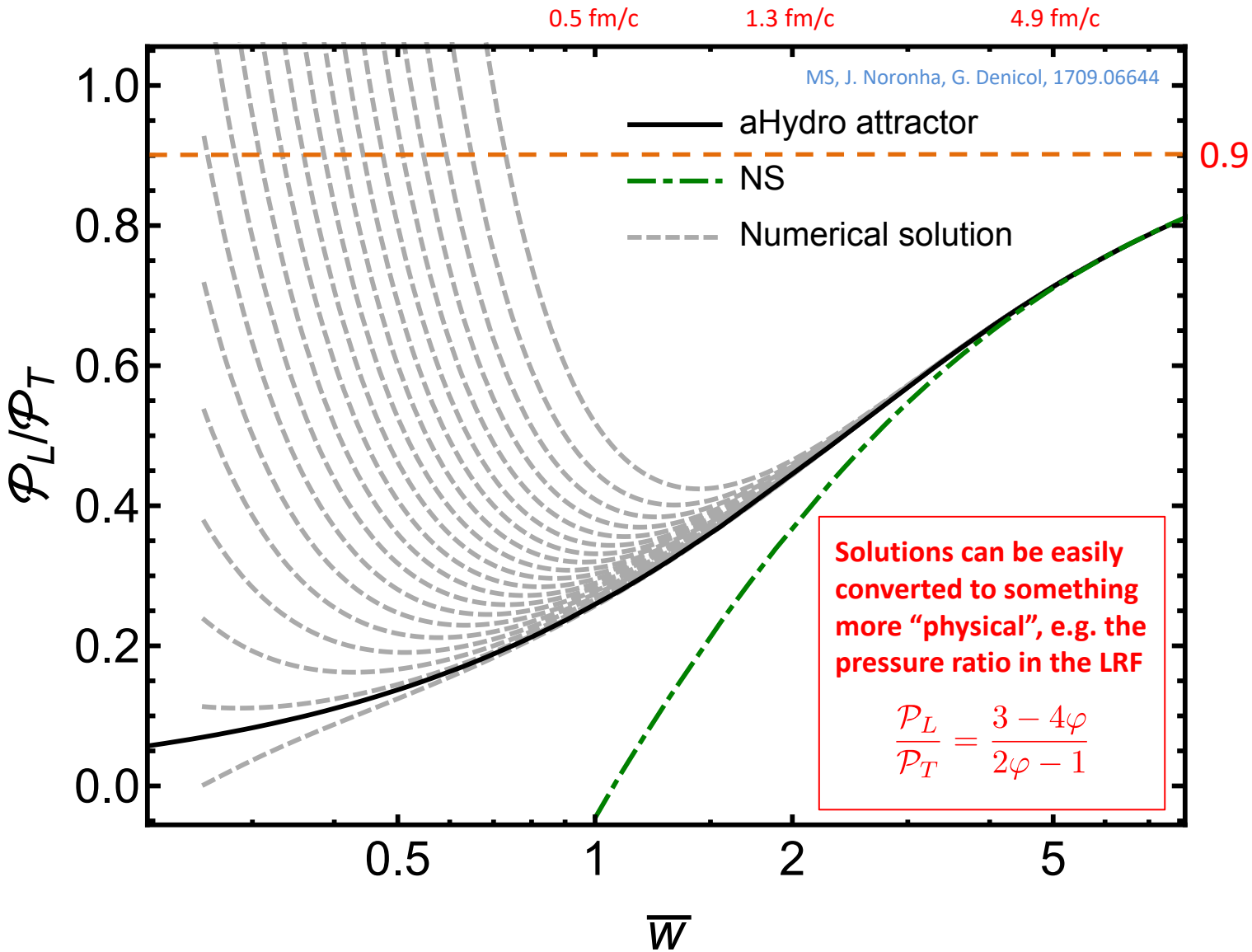


The attractor concept – 0+1d



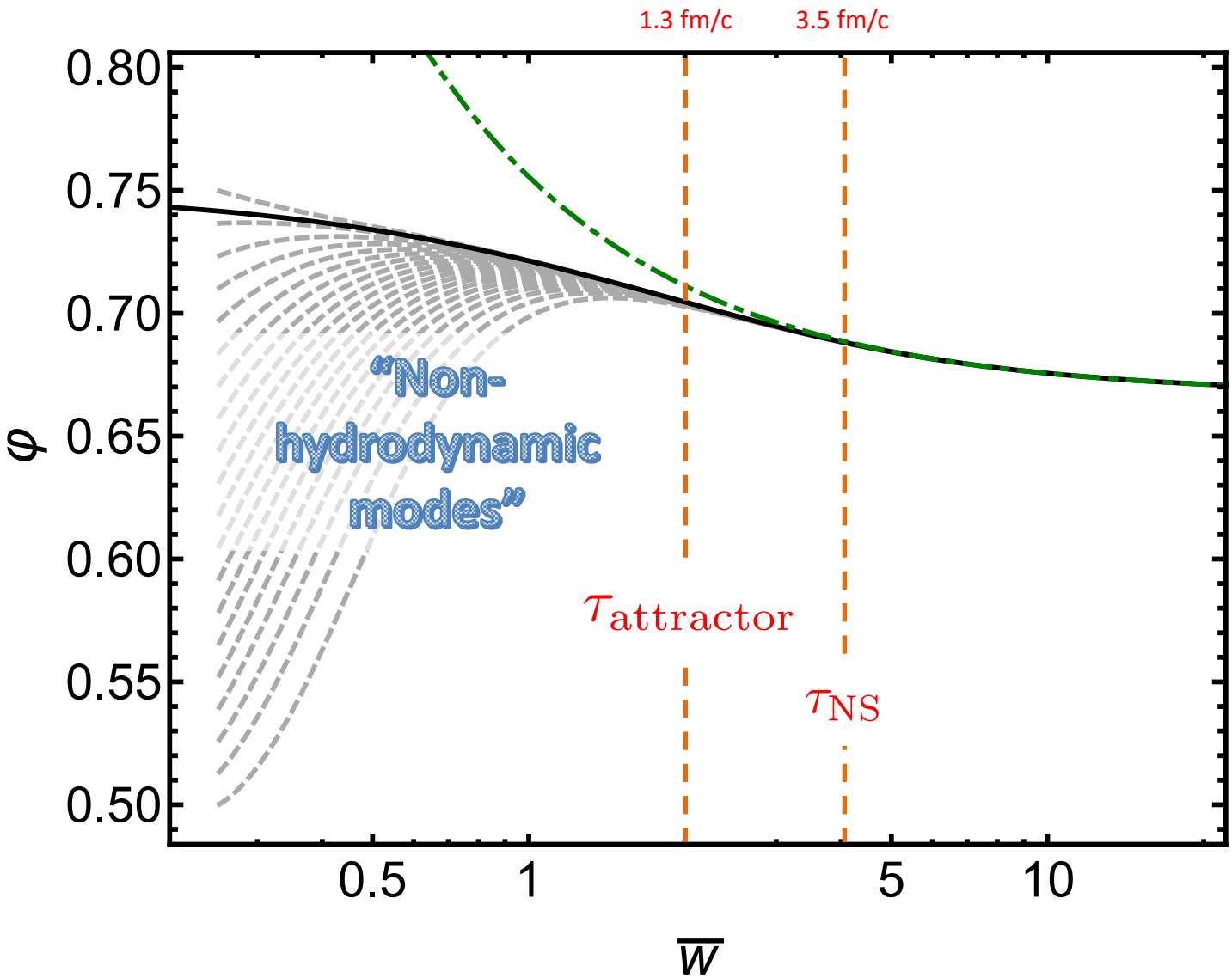
The attractor concept – 0+1d



Solutions can be easily converted to something more "physical", e.g. the pressure ratio in the LRF

$$\frac{\mathcal{P}_L}{\mathcal{P}_T} = \frac{3-4\varphi}{2\varphi-1}$$

The attractor concept – 0+1d



How does one obtain the attractor?

- Let's first look at 2nd order viscous hydrodynamics for simplicity (e.g. MIS amd DNMR, etc.)
- Start with the 0+1d energy conservation equation

$$\tau\dot{\epsilon} = -\frac{4}{3}\epsilon + \Pi \qquad \Pi = \Pi^{\varsigma}{}_{\varsigma}$$

- Change variables to

$$\boxed{w = \tau T} \qquad \boxed{\varphi(w) \equiv \tau\frac{\dot{w}}{w} = 1 + \frac{\tau}{4}\partial_\tau \log\epsilon}$$

$$\boxed{w\varphi\frac{\partial\varphi}{\partial w} = -\frac{8}{3} + \frac{20}{3}\varphi - 4\varphi^2 + \frac{\tau}{4}\frac{\dot{\Pi}}{\epsilon}}$$

How does one obtain the attractor?

- Need the evolution equation for the viscous correction.
- To linear order in the shear correction one has

$$\boxed{\dot{\Pi} = \frac{4\eta}{3\tau\tau_\pi} - \beta_{\pi\pi}\frac{\Pi}{\tau} - \frac{\Pi}{\tau_\pi}}$$

For DNMR in RTA $\beta_{\pi\pi} = \frac{38}{21}$

For MIS in RTA $\beta_{\pi\pi} = \frac{4}{3}$

- Plugging this into the energy-momentum conservation equation gives

$$\boxed{\overline{w}\varphi\varphi' + 4\varphi^2 + \left[\overline{w} + \left(\beta_{\pi\pi} - \frac{20}{3}\right)\right]\varphi - \frac{4c_{\eta/\pi}}{9} - \frac{2}{3}(\beta_{\pi\pi} - 4) - \frac{2\overline{w}}{3} = 0}$$

For DNMR in RTA $\overline{w} \equiv \frac{w}{c_\pi} = \frac{\tau T}{5\bar{\eta}}$ $\quad c_{\eta/\pi} \equiv \frac{c_\eta}{c_\pi} = \frac{1}{5}$

How does one <u>solve for</u> the attractor?

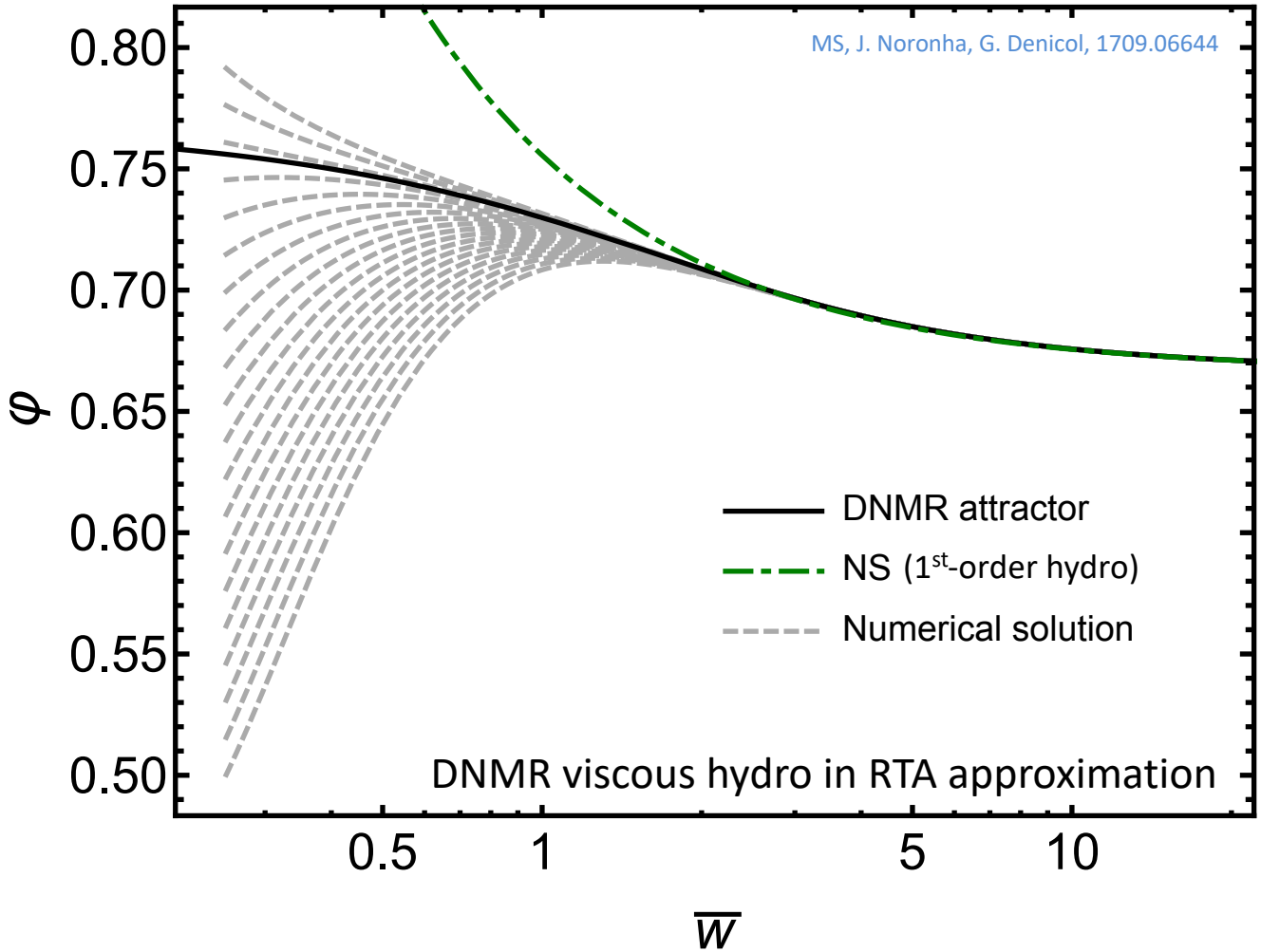
$$\boxed{\overline{w}\varphi\varphi' + 4\varphi^2 + \left[\overline{w} + \left(\beta_{\pi\pi} - \frac{20}{3}\right)\right]\varphi - \frac{4c_{\eta/\pi}}{9} - \frac{2}{3}(\beta_{\pi\pi} - 4) - \frac{2\overline{w}}{3} = 0}$$

- First try to approximate using "slow-roll" approx ($\varphi' = 0$)
- From this, we can read off the boundary condition as $w \to 0$

$$\lim_{\overline{w}\to 0} \varphi(\overline{w}) = \frac{1}{24}\left(-3\beta_{\pi\pi} + \sqrt{64c_{\eta/\pi} + (3\beta_{\pi\pi} - 4)^2} + 20\right)$$

- Then numerically solve the ODE at the top of the slide

The attractor concept – 0+1d



Conformal 0+1d aHydro attractor

- Use Romatschke-Strickland form

$$f(\tau, \mathbf{x}, \mathbf{p}) = f_{\rm eq}\left(\frac{1}{\Lambda(\tau, \mathbf{x})}\sqrt{p_T^2 + [1+\xi(\tau, \mathbf{x})]p_L^2}\right)$$

- Take 1st and 2nd moments of the Boltzmann equation (zz projection minus 1/3 (xx+yy+zz) projection)

$$\boxed{\begin{aligned} &\tau\dot{\epsilon} = -\frac{4}{3}\epsilon + \Pi \\ &\frac{1}{1+\xi}\dot{\xi} - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\rm eq}}\xi\sqrt{1+\xi} = 0 \end{aligned}}$$

$$\Pi = P_0 - \mathcal{P}_L$$

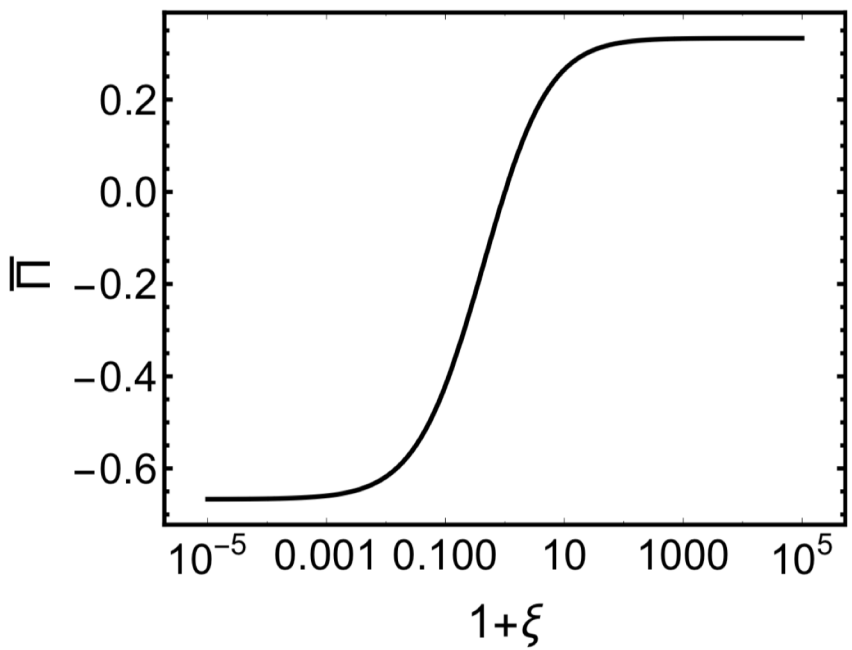
$$\mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$$

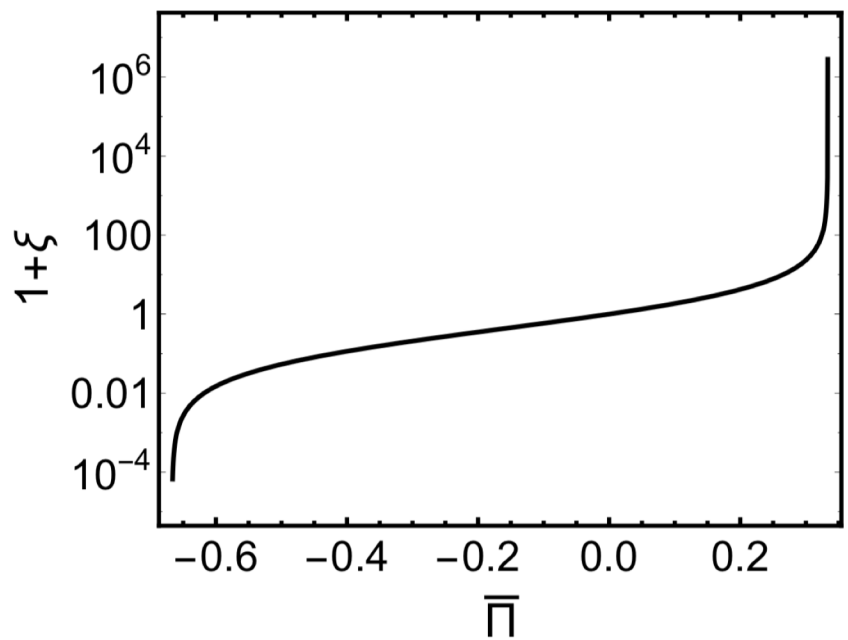
Conformal 0+1d aHydro attractor

- Convert 2nd moment equation to "vHydro" form using

$$\overline{\Pi}(\xi) \equiv \frac{\Pi}{\epsilon} = \frac{1}{3}\left[1 - \frac{\mathcal{R}_L(\xi)}{\mathcal{R}(\xi)}\right]$$

$$\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi) - 1}{\xi + 1}\right]$$





Conformal 0+1d aHydro attractor

aHydro 2nd moment eq in "vHydro form"

$$\frac{\dot{\Pi}}{\epsilon} + \frac{\Pi}{\epsilon\tau}\left(\frac{4}{3} - \frac{\Pi}{\epsilon}\right) - \left[\frac{2(1+\xi)}{\tau} - \frac{\mathcal{W}(\xi)}{\tau_{\rm eq}}\right]\overline{\Pi}'(\xi) = 0$$

$$\mathcal{W}(\xi) \equiv \xi(1+\xi)^{3/2}\mathcal{R}^{5/4}(\xi)$$

aHydro resums an infinite number of terms in the inverse Reynolds number.

$$R_\pi^{-1} = \frac{\sqrt{\Pi^{\mu\nu}\Pi_{\mu\nu}}}{P_0} = 3\sqrt{\frac{3}{2}}|\overline{\Pi}|$$

Note that if the aHydro equation above is expanded for small ξ it automatically reproduces the DNMR result.

$$\left.\begin{aligned}
\xi &= \frac{45}{8}\overline{\Pi}\left[1 + \frac{195}{56}\overline{\Pi} + \mathcal{O}(\overline{\Pi}^2)\right] \\
\overline{\Pi}' &= \frac{8}{45} - \frac{26}{21}\overline{\Pi} + \frac{1061}{392}\overline{\Pi}^2 + \mathcal{O}(\overline{\Pi}^3) \\
\mathcal{W} &= \frac{45}{8}\overline{\Pi}\left[1 + \frac{405}{56}\overline{\Pi} + \mathcal{O}(\overline{\Pi}^3)\right]
\end{aligned}\right\} \longrightarrow \dot{\Pi} - \frac{4\eta}{3\tau_\pi\tau} + \frac{38}{21}\frac{\Pi}{\tau} = -\frac{\Pi}{\tau_\pi}$$

Conformal 0+1d aHydro attractor

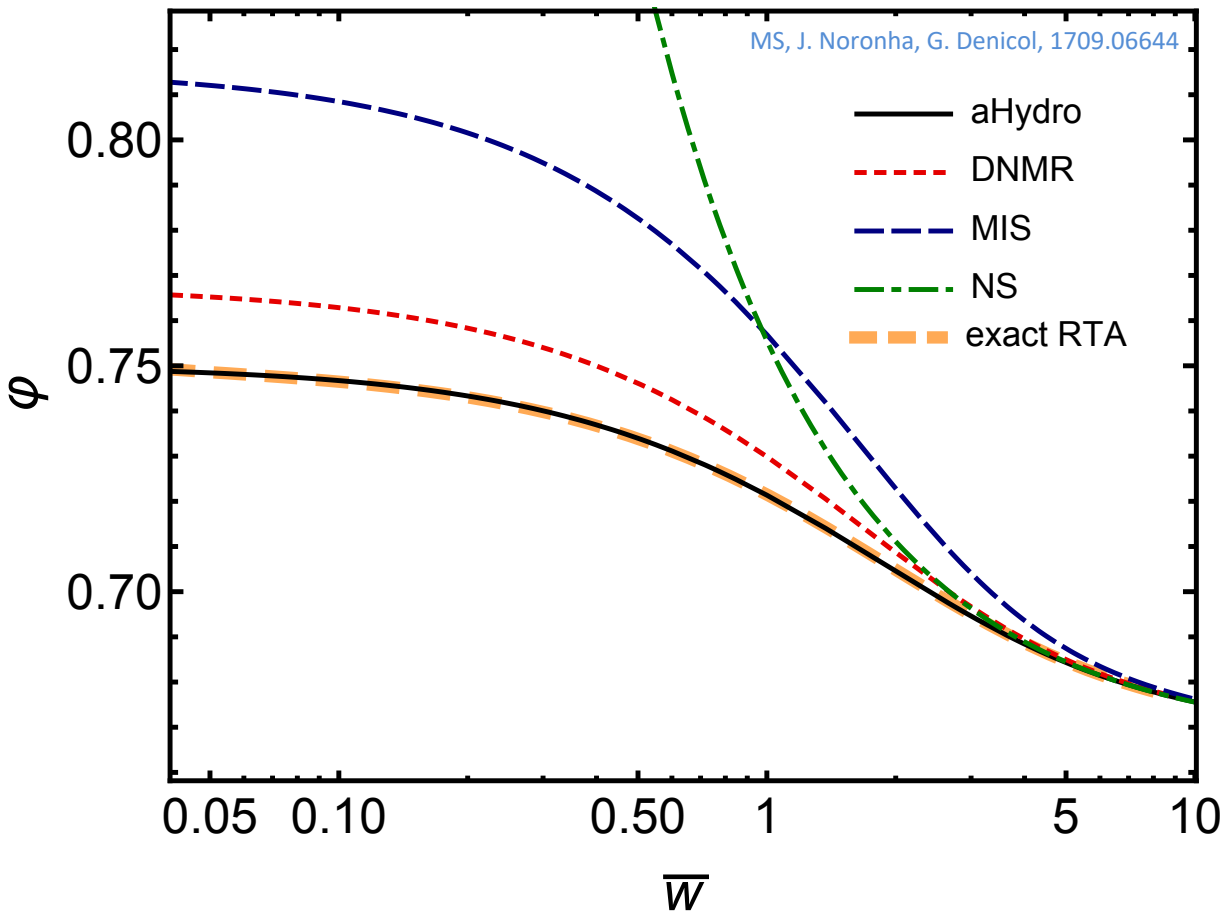
$$\boxed{\overline{w}\varphi\frac{\partial\varphi}{\partial\overline{w}} = \left[\frac{1}{2}(1+\xi) - \frac{\overline{w}}{4}\mathcal{W}\right]\overline{\Pi}'}$$

Slow-roll approximation → $\lim_{\overline{w}\to 0}\varphi(\overline{w}) = \frac{3}{4}$

Now just solve this ODE numerically….

Result

- Can **extract the attractor** for MIS, DNMR, aHydro, and the exact RTA solution [W. Florkowski, R. Ryblewski, and MS, 1304.0665]



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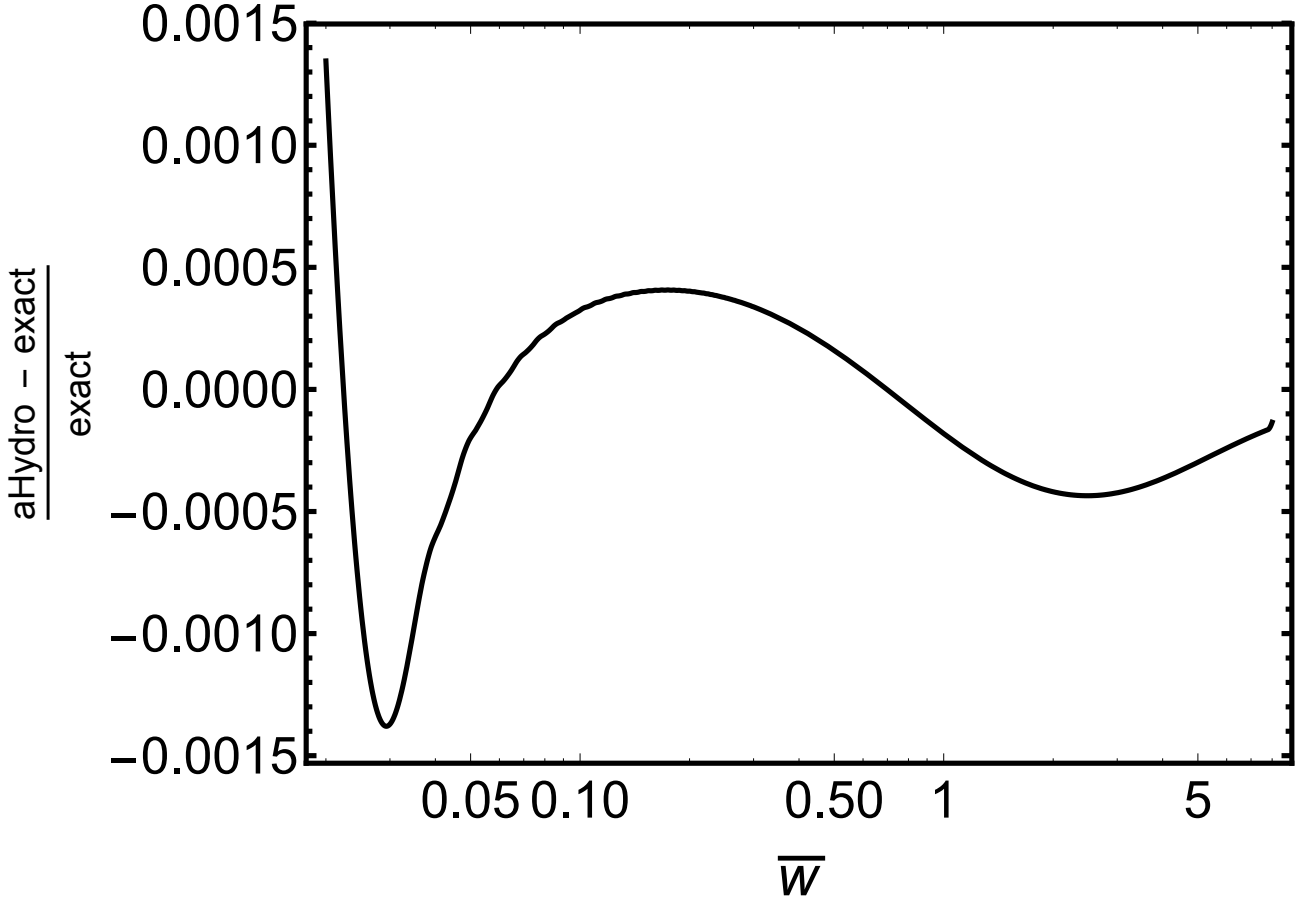


Part 2

Scalar collisional kernel

D. Almaalol and MS, 1801.10173

LO massless scalar scattering

- Change RTA → more realistic scattering kernel

$$C[f_p] = \frac{1}{32}\int dK dK' dP' \, |\mathcal{M}|^2 \, (2\pi)^4 \delta^{(4)}(k^\alpha + k'^\alpha - p^\alpha - p'^\alpha) \, \mathcal{F}(k,k',p,p')$$

$$\mathcal{F}(k,k',p,p') \equiv f_k f_{k'}(1+af_p)(1+af_{p'}) - (1+af_k)(1+af_{k'}) f_p f_{p'}$$

- a=0,1 correspond to classical, quantum stats

$$\partial_\lambda I^{\lambda\mu\nu} = \mathcal{C}^{\mu\nu}$$ ← 2nd moment of the Boltzmann equation

$$\boxed{\mathcal{C}^{\mu\nu} = \frac{1}{128\pi^2}\int dK dK' d\Omega_p \frac{p|\mathcal{M}|^2}{E_{p'}} \mathcal{F}(k,k',p,p') p^\mu p^\nu \Big|_{p\to\tilde{p}}}$$

$$I^{\mu\nu_1\nu_2\cdots\nu_n} \equiv \int dP \, p^\mu p^{\nu_1} p^{\nu_2} \cdots p^{\nu_n} f \qquad \mathcal{C}^{\nu_1\nu_2\cdots\nu_n} \equiv \int dP \, p^{\nu_1} p^{\nu_2} \cdots p^{\nu_n} \, C[f] \qquad \tilde{p} \equiv \frac{kk' - \mathbf{k}\cdot\mathbf{k}'}{k + k' - \mathbf{k}\cdot\hat{\mathbf{p}} - \mathbf{k}'\cdot\hat{\mathbf{p}}}$$

0+1d aHydro equations of motion

- 1st moment is as before

$$\partial_\tau \varepsilon = -\frac{\varepsilon + P_L}{\tau}$$

- 2nd moment is

$$\frac{1}{1+\xi}\partial_\tau \xi - \frac{2}{\tau} = \frac{\Lambda\lambda^2}{\kappa_a}\left[(1+\xi)^{1/2}\tilde{\mathcal{C}}^{xx}(\xi) - (1+\xi)^{3/2}\tilde{\mathcal{C}}^{zz}(\xi)\right]$$

$$\tilde{C}^{ii} = \frac{\mathcal{C}^{ii}}{\Lambda^6\lambda^2}$$

$$\kappa_a = \begin{cases} \frac{4}{\pi^2} & \text{if } a = 0 \ \ (\text{classical})\,, \\ \frac{4\zeta(5)}{\pi^2} & \text{if } a = 1 \ \ (\text{quantum})\,. \end{cases}$$

Small anisotropy limit

Classical

$$\mathcal{F}(k,k',p,p') = \frac{e^{-\frac{k+k'}{\Lambda}}}{2\Lambda p'}\,\mathcal{G}(\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}')\,\xi + \mathcal{O}(\xi^2)$$

Quantum

$$\mathcal{F}(k,k',p,p') = \frac{e^{\frac{k+k'}{\Lambda}} f_{\rm eq}(k/\Lambda) f_{\rm eq}(k'/\Lambda) f_{\rm eq}(p/\Lambda) f_{\rm eq}(p'/\Lambda)}{2\Lambda p'}\,\mathcal{G}(\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}')\,\xi + \mathcal{O}(\xi^2)$$

$$\begin{aligned}\mathcal{G}(\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}') = {} & 2k\cos\theta_k(k'\cos\theta_{k'} - p\cos\theta_p) \\ & + k(p-k')\cos^2\theta_k + k'(p-k)\cos^2\theta_{k'} \\ & + p(k+k')\cos^2\theta_p - 2k'p\cos\theta_{k'}\cos\theta_p\end{aligned}$$

→ Evaluate, e.g. C^{zz}, by performing the 8d integral using Monte Carlo

$$\lim_{\xi\to 0}\frac{\mathcal{C}^{zz}}{\Lambda^6} = \alpha_a\lambda^2\xi + \mathcal{O}(\xi^2)$$

$$\alpha_0 \simeq 0.4394 \pm 0.0002$$

$$\alpha_1 \simeq 0.7773 \pm 0.0008$$

Matching to RTA

- In RTA one has

$$\mathcal{C}^{zz}_{\rm RTA} = \frac{\kappa_a \Lambda^6}{5\bar{\eta}}\left[\mathcal{R}^{3/2}(\xi) - \frac{\mathcal{R}^{1/4}(\xi)}{(1+\xi)^{3/2}}\right] \qquad \kappa_a = \begin{cases} \frac{4}{\pi^2} & \text{if } a=0 \ \ (\text{classical})\,, \\ \frac{4\zeta(5)}{\pi^2} & \text{if } a=1 \ \ (\text{quantum})\,. \end{cases}$$

$$\boxed{\lim_{\xi\to 0}\frac{\mathcal{C}^{zz}_{\rm RTA}}{\Lambda^6} = \frac{2\kappa_a}{15\bar{\eta}}\xi + \mathcal{O}(\xi^2)}$$

Matching the scalar kernel and RTA kernel in the small anisotropy limit one obtains

$$\lambda^2 = \frac{2\kappa_a}{15\alpha_a\bar{\eta}}$$

Final 2nd moment equation

- Using this matching one obtains

$$\boxed{\partial_\tau \xi - \frac{2(1+\xi)}{\tau} + \frac{\mathcal{W}(\xi)}{\tau_{\rm eq}} = 0}$$

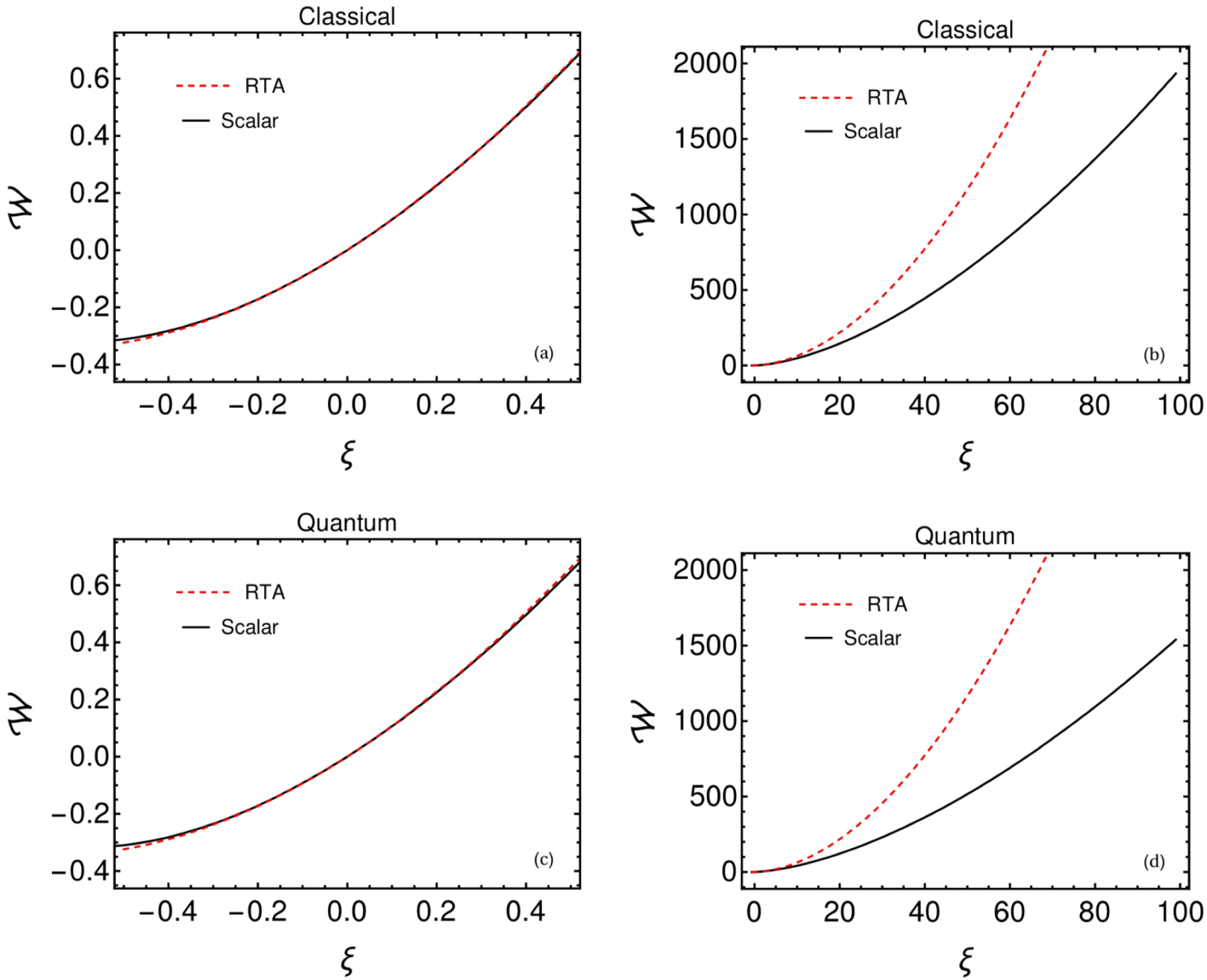
with $\tau_{\rm eq} = 5\bar{\eta}/T$ and

$$\mathcal{W}(\xi) \equiv \frac{2}{3\alpha_a \mathcal{R}^{1/4}(\xi)} \left[(1+\xi)^{5/2} \tilde{\mathcal{C}}^{zz}(\xi) - (1+\xi)^{3/2} \tilde{\mathcal{C}}^{xx}(\xi) \right]$$

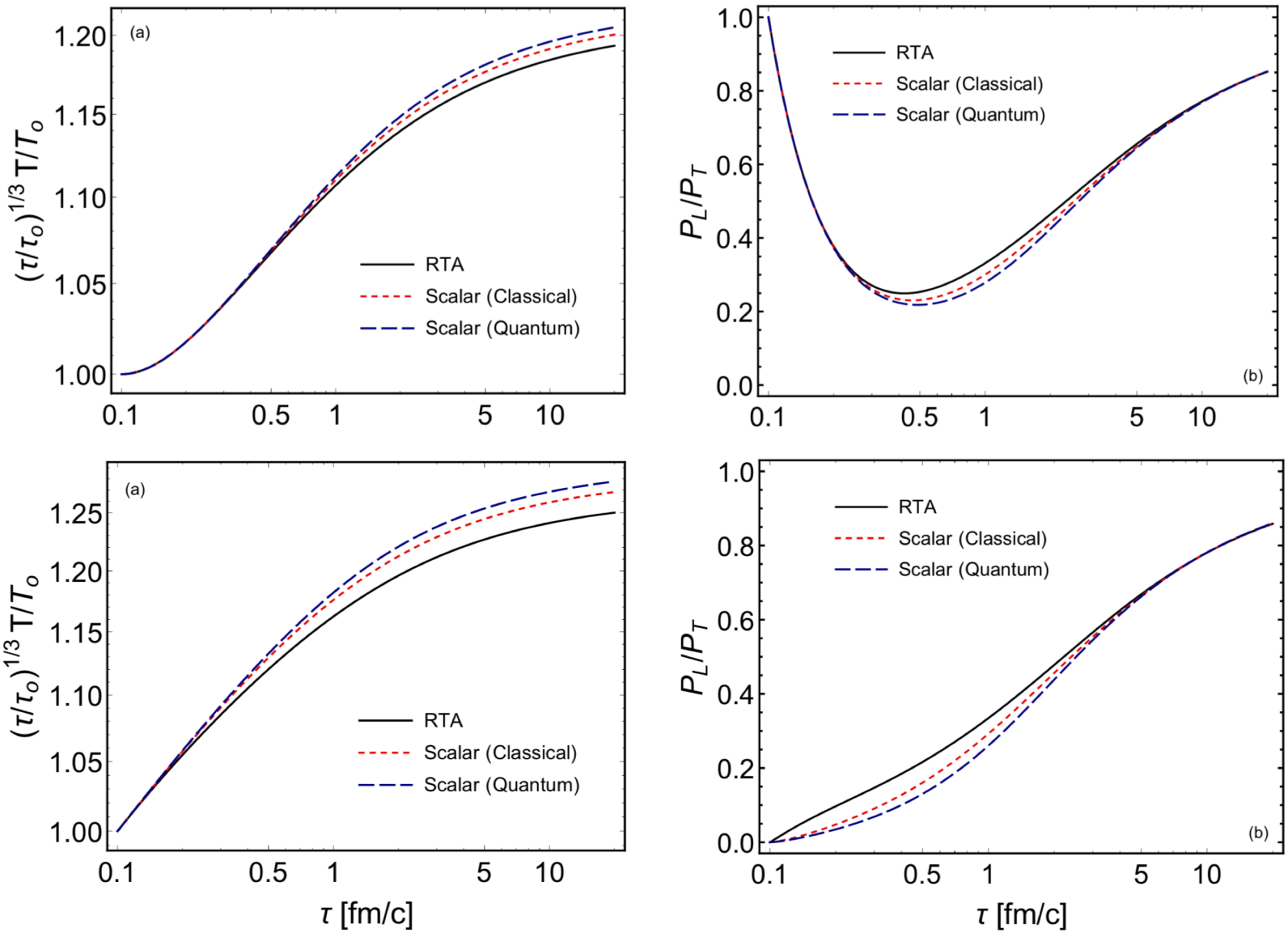
- Cast in same form as RTA 2nd moment equation. Recall that in RTA

$$\mathcal{W}(\xi) \to \mathcal{W}_{\rm RTA}(\xi) = \xi (1+\xi)^{3/2} \mathcal{R}^{5/4}(\xi)$$

Comparisons



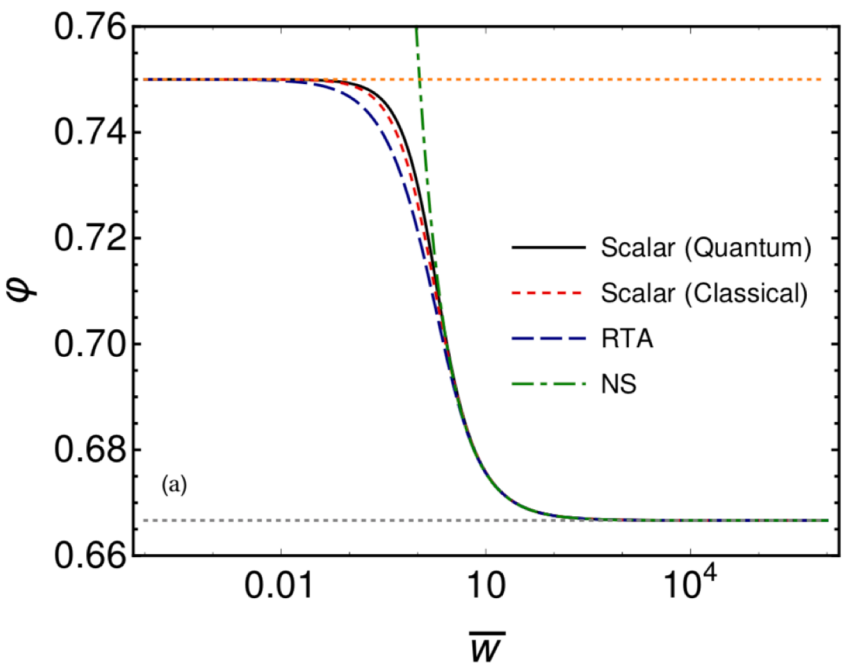
Comparisons

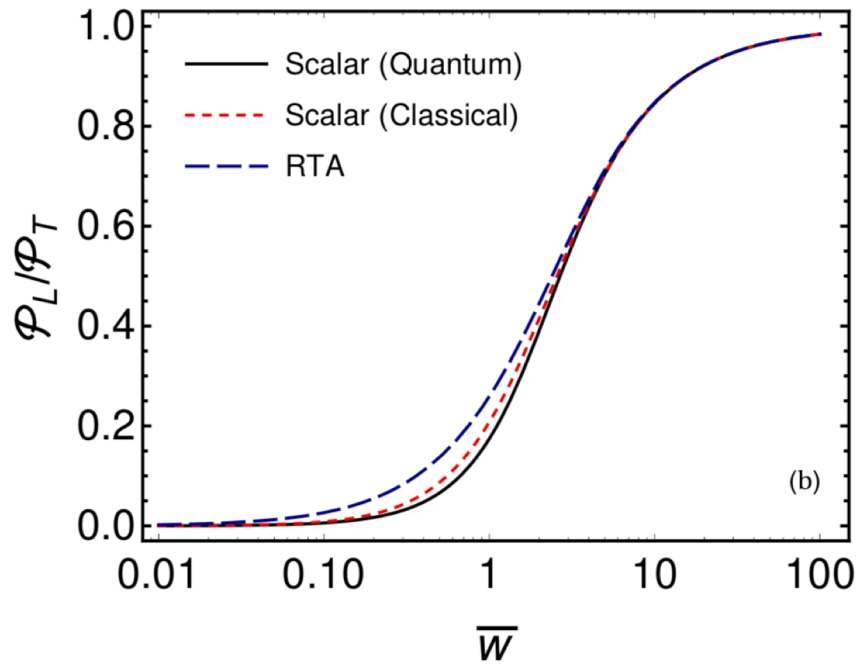


Attractor

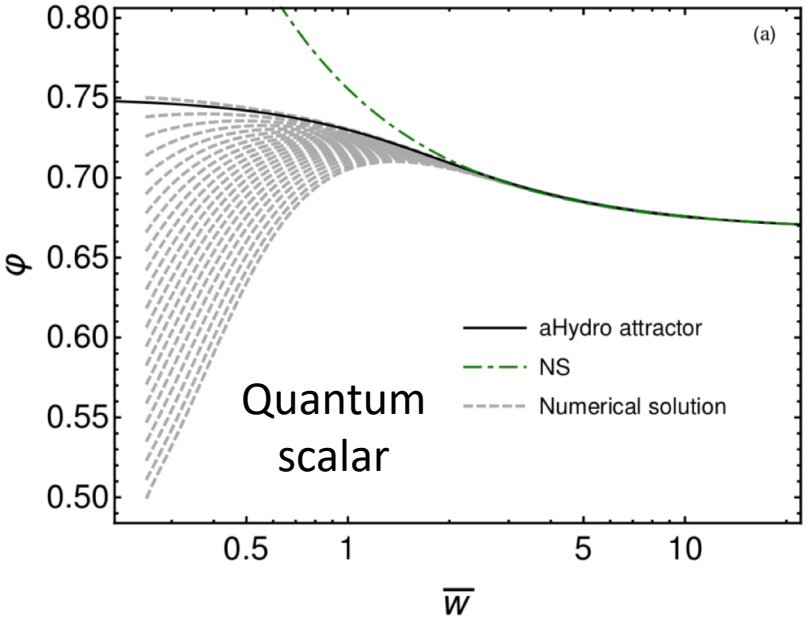
- Attractor equation is of same form; just need W function for the kernel you are considering

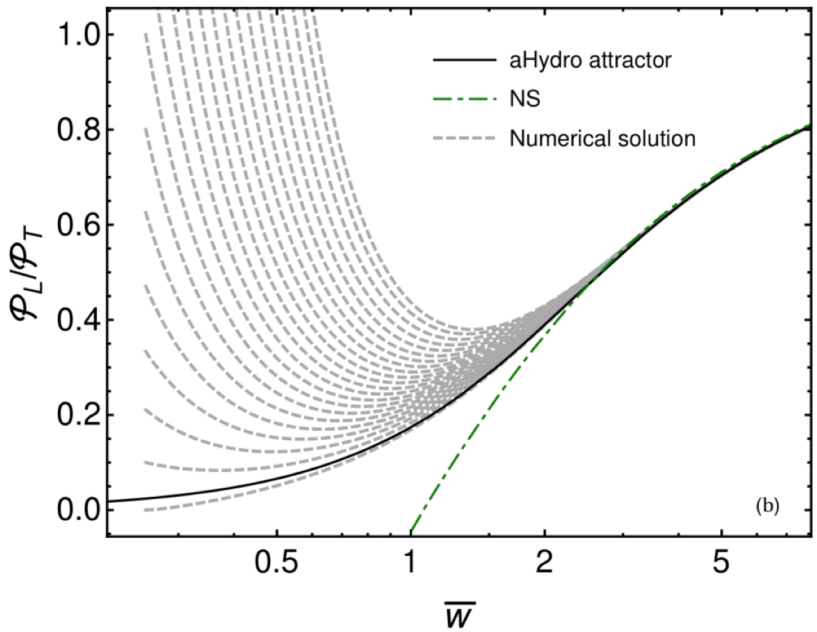
$$\overline{w}\varphi\frac{\partial\varphi}{\partial\overline{w}} = \left[\frac{1}{2}(1+\xi) - \frac{\overline{w}}{4}\mathcal{W}(\xi)\right]\bar{\pi}'$$





Approach to attractor





$$\frac{\mathcal{P}_L}{\mathcal{P}_T} - \left(\frac{\mathcal{P}_L}{\mathcal{P}_T}\right)_{\text{attractor}} \simeq A e^{-\gamma \overline{w}}$$

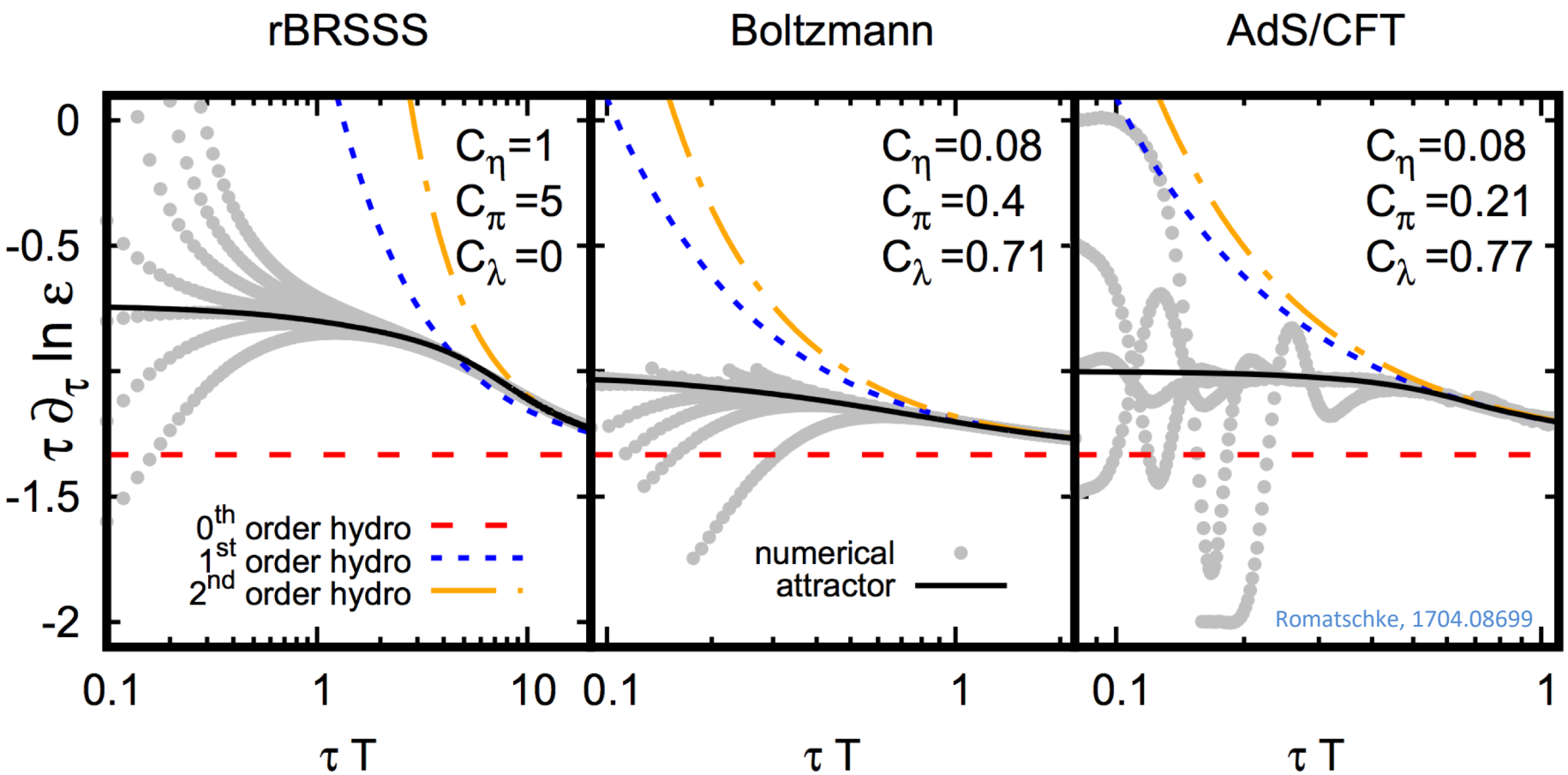
$$\gamma_0 = 1.73 \pm 0.01$$
$$\gamma_1 = 1.63 \pm 0.01$$
$$\gamma_{\text{RTA}} = 1.88 \pm 0.01$$

Conclusions

- aHydro attractor provides best approximation to exact RTA attractor
- Does this because of the resummation in inverse Reynolds number
- Scalar collisional kernel gives qualitatively similar results but larger differences in evolution as system gets farther away from equilibrium
- Scalar kernel results in larger degree of momentum-space anisotropy
- Scalar kernel results in slower approach to the attractor than RTA
- QCD comes next → moments of Kurkela AMY kernel
- Can be applied to full 3+1d simulations.

Backup slides

Attractor exists in many theories



Romatschke, 1704.08699; see also Keegan et al, 1512.05347

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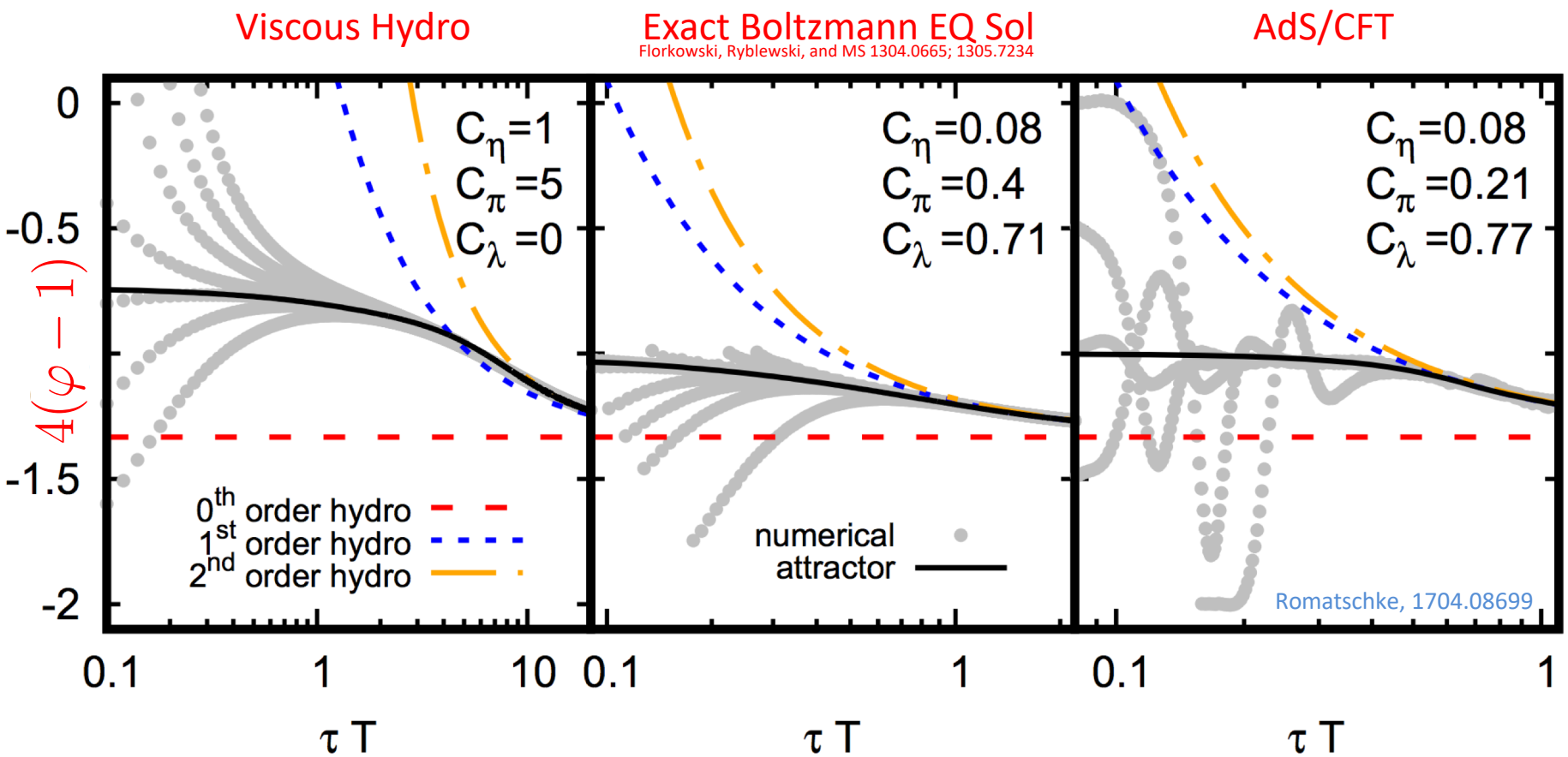

Romatschke, 1704.08699; see also Keegan et al, 1512.05347