A new approach to relativistic viscous hydrodynamics

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Bemfica, Disconzi, JN, arXiv:1708.06255 [gr-qc]
See also Disconzi, arXiv:1708.06572 [math.AP]

Foundational Aspects of Relativistic Hydrodynamics

ECT*, Trento, May 2018

Fluid dynamics in the relativistic regime

Energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0$$

Conserved charge

$$\nabla_\mu J^\mu = 0$$

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

An open problem in physics and mathematics since 1940's:

Prove that the system Einstein + viscous fluid:

- Causal + other properties (e.g, linear stability)

- Mathematically well-posed

Ideal (Euler) relativistic fluid dynamics

Energy-momentum tensor

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P(\epsilon, n)\Delta^{\mu\nu}$$

Conserved charge

$$J^\mu = n u^\mu$$

$\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

Assuming that speed of sound and $\epsilon + P > 0$

$u_\mu u^\mu = -1$

- System is well-posed (Sobolev space H^s with s > 5/2)

  - Existence, uniqueness, continuous dependence on initial data

- Causal (domain of dependence property in the sense of relativity)

Remember that a system of PDE's is well-posed (Hadamard) when:

http://www.math.ucla.edu/~tao/Dispersive/

1) A solution exists.

2) The solution is unique.

3) The solution depends continuously on initial data (e.g., initial conditions, boundary conditions).

Ex.: $u_{xx} + u = 0$

a) $u(0) = 0, u(\frac{\pi}{2}) = 1 \Rightarrow$ unique solution $u(x) = \sin(x)$
b) $u(0) = 0, u(\pi) = 1 \Rightarrow$ no solution
c) $u(0) = 0, u(\pi) = 0 \Rightarrow$ infinitely many solutions: $u(x) = A\sin(x)$

Ex.:

$$\left.\begin{cases} u_t = u_{xx} & \text{heat equation} \\ u(0,t) = u(1,t) = 0 & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{cases}\right\} \text{well-posed}$$

$$\left.\begin{cases} u_t = -u_{xx} & \text{backwards heat equation} \\ u(0,t) = u(1,t) & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{cases}\right\} \begin{array}{l}\text{no continuous dependence} \\ \text{on initial data}\end{array}$$

Mathematical definition of causality (relativity)

See, e.g., Choquet-Bruhat, Wald

Consider a system of (linear or nonlinear) PDE's

N unknowns

$$P^I_K \varphi^K = 0 \qquad \{\phi^K\}_{K=1}^N$$

The system is causal if for any point x in the future of Σ

$\varphi^K(x)$ depends only on $J^-(x) \cap \Sigma$

causal past

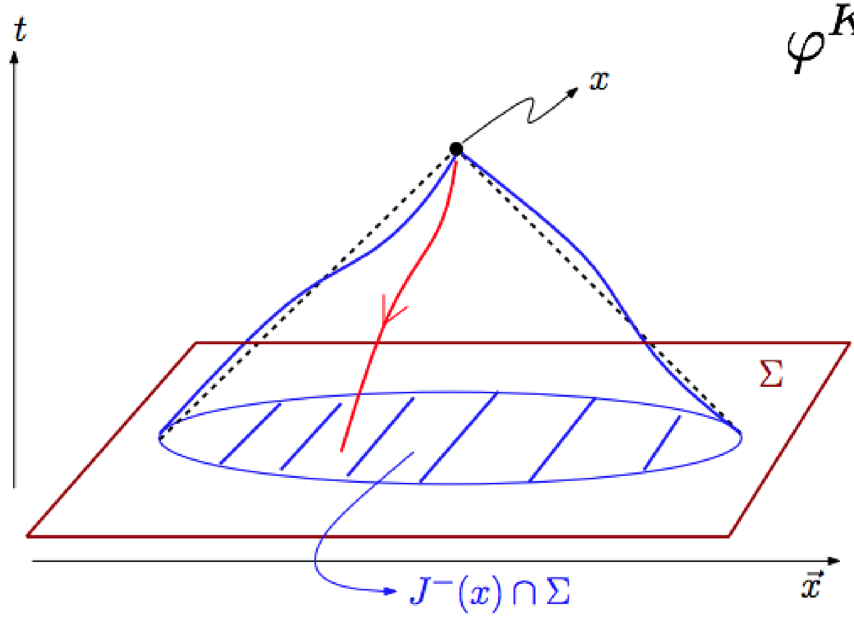


FIG. 1: (color online) Illustration of causality. In curved spacetime $J^-(x)$ looks like a distorted light-cone opening to the past (blue region); in flat spacetime the cone would be straight (dotted line). Points inside $J^-(x)$ can be joined to a point x in spacetime by a causal past directed curve (e.g. the red line). The Cauchy surface Σ supports the initial data and the value of the field $\varphi(x)$ depends only on the initial data on $J^-(x) \cap \Sigma$.

Fluid dynamics in the relativistic regime

- Einstein + Euler equations: Locally well-posed and causal

Choquet-Bruhat 1958, 1966
Lichnerowicz, 1967

- Einstein + Euler equations: Not globally well-posed (schocks occur)

Christodoulou, 2007

What about dissipative fluids? $T^{\mu\nu} = T^{\mu\nu}_{ideal} + \pi^{\mu\nu}$

(a) Einstein+viscous fluid admit existence + uniqueness of solutions?
(b) Causality?
(c) Stability (at least in the linear regime)?
(d) Does the solution really describe the physics of the system?

Eckart
1940's

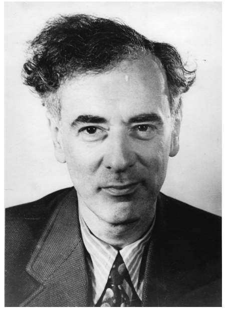
Relativistic Navier-Stokes equations

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu}$$

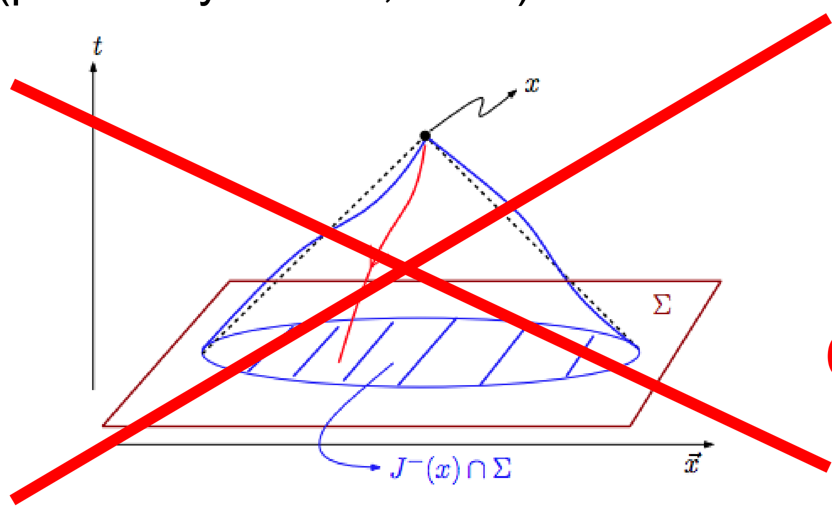
Conservation law

$$\nabla_\mu T^{\mu\nu} = 0$$



Landau
1950's

This theory* is **acausal**
(proven by Pichon, 1965)



Nonlinear diffusion-like equations

$$\partial_t u = \frac{\eta}{sT}\nabla^2 u + \ldots$$

also **unstable** around global equilibrium
(see Hiscock, Lindblom, 1984)

Coupling to Einstein's eqs???

Causality does not imply (linear) stability

$$T^{\alpha\beta} = T^{\alpha\beta}_{ideal} + \left(\lambda + \frac{\nu}{4}\right) g^{\alpha\beta}(\nabla_\lambda C^\lambda) - \frac{\nu}{2}\left(\nabla^\alpha C^\beta + \nabla^\beta C^\alpha\right)$$

Dynamic velocity $C^\alpha = (\epsilon + P)u^\alpha$

Choquet-Bruhat, 2006

$\nabla_\mu T^{\mu\nu} = 0$ $\rightarrow$ Principal part: $\frac{\nu}{2}\{\nabla_\alpha\nabla^\alpha C^\beta + \nabla_\alpha\nabla^\beta C^\alpha\} + \left(\lambda - \frac{1}{3}\nu\right)\nabla^\beta\nabla_\alpha C^\alpha$

Characteristic matrix: $\frac{\nu}{2}X_\alpha X^\alpha C^\beta + aX_\alpha X^\beta C^\alpha \quad a := \frac{\nu}{6} + \lambda.$

Characteristic determinant: $\frac{\nu}{2}\left(\frac{\nu}{2} + a\right)(X^\alpha X_\alpha)^4$ $\longrightarrow$ **CAUSAL**

However, this system is linearly unstable around global equilibrium

Bemfica, Disconzi, JN, 2017

A brief review of progress in relativistic fluids

- Divergence type theories (Geroch, Lindblom, etc).

- Israel-Stewart-type theories (see Denicol, Grossi, Rischke's talks).

  - Derivation from kinetic theory (see, e.g., DNMR 2012).
  - Causality only established in the linear regime.
  - No proof (yet?) of well-posedness + causality(nonlinear)+Einstein.
  - Display causality issues in the nonlinear regime (HL 1988).

- Lichnerowicz (Einstein+viscous fluid): Causality + well-posedness

  Proven by M. Disconzi, 2014

  - Linearly unstable

A new approach to relativistic viscous hydrodynamics

Bemfica, Disconzi, JN, arXiv:1708.06255 [gr-qc], Disconzi, arXiv:1708.06572 [math.AP]

In the following I will focus on conformally invariant systems and:

(a) Develop a new way to extract a viscous hydrodynamics from conformal kinetic theory.

(b) Prove existence and uniqueness of the solutions of this new Einstein+viscous fluid.

(c) Prove causality

(d) Prove linear stability around global equilibrium

(e) Discuss some simple initial applications within heavy ion collisions

Consider a conformal kinetic theory

BRSSS, JHEP 2008

Boltzmann equation transforms homogeneously under:

DHMNS, PRL 2014
DHMNS, PRD 2014

Weyl transformation: $g_{\mu\nu} \to e^{-2\Omega} g_{\mu\nu}$ (massless particles)

$$p^\mu \nabla_\mu f(x,p) = \mathcal{C}[f_p, f_p] \to e^{2\Omega} \left(p^\mu \nabla_\mu f(x,p) = \mathcal{C}[f_p, f_p] \right)$$

Perturbative approaches to the Boltzmann equation

D. Hilbert S. Chapman D. Enskog H. Grad

Perturbative approaches to the Boltzmann equation

Introduce a small book keeping parameter – identified with **Kn** or not

Knudsen number

$$f_k(x) = \sum_{n=0}^{\infty} \alpha^n f_k^{(n)}(x) \quad \Rightarrow \quad \mathcal{C}[f_k, f_k] = \sum_{n=0}^{\infty} \alpha^n \mathcal{C}^{(n)}$$

Different expansions according to the LHS of Boltzmann equation

Hilbert expansion

$$\alpha\, k^\mu \nabla_\mu f_k(x) = \mathcal{C}[f_k, f_k]$$

Chapman-Enskog expansion

$$E_k D f_k + \alpha\, k^{\langle\mu\rangle} \nabla_{\langle\mu\rangle} f_k = \mathcal{C}[f, f]$$

Zeroth order $\rightarrow$ $\mathcal{C}[f_k^{(0)}, f_k^{(0)}] = 0 \Longrightarrow f_k^{(0)} = f_k^{eq}$

equilibrium

Hilbert expansion for a conformal fluid

0th order solution: first approximation for the hydrodynamic fields

$$f_k^{(0)}(x) = e^{k^\mu u_\mu^0(x)/T_0(x)+\mu_0/T_0} \qquad \mu_0/T_0 \text{ is constant.}$$

$\{\mu_0, T_0, u_\mu^0\}$ $\rightarrow$ obey ideal hydrodynamic equations

$$\begin{aligned} \nabla_\mu J_0^\mu &= \nabla_\mu (n_0 u_0^\mu) = 0 \rightarrow \mathcal{D} n_0 = 0 \\ \nabla_\mu T_0^{\mu\nu} &= \nabla_\mu \left[\epsilon_0 \left(u_0^\mu u_0^\nu + \frac{\Delta_0^{\mu\nu}}{3} \right) \right] = 0 \rightarrow \mathcal{D} T_0 = 0 \text{ and } \mathcal{D}_{\langle\mu\rangle} T_0 = 0. \end{aligned}$$

Higher orders $\rightarrow$ $f_k^{(n)}(x) = f_k^{(0)}(x)\phi_k^{(n)}(x)$ (with $\phi_k^{(0)} = 1$)

$$\boxed{\frac{1}{f_k^{(0)}} k^\mu \nabla_\mu \left(f_k^{(0)} \phi_k^{(n-1)} \right) - F_k^{(n-1)} = \mathcal{L}[\phi_k^{(n)}]}$$

Hilbert expansion for a conformal fluid

Linearized collision operator:

$$\mathcal{L}[\phi_k^{(n)}] = \int_{k'pp'} W(kk'|pp')\, f_{k'}^{(0)} \left(\phi_p^{(n)} + \phi_{p'}^{(n)} - \phi_k^{(n)} - \phi_{k'}^{(n)}\right)$$

$\{1, E_k, k^{\langle\mu\rangle}\}$ is in the kernel of $\mathcal{L}$. $\qquad \mathcal{L}[\{1, E_k, k^{\langle\mu\rangle}\}] = 0$

Define the inner product: $\quad (f,h) = \int_k f_k^{(0)}\, h_1(x,k) h_2(x,k)$

Self-adjoint

$$(h_k, \mathcal{L}[\phi_k]) = (\phi_k, \mathcal{L}[h_k])$$

Non-positive

$$(\phi_k, \mathcal{L}[\phi_k]) \leq 0$$

Eigenfunctions with zero eigenvalues

$$\psi_1(x,k) = 1, \qquad \psi_2(x,k) = 2 + u_\mu^0 k^\mu / T_0, \qquad \psi_3^{\langle\mu\rangle}(x,k) = k^{\langle\mu\rangle}/T_0$$

Entropy production in perturbative expansions

In general, the non-equilibrium entropy current is

H-theorem
(full result)

$$\mathcal{S}^\mu(x) = -\int_k k^\mu \, f_k(x) \left(\ln f_k(x) - 1\right)$$

$$\nabla_\mu \mathcal{S}^\mu \geq 0$$

Entropy production

$$\nabla_\mu \mathcal{S}^\mu = -\int_k \mathcal{C}[f_k, f_k] \ln f_k$$

$$\Rightarrow \quad \nabla_\mu \mathcal{S}^\mu = \underbrace{-\alpha^2 (\phi_k^{(1)}, \mathcal{L}[\phi_k^{(1)}])}_{\geq 0} + \mathcal{O}(\alpha^3)$$

valid for the expansions considered here

Hilbert expansion for a conformal fluid – First order

$\{\mu_0, T_0, u^0_\mu\}$ → obey ideal hydrodynamic equations

$$\boxed{\mathcal{L}[\phi_k^{(1)}] = \frac{k^{\langle\alpha}k^{\beta\rangle}}{2T_0}\sigma^0_{\alpha\beta}}$$

Linear integral operator (Fredholm)

General solution: $\phi_k^{(1)} = \phi_k^h + \phi_k^{nh}$

homogeneous solution

$$\phi_k^h = \xi_0(x)\psi_1 + v_0(x)\psi_2 + v^0_{\langle\mu\rangle}\psi_3^{\langle\mu\rangle}$$

inhomogeneous solution

$$\phi_k^{nh} = \frac{k^{\langle\alpha}k^{\beta\rangle}}{2T_0^3}\sigma^0_{\alpha\beta}A\left(\frac{E_k^0}{T_0}\right)$$

$A(x)$ → obtained from inverting the operator in space orthogonal to kernel

Fluid always described by $\{n, \epsilon, u_\mu\}$

To 1st order: $$u_\mu = u_\mu^0 + \alpha\, \delta u_\mu + \mathcal{O}(\alpha^2)$$

$$n = n_0 + \alpha\, \delta n + \mathcal{O}(\alpha^2) \qquad \epsilon = \epsilon_0 + \alpha\, \delta\epsilon + \mathcal{O}(\alpha^2)$$

Fixing homogeneous: $\{\xi_0, v_0, v_{\langle\mu\rangle}\} \to \{\delta n, \delta\epsilon, \delta u_\mu\}$

Solution of Boltzmann to 1st order in the Hilbert expansion

$$f_k = f_k^{(0)} + f_k^{(0)}\left[\left(2\frac{\delta n}{n_0} - \frac{\delta\epsilon}{\epsilon_0}\right)\psi_1 + \left(\frac{\delta n}{n_0} - \frac{\delta\epsilon}{\epsilon_0}\right)\psi_2 + \delta u^\mu \psi_3^{\langle\mu\rangle}\right] + f_k^{(0)}\frac{k^{\langle\alpha}k^{\beta\rangle}}{2T_0^3}\sigma^0_{\alpha\beta}A\left(\frac{E_k^0}{T_0}\right)$$

- Solution asymptotes to a solution of the Boltzmann equation.
- Linear equations for $\{\delta n, \delta\epsilon, \delta u_\mu\}$ $\rightarrow$ not Navier-Stokes theory.
- Convergence of the series?

Chapman-Enskog expansion

De Groot, van Leeuwen, van Weert, 1980

Local hydro fields $\{\mu(x), T(x), u_\mu(x)\}$ in equilibrium are "exact"

$$f_k^{eq} = e^{\mu/T - E_k/T} \Longrightarrow f_k = f_k^{eq} + \delta f_k$$

See Israel, Tsumura, Hatsuda, Van, Biro

Define general conditions of fit:

$$\int_k E_k^n \delta f_k = 0 \qquad \int_k E_k^m \delta f_k = 0 \qquad \int_k E_k^\ell k^{\langle\mu\rangle} \delta f_k = 0$$

n, m, ℓ $\rightarrow$ non-negative

- For instance, for Landau n=1, m=2, l=1. Eckart's n=1,m=2, l=0.

Starting point: $\boxed{E_k D f_k + \alpha\, k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle} f_k = \mathcal{C}[f,f]}$

Perturbative series: $f_k = \sum_{n=0}^{\infty} \alpha^n f_k^{(n)}$ $\quad$ $D f_k = \sum_{n=1}^{\infty} \alpha^n (D f_k)^{(n)}$

Unknowns

General expression: $\boxed{\frac{1}{f_k^{eq}} E_k (D f_k)^{(n)} + \frac{1}{f_k^{eq}} k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle} f_k^{(n-1)} - F_k^{(n-1)} = \mathcal{L}[\phi_k^{(n)}]}$

Solubility conditions
(Fredholm)

$$
\begin{aligned}
\int_k E_k (D f_k)^{(n)} &= -\int_k k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle} f_k^{(n-1)} \\
\int_k E_k^2 (D f_k)^{(n)} &= -\int_k E_k k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle} f_k^{(n-1)} \\
\int_k E_k k^{\langle\nu\rangle} (D f_k)^{(n)} &= -\int_k k^{\langle\nu\rangle} k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle} f_k^{(n-1)}.
\end{aligned}
$$

Chapman-Enskog expansion – First order

$$\frac{1}{f_k^{eq}}E_k(Df_k)^{(1)} + \frac{k^{\langle\nu}k^{\lambda\rangle}}{2T}\sigma_{\nu\lambda} + \frac{E_k}{T}\frac{k^{\langle\mu\rangle}\nabla_{\langle\mu\rangle}T}{T} + \frac{\theta E_k^2}{3T} = \mathcal{L}[\phi_k^{(1)}]$$

Solubility conditions:

$$\boxed{\frac{(DT)^{(1)}}{T} + \frac{\theta}{3} = 0 \qquad (Du^\mu)^{(1)} + \frac{\nabla^{\langle\mu\rangle}T}{T} = 0}$$

The root of all (causal) evil !!! $\quad u^\mu\nabla_\mu \Longrightarrow \Delta^{\mu\nu}\nabla_\nu$

Must solve:

$$\mathcal{L}[\phi_k^{(1)}] = \frac{k^{\langle\nu}k^{\lambda\rangle}}{2T}\sigma_{\nu\lambda}$$

$\longrightarrow$

NAVIER-STOKES EQS.

Conditions of fit:

$$\xi = v = v_{\langle\mu\rangle} = 0$$

Perturbative expansions: a new hope

Based on DNNR, PRD 2011

Local hydro fields $\{\mu(x), T(x), u_\mu(x)\}$ in equilibrium are "exact"

$$f_k^{eq} = e^{\mu/T - E_k/T} \Longrightarrow f_k = f_k^{eq} + \delta f_k$$

Define general conditions of fit (as before):

$$\int_k E_k^n \delta f_k = 0 \qquad \int_k E_k^m \delta f_k = 0 \qquad \int_k E_k^\ell k^{\langle\mu\rangle} \delta f_k = 0$$

n, m, ℓ → non-negative

How do we bypass Chapman-Enskog's $D \longrightarrow \nabla_\perp$?

Derivatives of local equilibrium are “small”

$$\boxed{\alpha\, k^\mu \nabla_\mu f_k^{eq} + k^\mu \nabla_\mu \delta f_k - \mathcal{L}[\delta f_k] = \mathcal{C}[\delta f_k, \delta f_k]}$$

Perturbative series

$$\delta f_k = \sum_{n=1}^{\infty} \alpha^n \delta f_k^{(n)} \quad \Longrightarrow$$

First equation in the hierarchy

$$k^\mu \nabla_\mu \delta f_k - \mathcal{L}[\delta f_k] = -k^\mu \nabla_\mu f_k^{eq}$$

DNNR, PRD (2011)

General term:

$$k^\mu \nabla_\mu \delta f_k^{(n+2)} - \mathcal{L}[\delta f_k^{(n+2)}] = \sum_{m=0}^{n} \mathcal{C}[\delta f_k^{(n-m+1)}, \delta f_k^{(m+1)}]$$

This should be asymptotic to a solution of Boltzmann

Considering only the first equation in the hierarchy

$$k^\mu \nabla_\mu \delta f_k - \mathcal{L}[\delta f_k] = -k^\mu \nabla_\mu f_k^{eq}$$

Kernel of $\mathcal{L}$ $\longrightarrow$ $\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^\mu = 0$

Hilbert space orthogonal to kernel:

$$(\bar{\psi}_i, k^\mu \nabla_\mu \delta f_k - \mathcal{L}[\delta f_k]) = -(\bar{\psi}_i, k^\mu \nabla_\mu f_k^{eq})$$

+ derivative expansion at lowest order ("hydrodynamics")

$$\left(\bar{\psi}_i, \frac{k^{\langle\mu}k^{\nu\rangle}\sigma_{\mu\nu}}{2T} + \frac{\mathcal{D}T}{T^2}E_k^2 + \frac{E_k k^{\langle\mu\rangle}\mathcal{D}_{\langle\mu\rangle}T}{T^2}\right) = \left(\bar{\psi}_i, \mathcal{L}[\phi_k^{(1),1}]\right)$$

General solution:

New terms

$$\begin{aligned}\phi_k^{(1),1} &= \phi_k^h + \phi_k^{nh} \\ &= \left(\xi(x)\psi_1 + v(x)\psi_2 + v_{\langle\mu\rangle}\psi_3^{\langle\mu\rangle}\right) + \underbrace{\frac{k^{\langle\alpha}k^{\beta\rangle}}{2T^3}\sigma_{\alpha\beta}A_k + \frac{\mathcal{D}T}{T^2}B_k + \frac{k^{\langle\mu\rangle}\mathcal{D}_{\langle\mu\rangle}T}{T^3}C_k}\end{aligned}$$

A_k, B_k, C_k → Depend on properties of collision term

Energy-momentum tensor at this order is

$$T^{\mu\nu} = (\epsilon + \mathcal{A})\left(u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right) + \pi^{\mu\nu} + \mathcal{Q}^{\langle\mu\rangle}u^\nu + \mathcal{Q}^{\langle\nu\rangle}u^\mu$$

$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$ → Navier-Stokes stress

Non-equilibrium correction to equilibrium energy density

$$\mathcal{A} = (\xi - 2v)\epsilon + \frac{\mathcal{D}T}{T^2}(E_k^2, B_k)$$

obtained from collision term

Non-equilibrium correction to energy flow

$$\mathcal{Q}^{\langle\mu\rangle} = \frac{4\epsilon}{3}v^{\langle\mu\rangle} + \frac{\mathcal{D}^{\langle\mu\rangle}T}{3T^3}(E_k^3, C_k)$$

The specific form of these quantities depends on the conditions of fit

(definition of equilibrium fields)

For Landau's $\mathcal{A} = \mathcal{Q}^{\langle\mu\rangle} = 0$ $\longrightarrow$ causality and stability issues!!

$$\int_k E_k\, k^\mu \delta f_k = 0$$

Assume the simplest conformal kinetic theory → $\sigma \sim 1/T^2$

Among (infinitely) many choices, we will use the following "unorthodox", though still bonafide, conditions of fit

$$\int_k E_k^2\, k^\mu f_k = \int_k E_k^2\, k^\mu f_k^{eq} \quad \longrightarrow \quad u_\mu u_\nu \rho^{\mu\nu\lambda} = u_\mu u_\nu \rho_{eq}^{\mu\nu\lambda}$$

$$\int_k f_k = \int_k f_k^{eq}$$

In this case one finds:

$$\mathcal{A} = 3\chi \frac{\mathcal{D}T}{T} \qquad \mathcal{Q}^{\langle\mu\rangle} = \lambda \frac{\mathcal{D}^{\langle\mu\rangle}T}{T} \qquad \lambda, \chi > 4\eta$$

This system has remarkable properties

Existence, uniqueness, causality + linear stability proven for the 1st time in viscous general relativistic hydrodynamics

Bemfica, Disconzi, JN, arXiv:1708.06255 [gr-qc], Disconzi, arXiv:1708.06572 [math.AP]

$$T^{\mu\nu} = \left(\epsilon + 3\chi\frac{\mathcal{D}T}{T}\right)\left(u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right) - \eta\sigma^{\mu\nu} + \lambda\left(u^\mu\frac{\mathcal{D}^{\langle\nu\rangle}T}{T} + u^\nu\frac{\mathcal{D}^{\langle\mu\rangle}T}{T}\right)$$

$$\chi = a_1\eta, \;\; \lambda = a_2\eta$$

Theorem 1. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, \epsilon_0, \epsilon_1, v_0, v_1)$ be a sufficiently regular initial data set for Einstein's equations coupled to the energy-momentum tensor above. Assume the following: Σ is compact with no boundary, $\epsilon_0 > 0$, $\eta : (0,\infty) \to (0,\infty)$ is analytic, and either $a_1 = 4$ and $a_2 \geq 4$ or $a_1 > 4$ and $a_2 = 3a_1/(a_1-1)$. Then: (1) There exists a globally hyperbolic development M of $\mathcal{I}$. (2) Let (g, ϵ, u) be a solution of Einstein's equations provided by the globally hyperbolic development M. For any $x \in M$ in the future of Σ, $(g(x), u(x), \epsilon(x))$ depends only on $\mathcal{I}|_{i(\Sigma)\cap J^-(x)}$, where $J^-(x)$ is the causal past of x and $i : \Sigma \to M$ is the embedding associated with the globally hyperbolic development M.*

Theorem 2. *Under the same assumptions regarding a_1 and a_2, a statement similar to Theorem 1, i.e., existence, uniqueness, and causality holds for solutions of $\nabla_\mu T^{\mu\nu} = 0$, with $T^{\mu\nu}$ given by the tensor above, in Minkowski background.*

Theorem 3. *Consider Theorem 2. The system is also linearly stable around equilibrium (rest and boosted frame) in a Minkowski background.*

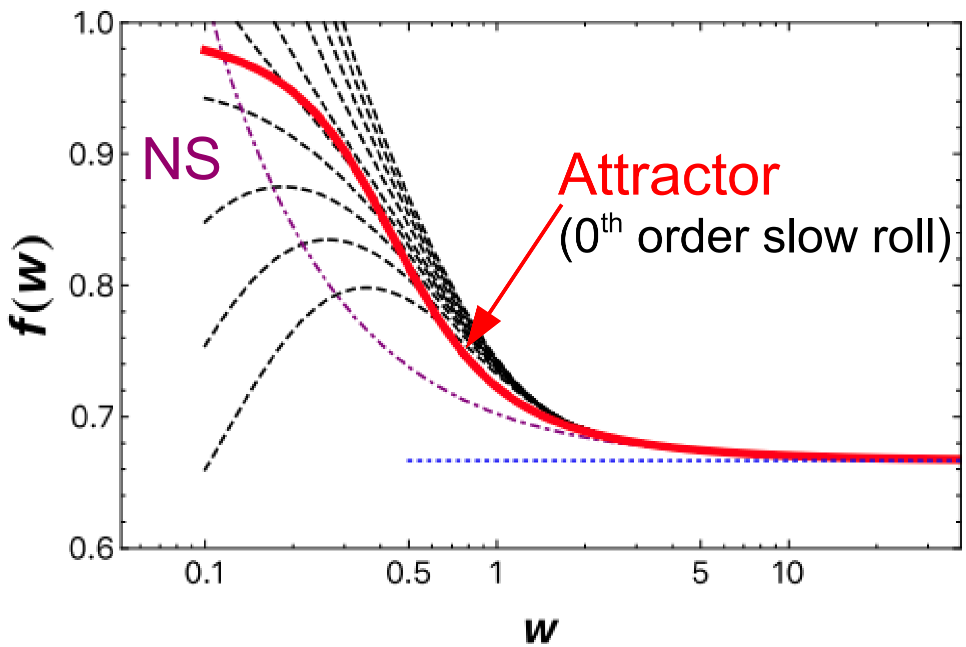
Applications: Bjorken flow

Milne coordinates $\quad u_\mu = (-1,0,0,0) \qquad T \to T(\tau)$

Equation of motion: $\quad w = \tau T$ and $f = 1 + \tau \partial_\tau T / T$

Heller-Spalinski form

$$\bar{\chi} w f(w) \frac{df(w)}{dw} + 3\bar{\chi} f(w)^2 + f(w)\left(w - \frac{14}{3}\bar{\chi}\right) + \frac{16\bar{\chi}}{9} - \frac{4\bar{\eta}}{9} - \frac{2w}{3} = 0$$



$\eta / s = 0.08$

$\chi = 4\eta$

Applications: Gubser flow

$dS_3 \otimes \mathbb{R}$ spacetime $\quad u_\mu = (-1,0,0,0) \qquad T \to \hat{T}(\rho)$

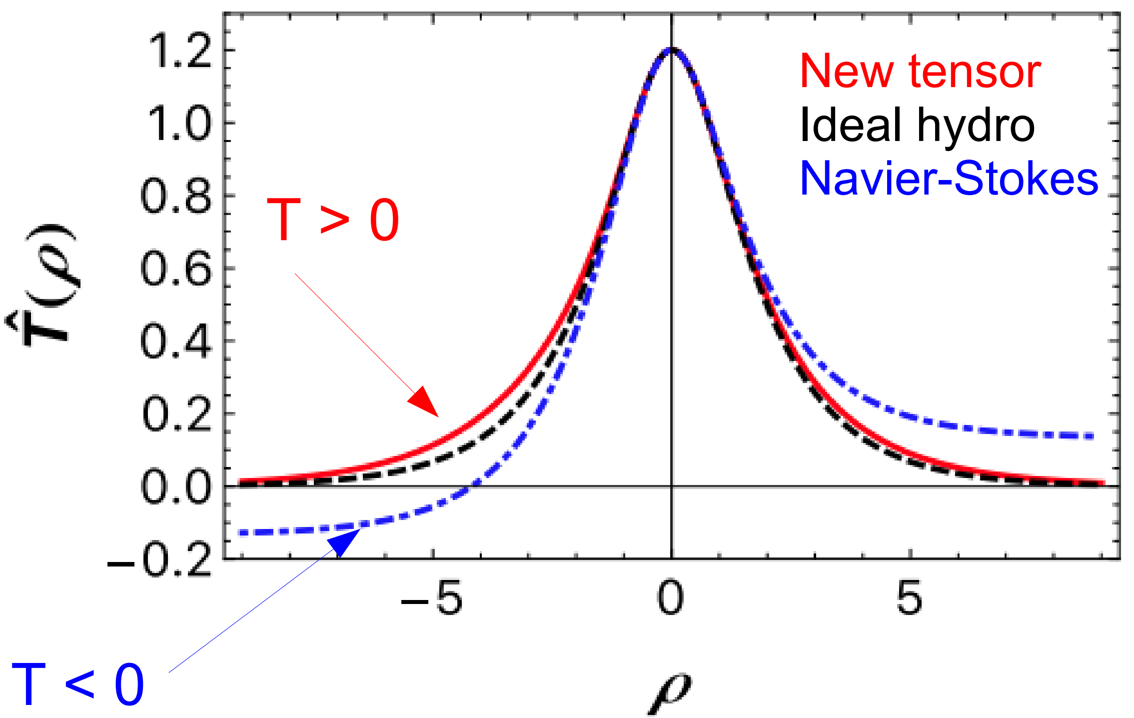
Equation of motion (written in 1st order form):

$$\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \hat{\mathcal{F}}(\rho) \qquad \bar{\chi}\frac{d\hat{\mathcal{F}}}{d\rho} + 3\bar{\chi}\hat{\mathcal{F}}^2 + \frac{2}{3}\bar{\chi}\hat{\mathcal{F}}\tanh\rho + \hat{T}\hat{\mathcal{F}} - \frac{4}{9}\bar{\eta}(\tanh\rho)^2 = 0$$

$\eta/s = 0.2$

$\chi = 4\eta$

$\hat{T}_0 = 1.2$



Conclusions and Outlook

- New perturbative expansion of Boltzmann equation
- New viscous relativistic energy-momentum tensor
- Existence, uniqueness, causality, linear stability proven for the 1st time in general relativistic viscous hydrodynamics
- Extension to non-conformal fluids + baryon charge
- Numerical applications in heavy ions and general relativity

ADDITIONAL SLIDES

Definition 8.6 (Strongly causal spacetime)
A spacetime $\mathscr{M}$ is strongly causal *if given an arbitrarily chosen event* $\mathbf{p} \in \mathscr{M}$ *for each $U \subset \mathscr{M}$ open neighborhood of* $\mathbf{p}$ *there exist another open neighborhood of* $\mathbf{p}$*, $V \subset U$, such that no casual curve intersects it more than once.*

Definition 8.7 (Inextendible causal curve)
A causal curve γ_C is called future *(resp.* past*)* inextendible *if it is impossible to find an event $\mathbf{p} \in \mathscr{M}$ such that for all $U \subset \mathscr{M}$, U neighborhood of $\mathbf{p}$, there exist a t' such that $\gamma_C(t) \in U$ for all $t > t'$ (resp $t < t'$)*

In more concrete words, this means that γ_C has no *future* (resp. *past*) *endpoint.*

Definition 8.10 (Domains of dependence)
Let $\mathscr{A}$ be a closed achronal set. The set $D^+(\mathscr{A})$ (resp. $D^-(\mathscr{A})$) of all spacetime events $\mathbf{p}$ *such that every past (resp. future) inextendible causal curve passing through* $\mathbf{p}$ *intersects $\mathscr{A}$ is called the* future *(resp.* past*)* domain of dependence of $\mathscr{A}$. *The set $D(\mathscr{A}) = D^+(\mathscr{A}) \cup D^-(\mathscr{A})$, union of the past and of the future domains of dependence is the* domain of dependence *of $\mathscr{A}$.*

Definition 8.11 (Cauchy surface and global hyperbolicity)
Let $\mathscr{A} \subset \mathscr{M}$ be an achronal set such that $D(\mathscr{A}) = \mathscr{M}$. Then $\mathscr{A}$ is called a Cauchy surface *(we instead use the denomination* partial Cauchy surface *for a closed achronal set without edge). A spacetime $\mathscr{M}$ which admits a Cauchy surface is called* globally hyperbolic.

Theorem 2.2. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, \epsilon_0, \epsilon_1, v_0, v_1)$ be an initial data set for the VECF system. Assume that Σ is compact with no boundary, and that $\epsilon_0 > 0$. Suppose that χ and λ are given by (1.5), where $\eta : (0,\infty) \to (0,\infty)$ is analytic, and assume that $a_1 = 4$ and $a_2 \geq 4$. Finally, assume that the initial data is in $G^{(s)}(\Sigma)$ for some $1 \leq s < \frac{17}{16}$. Then:*

1) There exists a globally hyperbolic development M of $\mathcal{I}$.

2) M is causal, in the following sense. Let (g, ϵ, u) be a solution to the VECF system provided by the globally hyperbolic development M. For any $p \in M$ in the future of Σ, $(g(p), u(p), \epsilon(p))$ depends only on $\mathcal{I}|_{i(\Sigma)\cap J^-(p)}$, where $J^-(p)$ is the causal past of p and $i : \Sigma \to M$ is the embedding associated with the globally hyperbolic development M.

Theorem 2.3. *Let T be given by (1.1) with g being the Minkowski metric. Suppose that χ and λ satisfy (1.5), with $a_1 = 4$, $a_2 \geq 4$, where $\eta : (0,\infty) \to (0,\infty)$ is a given analytic function. Let $\epsilon_0, \epsilon_1 : \mathbb{R}^3 \to \mathbb{R}$ and $v_0, v_2 : \mathbb{R}^3 \to \mathbb{R}^3$ belong to $G^{(s)}(\mathbb{R}^3)$ for some $1 \leq s < \frac{7}{6}$, and assume that $\epsilon_0 \geq C_0 > 0$, where C_0 is a constant.*

Then, there exists a $\mathcal{T} > 0$, a function $\epsilon : [0, \mathcal{T}) \times \mathbb{R}^3 \to (0,\infty)$, and a vector field $u : [0,\mathcal{T}) \times \mathbb{R}^3 \to \mathbb{R}^4$, such that (ϵ, u) satisfies equations (1.2) and (1.4) in $[0,\mathcal{T}) \times \mathbb{R}^3$, $\epsilon(0,\cdot) = \epsilon_0$, $\partial_0 \epsilon(0,\cdot) = \epsilon_1$, $u(0,\cdot) = u_0$, and $\partial_0 u(0,\cdot) = u_1$, where ∂_0 is the derivative with respect to the first coordinate in $[0,\mathcal{T}) \times \mathbb{R}^3$. This solution belongs to $G^{2,(s)}([0,\mathcal{T}) \times \mathbb{R}^3)$ and is unique in this class. Finally, the solution is causal, in the following sense. For any $p \in [0, T) \times \mathbb{R}^3$, $(\epsilon(p), u(p))$ depends only on $(\epsilon_0, \epsilon_1, v_0, v_1)|_{\{x^0=0\}\cap J^-(p)}$, where $J^-(p)$ is the causal past of p (with respect to the Minkowski metric).

See Disconzi, arXiv:1708.06572 [math.AP]

Analytic functions obey (α = multi-index):

$$|\partial^\alpha f| \leq C^{|\alpha|+1}\,\alpha!$$

The Gevrey class $\gamma^{(\sigma)}$, $\sigma > 1$, consists of C^∞ functions that obey the weaker inequality:

$$|\partial^\alpha f| \leq C^{|\alpha|+1}(\alpha!)^\sigma.$$

Advantage: large class of functions, including compactly supported (not determined by values on an open set).

The larger the σ, the larger the space. Larger σ: more general results.

$\gamma^{(\infty)}$ = Sobolev space.

$\gamma^{(\sigma)}$: used in the study of non-relativistic viscous fluids; also have had applications in General Relativity (magneto-hydrodynamics).

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Sobolev H_s

$$||f||_s^2 = \sum_{|j|\leq s} ||\partial^j u||_{L^2}^2$$

Characteristics.

Consider the linear differential operator:

$$Lu = a^{\mu\nu}(x)\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b(x, \partial u)$$

or, more generally (α=multi-index),

$$Lu = \sum_{|\alpha|=m} a^\alpha(x)\partial_\alpha^{|\alpha|}u + b(x, \partial^{m-1}u, \dots, \partial u, u).$$

We define the characteristic cone V_x of L at T_x^*M by

$$h(x,\xi) \equiv \sum_{|\alpha|=m} a^\alpha(x)\xi_\alpha = 0.$$

$h(x,\xi)$ (=characteristic polynomial) is a homogeneous polynomial of degree m.

Hyperbolic polynomials (Leray).

$h(x,\xi)$ is called a hyperbolic polynomial (at x) if there exists $\zeta \in T_x^* M$ such that every line through ζ that does not pass through the origin intersects V_x at m real distinct points (m =degree of h= order of L).

In this case, the set of $\zeta \in T_x^* M$ with this property forms the interior of two opposite convex half-cones $\Gamma_x^{\pm}$.

The differential operator L is called hyperbolic (at x) if $h(x,\xi)$ is hyperbolic.

Dualizing, one obtains $C_x^{\pm} \subset T_x M$. For example

$$C_x^{+} = \{v \in T_x M \,|\, \zeta(v) \geq 0 \text{ for all } \zeta \in \Gamma_x^{+}\}.$$

$\Sigma^n = \{\varphi(x) = 0\} \subset M^{n+1}$ is characteristic for L if

$$\sum_{|\alpha|=m} a^{\alpha}(x) \partial_{\alpha} \varphi = 0.$$

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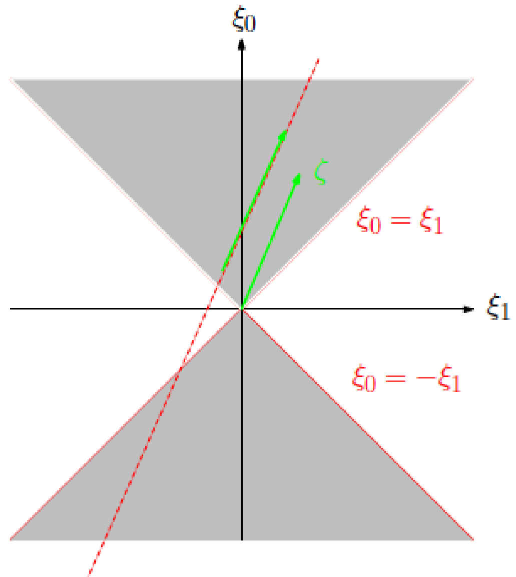
Wave equation: characteristics.

Consider $Lu = u_{tt} - u_{xx}$, $\xi = (\xi_0, \xi_1)$. Then:

$$\xi_0^2 - \xi_1^2 = 0 \Rightarrow \xi_0 = \pm\xi_1.$$



Γ_x and C_x are both given by the "light-cone".

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Hyperbolic and weakly hyperbolic operators.

Hyperbolic operators (sometimes called strictly hyperbolic) have a Cauchy problem that is well-posed in Sobolev spaces.

When the definition of a hyperbolic polynomial is weakened to:

there exists $\zeta \in T_x^* M$ such that every line through ζ that does not pass through the origin intersects V_x at m, not necessarily distinct, real points, we obtain weakly hyperbolic polynomials and operators (m =degree of h=order of L).

Weakly hyperbolic operators are well-posed in Gevrey spaces, but there are counter-examples to well-posed in Sobolev spaces.

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Definition 2.1. An initial data set for the VECF system consists of a three-dimensional smooth manifold Σ, a Riemannian metric g_0 on Σ, a symmetric two-tensor κ on Σ, two real-valued functions ϵ_0 and ϵ_1 defined on Σ, and two vector fields v_0 and v_1 on Σ, such that the Einstein constraint equations are satisfied.

As is customary, we shall write (1.3) in trace-reversed form and in harmonic coordinates. More precisely, we consider the reduced Einstein equations given by

$$g^{\mu\nu}\partial^2_{\mu\nu}g_{\alpha\beta} = B_{\alpha\beta}(\partial\epsilon, \partial u, \partial g), \tag{3.1}$$

where above and henceforth we adopt the following:

A.2. **The Cauchy problem.** Let $a = a(x, \partial^k)$, $x \in X$, be a linear differential operator of order k. We can write

$$a(x, \partial^k) = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index. Let $p(x, \partial^k)$ be the principal part of $a(x, \partial^k)$, i.e.,

$$p(x, \partial^k) = \sum_{|\alpha| = k} a_\alpha(x)\partial^\alpha.$$

At each point $x \in X$ and for each co-vector $\xi \in T^*_x X$, where T^*X is the cotangent bundle of X, we can associate a polynomial of order k in the cotangent space $T^*_x X$ obtained by replacing the derivatives by $\xi \in T^*_x X$. More precisely, for each k^{th} order derivative in $a(x, \partial^k)$, i.e.,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0}\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$$

$|\alpha| = k$, we associate the polynomial

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$$\xi^\alpha \equiv \xi_0^{\alpha_0}\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n},$$

where $\xi = (\xi_0, \xi_1, \xi_2, \ldots, \xi_n) \in T^*_x X$, forming in this way the polynomial

$$p(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x)\xi^\alpha.$$

Clearly, $p(x, \xi)$ is a homogeneous polynomial of degree k. It is called the characteristic polynomial (at x) of the operator a.

The cone $V_x(p)$ of p in $T^*_x X$ is defined by the equation

$$p(x, \xi) = 0.$$

Definition A.5. With the above notation, $p(x,\xi)$ is called a hyperbolic polynomial (at x) if there exists $\zeta \in T_x^*X$ such that every straight line through ζ that does not contain the origin intersects the cone $V_x(p)$ at k real distinct points. The differential operator $a(x,\partial^k)$ is called a hyperbolic operator (at x) if $p(x,\xi)$ is hyperbolic.

Leray proved in [16] that if $p(x,\xi)$ is hyperbolic at x, then the set of points ζ satisfying the condition of Definition A.5 forms the interior of two opposite half-cones $\Gamma_x^{*,+}(a)$, $\Gamma_x^{*,-}(a)$, with $\Gamma_x^{*,\pm}(a)$ non-empty, with boundaries that belong to $V_x(p)$.

Remark A.6. An equivalent definition of hyperbolic polynomials is as follows [4]: $p(x,\xi)$ is hyperbolic at x if for each non-zero $\xi = (\xi_0,\dots,\xi_n) \in T_x^*X$, the equation $p(x,\xi) = 0$ has k distinct real roots $\xi_0 = \xi_0(\xi_1,\dots,\xi_n)$.

With applications to systems in mind, we next consider the $N \times N$ diagonal linear differential operator matrix

$$A(x,\partial) = \begin{pmatrix} a^1(x,\partial^{k_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^N(x,\partial^{k_N}) \end{pmatrix}.$$

Each $a^J(x,\partial^{k_J})$, $J = 1,\dots,N$ is a linear differential operator of order k_J.

Definition A.7. The operator $A(x,\partial)$ is called Leray-Ohya hyperbolic (at x) if:

(i) The characteristic polynomial $p^J(x,\xi)$ of each $a^J(x,\partial^{k_J})$ is a product of hyperbolic polynomials, i.e.

$$p^J(x,\xi) = p^{J,1}(x,\xi)\cdots p^{J,r_J}(x,\xi),\ J = 1,\dots,N,$$

where each $p^{J,q}(x,\xi)$, $q = 1,\dots,r_J$, $J = 1,\dots,N$, is a hyperbolic polynomial.

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(ii) The two opposite convex half-cones,

$$\Gamma_x^{*,+}(A) = \bigcap_{J=1}^{N}\bigcap_{q=1}^{r_J}\Gamma_x^{*,+}(a^{J,q}), \text{ and } \Gamma_x^{*,-}(A) = \bigcap_{J=1}^{N}\bigcap_{q=1}^{r_J}\Gamma_x^{*,-}(a^{J,q}),$$

have a non-empty interior. Here, $\Gamma_x^{*,\pm}(a^{J,q})$ are the half-cones associated with the hyperbolic polynomials $p^{J,q}(x,\xi)$, $q = 1,\dots,r_J$, $J = 1,\dots,N$.

Remark A.8. When the above hyperbolicity properties hold for every x, we call the corresponding operators hyperbolic (we can also talk about hyperbolicity in an open set, a certain region, etc.). When we say that an operator is Leray-Ohya hyperbolic on the whole space (or in an open set, etc.), this means not only that Definition A.7 applies for every x, but also that the numbers r_J and the degree of the polynomials $p^{J,q}(x,\xi)$, $q = 1,\dots,r_J$, $J = 1,\dots,N$, do not change with x.

NONLINEAR PATHOLOGIES IN RELATIVISTIC HEAT-CONDUCTING FLUID THEORIES

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Hyperbolicity and stability are analyzed in the nonlinear regimes of two theories of relativistic heat-conducting fluids. Both theories are found to be unstable and non-hyperbolic for sufficiently large deviations from equilibrium. One of these theories (an extended hydrodynamic theory) is well behaved for small (but finite) deviations from equilibrium.

In summary, we have examined the properties of hyperbolicity and stability in extremely simplified but fully nonlinear versions of Eckart's theory and the Israel–Stewart theory of relativistic dissipative fluids. We have shown that Eckart's theory continues to display the generic instability and acausal, non-hyperbolic behavior that first appeared in the analysis of the linear equations. The Israel–Stewart theory (with β given by its kinetic theory value) fails to be hyperbolic for states of the fluid that are not sufficiently close to equilibrium $|q|/\rho > 0.08898$. In addition the spatially homogeneous solutions in this theory are thermodynamically unstable for all initial values of $|q|/\rho > 0.50308$. Further investigation of the stability of the entire class of solutions will have to be performed before it will be possible to determine whether or not there is as close a relationship between hyperbolicity and stability in the nonlinear theory as there was in the linear regime.

High density QCD matter: From the lab to the sky
Neutron star mergers

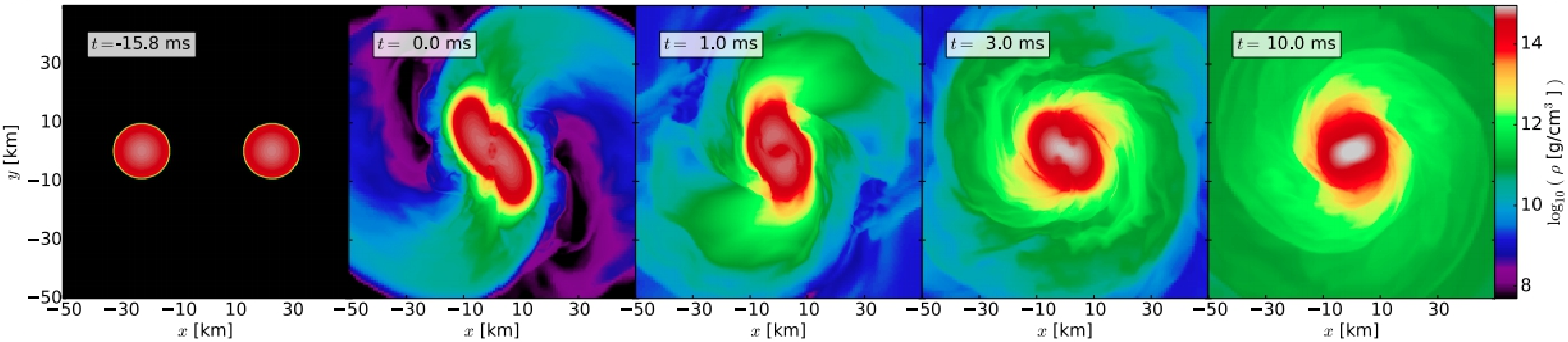


Fig. by L. Rezzolla

Viscous fluid dynamics + strong gravitational fields

Important problem in physics and mathematics
(Relativistic Navier-Stokes equations are acausal and unstable)

Viscous effects in neutron star mergers

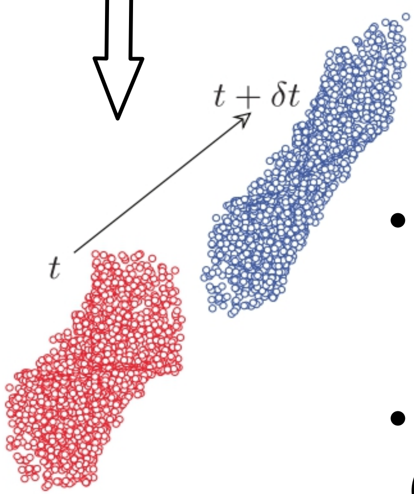
Duez et al PRD (2004), Shibata et al. PRD (2017), Alford et al. PRL (2018)

Relativistic Boltzmann Equation

Special (general) relativity

classical statistics

$$k^{\mu}\nabla_{\mu} f_k = \underbrace{\int_{k'pp'} W(pp'|kk') f_p f_{p'}}_{\text{Gain}} - \underbrace{\int_{k'pp'} W(pp'|kk') f_k f_{k'}}_{\text{Loss}}$$



- Used in heavy-ion collisions, cosmology/astrophysics.

- 1st analytical solution, since 1872, for an expanding gas (also in curved spacetime).

Bazow, Denicol, Heinz, Martinez, JN, PRL (2016)