Relativistic hydrodynamics in strong gravitational fields

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Plan of the talk

- When hydrodynamics is general relativistic
- Application: binary neutron star mergers
- Where we are now:
  - fundamental aspects of GRHD
- Where we are going:
  - away from perfect fluids and ideal MHD
  - new advanced methods

When hydrodynamics is general relativistic

The goals of general-relativistic hydrodynamics are to describe the dynamics of a **self-gravitating fluid** in regions with **strong** and **dynamical** gravitational fields.

As a result:

- spacetime curvature needs to be taken into **account**
- spacetime curvature varies with **time** (Einstein eqs.)
- dynamics is necessarily **relativistic**: $v \sim c$

The equations of GRHD/MHD

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \,, \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu}_{\rm fluid} = \mathcal{R}^\nu_{\rm rad} \qquad \text{(cons. energy/momentum)}$$

$$\nabla_\mu (\rho u^\mu) = 0 \,, \qquad \text{(cons. rest mass)}$$

$$p = p(\rho, \epsilon, Y_e, \ldots) \,, \qquad \text{(equation of state)}$$

$$\nabla_\nu F^{\mu\nu} = I^\mu \,, \quad \nabla^*_\nu F^{\mu\nu} = 0 \,, \qquad \text{(Maxwell equations)}$$

$$T_{\mu\nu} = T^{\rm fluid}_{\mu\nu} + T^{\rm EM}_{\mu\nu} + \ldots \qquad \text{(energy – momentum tensor)}$$

We normally solve the same set of equations:

$$\nabla_\mu T^{\mu\nu} = 0$$

Overall, in general-**relativistic hydrodynamics** most of the **headaches** come from

$$\nabla_\mu$$

Instead, in **special-relativistic hydrodynamics** most of the **headaches** come from

$$T^{\mu\nu}$$

Hereafter, I will assume a **perfect fluid**.

What are the challenges?

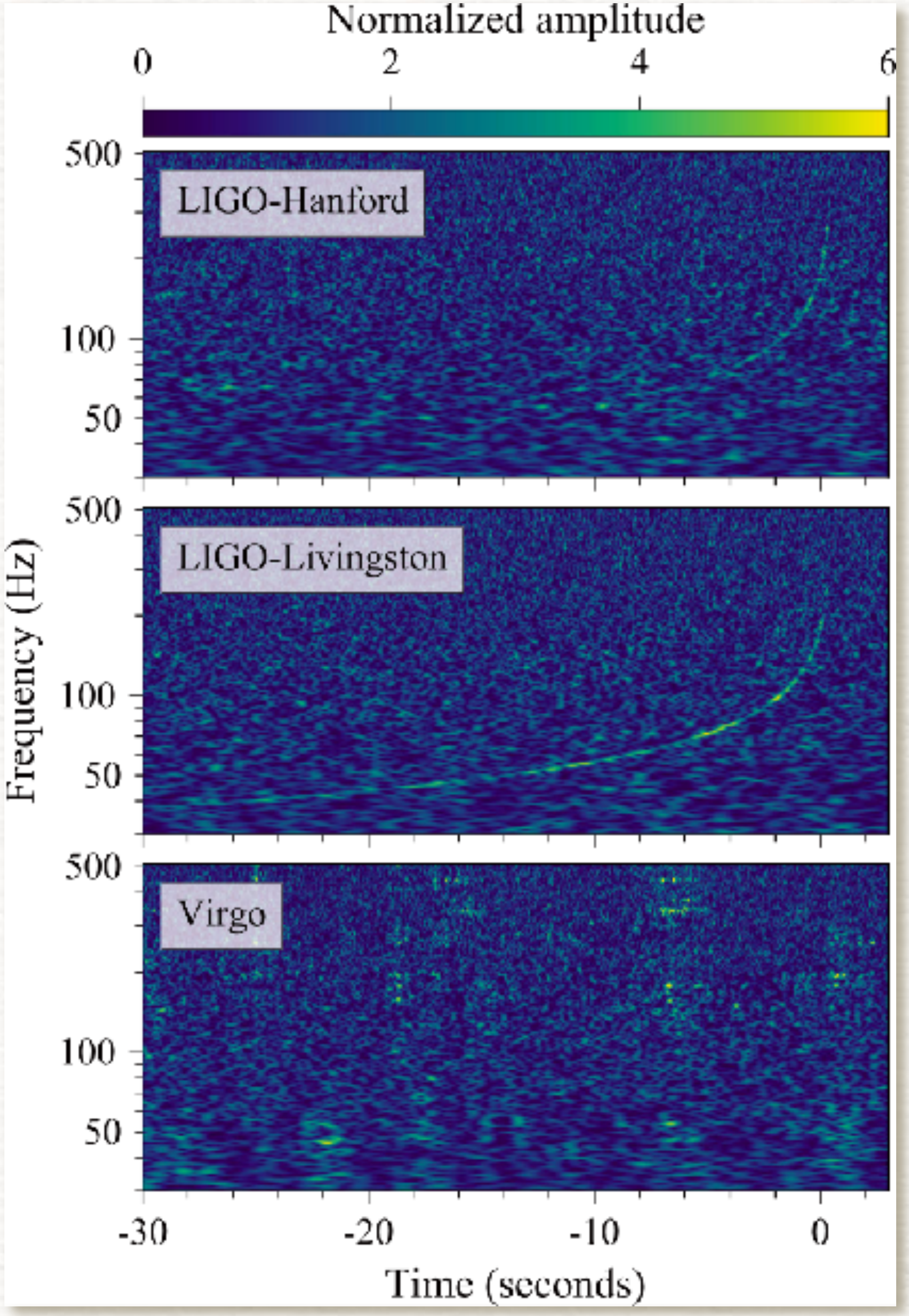
- On 16 October 2017 the LSC/Virgo collaboration announced detection of the gravitational signal from merging binary neutron-star system: **GW170817**.
- Total mass:

$$M_1 + M_2 = 2.74^{+0.04}_{-0.01} M_\odot$$

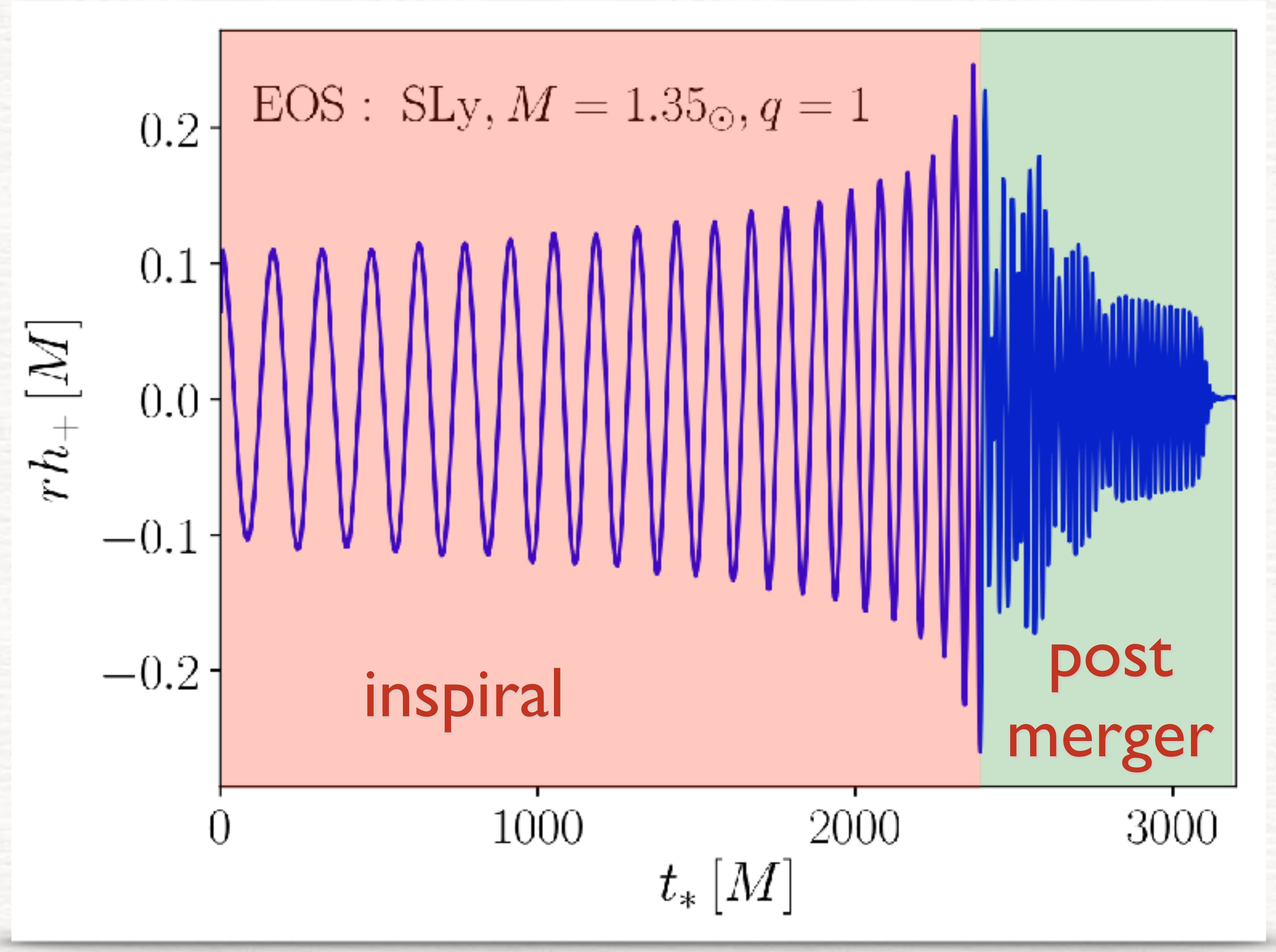
- Individual masses:

$$M_1 = 1.36 - 1.60 M_\odot$$

$$M_2 = 1.17 - 1.36 M_\odot$$

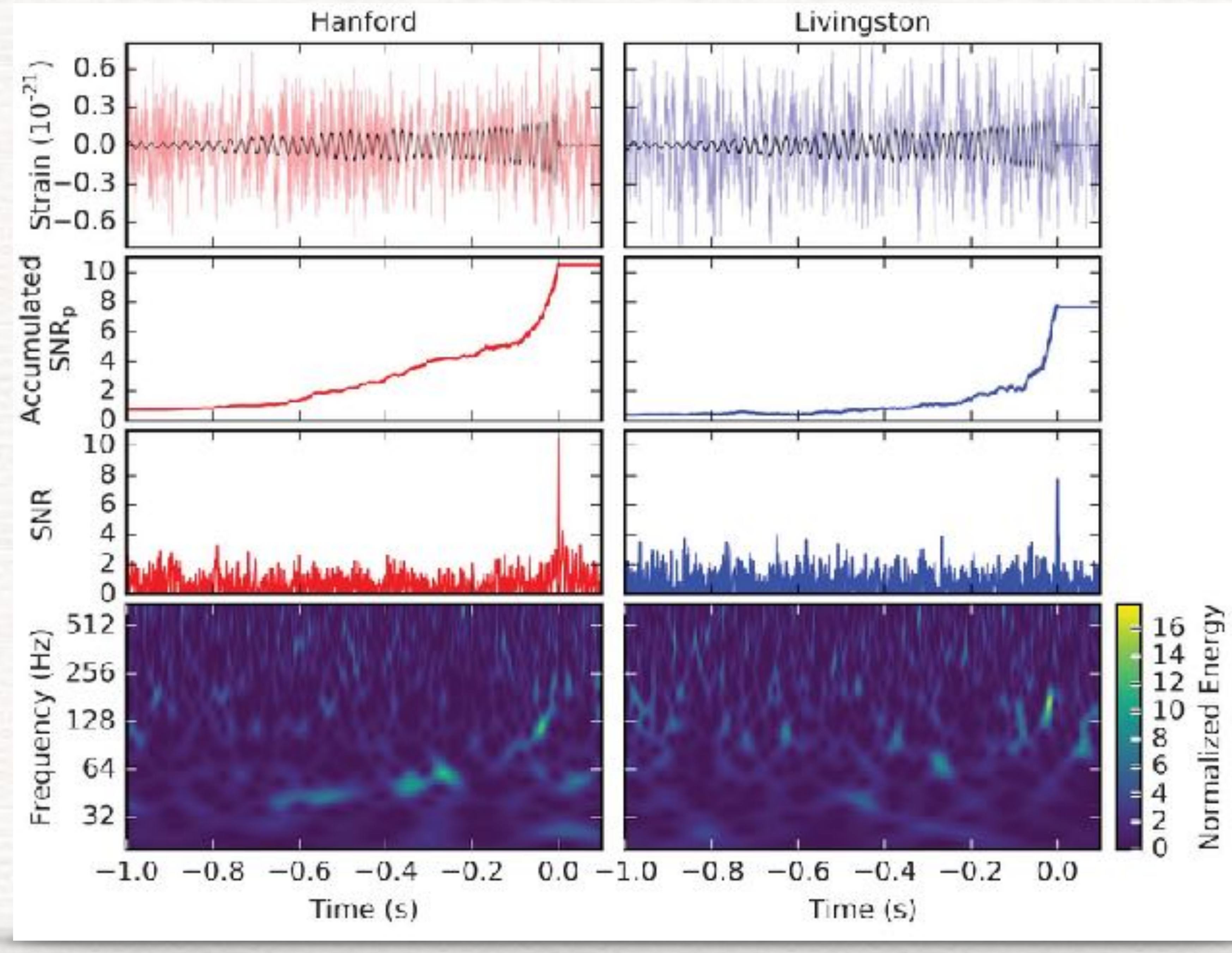


What we want to model: GWs



Winning over the noise

Detection in most cases possible only when signal can be extracted from noise: matched filtering



What we want to model: SGRBs

GW170817 has confirmed expectation that merger is accompanied by electromagnetic counterpart: **short gamma-ray burst (SGRBs)**.

short GRB,
artist impression
(NASA)

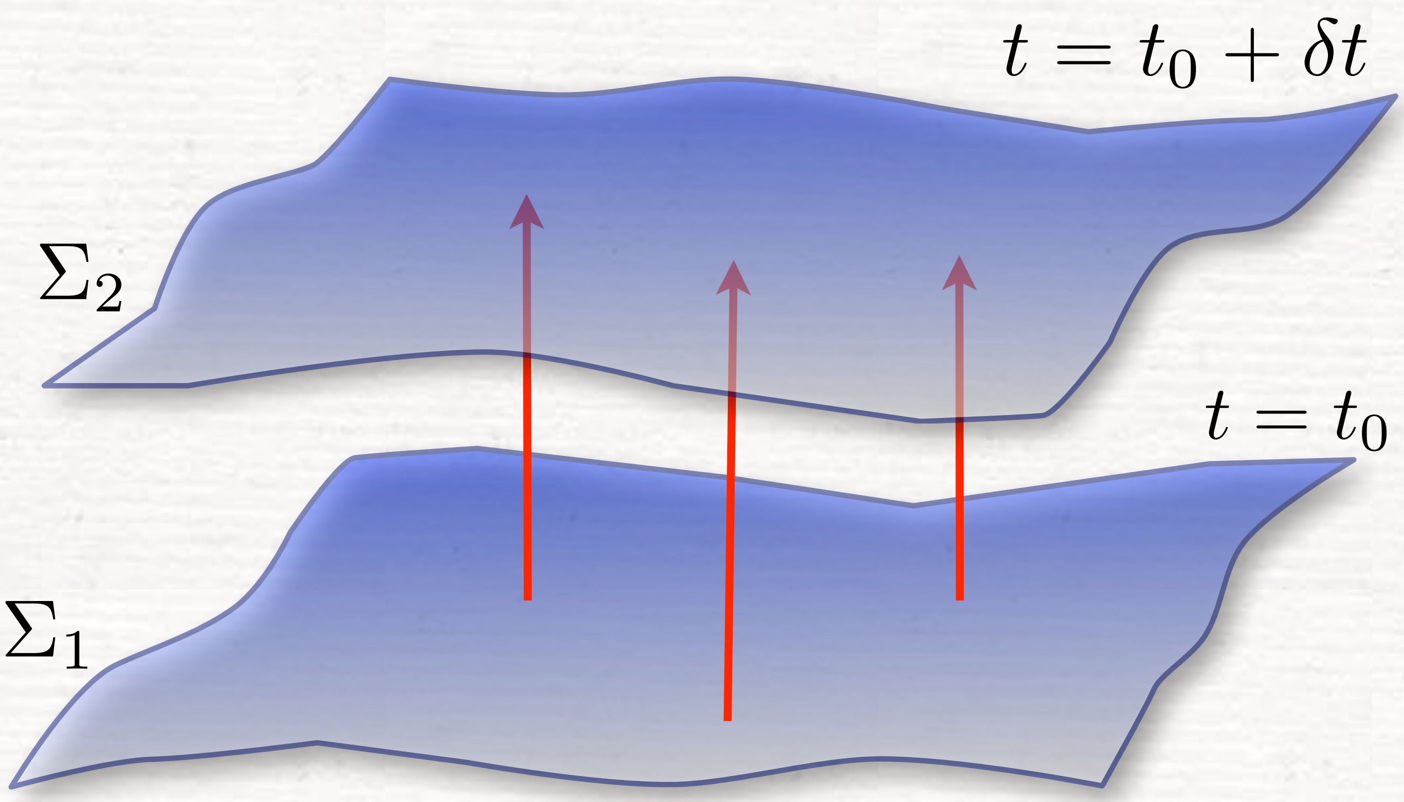
The techniques of numerical relativity

The basics first

- Einstein equations provide a **local** solution of spacetime curvature (not of topology)
- Einstein equations are **covariant**:
  - before finding a solution need to find a set of coordinates where to find this solution: 3+1 standard choice.
  - given a set of coordinates, you still have total freedom on how to write equations
  - perfectly reasonable choices can lead to weakly hyperbolic systems and hence to problems

First step: foliate the 4D spacetime

Given a manifold $\mathcal{M}$ with with 4-metric $g_{\mu\nu}$, we want to foliate it via space-like, three-dimensional hypersurfaces, i.e., $\Sigma_1, \Sigma_2, \ldots$ levelled by a scalar function. The time coordinate t is obvious choice



Define therefore

$$\Omega_\mu \equiv \nabla_\mu t$$

such that

$$|\Omega|^2 \equiv g^{\mu\nu} \nabla_\mu t \nabla_\nu t = -\alpha^{-2}$$

This defines the "lapse" function which is strictly positive for spacelike hypersurfaces

$$\alpha(t, x^i) > 0$$

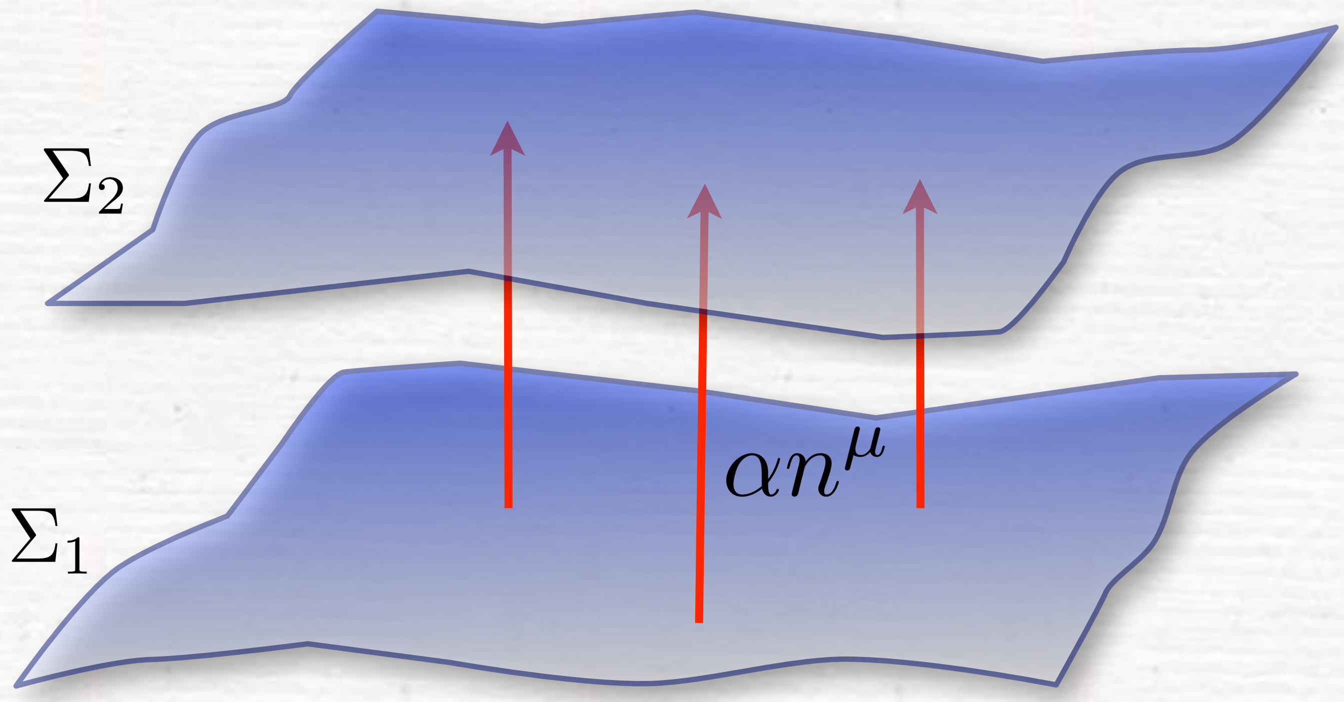
The lapse function allows then to do two important things:

i) define the unit normal vector to the hypersurface Σ

$$n^\mu \equiv -\alpha g^{\mu\nu}\Omega_\nu = -\alpha g^{\mu\nu}\nabla_\nu t$$

where

$$n^\mu n_\mu = -1$$



ii) define the spatial metric

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$$

Finding a direction for evolutions

The unit normal $\boldsymbol{n}$ to a spacelike hypersurface Σ is not good time vector because $\boldsymbol{n}$ is not dual to surface one-norm $\boldsymbol{\Omega}$

$$n^\mu \Omega_\mu = n^\mu \nabla_\mu t = -\alpha \Omega^\mu \Omega_\mu = \frac{1}{\alpha}$$

Need a vector along which to carry out the time evolutions dual to the surface one-norm. Such a vector is defined as

$$t^\mu \equiv \alpha n^\mu + \beta^\mu$$

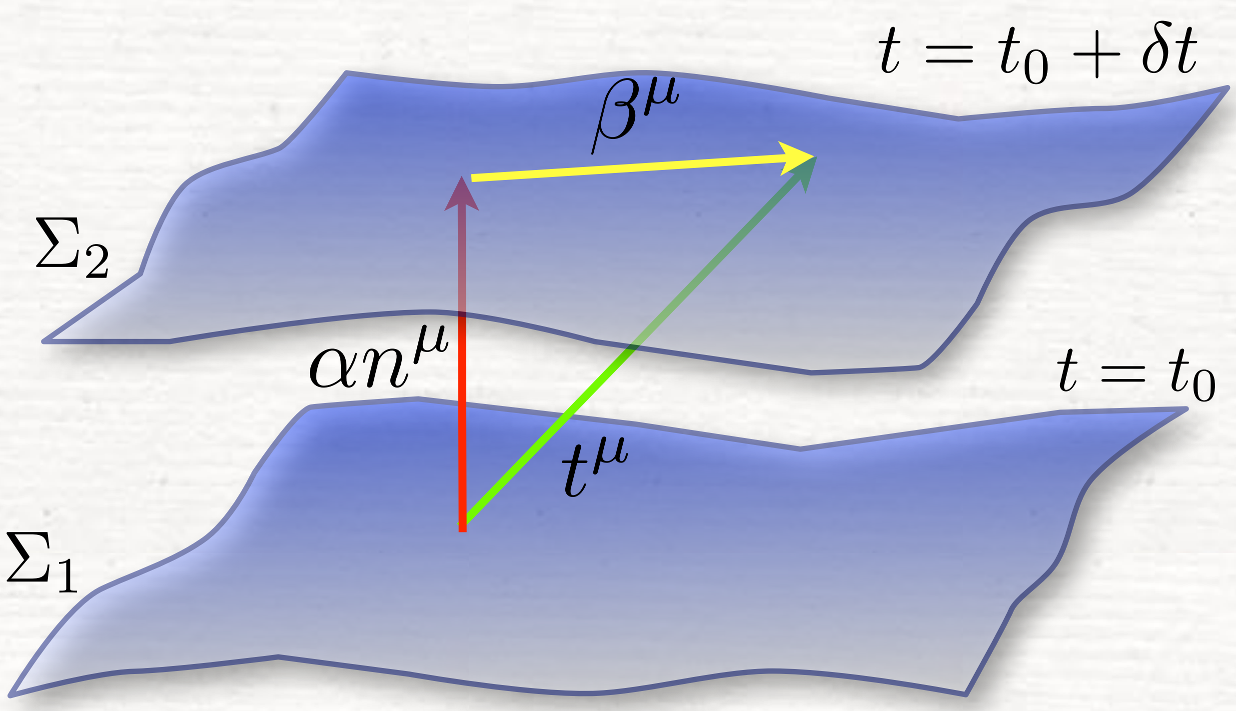
where $\boldsymbol{\beta}$ is any spatial "shift" vector.

Clearly now the two tensors are dual to each other, ie

$$t^\mu \Omega_\mu = \alpha n^\mu \Omega_\mu + \beta^\mu \Omega_\mu = \alpha/\alpha = 1$$

Using the expression for the covariant 4-dim covariant metric, the line element is given

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\mu = -(\alpha^2 - \beta^i\beta_i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$



The **lapse** measures proper time between two adjacent hypersurfaces

$$\mathrm{d}\tau^2 = -\alpha^2(t, x^j)\mathrm{d}t^2$$

The **shift** relates spatial coordinates between two adjacent hypersurfaces

$$x^i_{t_0+\delta t} = x^i_{t_0} - \beta^i(t, x^j)\mathrm{d}t$$

spatial metric measures distances between points on hypersurface

$$\mathrm{d}l^2 = \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

Second step: decompose 4-dim tensors

$\boldsymbol{n}$ and $\boldsymbol{\gamma}$ allow to decompose any 4D tensor into a purely spatial part (hence in Σ) and a purely timelike part

The spatial part is obtained after contracting with the spatial projection operator

$$\gamma^{\mu}{}_{\nu} = g^{\mu\alpha}\gamma_{\alpha\nu} = g^{\mu}{}_{\nu} + n^{\mu}n_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu}$$

while the timelike part is obtained after contracting with the timelike projection operator

$$N^{\mu}{}_{\nu} = -n^{\mu}n_{\nu}$$

The two projectors are obviously orthogonal

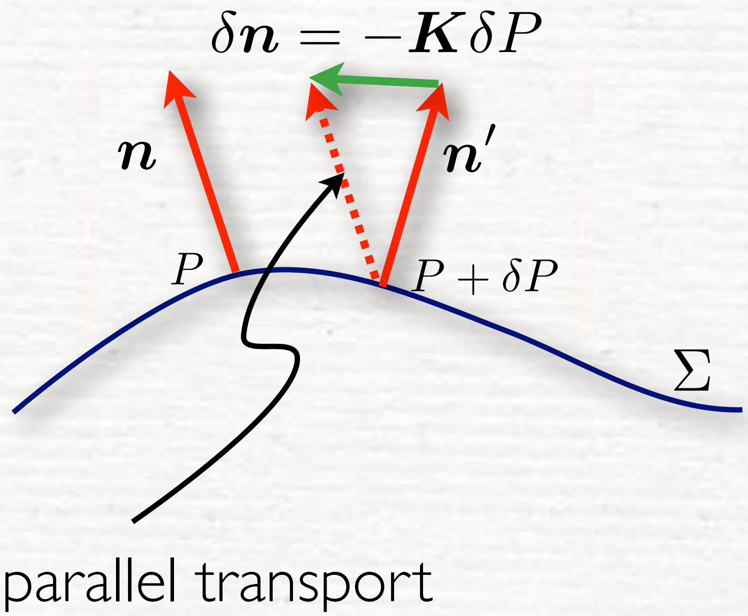
$$\gamma^{\nu}{}_{\mu}N^{\mu}{}_{\nu} = 0$$

It is important not to confuse the **3-dim Riemann tensor** ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ with the corresponding **4-dim** one $R^{\mu}{}_{\delta\alpha\beta}$

${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ is purely spatial (spatial derivatives of the spatial metric $\boldsymbol{\gamma}$)

$R^{\mu}{}_{\delta\alpha\beta}$ is a full 4-dimensional object containing also time derivatives of the full 4-dim metric $\boldsymbol{g}$

Information present in $R^{\mu}{}_{\delta\alpha\beta}$ and "missing" in ${}^{(3)}R^{\mu}{}_{\delta\alpha\beta}$ is contained in another spatial tensor: the **extrinsic curvature**.



Decomposing the Einstein equations

- The 3+1 naturally "splits" the Einstein equations into:
  - ✴ a set which is fully defined on each spatial hypersurfaces (and does not involve therefore time derivatives): "**constraint equations**"
  - ✴ a set which instead relates quantities (i.e. the spatial metric and the extrinsic curvature) between two hypersurfaces: "**evolution equations**"

The (ADM) Einstein eqs in 3+1

$$\boldsymbol{n} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \ + \ \text{Gauss eqs} \Longrightarrow$$

$$\boxed{R + K^2 - K_{ij}K^{ij} = 16\pi e}$$

Hamiltonian Constraint [1]

$$\boldsymbol{\gamma} \cdot \boldsymbol{n} \cdot (\text{Einstein eqs}) \ + \ \text{Codazzi eqs} \Longrightarrow$$

$$\boxed{D_j K^j{}_i - D_i K = 8\pi j_i}$$

Momentum Constraints [3]

$$S_{\alpha\beta} = \gamma^\mu{}_\alpha \gamma^\nu{}_\beta T_{\mu\nu} \qquad S = S^\mu{}_\mu$$

$$e = n^\mu n^\nu T_{\mu\nu} \qquad j_\mu = -\gamma^\alpha{}_\mu n^\beta T_{\alpha\beta}$$

These are 1+3 elliptic (second-order in space), nonlinear partial differential equations: **constraint equations**

The (ADM) Einstein eqs in 3+1

Similarly

$$\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}\cdot(\text{Einstein eqs}) \;+\; \text{Ricci eqs} \Longrightarrow$$

$$\begin{aligned}\partial_t K_{ij} = &-D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^{kj} + KK_{ij}) \\ &-8\pi\alpha(R_{ij} - \frac{1}{2}\gamma_{ij}(S-e)) + \mathcal{L}_{\boldsymbol{\beta}} K_{ij}\end{aligned}$$

[6]

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_{\boldsymbol{\beta}} \gamma_{ij}$$

[6]

These are 12 first-order in time, second-order in space, nonlinear partial differential equations: **evolution equations**

In practice we do not solve them because **weakly hyperbolic**.

New evolution variables are introduced to obtain a set of equations that is **strongly hyperbolic**

$$\phi = \frac{1}{12}\ln(\det(\gamma_{ij})) = \frac{1}{12}\ln(\gamma),$$

ϕ : conformal factor

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij},$$

$\tilde{\gamma}_{ij}$: conformal 3-metric

$$K = \gamma^{ij}K_{ij},$$

K : trace of extrinsic curvature

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3}\gamma_{ij}K),$$

$\tilde{A}_{ij}$: trace-free conformal extrinsic curvature

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

$\tilde{\Gamma}^i$:"Gammas" (aux. variables)

are our new **evolution variables**

The ADM equations are then rewritten as

$$
\begin{aligned}
\partial_t \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\,\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\,\beta^k + \beta^k\partial_k\tilde{\gamma}_{ij}\,, \\
\partial_t \tilde{A}_{ij} &= \phi^2\left[-\nabla_i\nabla_j\alpha + \alpha\left(R_{ij} + \nabla_i Z_j + \nabla_j Z_i\right)\right]^{\mathrm{TF}} + \alpha\tilde{A}_{ij}\left(K - 2\Theta\right) \\
&\quad -2\alpha\tilde{A}_{il}\tilde{A}^l_j + 2\tilde{A}_{k(i}\partial_{j)}\,\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\,\beta^k + \beta^k\partial_k\tilde{A}_{ij}\,, \\
\partial_t\phi &= \frac{1}{3}\alpha\phi K - \frac{1}{3}\phi\partial_k\beta^k + \beta^k\partial_k\phi\,, \\
\partial_t K &= -\nabla^i\nabla_i\alpha + \alpha\left(R + 2\nabla_i Z^i + K^2 - 2\Theta K\right) + \beta^j\partial_j K - 3\alpha\kappa_1\left(1+\kappa_2\right)\Theta\,, \\
\partial_t\Theta &= \frac{1}{2}\alpha\left(R + 2\nabla_i Z^i - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 - 2\Theta K\right) - Z^i\partial_i\alpha + \beta^k\partial_k\Theta - \alpha\kappa_1\left(2+\kappa_2\right)\Theta\,, \\
\partial_t\hat{\Gamma}^i &= 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - 3\tilde{A}^{ij}\frac{\partial_j\phi}{\phi} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K\right) + 2\tilde{\gamma}^{ki}\left(\alpha\partial_k\Theta - \Theta\partial_k\alpha - \frac{2}{3}\alpha K Z_k\right) - 2\tilde{A}^{ij}\partial_j\alpha + \beta^k\partial_k\hat{\Gamma}^i \\
&\quad +\tilde{\gamma}^{kl}\partial_k\partial_l\beta^i + \frac{1}{3}\tilde{\gamma}^{ik}\partial_k\partial_l\beta^l + \frac{2}{3}\tilde{\Gamma}^i\partial_k\beta^k - \tilde{\Gamma}^k\partial_k\beta^i + 2\kappa_3\left(\frac{2}{3}\tilde{\gamma}^{ij}Z_j\partial_k\beta^k - \tilde{\gamma}^{jk}Z_j\partial_k\beta^i\right) - 2\alpha\kappa_1\tilde{\gamma}^{ij}Z_j\,, \\
\partial_t\alpha &= -\alpha^2 g(\alpha)\left(K - K_0 - 2\Theta\right) + \beta^k\partial_k\alpha\,, \\
\partial_t\beta^i &= f b^i + \beta^k\partial_k\beta^i\,, \\
\partial_t b^i &= \partial_t\hat{\Gamma}^i - \beta^k\partial_k\hat{\Gamma}^i + \beta^k\partial_k b^i - \eta b^i\,,
\end{aligned}
$$

These equations are known as the conformal covariant Z4 formulation (CCZ4) of the Einstein equations.

$$
\begin{aligned}
\partial_t \tilde{\gamma}_{ij} &= \beta^k 2D_{kij} + \tilde{\gamma}_{ki} B^k_j + \tilde{\gamma}_{kj} B^k_i - \frac{2}{3}\tilde{\gamma}_{ij} B^k_k - 2\alpha\left(\tilde{A}_{ij} - \frac{1}{3}\tilde{\gamma}_{ij}\mathrm{tr}\tilde{A}\right) - \tau^{-1}(\tilde{\gamma}-1)\,\tilde{\gamma}_{ij}, \\
\partial_t \ln\alpha &= \beta^k A_k - \alpha g(\alpha)(K - K_0 - 2\Theta c)\,, \\
\partial_t \beta^i &= s\beta^k B^i_k + sfb^i \\
\partial_t \ln\phi &= \beta^k P_k + \frac{1}{3}\left(\alpha K - B^k_k\right)\,, \\
\partial_t \tilde{A}_{ij} - \beta^k\partial_k \tilde{A}_{ij} &- \phi^2\Big[-\nabla_i\nabla_j\alpha + \alpha\left(R_{ij} + \nabla_i Z_j + \nabla_j Z_i\right)\Big] + \phi^2\frac{1}{3}\frac{\tilde{\gamma}_{ij}}{\phi^2}\Big[-\nabla^k\nabla_k\alpha + \alpha(R + 2\nabla_k Z^k)\Big] \\
&= \tilde{A}_{ki}B^k_j + \tilde{A}_{kj}B^k_i - \frac{2}{3}\tilde{A}_{ij}B^k_k + \alpha\tilde{A}_{ij}(K - 2\Theta c) - 2\alpha\tilde{A}_{il}\tilde{\gamma}^{lm}\tilde{A}_{mj} - \tau^{-1}\,\tilde{\gamma}_{ij}\,\mathrm{tr}\tilde{A}\,, \\
\partial_t K - \beta^k\partial_k K &+ \nabla^i\nabla_i\alpha - \alpha(R + 2\nabla_i Z^i) = \alpha K(K - 2\Theta c) - 3\alpha\kappa_1(1+\kappa_2)\Theta \\
\partial_t \Theta - \beta^k\partial_k\Theta &- \frac{1}{2}\alpha e^2(R + 2\nabla_i Z^i) = \frac{1}{2}\alpha e^2\left(\frac{2}{3}K^2 - \tilde{A}_{ij}\tilde{A}^{ij}\right) - \alpha\Theta K c - Z^i\alpha A_i - \alpha\kappa_1(2+\kappa_2)\Theta\,, \\
\partial_t\hat{\Gamma}^i - \beta^k\partial_k\hat{\Gamma}^i &+ \frac{4}{3}\alpha\tilde{\gamma}^{ij}\partial_j K - 2\alpha\tilde{\gamma}^{ki}\partial_k\Theta - \tilde{\gamma}^{kl}\partial_{(k}B^i_{l)} - \frac{1}{3}\tilde{\gamma}^{ik}\partial_{(k}B^l_{l)} - s2\alpha\tilde{\gamma}^{ik}\tilde{\gamma}^{nm}\partial_k\tilde{A}_{nm} \\
&= \frac{2}{3}\tilde{\Gamma}^i B^k_k - \tilde{\Gamma}^k B^i_k + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - 3\tilde{A}^{ij}P_j\right) - 2\alpha\tilde{\gamma}^{ki}\left(\Theta A_k + \frac{2}{3}KZ_k\right) - 2\alpha\tilde{A}^{ij}A_j \\
&\quad -4s\,\alpha\tilde{\gamma}^{ik}D_k{}^{nm}\tilde{A}_{nm} + 2\kappa_3\left(\frac{2}{3}\tilde{\gamma}^{ij}Z_j B^k_k - \tilde{\gamma}^{jk}Z_j B^i_k\right) - 2\alpha\kappa_1\tilde{\gamma}^{ij}Z_j \\
\partial_t b^i - s\beta^k\partial_k b^i &= s\left(\partial_t\hat{\Gamma}^i - \beta^k\partial_k\hat{\Gamma}^i - \eta b^i\right),
\end{aligned}
$$

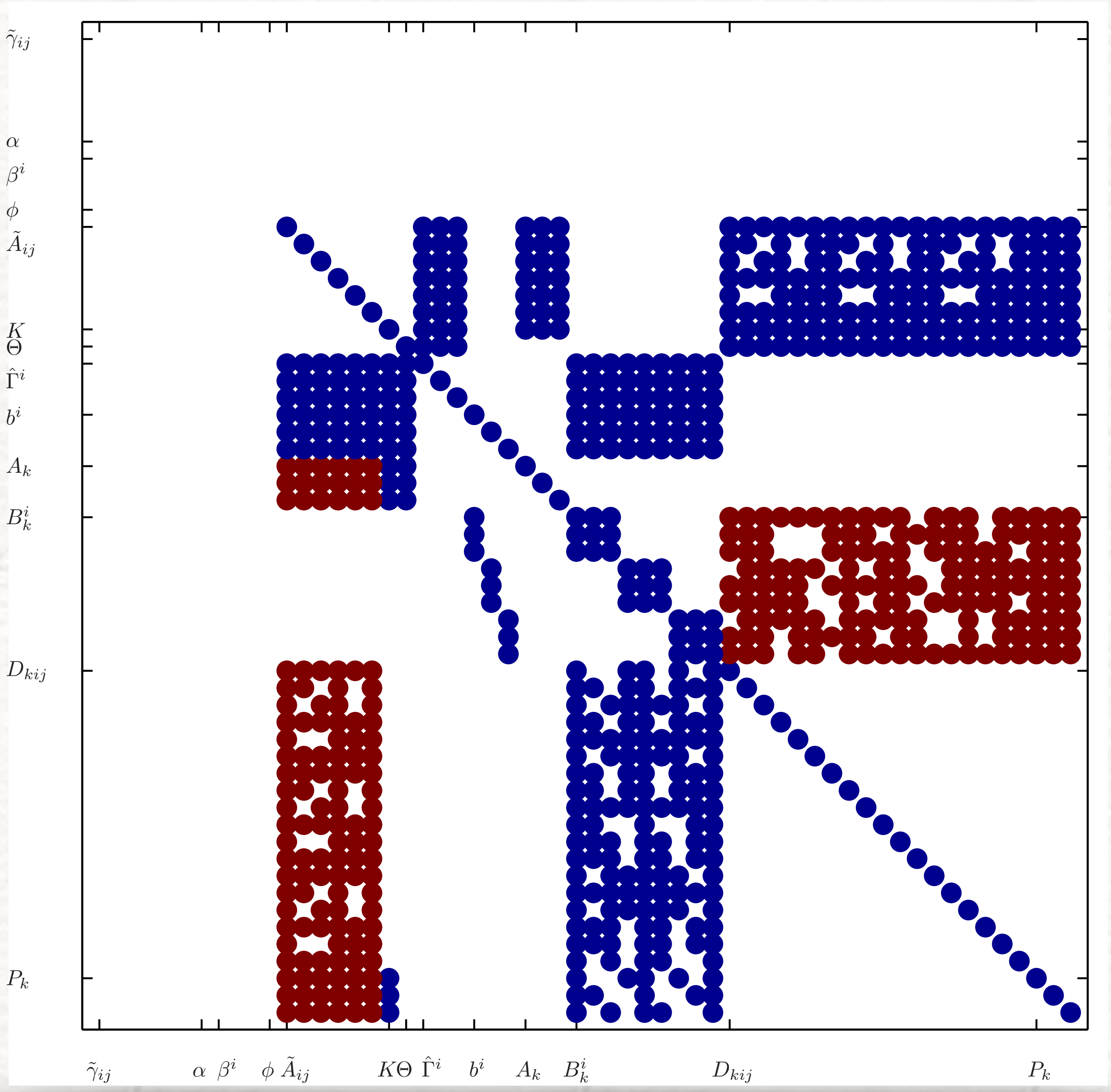
These equations are known as the **first-order** conformal covariant Z4 formulation (FO-CCZ4) of the Einstein equations.

$$
\begin{aligned}
\partial_t A_k - \beta^l \partial_l A_k &+ \alpha g(\alpha)\left(\partial_k K - \partial_k K_0 - 2c\partial_k \Theta\right) + s\alpha g(\alpha)\tilde{\gamma}^{nm}\partial_k \tilde{A}_{nm} \\
&= +2s\,\alpha g(\alpha) D_k{}^{nm}\tilde{A}_{nm} - \alpha A_k\left(K - K_0 - 2\Theta c\right)\left(g(\alpha) + \alpha g'(\alpha)\right) + B_k^l\, A_l\,, \\
\partial_t B_k^i - s\beta^l \partial_l B_k^i &- s\left(f\partial_k b^i + \alpha^2\mu\,\tilde{\gamma}^{ij}\left(\partial_k P_j - \partial_j P_k\right) - \alpha^2\mu\,\tilde{\gamma}^{ij}\tilde{\gamma}^{nl}\left(\partial_k D_{ljn} - \partial_l D_{kjn}\right)\right) \\
&= sB_k^l\, B_l^i\,, \\
\partial_t D_{kij} - \beta^l \partial_l D_{kij} &+ s\left(-\frac{1}{2}\tilde{\gamma}_{mi}\partial_{(k}B_{j)}^m - \frac{1}{2}\tilde{\gamma}_{mj}\partial_{(k}B_{i)}^m + \frac{1}{3}\tilde{\gamma}_{ij}\partial_{(k}B_{m)}^m\right) + \alpha\partial_k \tilde{A}_{ij} - \alpha\frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{nm}\partial_k \tilde{A}_{nm} \\
&= B_k^l D_{lij} + B_j^l D_{kli} + B_i^l D_{klj} - \frac{2}{3}B_l^l D_{kij} - \alpha\frac{2}{3}\tilde{\gamma}_{ij} D_k{}^{nm}\tilde{A}_{nm} - \alpha A_k\left(\tilde{A}_{ij} - \frac{1}{3}\tilde{\gamma}_{ij}\mathrm{tr}\tilde{A}\right), \\
\partial_t P_k - \beta^l \partial_l P_k &- \frac{1}{3}\alpha\partial_k K + \frac{1}{3}\partial_{(k}B_{i)}^i - s\frac{1}{3}\alpha\tilde{\gamma}^{nm}\partial_k \tilde{A}_{nm} \\
&= \frac{1}{3}\alpha A_k K + B_k^l P_l - s\frac{2}{3}\alpha\, D_k{}^{nm}\tilde{A}_{nm}.
\end{aligned}
$$

These equations are known as the **first-order** conformal covariant Z4 formulation (FO-CCZ4) of the Einstein equations.

The 12 Einstein eqss are therefore written as a system of **58 fields**. These eqs are only for the spacetime part…

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \boldsymbol{A_1}(\boldsymbol{Q})\frac{\partial \boldsymbol{Q}}{\partial x_1} + \boldsymbol{A_2}(\boldsymbol{Q})\frac{\partial \boldsymbol{Q}}{\partial x_2} + \boldsymbol{A_3}(\boldsymbol{Q})\frac{\partial \boldsymbol{Q}}{\partial x_3} = \boldsymbol{S}(\boldsymbol{Q}),$$



Still don't know anything about properties of these equations

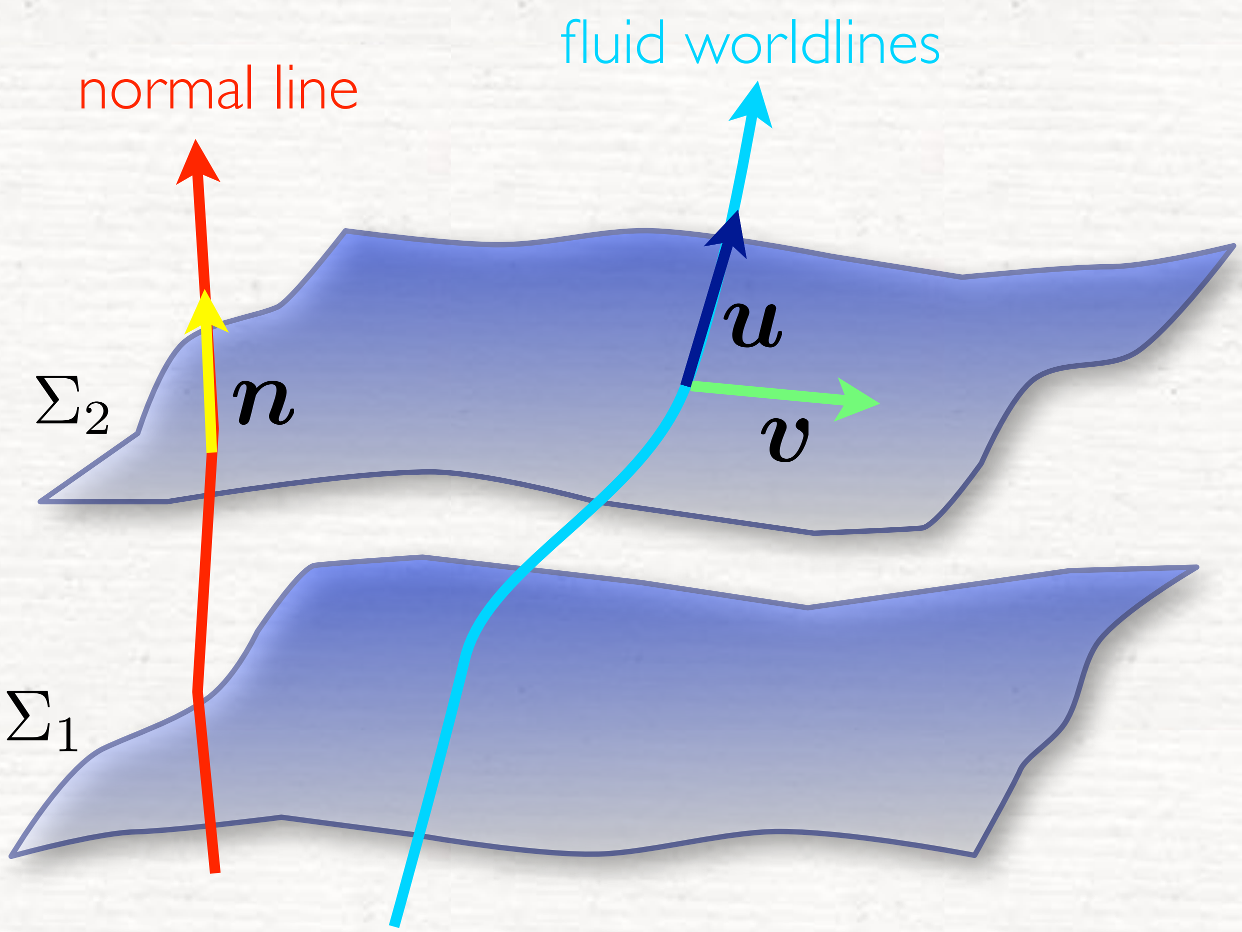
With a proper combination of the constraints on the auxiliary variables the set is cast in a **strongly hyperbolic** a

Sparsity pattern of the system matrix $\boldsymbol{A} \cdot \boldsymbol{n}$

Solving the hydrodynamics equations

3+1 splitting also for the matter

We are not interested in the 4-velocity $\boldsymbol{u}$ but rather its projection on the spatial slice, ie the 3-velocity $\boldsymbol{v}$



Those observers with $\boldsymbol{u}$ parallel to $\boldsymbol{n}$ move from one slice to the next along the normal to the slice: Eulerian observers.

They measure a fluid 3-velocity

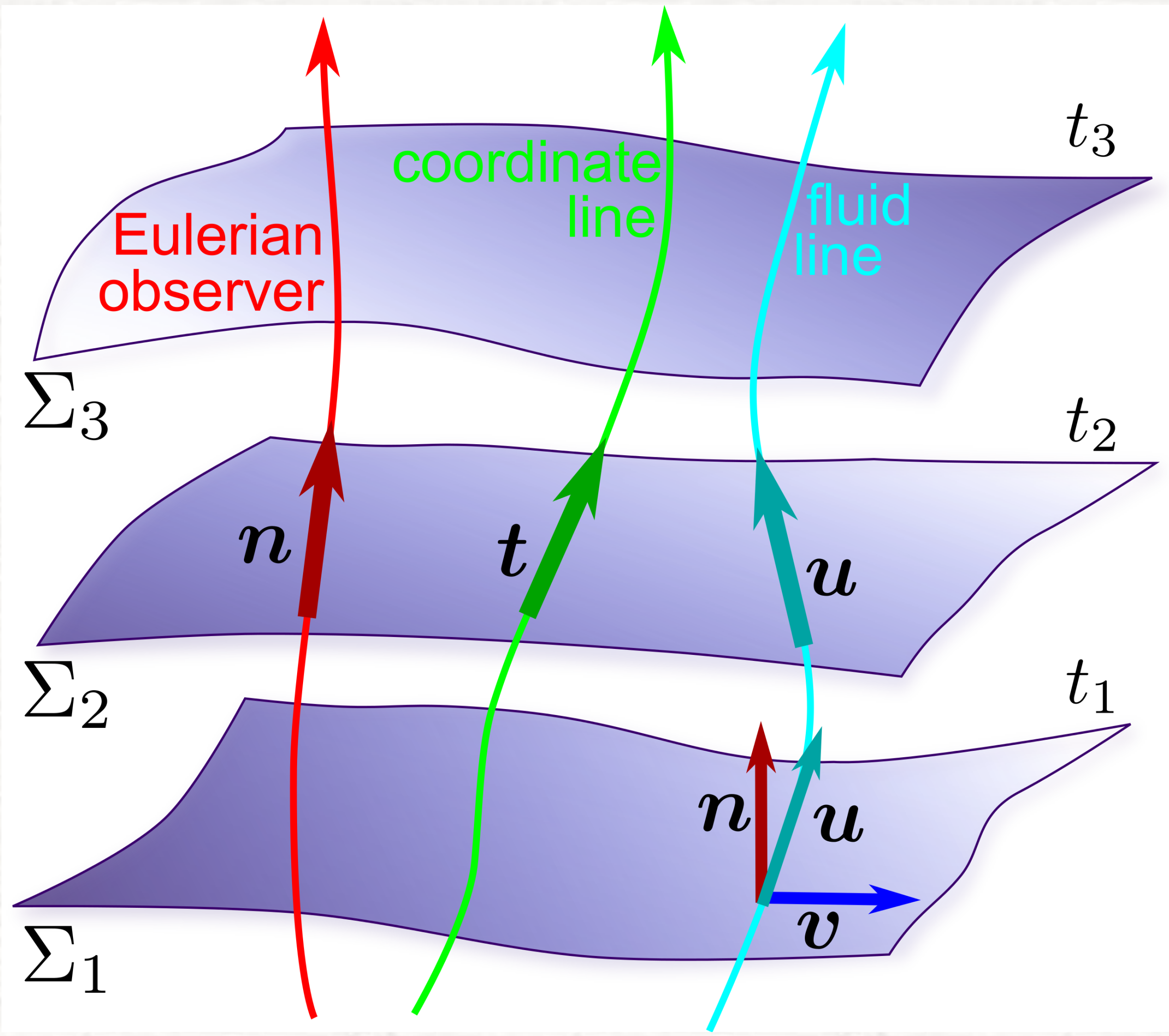
$$\boldsymbol{v} = \frac{\boldsymbol{\gamma} \cdot \boldsymbol{u}}{-\boldsymbol{n} \cdot \boldsymbol{u}}$$

Remember that in special relativity

$$u^i = \frac{dx^i}{d\tau}, \qquad v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau}\frac{d\tau}{dt} = \frac{u^i}{u^0}$$

3+1 splitting also for the matter

To aid comparison with what you are more familiar with, the contravariant (upstairs) components of this vector are



$$v^i = \frac{\boldsymbol{\gamma}^i \cdot \boldsymbol{u}}{-\boldsymbol{n} \cdot \boldsymbol{u}} = \frac{\gamma^i{}_\mu u^\mu}{\alpha u^0} = \frac{1}{\alpha}\left(\frac{u^i}{u^0} + \beta^i\right)$$

$$v_i = \gamma_{ij} v^j = \gamma_{ij} \frac{1}{\alpha}\left(\frac{u^j}{u^0} + \beta^j\right)$$

Using the normalization condition $u^\mu u_\mu = -1$ one obtains

$$\alpha u^0 = \frac{1}{\sqrt{1 - v^i v_i}} = \frac{1}{\sqrt{1 - v^2}} \equiv W$$

W is the Lorentz factor

The hydrodynamic equations

$$\boldsymbol{\gamma} \cdot (\nabla \boldsymbol{T}) = 0 \;, \quad \text{(spacelike projection of divergence of } \boldsymbol{T}\text{)}$$

$$\boldsymbol{n} \cdot (\nabla \boldsymbol{T}) = 0 \;, \quad \text{(timelike projection of divergence of } \boldsymbol{T}\text{)}$$

$$(\nabla \cdot \rho \boldsymbol{u}) = 0 \;, \quad \text{(divergence of mass flux)}$$

$$p = p(\rho, \epsilon) \;, \quad \text{(equation of state EOS)}$$

Covariant form of the equations does not fix a **formulation**, which needs to be **conservative** for a numerical solution.

Conservative form of the equations

The homogeneous partial differential equation

$$\partial_t u(x,t) + a[u(x,t)]\partial_x u(x,t) = 0$$

is said to be in **flux-conservative (FC) form** if written as

$$\partial_t u(x,t) + \partial_x F[u(x,t)] = 0$$

Theorems (Lax, Wendroff; Hou, LeFloch)

- FC formulation converges to **weak solution** of the problem
- NFC converges to the wrong **weak solution** of the problem

The Valencia (conservative) formulation

(Banyuls et al. 97)

$$\frac{1}{\sqrt{-g}}\left\{\partial_t\left[\sqrt{\gamma}\mathbf{F}^0(\mathbf{U})\right]+\partial_i\left[\sqrt{\gamma}\mathbf{F}^i(\mathbf{U})\right]\right\}=\mathbf{s}(\mathbf{U})\ ,$$

where $\sqrt{-g}=\sqrt{\det(g_{\mu\nu})}=\alpha\sqrt{\det(\gamma_{\mu\nu})}=\alpha\sqrt{\gamma}$ and

$$\mathbf{F}^0(\mathbf{U})=(D,S_j,\tau)^T\ ,$$

$$\mathbf{F}^i(\mathbf{U})=[D(\alpha v^i-\beta^i),S_j(\alpha v^i-\beta^i)+p\delta^i_j,\tau(\alpha v^i-\beta^i)+pv^i]^T$$

$$\mathbf{s}(\mathbf{U})=\left[0,T^{\mu\nu}\left(\partial_\mu g_{\nu j}+\Gamma^\delta_{\mu\nu}g_{\delta j}\right),\alpha\left(T^{\mu 0}\partial_\mu\ln\alpha-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right]\ .$$

Source terms do not contain derivatives of hydrodynamical quantities and vanish in a flat spacetime

The Valencia (conservative) formulation

The first step is the identification of suitable ***"conserved"*** quantities in place of the ***"primitive"*** variables (ρ, ϵ, v^j). After lot of algebra…

$$\begin{aligned} D &= \rho W \ , \\ S_j &= \rho h W^2 v_j \ , \\ \tau &= \rho h W^2 - \rho W - p \end{aligned}$$

- Transformation primitive-to-conserved is **algebraic**.
- Transformation conserved-to-primitive is **not**.
- Transformation conserved-to-primitive requires numerical solution (root finding) at each cell:
- This is considerable **bottleneck** and source of errors.

Follow these instructions, work for a decade and…

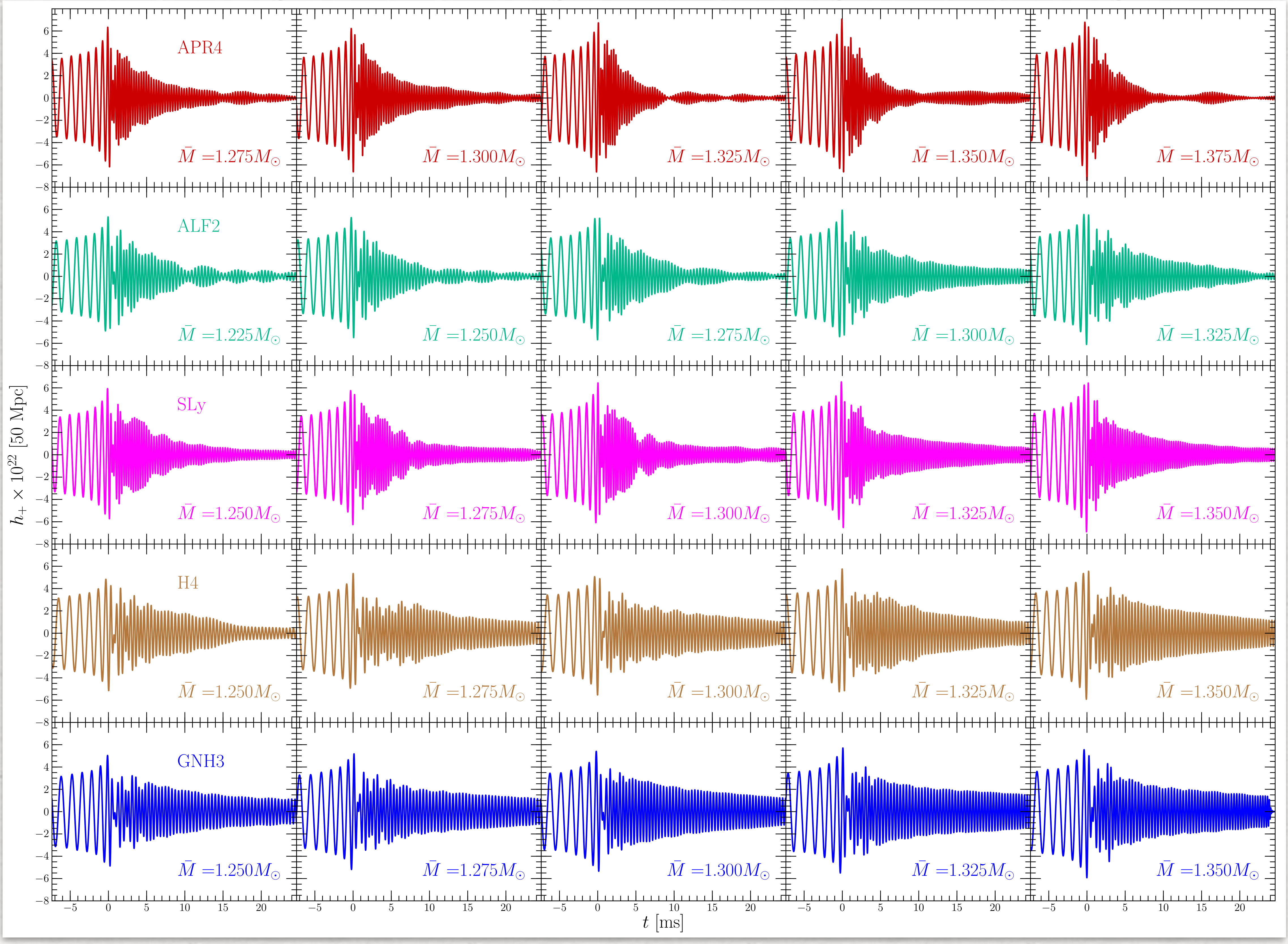
Animations: Breu, Radice, LR

$$M = 2 \times 1.35\, M_\odot$$

LS220 EOS

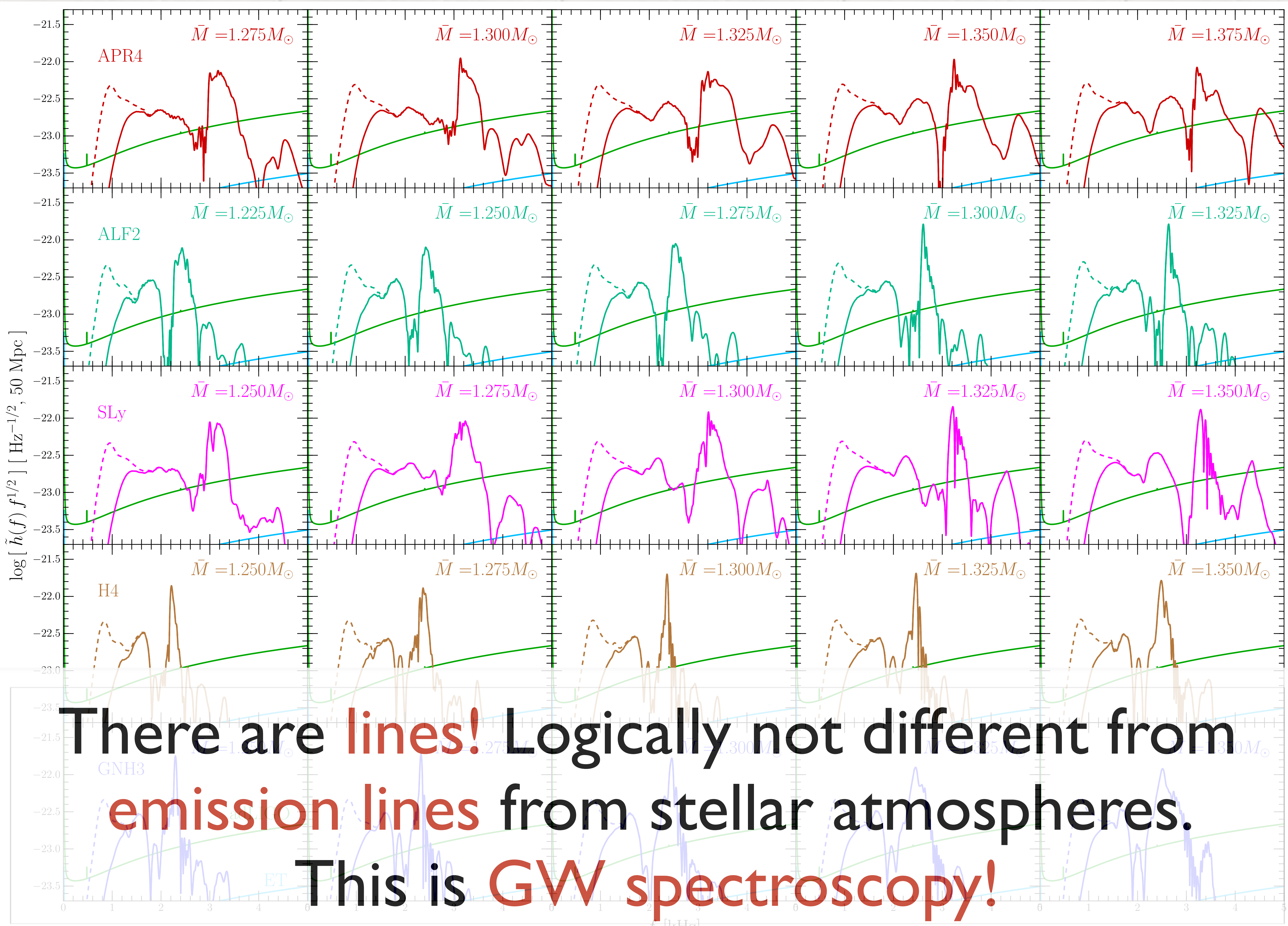
What we can do nowadays

Takami, LR, Baiotti (2014, 2015), LR+ (2016)



Extracting information from the EOS

Takami, LR, Baiotti (2014, 2015), LR+ (2016)



There are lines! Logically not different from emission lines from stellar atmospheres. This is GW spectroscopy!

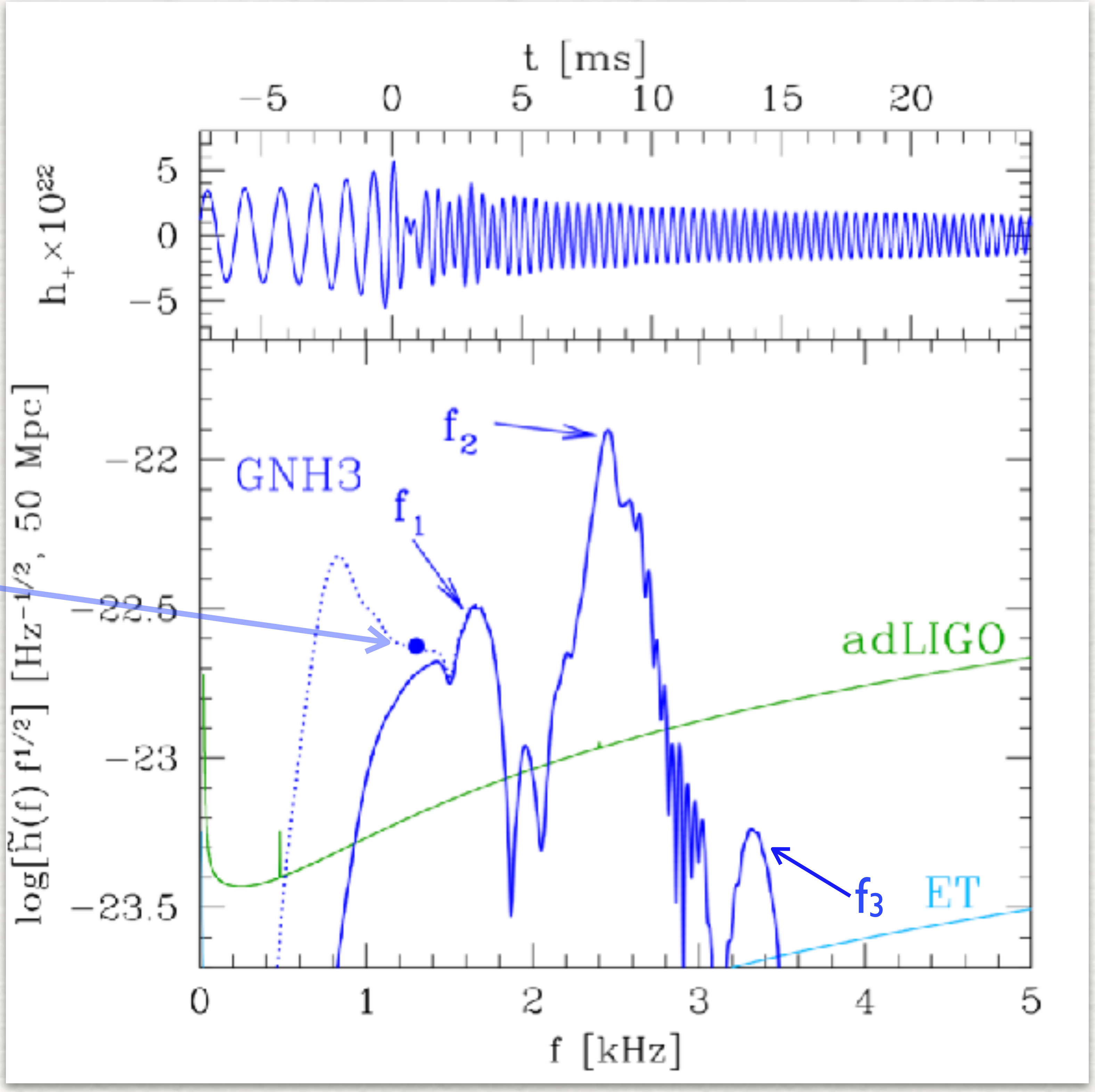
Moving away from perfect fluids

Neutron star matter

- Astronomical observations show viscosity in neutron stars is very small: superfluidity needed to explain pulsar glitches.
- During inspiral stars interact gravitationally only: fluid is tidal distorted but only before merger.
- After merger temperatures increase (10-50 MeV) and a number of dissipative effects can become important.
- Viscous dissipation is normally neglected in numerical modelling on assumption microscopic viscosity too small.
- Should we **worry?**

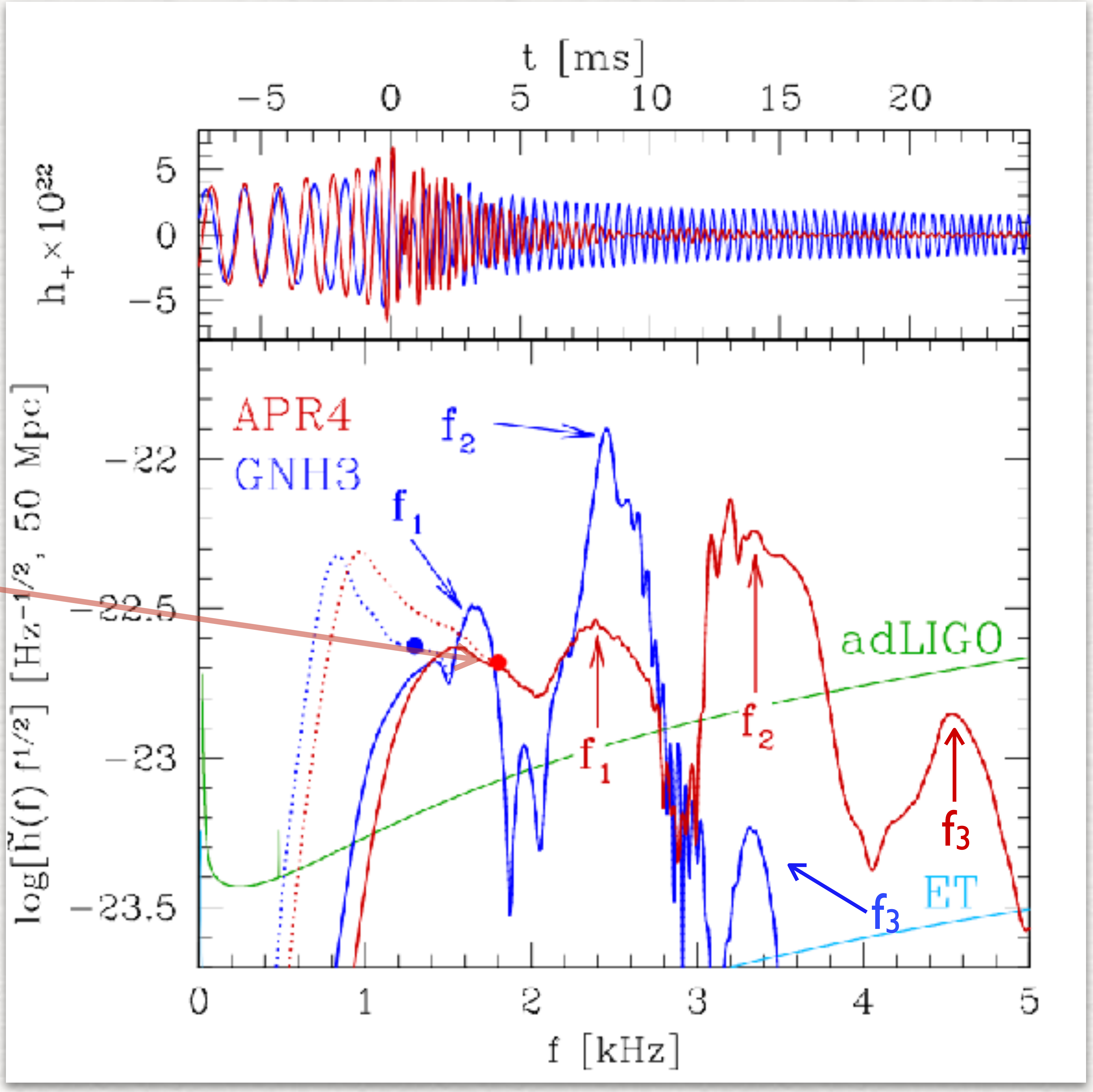
GW spectroscopy

merger frequency



GW spectroscopy

merger frequency



Potential viscous contributions

Alford+(2018)

- Possible channels of micro/macroscopic viscosity are:
  1. nuclear-matter **shear** viscosity
  2. nuclear-matter **bulk** viscosity
  3. neutrino **shear** viscosity (Guilet+ 2016)
  4. "**MRI**-induced" **viscosity** (Radice2017, Shibata+2017a, b)
- Channels 3. and 4. act on timescales typical of MRI, which depends on B-field and very **uncertain** still.
- Impact of MRI on GWs depends on the value for viscous angular momentum transport.
- This is presently essentially unknown: $\tau \gtrsim 10 - 100\,\mathrm{ms}$?

Viscous contributions: 1. **shear viscosity**

- Low-temperature, **electron-dominated** regime, i.e.

  $T \lesssim 10\,\mathrm{MeV}$

  $$\tau_\eta^{(e)} \approx 1.6\times 10^8\,\mathrm{s}\left(\frac{z_{\rm typ}}{1\,\mathrm{km}}\right)^2\left(\frac{T}{1\,\mathrm{MeV}}\right)^{\frac{5}{3}}\left(\frac{n_0}{n_B}\right)^{\frac{5}{9}}\left(\frac{0.1}{x_p}\right)^{\frac{14}{9}},$$

- High-temperature, **neutrino-dominated** regime, i.e.

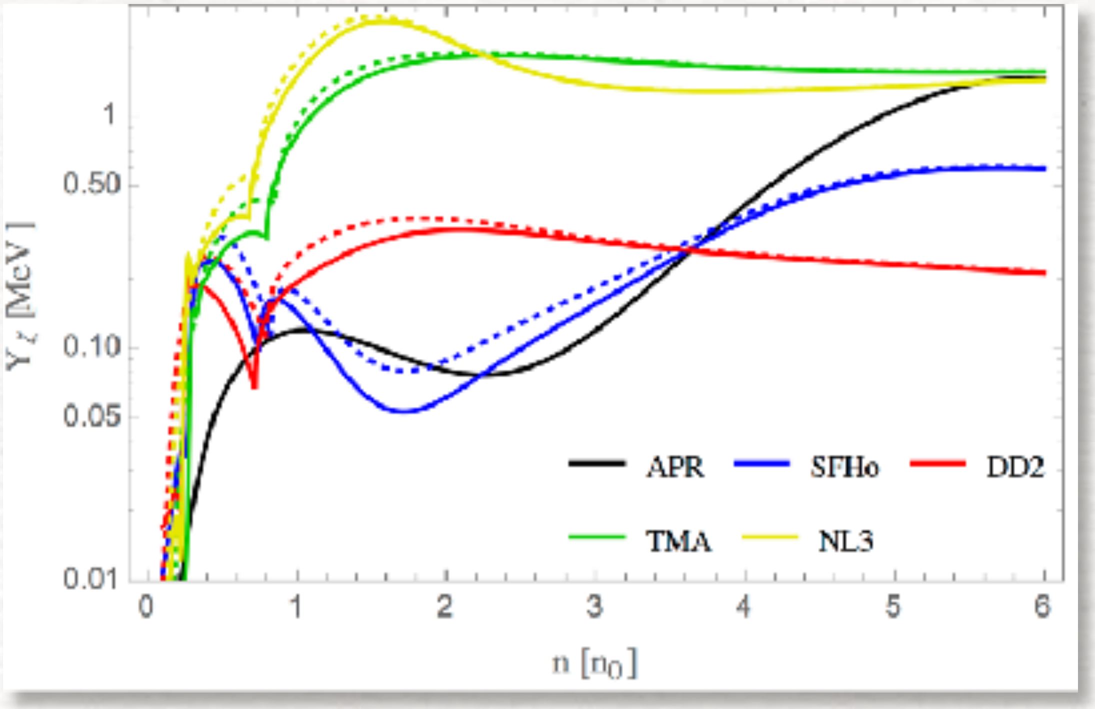
  $T \gtrsim 10\,\mathrm{MeV}$

  $$\tau_\eta^{(\nu)} \approx 54\,\mathrm{s}\left(\frac{0.1}{x_p}\right)\left(\frac{m_n^*}{0.8\,m_n}\right)^2\left(\frac{\mu_e}{2\,\mu_\nu}\right)^4\left(\frac{z_{\rm typ}}{1\,\mathrm{km}}\right)^2\left(\frac{T}{10\,\mathrm{MeV}}\right)^2,$$

Hence, shear viscosity not relevant unless neutrinos dominate and flow is turbulent with $z_{\rm typ} \sim 10 - 100\,\mathrm{m}$; not likely.

Viscous contributions: 2. **bulk viscosity**

- Impact of bulk viscosity depends sensitively on process responsible for flavor re-equilibration.
- If **direct-Urca** dominates, bulk viscosity will be very **small**: never possible for softer EOSs, hard for stiff EOS at small T.
- If **modified-Urca** dominates, then bulk viscosity



$\mathcal{E}_{\rm comp}$: en. density variation due comp.

$$\mathcal{E}_{\rm comp} \approx K\bar{n}(\Delta n/\bar{n})^2/18$$

$$\tau_\zeta \equiv \mathcal{E}_{\rm comp}/\,(d\mathcal{E}/dt)_{\rm bulk} \approx K\bar{n}\,t^2_{\rm exp}/(36\pi^2\,\bar{\zeta})$$

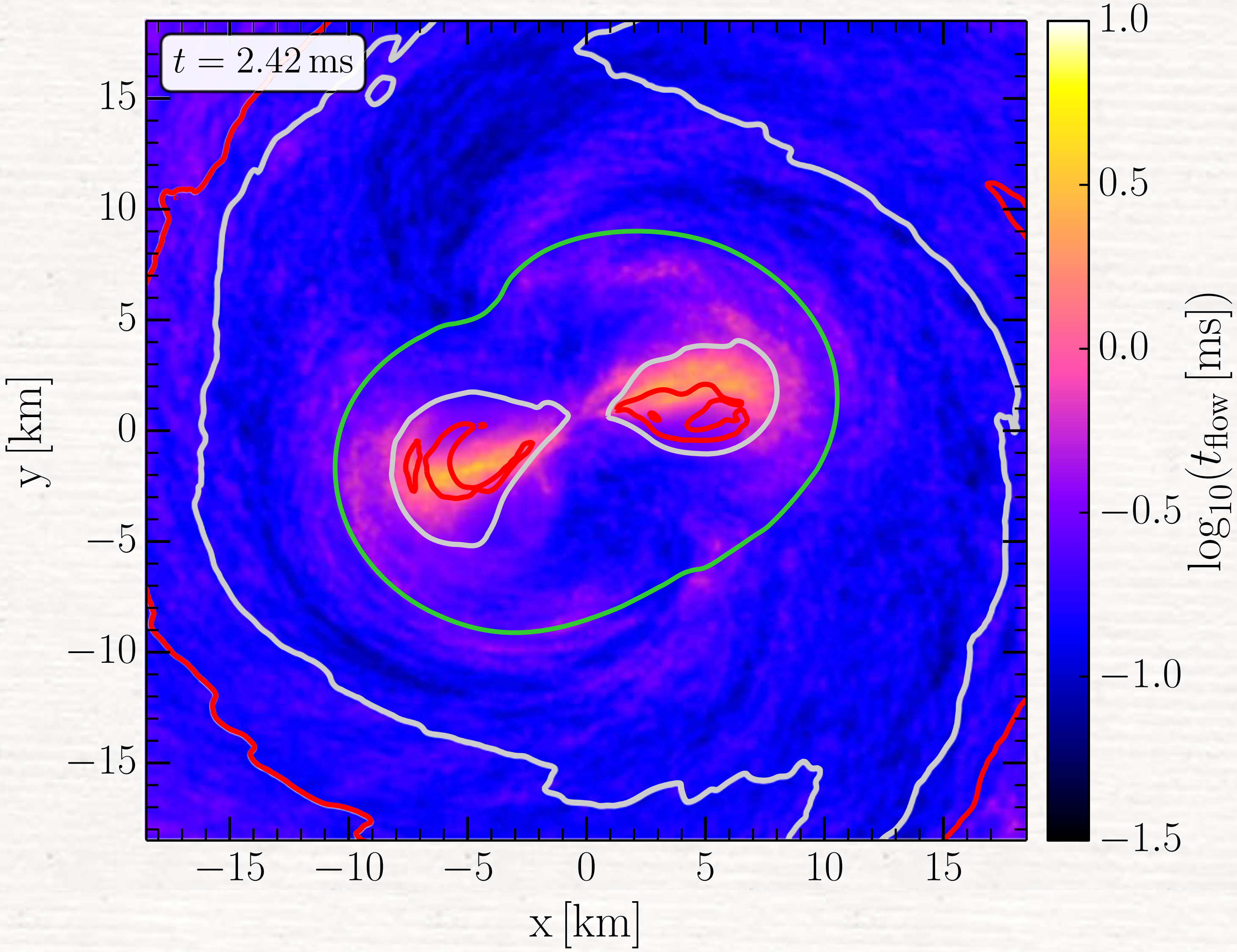
$$\approx 7\,{\rm ms}\left(\frac{t_{\rm exp}}{1\,{\rm ms}}\right)\left(\frac{K}{250\,{\rm MeV}}\right)\left(\frac{0.1\,{\rm MeV}}{Y_\zeta}\right)$$

key!

$t_{\rm exp} \sim$ bulk-dissipation timescale of internal energy

K : nuclear compressibility at n_0 ; Y_ζ : bulk viscosity prefactor

Viscous contributions



$$\langle t_{\rm flow}\rangle := \left\langle \frac{\rho}{D_t \rho} \right\rangle = \left\langle \frac{1}{\nabla\cdot\vec{v}} \right\rangle$$

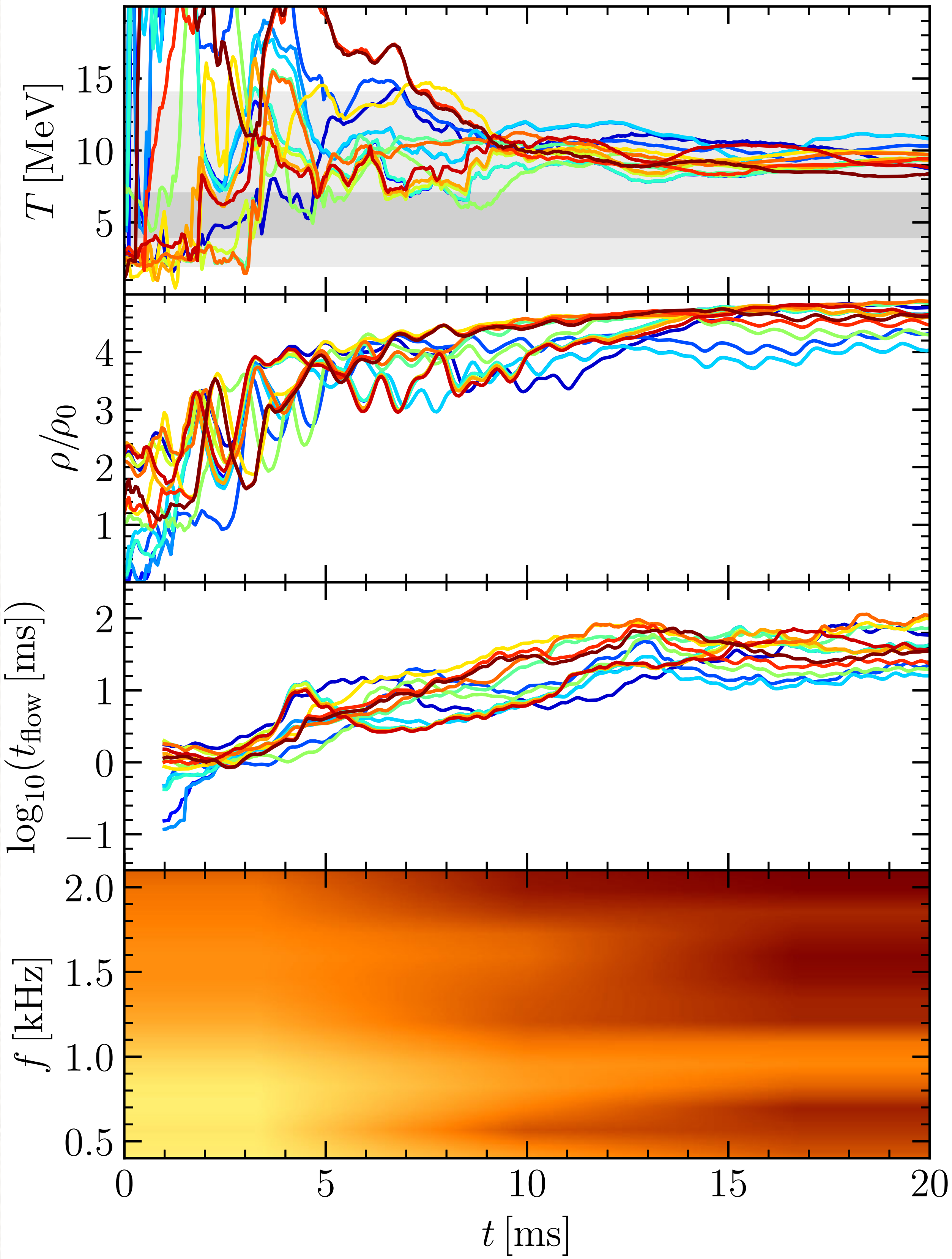
right after merger

$$t_{\rm flow} \lesssim \tau_{\rm dyn} = \frac{R}{c_s}$$

instantaneous bulk-dissipation timescale can be measured in simulations.

Soon after merger ***bulk-dissipation timescale*** comparable with **dynamical timescale** in large portions of the object: cannot be ignored.

Viscous contributions



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Soon after merger ***bulk-dissipation timescale*** comparable with **dynamical timescale** in large portions of the object: cannot be ignored.

Moving away from hydrodynamics: ideal-MHD

Electromagnetic counterparts

- EM counterparts can boost understanding of BNSs and SGRBs.
- EM counterparts via B-fields or radioactive decay (kilonova).
- B- fields may be too weak to be "visible" in inspiral waveforms.
- Pre-merger interaction of magnetosphere may be too weak.
- Best chances are after BH formation: jet launching.

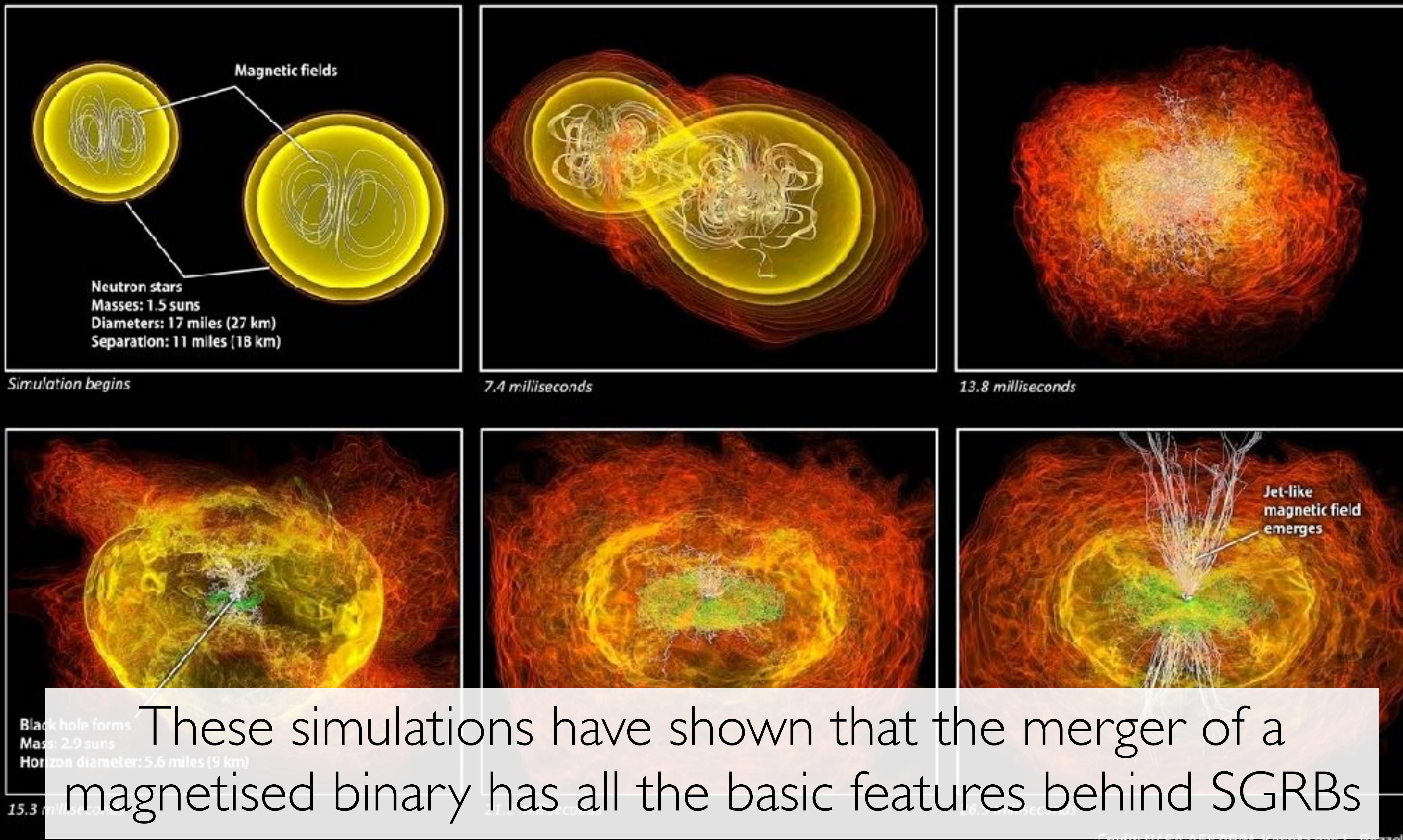
This requires extending the equations to **ideal-MHD (IMHD)**

$$T_{\mu\nu} = (e+p)\,u_\mu u_\nu + p g_{\mu\nu} + F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu}\, F^{\lambda\alpha} F_{\lambda\alpha},$$

$$\nabla^\nu T_{\mu\nu} = 0 \quad E^i = -\epsilon^{ijk} v_j B_k$$

$$\nabla_\nu (F^{\mu\nu} + g^{\mu\nu}\psi) = I^\mu - \kappa n^\mu \psi, \quad \nabla_\nu ({}^*F^{\mu\nu} + g^{\mu\nu}\phi) = -\kappa n^\mu \phi,$$

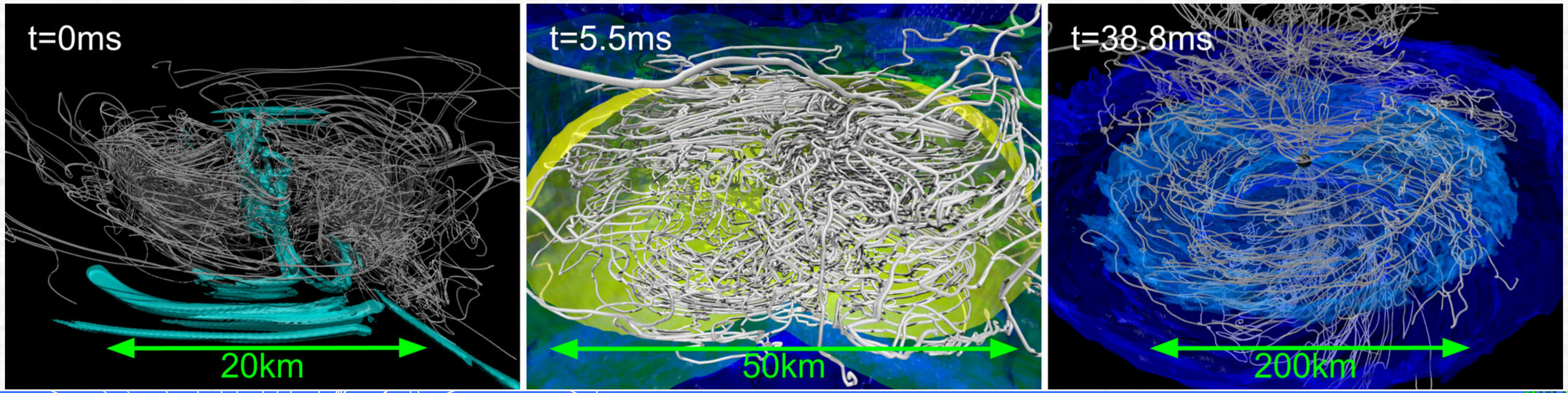
LR+ 2011



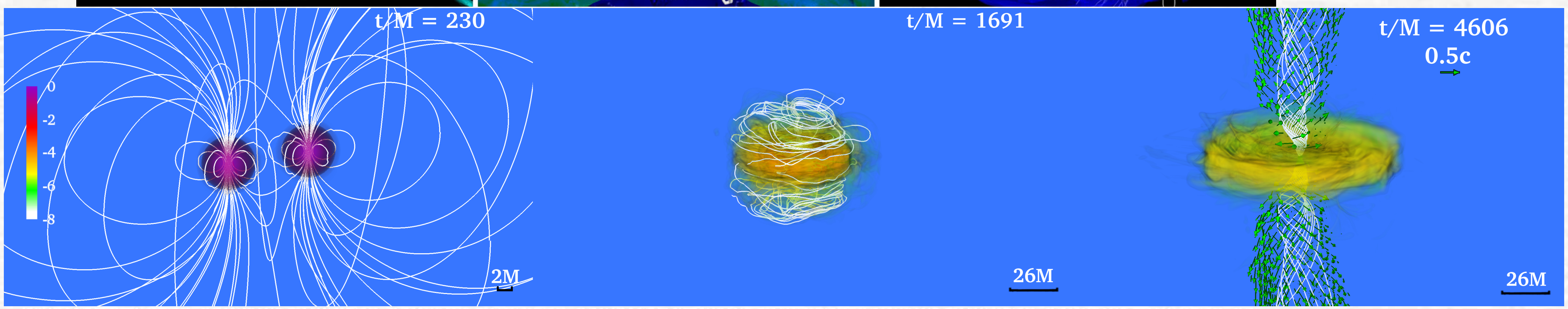
These simulations have shown that the merger of a magnetised binary has all the basic features behind SGRBs

$J/M^2 = 0.83$ $\mathrm{M}_{\mathrm{tor}} = 0.063 M_{\odot}$ $\mathrm{t}_{\mathrm{accr}} \simeq M_{\mathrm{tor}}/\dot{M} \simeq 0.3\ \mathrm{s}$

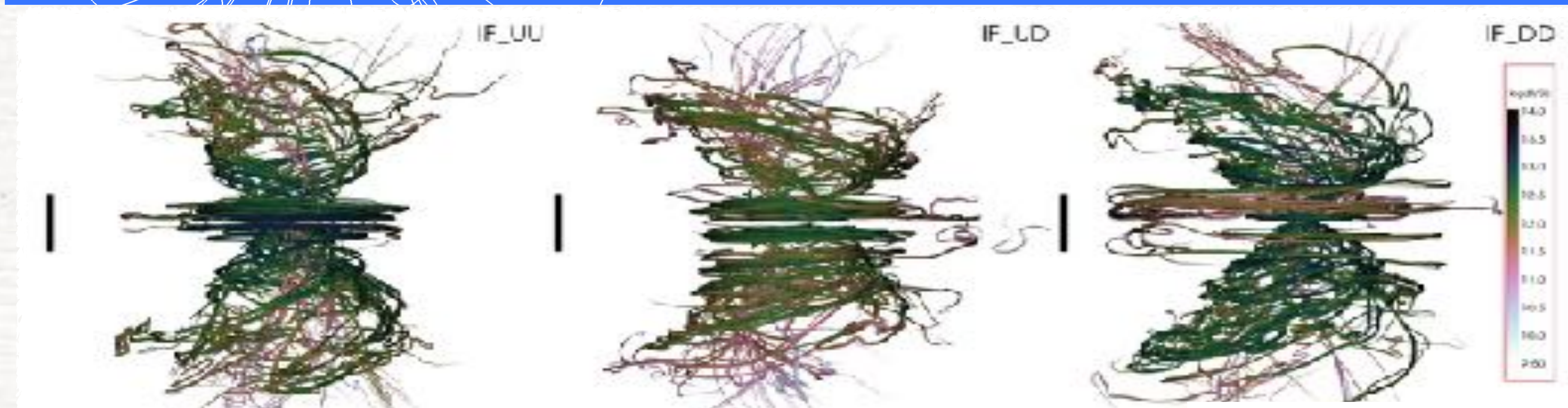
With due differences, other groups confirm this picture



Kiuchi+ 2014

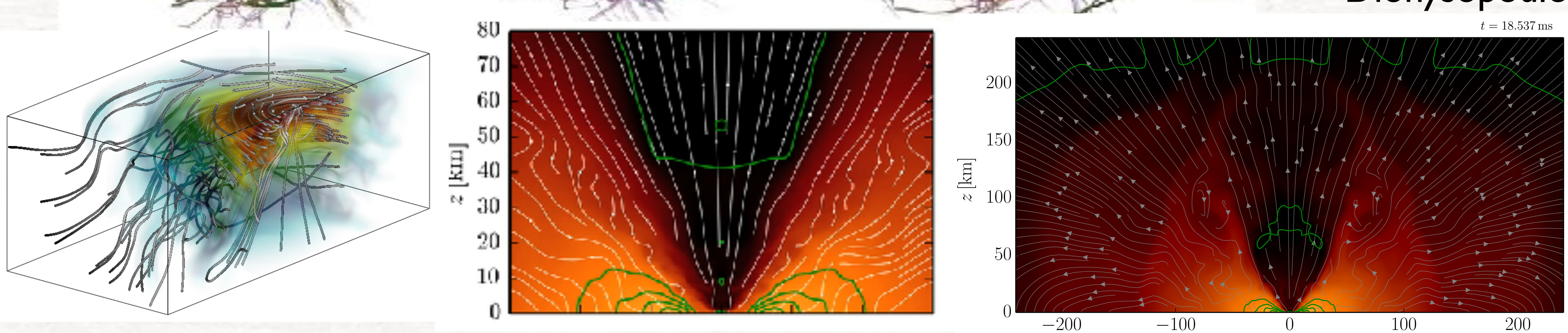


Ruiz+ 2016



Kawamura+2016

Dionysopoulou+ 2015



RMHD

Beyond IMHD: **Resistive Magnetohydrodynamics**

Dionysopoulou, Alic, LR (2015)

- Ideal MHD is a good approximation in the inspiral, but not after the merger; match to **electro-vacuum** not possible.
- Main difference in resistive regime is the current, which is dictated by Ohm's law but microphysics is **poorly** known.
- We know conductivity σ is a **tensor** but hardly know it as a scalar (prop. to density and inversely prop. to temperature).
- A simple prescription with scalar (isotropic) conductivity:

$$J^i = qv^i + W\sigma[E^i + \epsilon^{ijk}v_j B_k - (v_k E^k)v^i]\,,$$

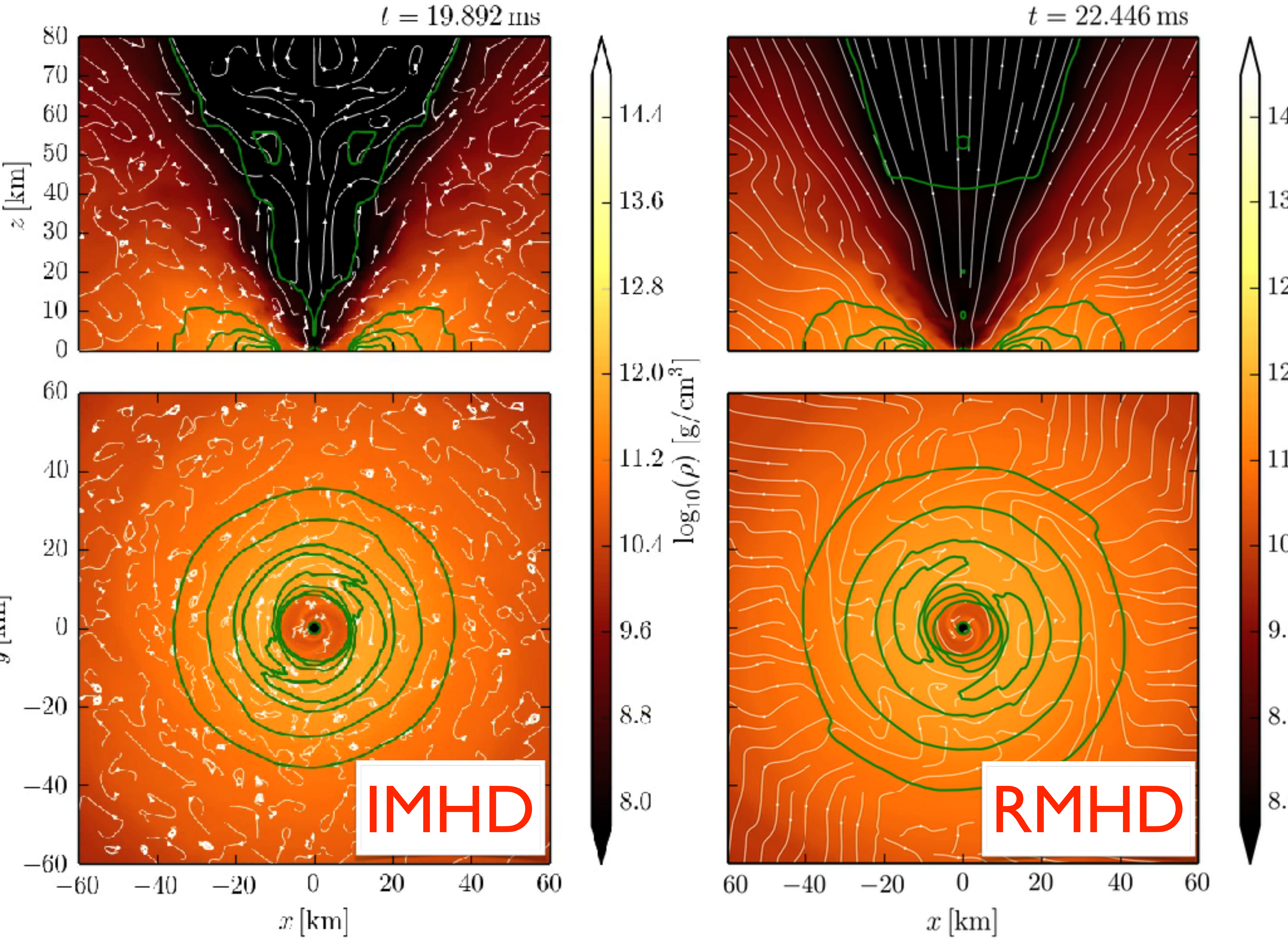
$\sigma \to \infty$ ideal-MHD (IMHD)

$\sigma \neq 0$ resistive-MHD (RMHD)

$\sigma \to 0$ electrovacuum

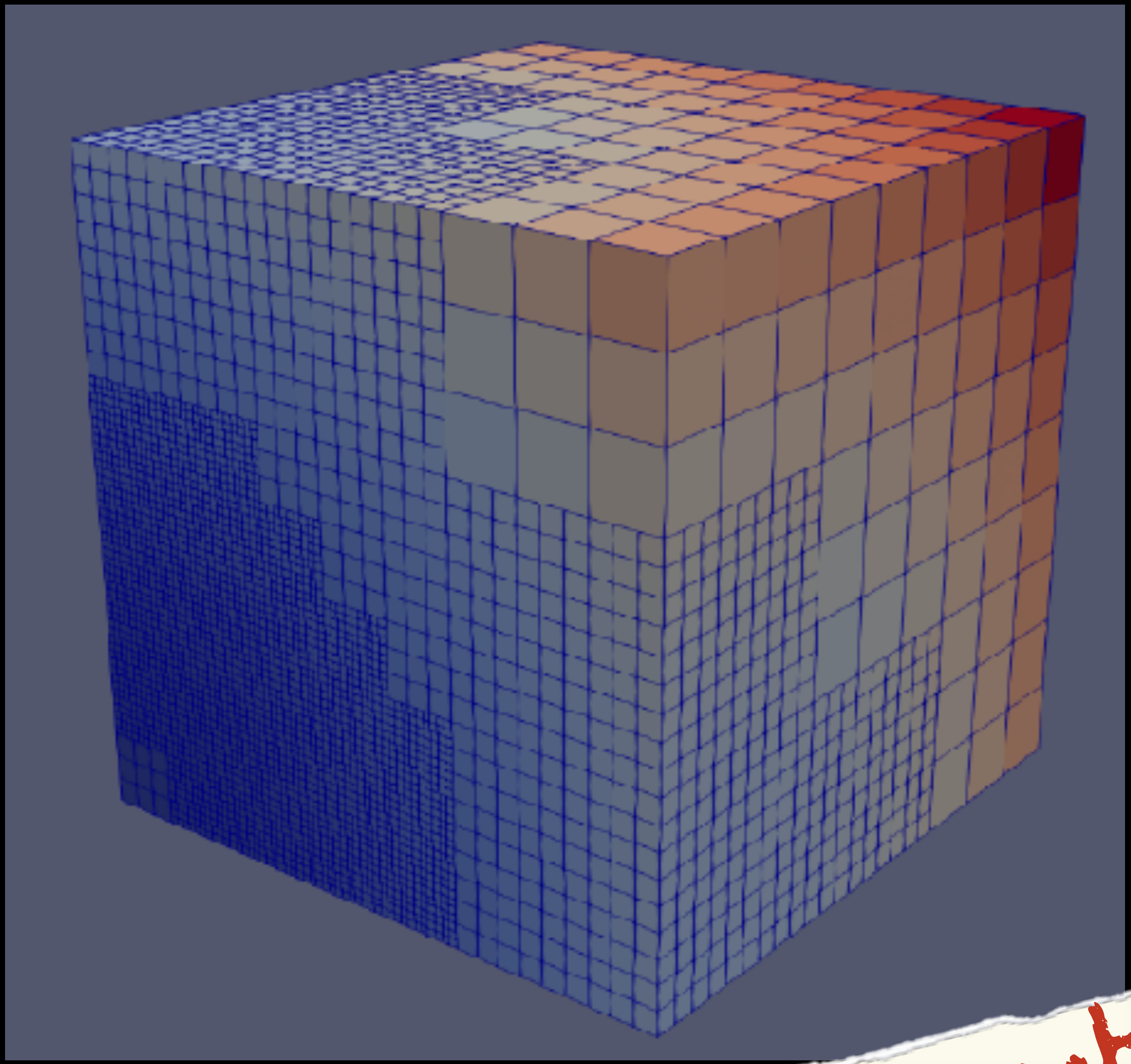
$$\sigma = f(\rho, \rho_{\min})$$

phenomenological prescription



$t = 19.892$ ms
$t = 22.446$ ms
z [km]
y [km]
x [km]
x [km]
$\log_{10}(\rho)$ [g/cm^3]
IMHD
RMHD

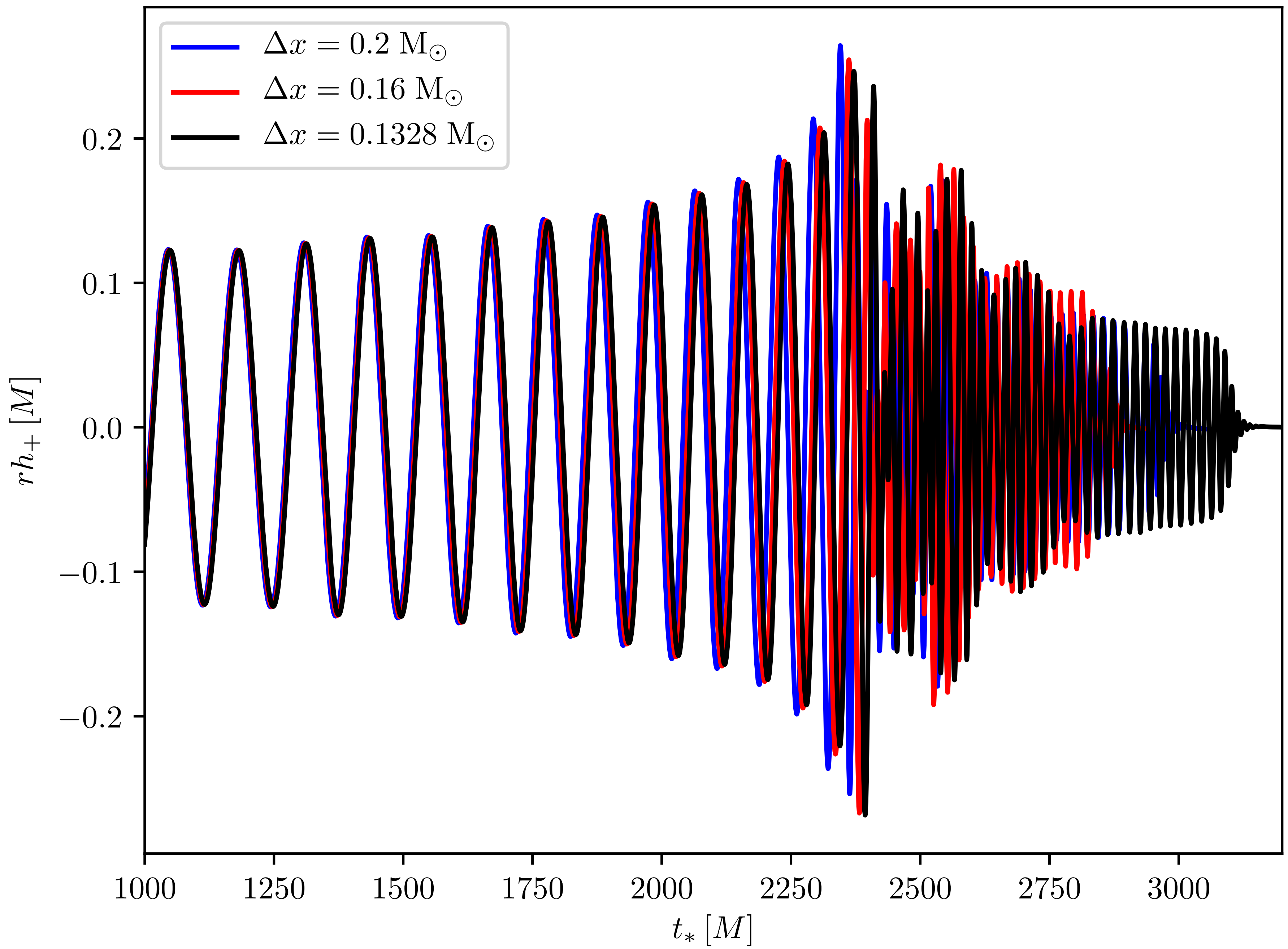
New methods in relativistic hydrodynamics/MHD



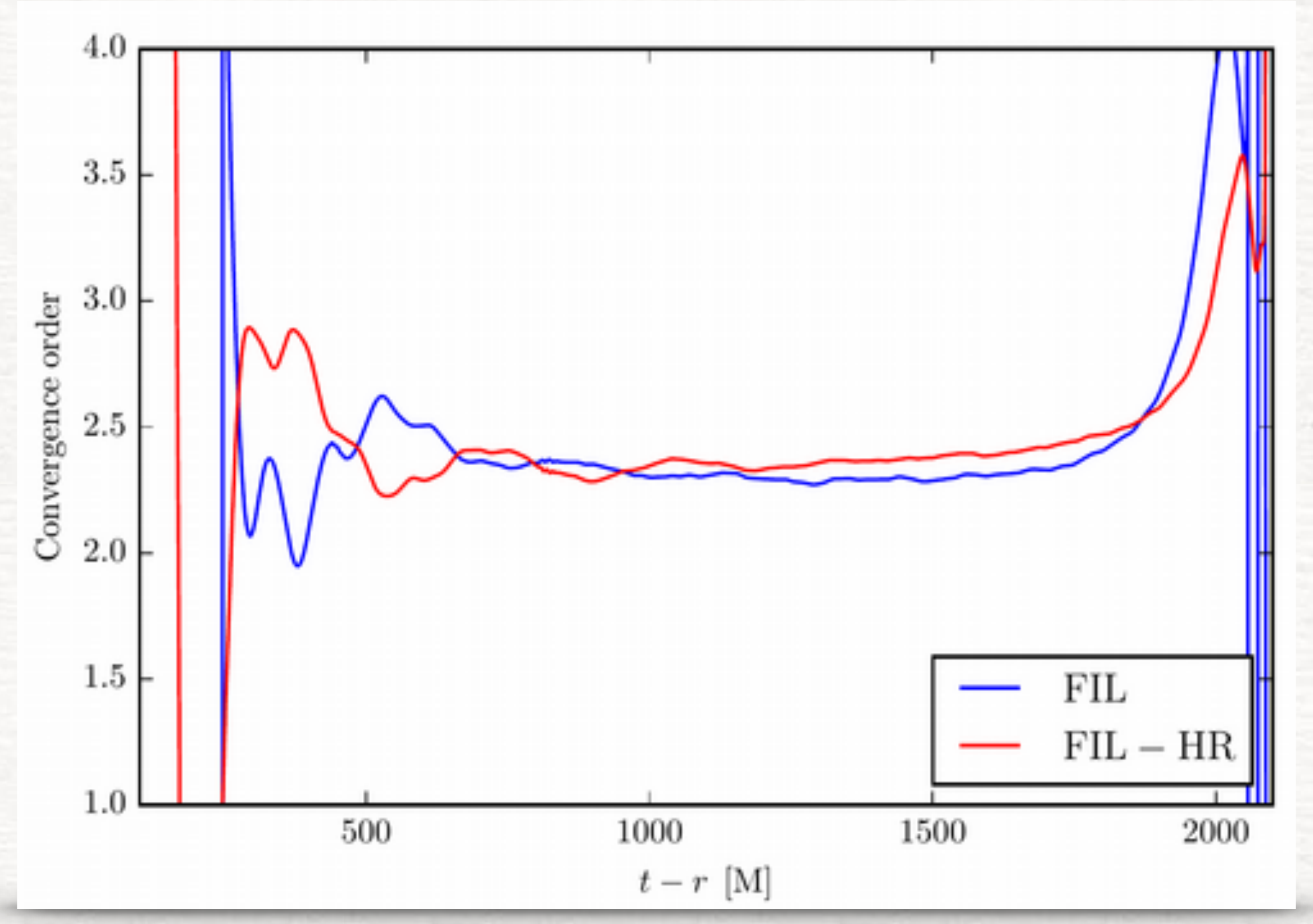
see Dumbser's talk!

State of the art

- Present state-of-the-art codes can handle in 3D:
  - ✳ GRHD
  - ✳ GRMHD (ideal and resistive)
  - ✳ Sophisticated equations of state: $p = p(\rho, \epsilon, Y_e, \ldots)$
  - ✳ Neutrino emission/absorption
- Standard numerical methods are:
  - ✳ finite difference for Einstein eqs. (also spectral methods)
  - ✳ finite-difference, finite-volume for HD/MHD



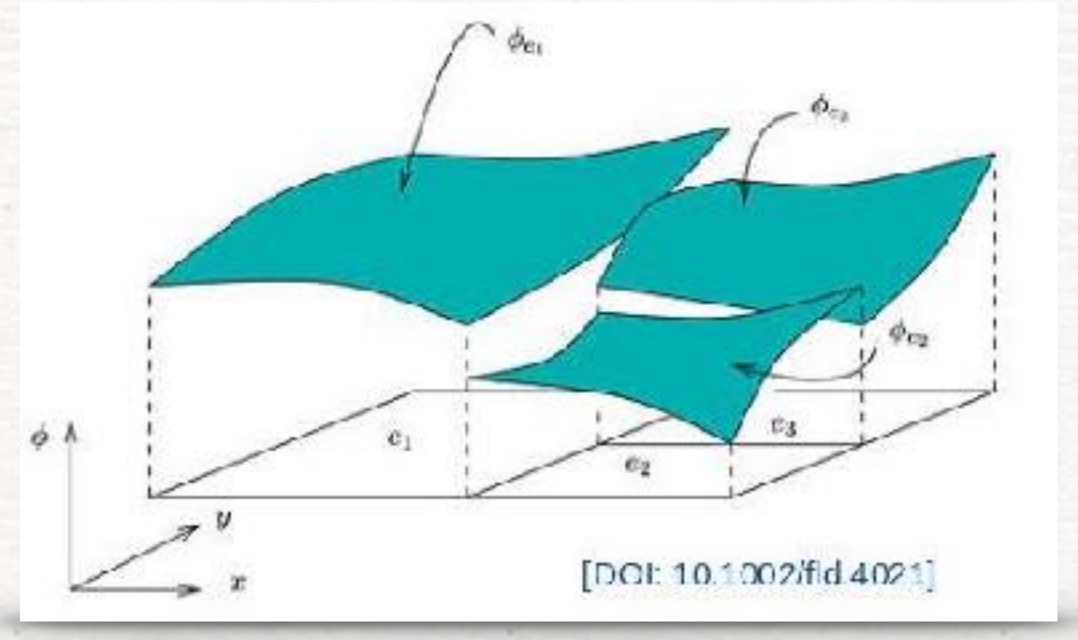
- Example of dependence of waveform on resolution: clear phase difference is observable during inspiral.
- Even larger differences appear after merger.



- Modern codes have "low" convergence orders before the inspiral and very low convergence orders after merger.
- Numerical methods are sophisticated and require intense exchange of information across neighbouring cells.
- As a result: codes scale poorly beyond few thousands cores

More advanced methods

- Slowly but surely the consensus is that most promising methods are **discontinuous Galerkin** methods
- Advantages are:
  - ✳ arbitrary order of accuracy
  - ✳ exponential convergence
  - ✳ natural treatment of shocks
  - ✳ in **ADER** approach algebra is fully local (boundary exchanges limited to the minimum)



- At least four codes have been developed since initial work in 1+1 D (Radice, LR 2011)

DG methods in a nutshell

$$\partial_t \boldsymbol{U} + \partial_\xi \boldsymbol{F}^*(\boldsymbol{U}) = 0$$

Take representative hyperbolic eq. in 1+1D

$$\boldsymbol{U}_j(\xi, t) = \sum_{k=0}^{p-1} \hat{\boldsymbol{U}}_k(t)\, \Psi_k(\xi)\,,$$

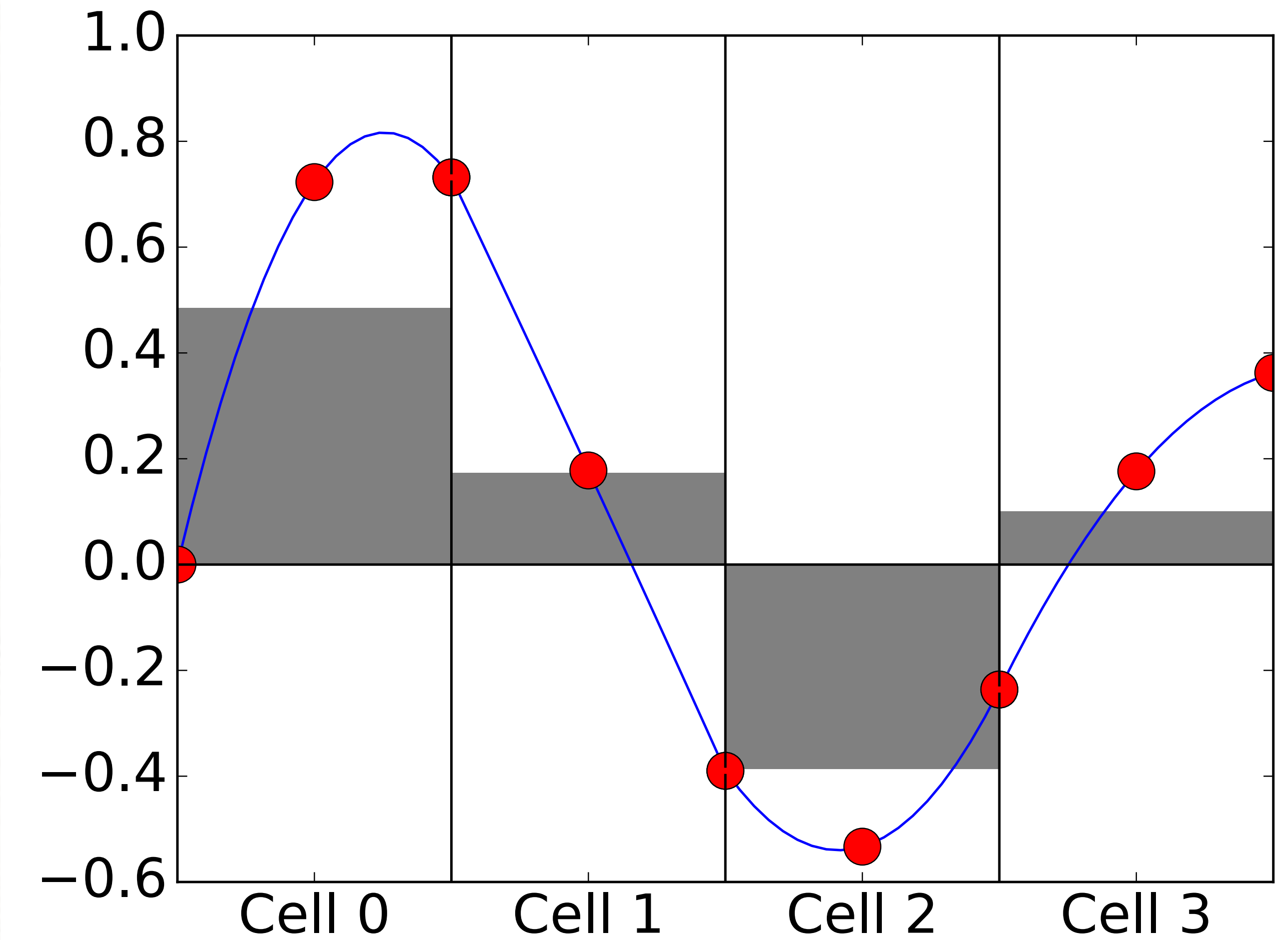
Expand solution $\boldsymbol{U}(t)$ at each cell $\boldsymbol{U}_j(t)$ in terms of known polynomial basis (Legendre)

$$\int_0^1 (\Psi_l \partial_t \boldsymbol{U} + \Psi_l \partial_\xi \boldsymbol{F}^*) d\xi = 0$$

Integrate over reference spacetime volume and integrate by parts

$$\sum_{k=0}^{p-1} \left(\int_0^1 \Psi_l \Psi_k \, d\xi \right) d_t \hat{\boldsymbol{U}}_k + \left[\Psi_l \boldsymbol{F}^* \right]_0^1 - \int_0^1 \boldsymbol{F}^*(\boldsymbol{U}(\xi, t))\, d_\xi \Psi_l \, d\xi = 0$$

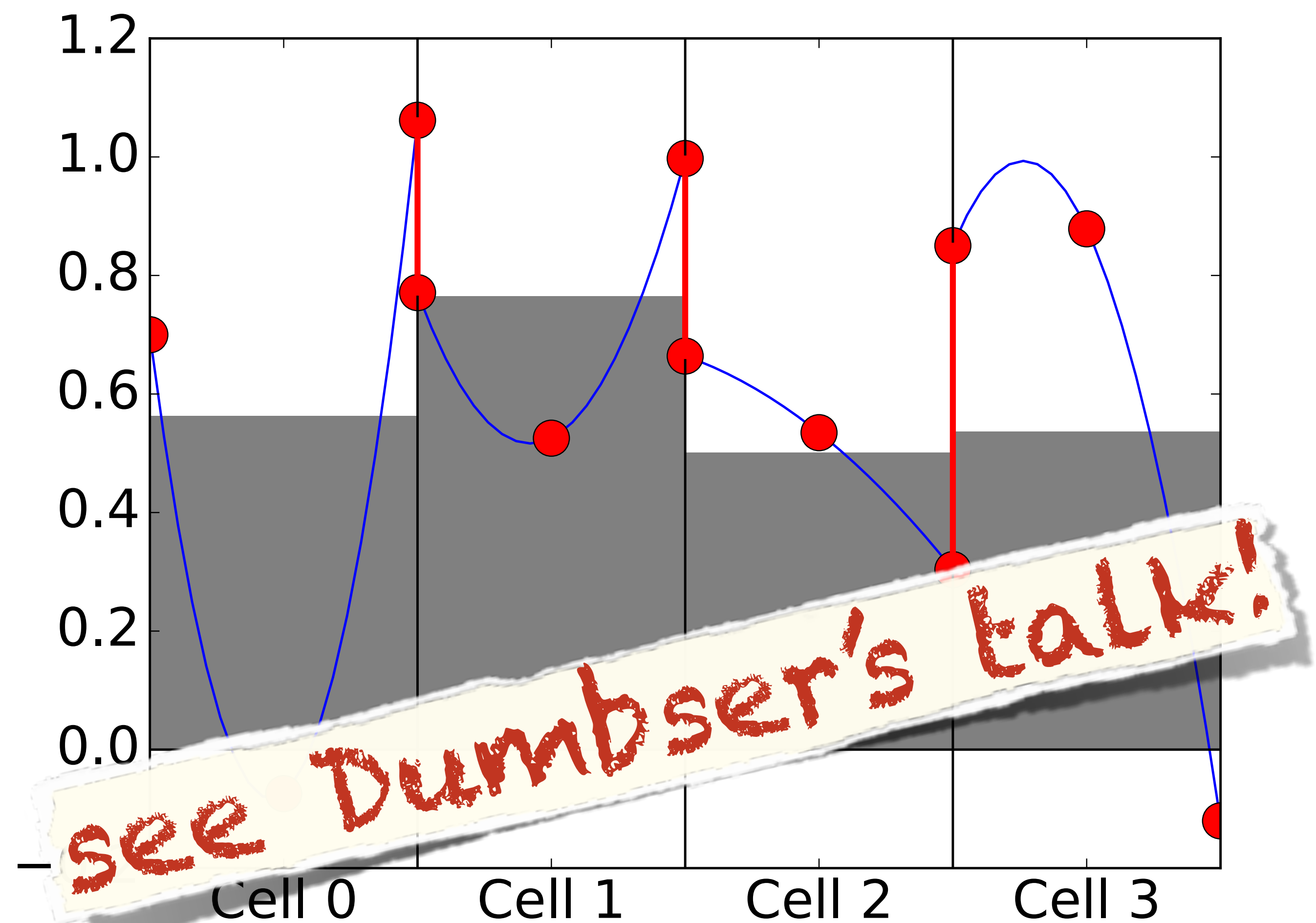
This is a system of (coupled) ordinary differential equations in time for the degrees of freedom $\hat{\boldsymbol{U}}_l(t)$



- Shown in grey is standard finite volume representation
- Circles are nodes of polynomial basis
- Blue line is reconstructed polynomial inside the cell

- Polynomials across different cells are naturally discontinuous
- Jumps are initial conditions for Riemann problems.

$$\sum_{k=0}^{p-1}\left(\int_0^1 \Psi_l \Psi_k \, d\xi\right) d_t \boldsymbol{U}_k + \left[\Psi_l \boldsymbol{F}^*\right]_0^1 - \int_0^1 \boldsymbol{F}^*(\boldsymbol{U}(\xi,t))\, d_\xi \Psi_l \, d\xi = 0$$



Conclusions

- ✳ General-relativistic HD and MHD combine complexity of solution of Einstein eqs with those of hydrodynamics.
- ✳ As in heavy-ion collisions: developments are driven by comparison with observations (now possible!).
- ✳ Several challenges have been tackled: 3D, HD/MHD, resistive effects, EOSs, neutrinos, etc.
- ✳ Dynamics of binary neutron stars is sufficiently accurate and robust to compare with observations and make predictions.
- ✳ Many points remain open:
  - ✦ improve formulation of Einstein/HD/MHD eqs.
  - ✦ improve description of **dissipative** effects: could be important.
  - ✦ improve description of **EM** effects: essential for astrophysics.
  - ✦ improve **convergence order** and **scalability** of codes: DG

Future is exciting!