Rheological properties of a rapidly expanding plasma

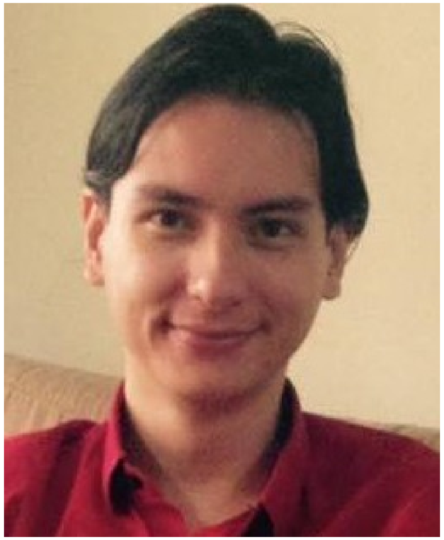
Mauricio Martinez Guerrero

NC STATE UNIVERSITY

BEST COLLABORATION

Foundational aspects of relativistic hydrodynamics

May 7-11, 2018,Trento

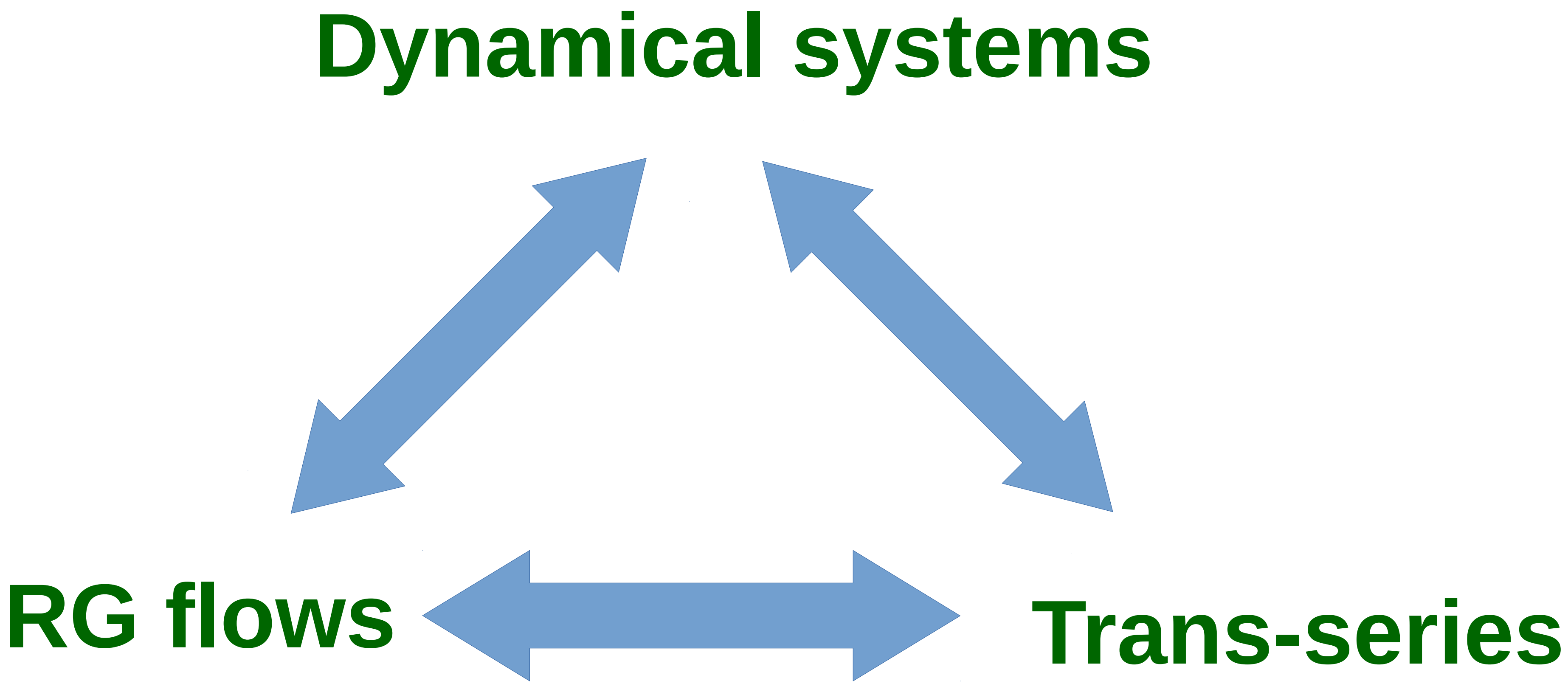


Collaborators: *A. Behtash* *C. N. Camacho* *S. Kamata*

Carlos A. Salgado
@CASsalgado

How many fishes are collective? #InitialStages2016

4:20 AM - 23 May 2016


Dynamical systems
RG flows
Trans-series

Dynamical systems

- Stability analysis around fixed points in UV and IR (Lyapunov)

RG flows

- Analyze the change of the coefficients of trans-series as a function of the ‘energy’ scale

Trans-series

- Non-perturbative contributions to the solution by knowing the IR fixed point

Results

For Bjorken flow:
The shear viscous tensor can be written as a trans-series

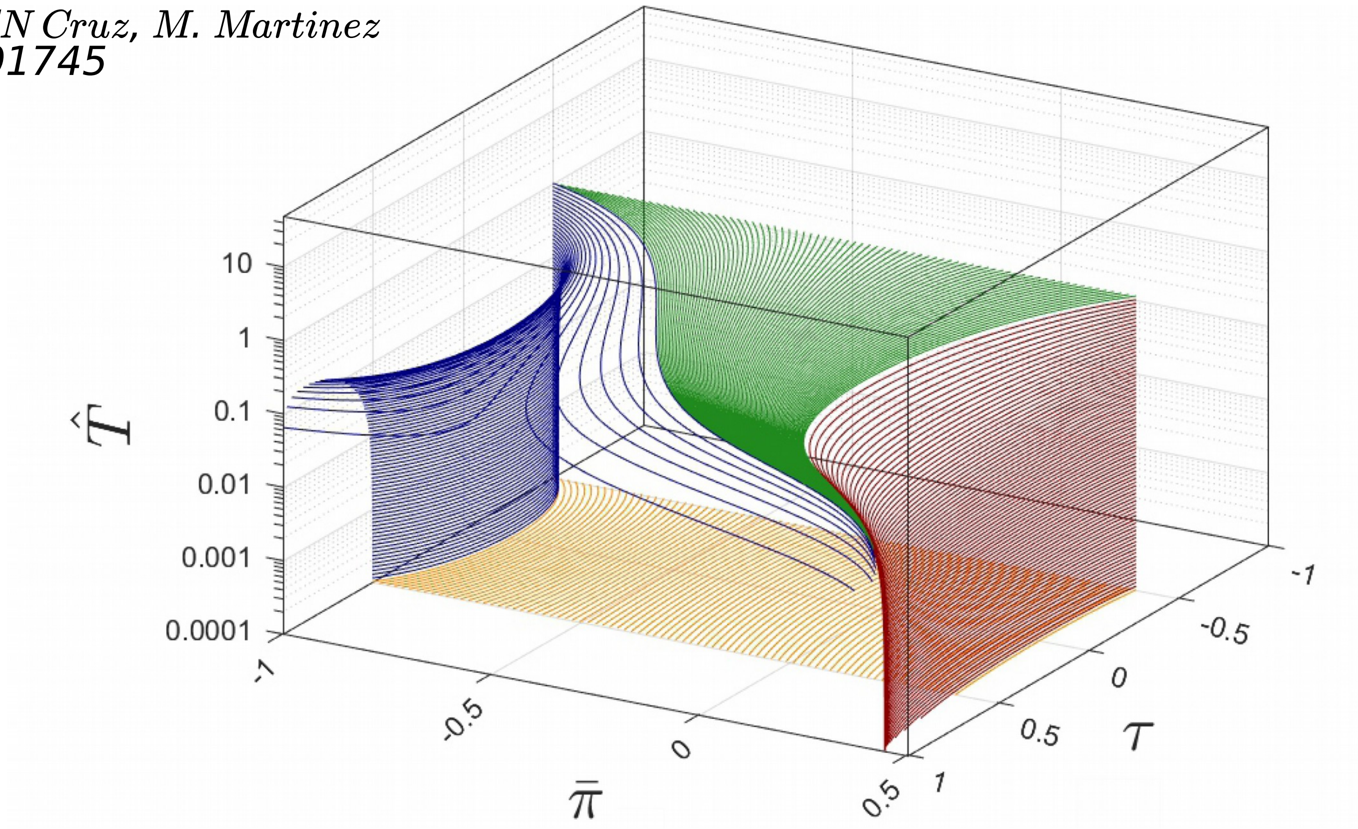
$$\bar{\pi} = \frac{\pi_\varsigma^\varsigma}{\epsilon} = \sum_{k=0}^{\infty} F_k(\sigma e^{-Sw} w^\beta)\, w^{-k}$$

Each 'coefficient' F_k is the summation of non-perturbative contributions of the inverse Knudsen number $w=1/(T\tau)$ (non-hydrodynamical series)

The differential equation for F_k admits a Renormalization Group interpretation for a particular transport coefficient

Dynamical systems point of view: Gubser flow for IS theory

A. Behtash, CN Cruz, M. Martinez
arXiv:1711.01745
PRD in press



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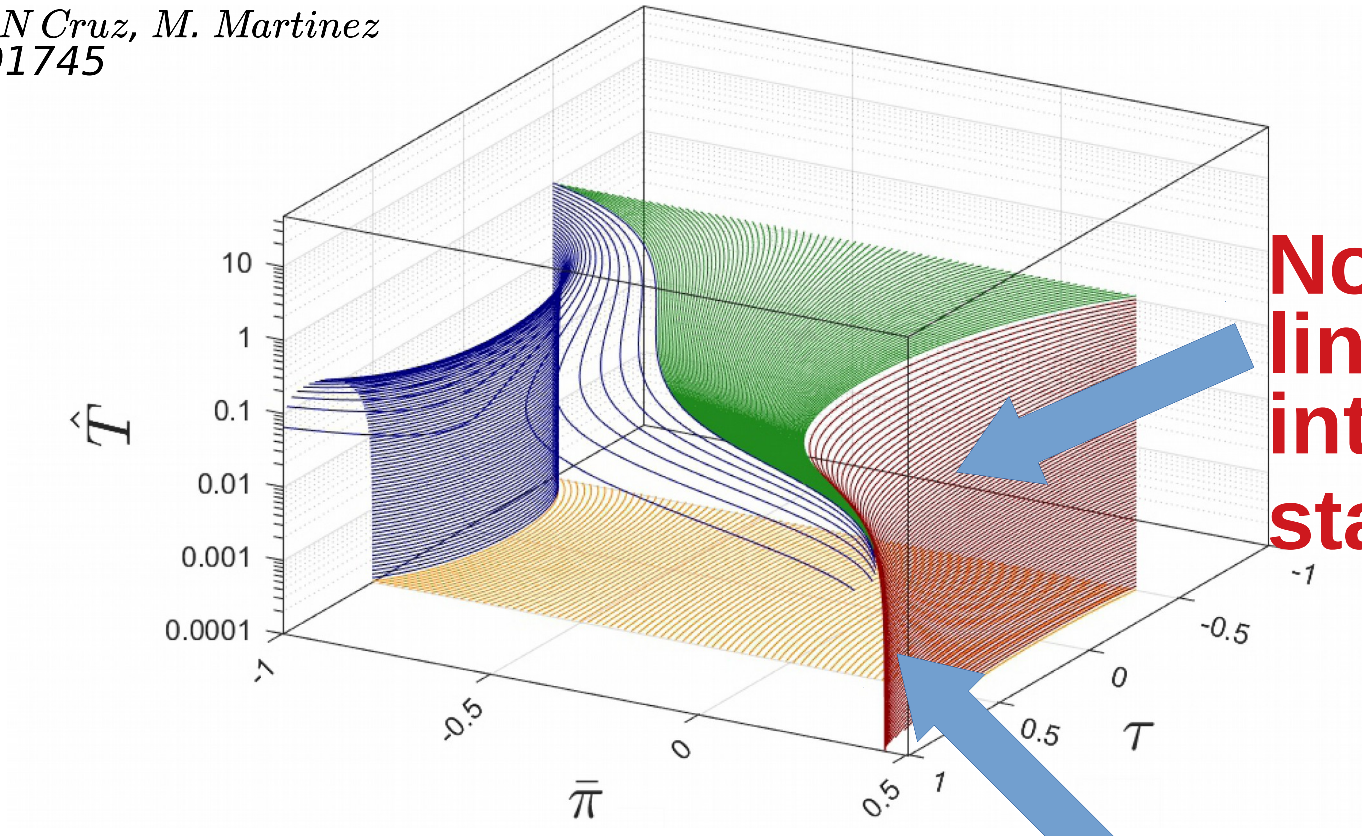

No universal line during intermediate stages

Late time asymptotic attractor

Dynamical systems point of view: Gubser flow for IS theory

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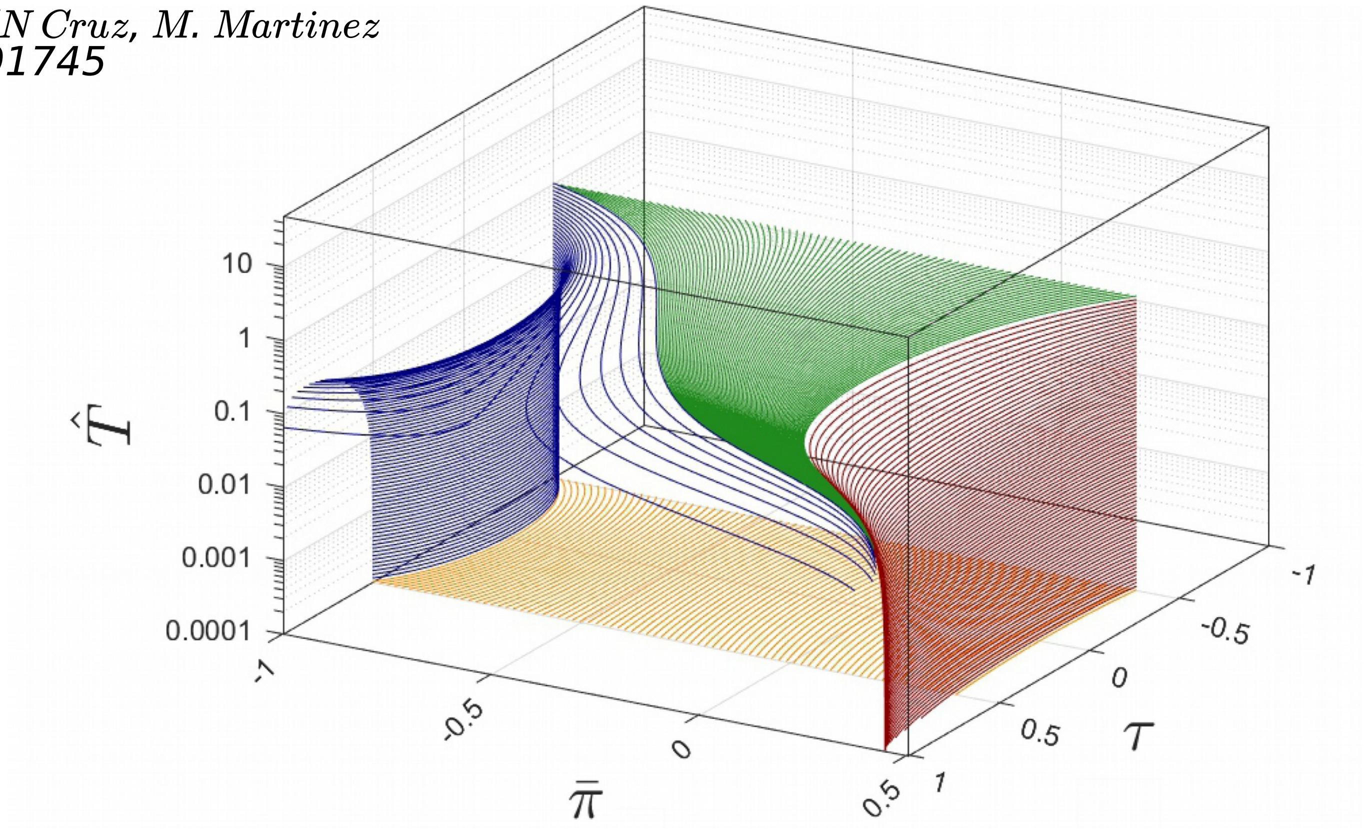

- Attractor is a 1-d non planar manifold
- In Bjorken you see a unique line cause the attractor is a 1d planar curve

Dynamical systems point of view: Gubser flow for IS theory

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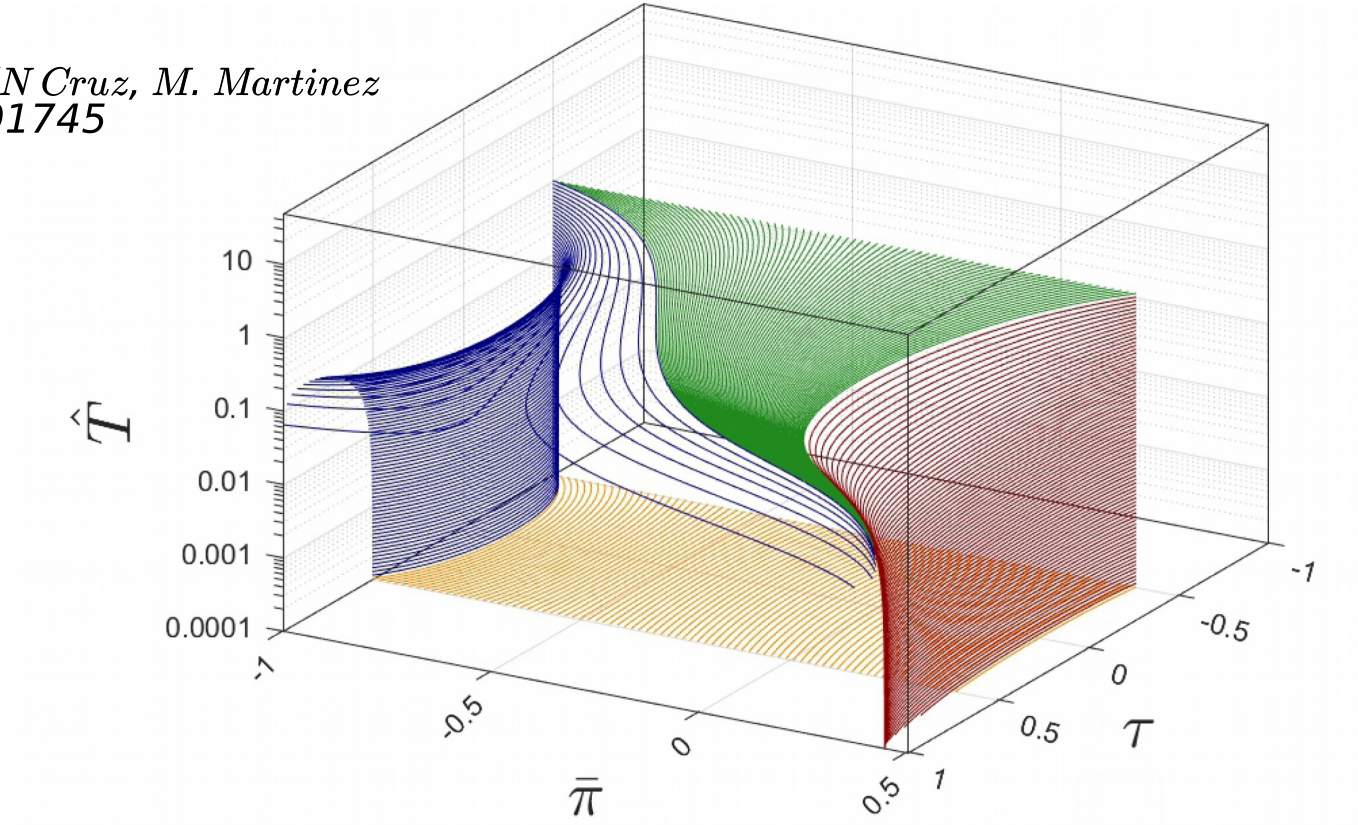
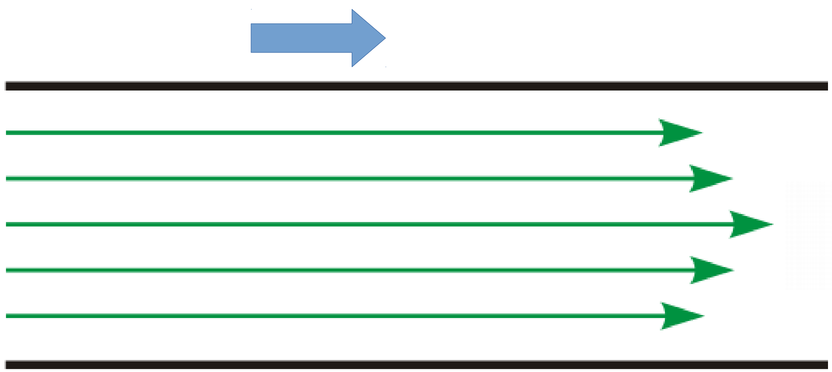
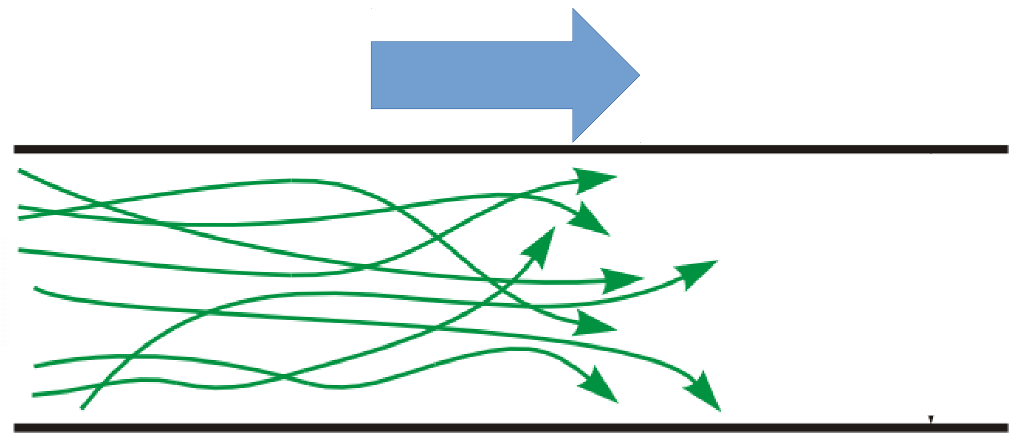


- **Asymptotic behavior of temperature is not determined by the Knudsen number**

Non-newtonian fluids and rheology





$$\pi_{yx} \sim \eta\, \partial_y v_x$$

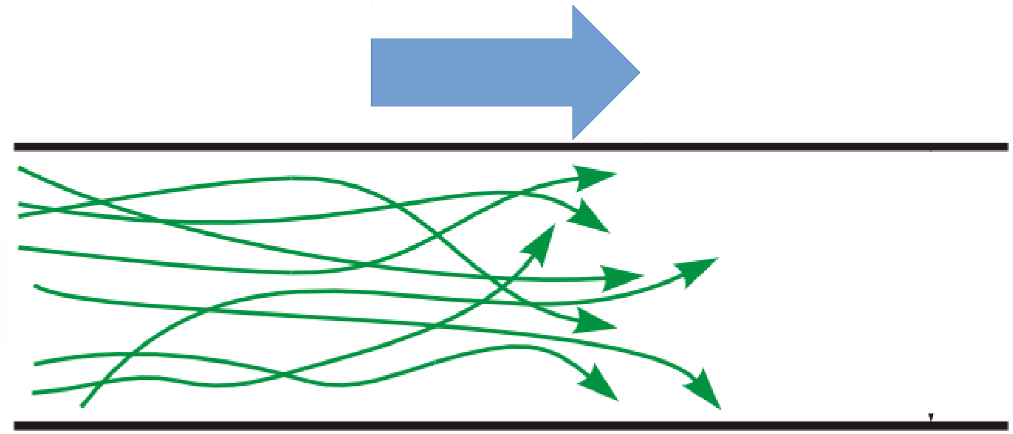
$$\pi_{yx} \sim \eta(\partial_y v_x)\, \partial_y v_x$$

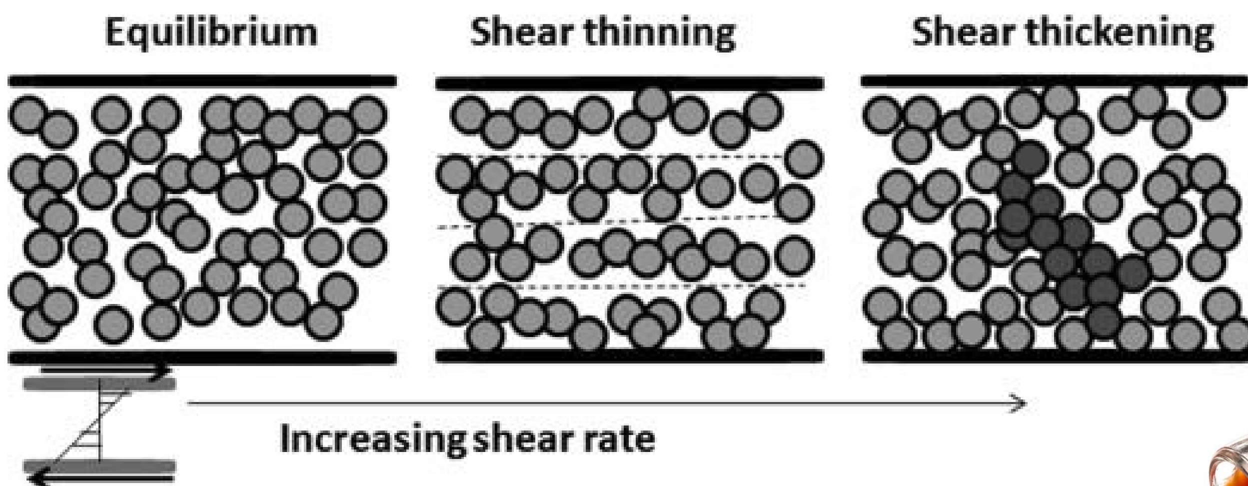
This is called shear
thinning and shear thickening

Shear viscosity

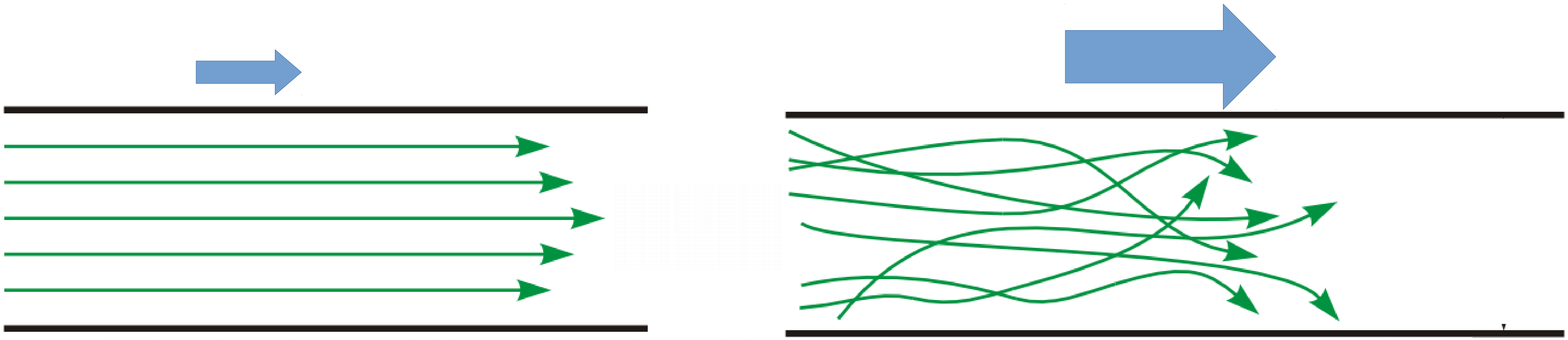
- Becomes a **function** of the **gradient of the flow velocity**
- can **increase** or **decrease depending** on the **size** of the **gradient of the flow velocity**

Non-newtonian fluids and rheology





Non-newtonian fluids and rheology



Selected for a Viewpoint in *Physics*
PRL **101,** 138301 (2008) PHYSICAL REVIEW LETTERS week ending 26 SEPTEMBER 2008

First-Principles Constitutive Equation for Suspension Rheology

J. M. Brader,[1] M. E. Cates,[2] and M. Fuchs[1]
[1]*Fachbereich Physik, Universität Konstanz, D-78457 Konstanz, Germany*
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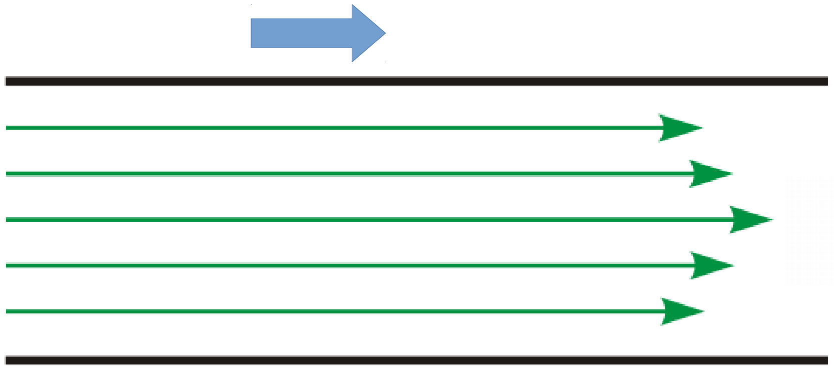
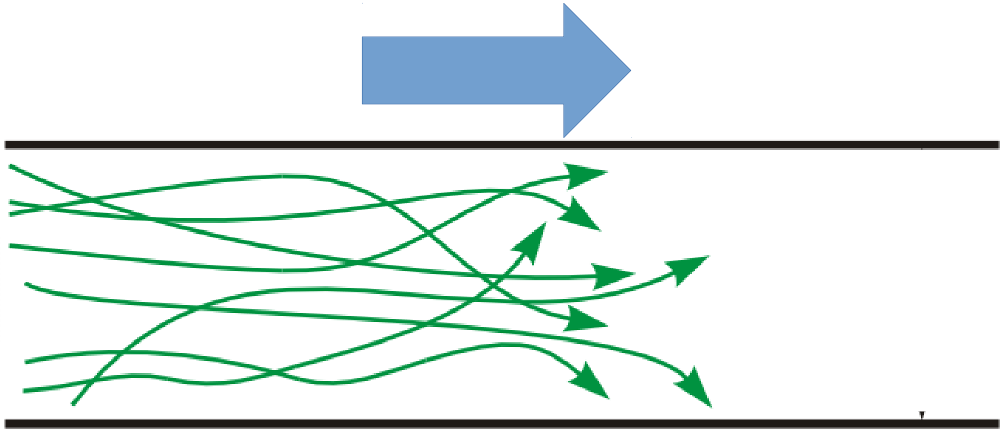
(Received 30 April 2008; published 22 September 2008)

A central aim of theoretical rheology is thus to derive from the underlying microscopic interactions the constitutive equations that relate the stress tensor to the macroscopic deformation history of a material. For entangled

Non-newtonian fluids and rheology





$$\pi_{yx} \sim \eta \, \partial_y v_x$$

$$\pi_{yx} \sim \eta(\partial_y v_x) \, \partial_y v_x$$

Effective Shear viscosity
Shuryak, Lublinski, Strickland, Florkowski, Ryblewski, Romatschke, Yan, Blaizot

Kinetic theory model

RTA Boltzmann

$$\partial_\tau f = -\frac{1}{\tau_r(\tau)}\left(f - f_{eq.}\right)$$

$$\tau_r = \theta_0 / T(\tau)$$

Ansatz for f inspired in Lattice Boltzmann (Romatschke et. al PRC84, 034903, 2011)

$$f\left(\tau, p_T, p_\varsigma\right) = f_{eq.}\left(\frac{p^\tau}{T}\right)\left[\sum_{n=0}^{N_n}\sum_{l=0}^{N_l} c_{nl}(\tau)\,\mathcal{P}_{2l}\left(\frac{p_\varsigma}{\tau\, p^\tau}\right)\mathcal{L}_n^{(3)}\left(\frac{p^\tau}{T}\right)\right]$$

$$p^\tau = \sqrt{p_T^2 + (p_\varsigma/\tau)^2}$$

Asymptotic behaviour of c_1 and c_2

The distribution function can be expanded asymptotically (Yan and Blaizot) in w= 1/(t T)

$$\delta f = \left[-\tilde{\chi}_p \tilde{p}^2 \left(\frac{2}{3tT} \right) + \tilde{\chi}'_p \tilde{C}_p \tilde{p}^4 \left(\frac{8}{63t^2T^2} \right) - \tilde{\chi}_p \tilde{C}_p \tilde{p}^3 \left(\frac{8}{9t^2T^2} \right) + \ldots \right] P_2(\cos\theta)$$

$$+ \left[\tilde{\chi}'_p \tilde{C}_p \tilde{p}^4 \left(\frac{8}{35t^2T^2} \right) + \ldots \right] P_4(\cos\theta) + \ldots ,$$

Thus asymptotically

$$c_1 = \frac{5}{2}\left(-\frac{4}{3}\frac{1}{w}\left(\frac{\eta}{s}\right) - \frac{8}{9}\frac{1}{w^2}\left(T\tau_\pi \frac{\eta}{s} - T\frac{\lambda_1}{s} \right) \right)$$

$$c_2 = -\frac{20}{9}\frac{1}{w^2}\left(T\tau_\pi \frac{\eta}{s} + T\frac{\lambda_1}{s} \right)$$

Kinetic theory model

Conservation laws + RTA Boltzmann

$$\frac{dT}{d\tau} + \frac{T}{3\,\tau} = -\frac{T\,c_1}{30\,\tau}\,,$$

$$\partial_\tau c_l + \frac{1}{\tau}\left[\alpha_l\, c_{l+1} + \beta_l\, c_l - \frac{2}{15} c_1\, c_l + \gamma_l\, c_{l-1}\right] + \frac{1}{\tau_r(\tau)} c_l = 0\,.$$

$$\tilde{\mathbf{c}} = U\mathbf{c}$$

$$\left(1 - \frac{c_1}{20}\right)\frac{d\tilde{\mathbf{c}}}{dw} + \hat{\Lambda}\tilde{\mathbf{c}} + \frac{1}{w}\hat{\mathcal{B}}_D\tilde{\mathbf{c}} - \frac{c_1}{5w}\tilde{\mathbf{c}} + \frac{3}{2w}\tilde{\gamma} = 0$$

O. Coustin (Ohio State University)
Duke Math. J. vol 93, No 2, 1998

If you have a non-linear differential equation of the form

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y})$$

Then

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0; |\mathbf{k}|>0} C_1^{k_1} \cdots C_n^{k_n} e^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}}$$

$$\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\beta+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$$

1. Non-resonance condition: *Λ does not have null eigenvalues*

2. *Regularity when $x \to \infty$*

Kinetic theory model

$$\tilde{\mathbf{c}} = U\mathbf{c}$$

$$\left(1 - \frac{c_1}{20}\right)\frac{d\tilde{\mathbf{c}}}{dw} + \hat{\Lambda}\tilde{\mathbf{c}} + \frac{1}{w}\hat{\mathcal{B}}_D\tilde{\mathbf{c}} - \frac{c_1}{5w}\tilde{\mathbf{c}} + \frac{3}{2w}\tilde{\gamma} = 0$$

After some transformations one can show that (Costin, 2006)

$$c_{0l}(w) = \sum_{l'=1}^{L}\sum_{k=0}^{\infty} U^{-1}_{ll'}\tilde{C}_{l,k}\left(\boldsymbol{\sigma}\boldsymbol{\zeta}(w)\right)w^{-k}$$

$$\tilde{C}_{l,k}(\boldsymbol{\sigma}\boldsymbol{\zeta}(w)) = \sum_{\mathbf{n}\geq 0}^{\infty}\boldsymbol{\sigma}^{\mathbf{n}}\boldsymbol{\zeta}^{\mathbf{n}}(w)\,\tilde{u}^{(\mathbf{n})}_{l,k}.$$

$$\boldsymbol{\sigma}^{\mathbf{n}}\boldsymbol{\zeta}^{\mathbf{n}}(w) = [\sigma_1\zeta_1(w)]^{n_1}\cdots[\sigma_L\zeta_L(w)]^{n_L}$$

$$\zeta_l(w) = e^{-S_l w}w^{\tilde{b}_l}$$

Comparing asymptotic behaviour

$$c_{0l}(w) = \sum_{l'=1}^{L} \sum_{k=0}^{\infty} U_{ll'}^{-1} \tilde{C}_{l,k}\left(\boldsymbol{\sigma}\boldsymbol{\zeta}(w)\right) w^{-k}$$

$$\tilde{C}_{l,k}(\boldsymbol{\sigma}\boldsymbol{\zeta}(w)) = \sum_{\mathbf{n}\geq 0}^{\infty} \boldsymbol{\sigma}^{\mathbf{n}} \boldsymbol{\zeta}^{\mathbf{n}}(w)\, \tilde{u}_{l,k}^{(\mathbf{n})}.$$

Consider only c1

At O(1/w) the dominant term of the trans-series is:

$$\mathcal{O}(w^{-1}): \qquad c_1 = \frac{\tilde{u}_{1,1}^{(0)}}{w}$$

On the other hand, the asymptotic expansion of the distribution function

$$\mathcal{O}(w^{-1}): \qquad c_1 = -\frac{40}{3}\frac{1}{w}\left(\frac{\eta}{s}\right)_0$$

Comparing asymptotic behaviour

$$c_{0l}(w) = \sum_{l'=1}^{L} \sum_{k=0}^{\infty} U_{ll'}^{-1} \tilde{C}_{l,k} \left(\boldsymbol{\sigma} \boldsymbol{\zeta}(w) \right) w^{-k}$$

$$\tilde{C}_{l,k}(\boldsymbol{\sigma}\boldsymbol{\zeta}(w)) = \sum_{\mathbf{n} \geq 0}^{\infty} \boldsymbol{\sigma}^{\mathbf{n}} \boldsymbol{\zeta}^{\mathbf{n}}(w)\, \tilde{u}_{l,k}^{(\mathbf{n})} .$$

One then finds

$$\rightarrow \left(\frac{\eta}{s} \right)_0 = -\frac{3}{40} \tilde{u}_{1,1}^{(0)}$$

this is not a matching!!!!
it comes from the differential equation itself

So the asymptotic value of the η/s is determined by the asymptotic value of the associated trans-series

Comparing asymptotic behaviour

$$c_{0l}(w) = \sum_{l'=1}^{L} \sum_{k=0}^{\infty} U^{-1}_{ll'} \tilde{C}_{l,k}\left(\boldsymbol{\sigma\zeta}(w)\right) w^{-k}$$

$$\tilde{C}_{l,k}(\boldsymbol{\sigma\zeta}(w)) = \sum_{\mathbf{n}\geq 0}^{\infty} \boldsymbol{\sigma}^{\mathbf{n}} \boldsymbol{\zeta}^{\mathbf{n}}(w)\, \tilde{u}^{(\mathbf{n})}_{l,k}.$$

Let's be brave and promote the renormalized η/s with the associated non-hydrodynamical series

$$\rightarrow \left(\frac{\eta}{s}\right)_R = -\frac{3}{40} \tilde{C}_{1,1}(\sigma e^{-Sw} w^{b_1})$$

The non-hydrodynamical series keeps track of the relaxation of non-hydro modes which change the value of η/s $\Rightarrow$ Rheology

Comparing asymptotic behaviour

$$c_{0l}(w) = \sum_{l'=1}^{L} \sum_{k=0}^{\infty} U_{ll'}^{-1} \tilde{C}_{l,k}\left(\boldsymbol{\sigma\zeta}(w)\right) w^{-k}$$

$$\tilde{C}_{l,k}(\boldsymbol{\sigma\zeta}(w)) = \sum_{\mathbf{n}\geq 0} \boldsymbol{\sigma}^{\mathbf{n}} \boldsymbol{\zeta}^{\mathbf{n}}(w)\, \tilde{u}_{l,k}^{(\mathbf{n})}.$$

More importantly:

$$\frac{d}{dw}\left(\frac{\eta}{s}\right)_R \equiv -\frac{3}{40}\frac{d}{dw}\tilde{C}_{1,1}(\sigma e^{-Sw} w^{b_1})$$

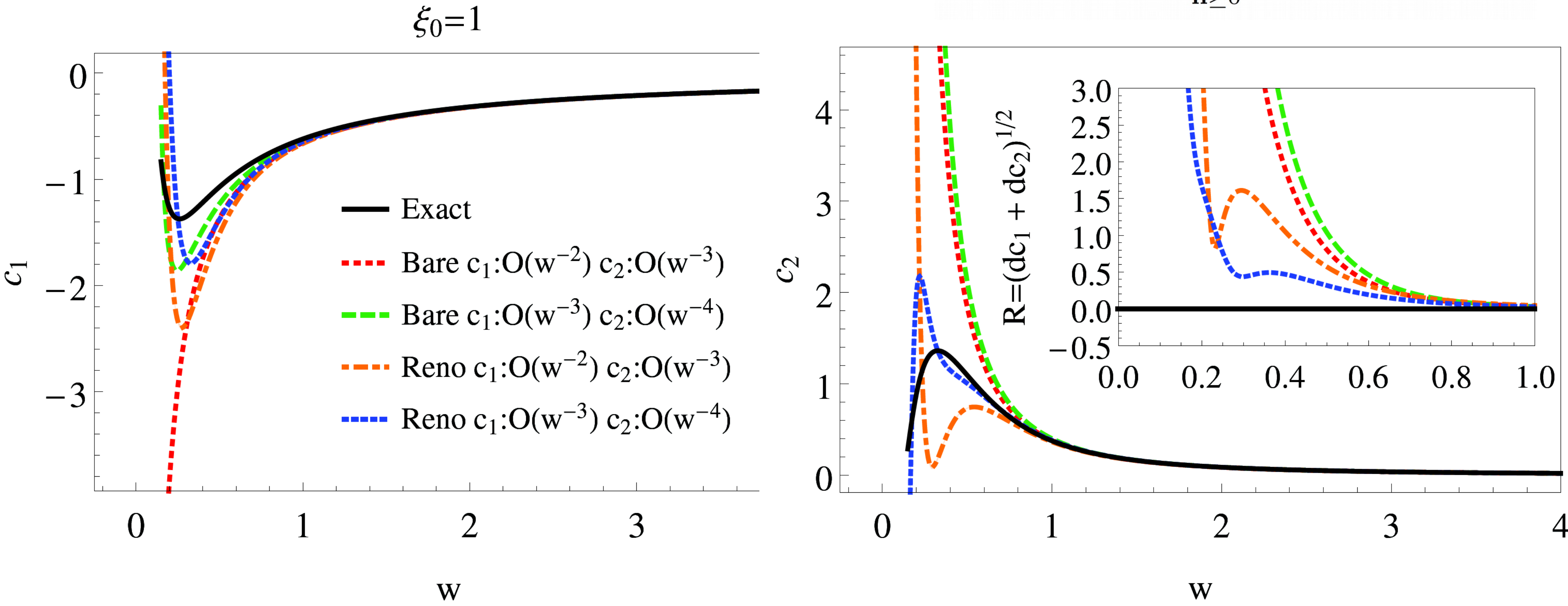
RG equation for η/s is understood as the evolution equation of the associated non-hydrodynamical series

Preliminary results

A. Behtash, CN Cruz, S. Kamata, M. Martinez
Forthcoming

$$c_{0l}(w) = \sum_{l'=1}^{L} \sum_{k=0}^{\infty} U_{ll'}^{-1} \tilde{C}_{l,k}\left(\boldsymbol{\sigma\zeta}(w)\right) w^{-k}$$

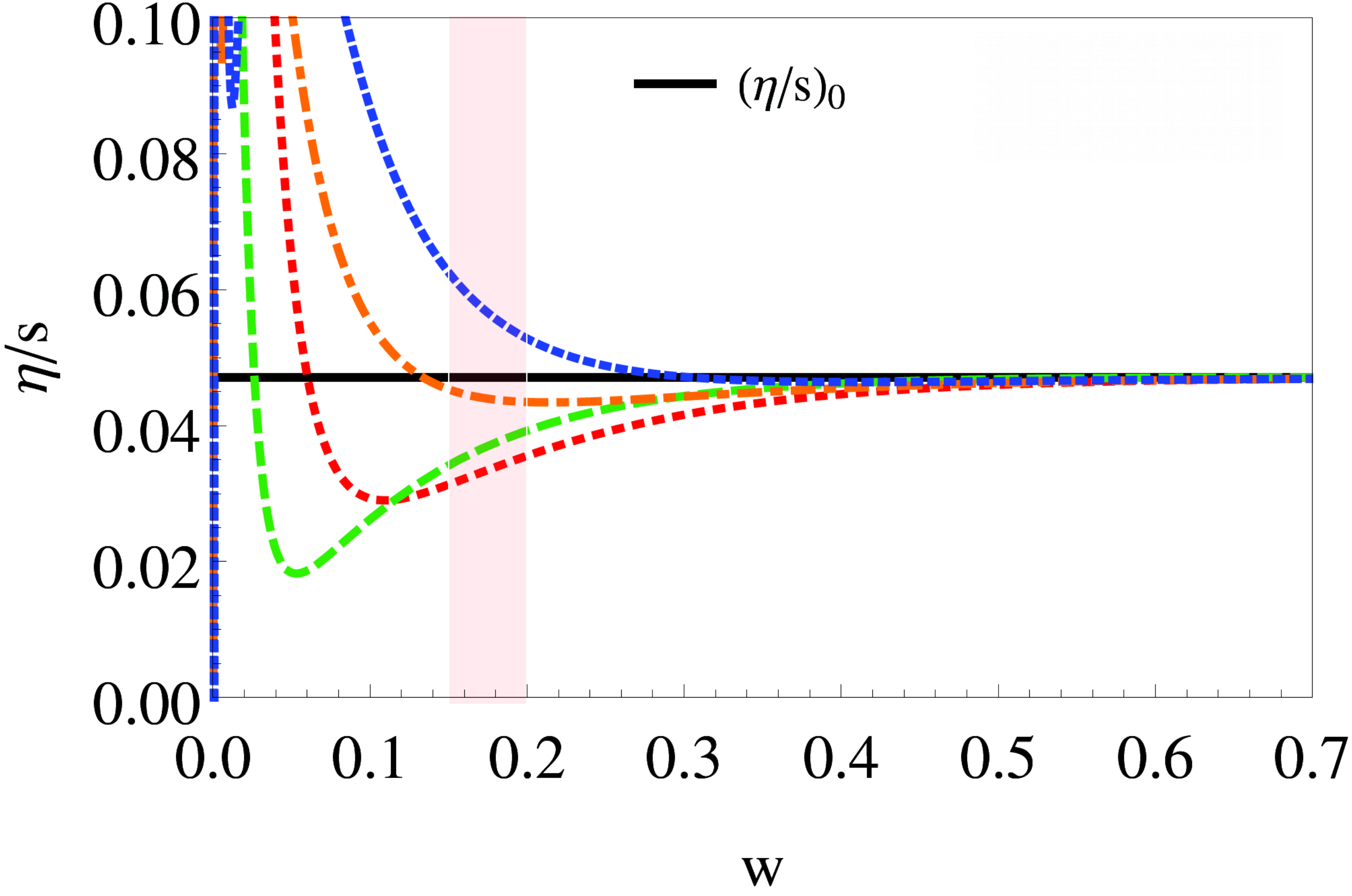
$$\tilde{C}_{l,k}(\boldsymbol{\sigma\zeta}(w)) = \sum_{\mathbf{n}\geq 0}^{\infty} \boldsymbol{\sigma}^{\mathbf{n}} \boldsymbol{\zeta}^{\mathbf{n}}(w)\, \tilde{u}_{l,k}^{(\mathbf{n})}.$$



The UV completion of your effective theory depends on the finer structure (more moments)

Preliminary results

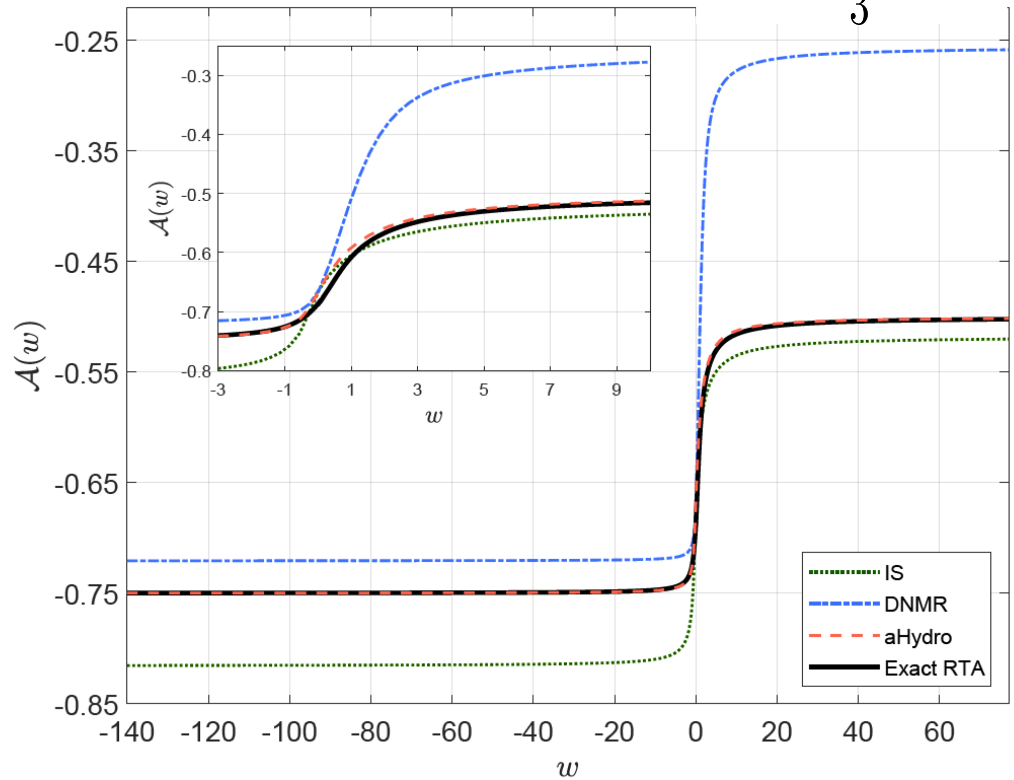
A. Behtash, CN Cruz, S. Kamata, M. Martinez
Forthcoming



Relating Anisotropic hydrodynamics and trans-series

$$\mathcal{A}(w) = \frac{1}{3}\left(\bar{\pi} - 2\right)$$

- **Ahydro has successfully described and reproduce to high accuracy the results obtained from exact solutions of the Boltzmann equation.**
- **Nonetheless, it does not describe higher order modes**
  Molnar et. al., Heinz et. al



A. Behtash, CN Cruz, M. Martinez
PRD in press

- **For conformal systems Ahydro turns out to predict the same results obtained from the trans-series of the first non-hydrodynamical mode, i.e., shear viscous tensor (Conjecture?)**

Conclusions

- **For the Bjorken flow and RTA Boltzmann**

  1. The solutions of the moments are written as multi-parameter trans-series.

- **We identify the transport coefficient with the associated non-hydrodynamical series**

  1. The evolution equation of the non-hydrodynamical series is understood as a RG equation for the associated transport coefficient

- **The comparisons with numerical solutions indicate a remarkable improvement due to the inclusion of non-perturbative contributions.**

Outlook

- Resurgence analysis of other kinetic models
  A. Behtash et. al.
  1. Non-conformal systems
  2. Finite chemical potential

- Challenges:
  1. How to generalize to arbitrarily expanding geometries in kinetic theory?
  2. Phase transitions?
  3. Effective action (Lyapunov functionals)
  For Gubser flow: Behtash. et. al. PRD 97 044041 (2018)

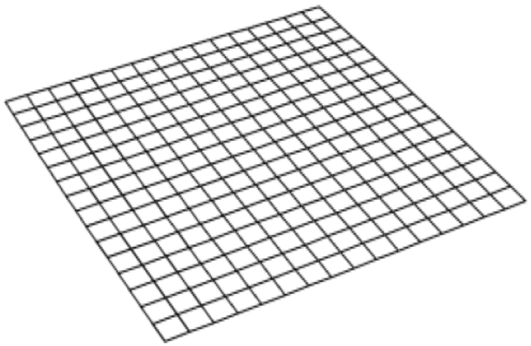
Backup slides

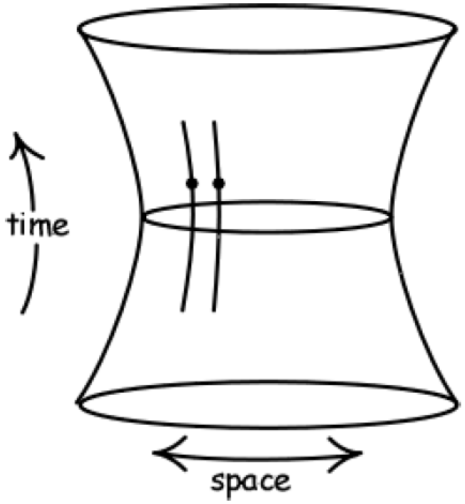
Gubser flow

$$g_{\mu\nu}(x) \to e^{-2\Omega(x)} g_{\mu\nu}(x)$$

Flat Minkowski space

dS$_3$ x R





$$\sinh\rho = -\frac{1-\tilde{\tau}^2+\tilde{r}^2}{2\tilde{\tau}}, \qquad \tan\theta = \frac{2\tilde{r}}{1+\tilde{\tau}^2-\tilde{r}^2}.$$

Complicated dynamics

3d de Sitter space

$$x^\mu = (\tau, r, \phi, \eta) \Rightarrow \hat{x}^\mu = (\rho, \theta, \phi, \eta)$$

$$ds^2 = -d\tau^2 + dr^2 + r^2 d\phi^2 + d\eta^2 \Rightarrow d\hat{s}^2 = -d\rho^2 + \cosh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right) + d\eta^2$$

$$u^\mu = (u^\tau(\tau,r), u^r(\tau,r), 0, 0) \Rightarrow \hat{u}^\mu = (1,0,0,0)$$

$$\epsilon(\tau, r) \Rightarrow \hat{\epsilon}(\rho)$$

Exact Gubser solution

- *In $dS_3 \otimes R$ the dependence of the distribution function is restricted by the symmetries of the Gubser flow*

$$f(\hat{x}^\mu, \hat{p}_i) = f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right)$$

$$\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta}$$ ⟶ *Total momentum in the (θ,ϕ) plane*

$\hat{p}_\eta$ ⟶ *Momentum along the η direction*

- *The RTA Boltzmann equation gets reduced to*

$$\frac{\partial}{\partial\rho} f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = -\frac{\hat{T}(\rho)}{c}\left(f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) - f_{eq}\left(\hat{p}^\rho / \hat{T}(\rho)\right)\right)$$

$$c = 5\frac{\eta}{\mathcal{S}}$$

- *The exact solution to this equation is*

$$f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) = D(\rho, \rho_0)\, f_0\left(\rho, \hat{p}_\Omega^2, \hat{p}_\eta\right) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, \hat{T}(\rho')\, D(\rho, \rho')\, f_{eq}\left(\hat{p}^\rho / \hat{T}(\rho)\right)$$

Fluid models for the Gubser flow

E-M conservation law

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

DNMR theory

$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho\bar{\pi} + \frac{4}{3}(\bar{\pi})^2\tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho + \frac{10}{7}\hat{\tau}_{\hat{\pi}}\bar{\pi}\tanh\rho$$

IS theory

Anisotropic hydrodynamics

$$\partial_\rho\bar{\pi} + \frac{\bar{\pi}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{5}{16} + \bar{\pi} - \bar{\pi}^2 - \frac{9}{16}\mathcal{F}(\bar{\pi})\right)$$

Non-linear dynamical system analysis of the IS theory

arXiv:1711.01745

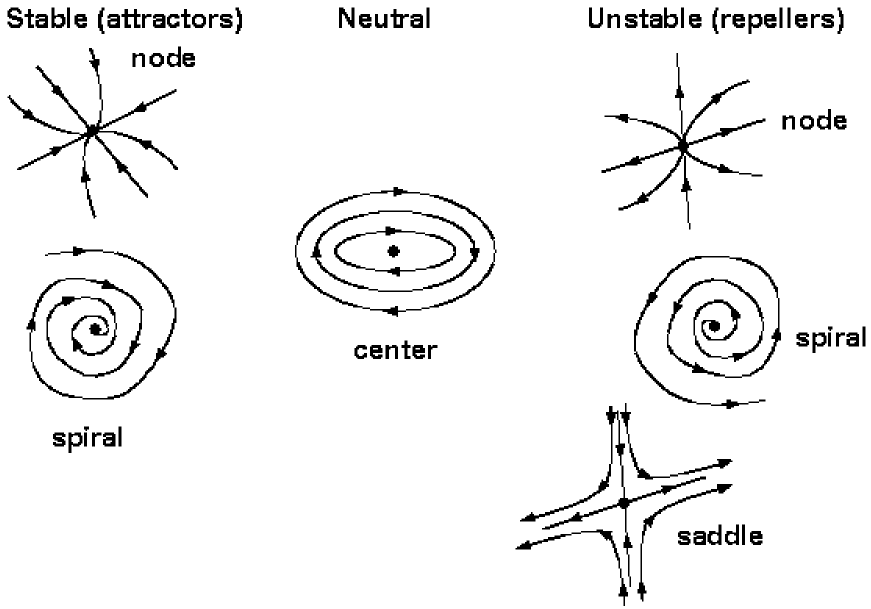
IS theory as a 2d non-autonomous system

IS evolution eqs. can be re-written as $\tau = \tanh \rho$

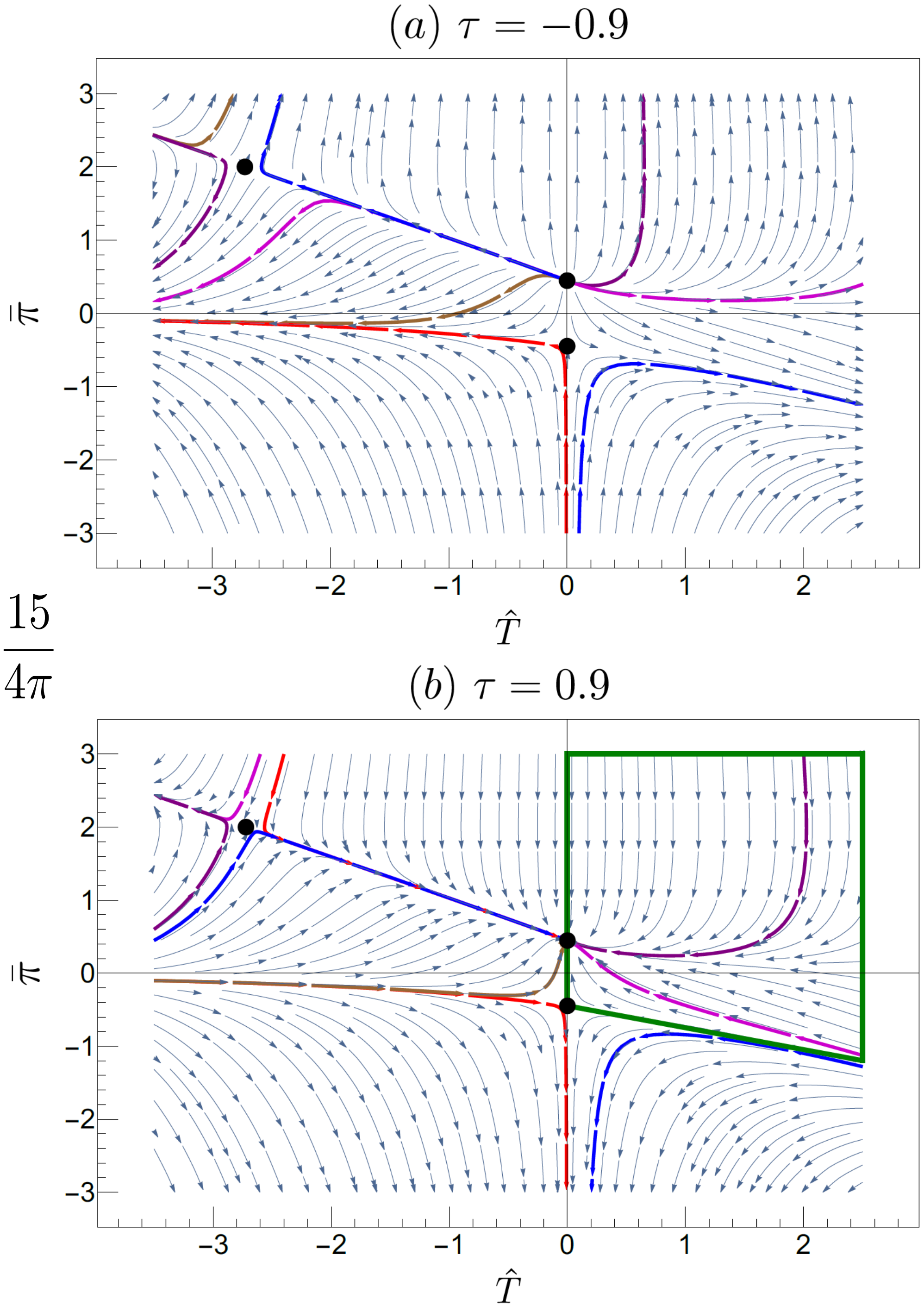
$$\frac{d\hat{T}}{d\tau} = \frac{\tau \hat{T}}{3(1-\tau^2)} \left(\bar{\pi}(\tau) - 2\right),$$

$$\frac{d\bar{\pi}}{d\tau} = -\frac{1}{1-\tau^2} \left(\frac{4}{3}\,\bar{\pi}^2(\tau)\,\tau + \frac{1}{c}\,\bar{\pi}(\tau)\,\hat{T}(\tau) - \frac{4}{15}\,\tau \right).$$

Before continuing, let's remember some basic of flow lines in the phase space of the dynamical variables



IS theory as a 2d non-autonomous system



$$c = \frac{15}{4\pi}$$

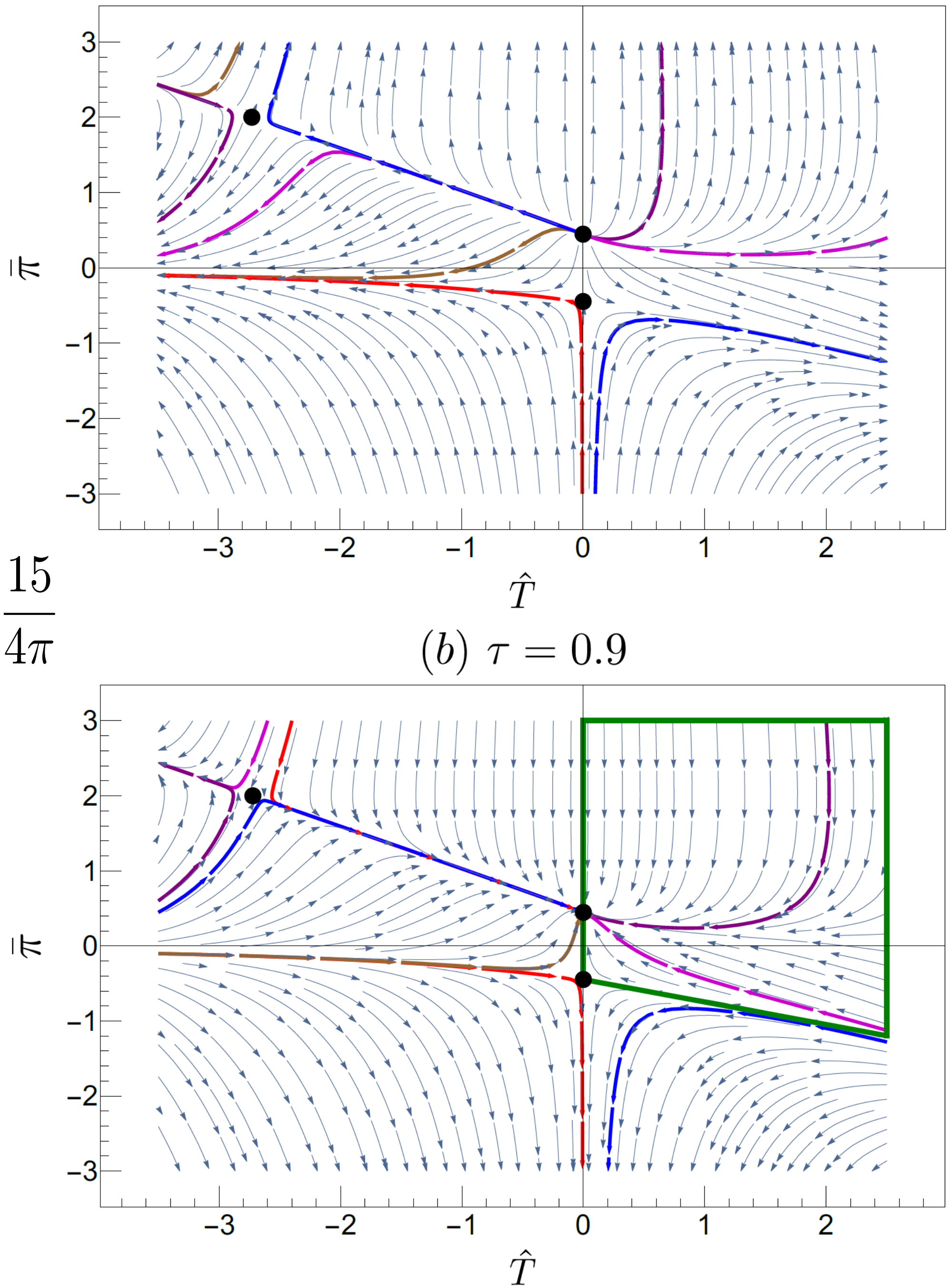
Fixed points are determined from the null-line conditions:

$$\frac{d\hat{T}}{d\tau} = 0$$

$$\frac{d\bar{\pi}}{d\tau} = 0$$

IS theory as a 2d non-autonomous system

$(a)\ \tau = -0.9$



$$c = \frac{15}{4\pi}$$

$(b)\ \tau = 0.9$

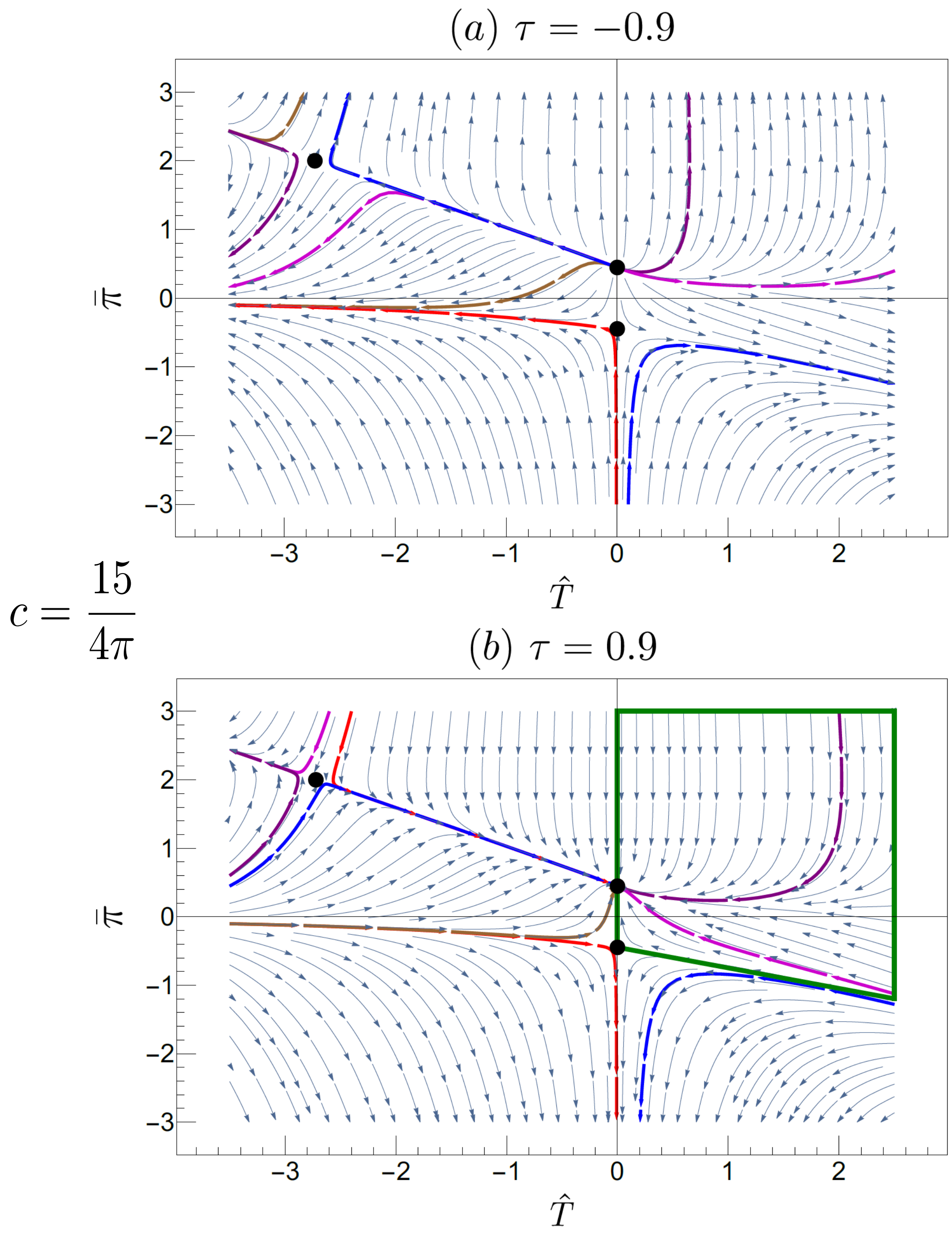
Fixed points : $\bar{\pi}_c^{\pm} = \pm\frac{1}{\sqrt{5}}, \quad \hat{T}_c = 0,$

$$\bar{\pi}_c = 2, \quad \hat{T}_c = -\frac{38c}{15}.$$

Early times:

- *Three unstable fixed points*: 2 *saddle fixed point and one source*

IS theory as a 2d non-autonomous system



$$\text{Fixed points}: \quad \bar{\pi}_c^{\pm} = \pm\frac{1}{\sqrt{5}}, \quad \hat{T}_c = 0,$$

$$\bar{\pi}_c = 2, \quad \hat{T}_c = -\frac{38c}{15}.$$

Late times:

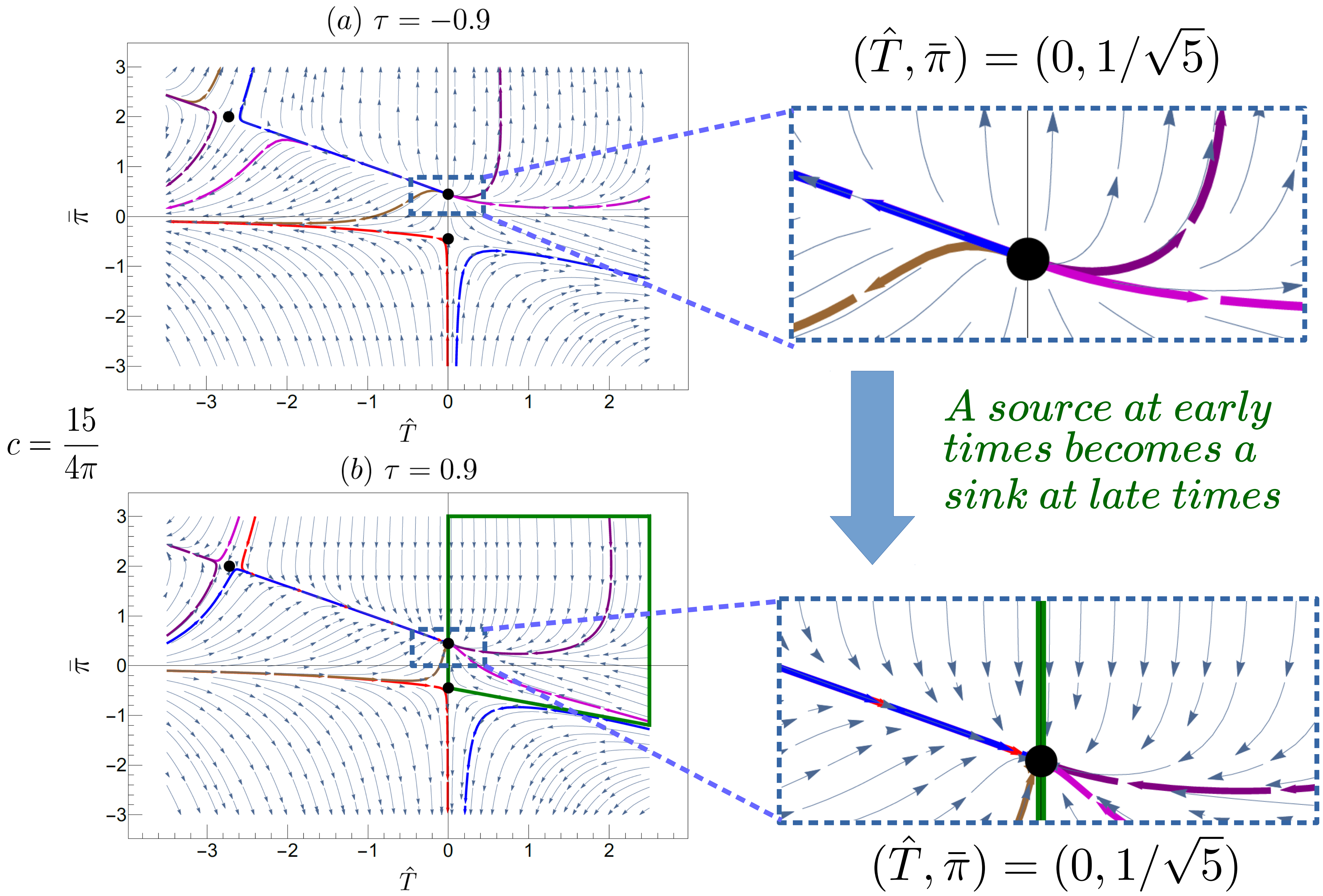
- *Two unstable fixed points (saddle) and one stable fixed point (sink)*
- *Stable point correspond to*

$$(\hat{T}, \bar{\pi}) = (0, 1/\sqrt{5})$$

$\Rightarrow$ *system never reaches thermal equilibrium.*

Steady non-equilibrium state!!!

IS theory as a 2d non-autonomous system



Subtle issue of IS theory for Gubser flow

For the Gubser flow IS can be combined into one equation

$$\frac{\partial_\rho \hat{T}}{\hat{T}} + \frac{2}{3}\tanh\rho = \frac{\bar{\pi}}{3}\tanh\rho$$

$$\hat{\tau}_{\hat{\pi}}\left(\partial_\rho \bar{\pi} + \frac{4}{3}(\bar{\pi})^2 \tanh\rho\right) + \bar{\pi} = \frac{4}{3}\frac{\eta}{s\,\hat{T}}\tanh\rho$$

$$\mathcal{A}(w) = \frac{1}{\tanh\rho}\frac{\partial_\rho \hat{T}}{\hat{T}} = \frac{d\log(\hat{T})}{d\log(\cosh\rho)}$$

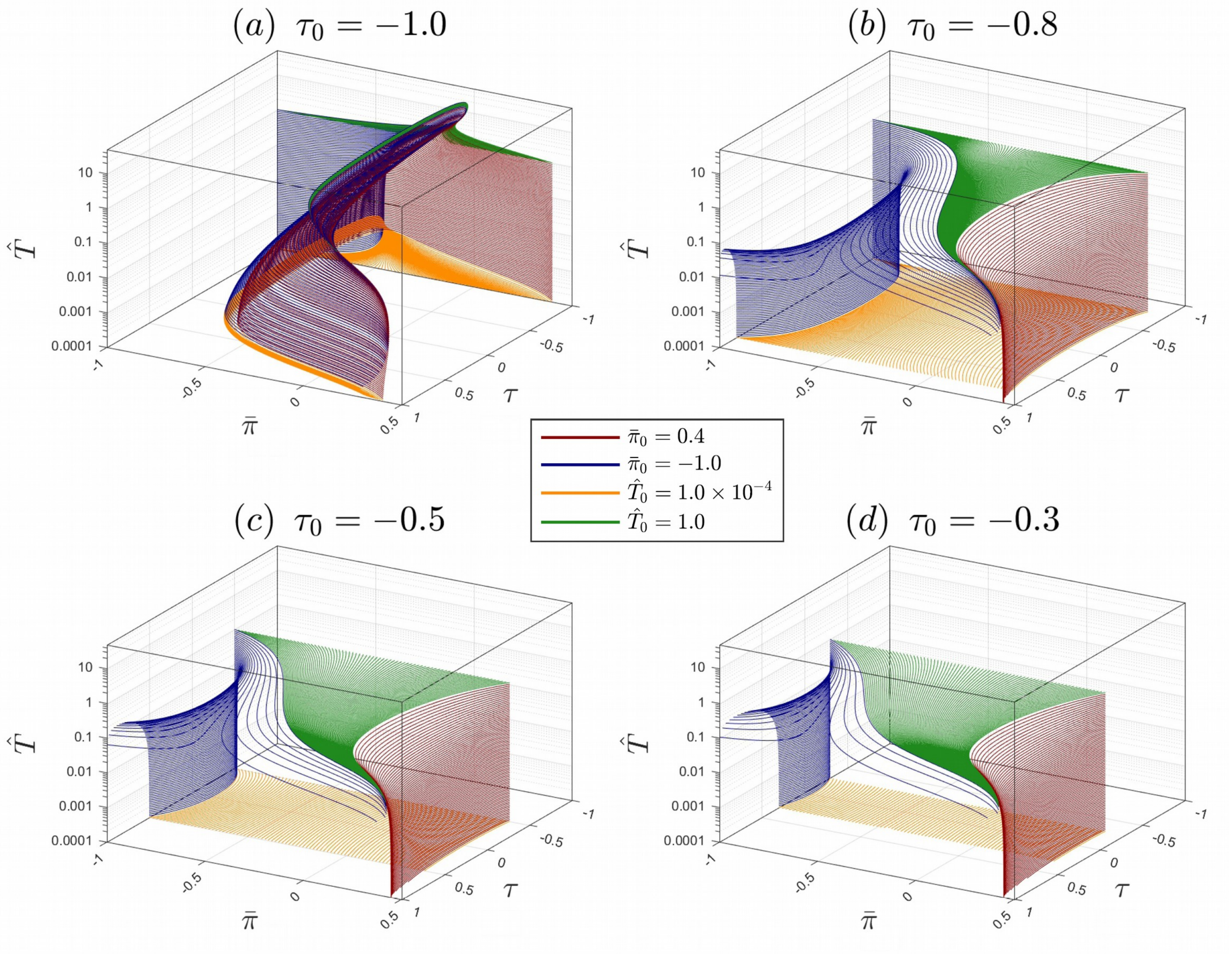
$$3w\left(\coth^2\rho - 1 - \mathcal{A}(w)\right)\frac{d\mathcal{A}(w)}{dw} + \frac{4}{3}(3\mathcal{A}(w)+2)^2 + \frac{3\mathcal{A}(w)+2}{cw} - \frac{4}{15} = 0$$

The solution of this ODE depends on ρ

- *$dS_3 \otimes R$ is a curved space whose expansion rate does not vanish asymptotically (non-equilibrium steady state)*
- *This did not happen for the 0+1 dim. system (Bjorken)*

Rethinking IS Eqs. as a 3d DOE system

$$\frac{d\hat{T}}{d\rho} = \frac{1}{3}\hat{T}(\bar{\pi} - 2)\tau, \qquad \frac{d\bar{\pi}}{d\rho} = \frac{4}{3}\left(\frac{1}{5} - \bar{\pi}^2\right)\tau - \frac{1}{c}\pi\hat{T}, \qquad \frac{d\tau}{d\rho} = 1 - \tau^2.$$



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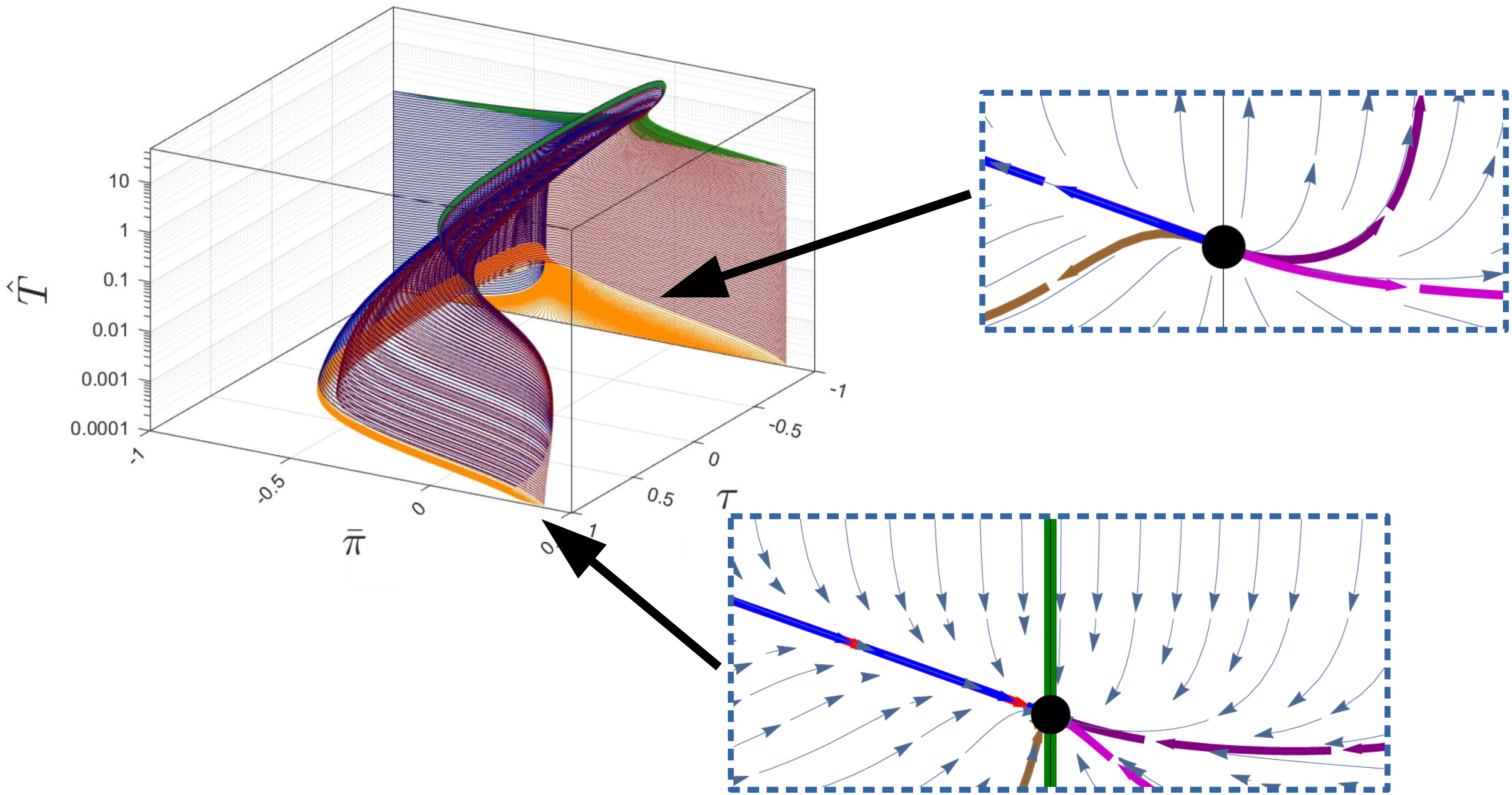
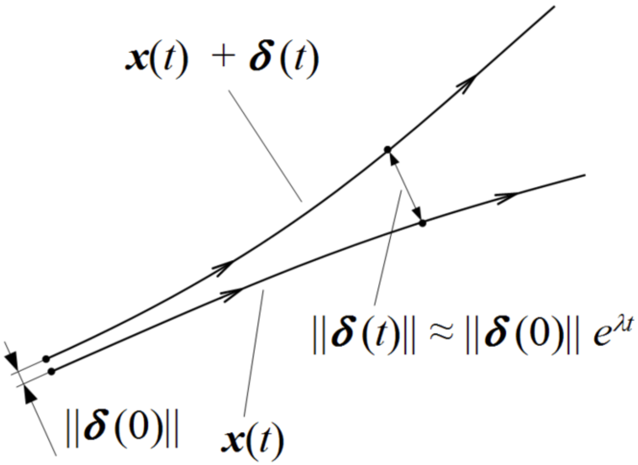
***Basin of attraction for the Gubser flow is* 3 *dim*.**



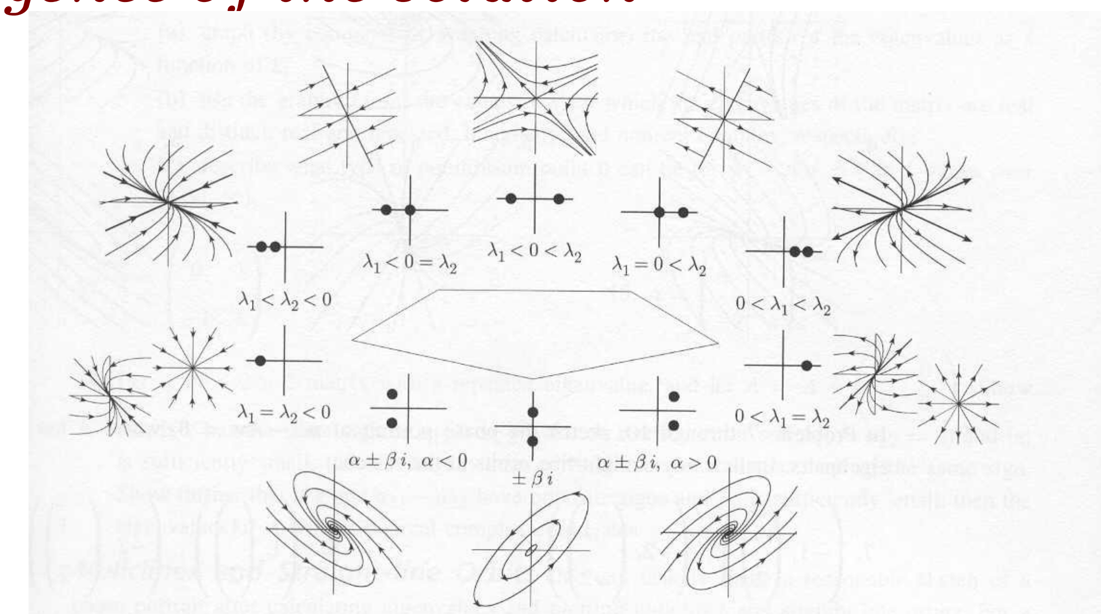
Lyapunov exponents of IS theory



Lyapunov exponent *measures the distance between two trajectories in the phase space*
Stability of the DOE's *depend on the value of the Lyapunov exponent*

$$\frac{dx}{dt} = Ax$$

Eigenvalues of matrix A determine the stability and convergence of the solution



Lyapunov exponents of IS theory

We can linearize our 3d system around the fixed points of the IS theory for the Gubser flow

$$\begin{pmatrix} \partial_\rho \hat{T} \\ \partial_\rho \bar{\pi} \\ \partial_\rho \tau \end{pmatrix} = \begin{pmatrix} \frac{1(\bar{\pi}-2)}{3}\tau & \frac{\hat{T}\tau}{3} & \frac{\hat{T}(\bar{\pi}-2)}{3} \\ -\frac{\bar{\pi}}{c} & -\frac{\hat{T}}{c} - \frac{8\bar{\pi}\tau}{3} & \frac{4}{15} - \frac{4\bar{\pi}^2}{3} \\ 0 & 0 & -2\tau \end{pmatrix}_{(\hat{T}_c,\bar{\pi}_c,\tau_c)} \begin{pmatrix} \hat{T} - \hat{T}_c \\ \bar{\pi} - \bar{\pi}_c \\ \tau - \tau_c \end{pmatrix}$$

The eigenvalues of this matrix at $\tau \longrightarrow 1$

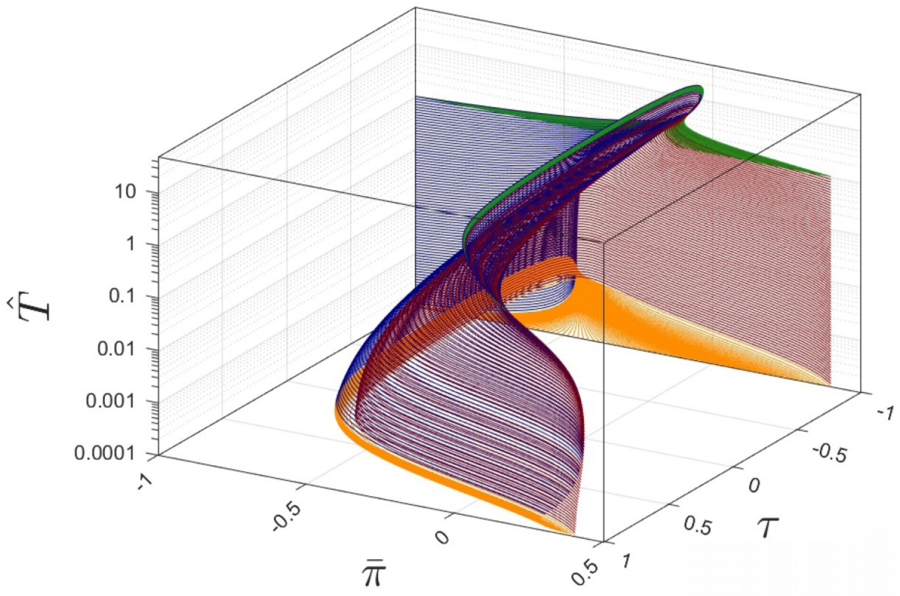
$$A : \left\{-2, -\frac{8}{3\sqrt{5}}, -\frac{2}{3} + \frac{1}{3\sqrt{5}}\right\} \longrightarrow \textit{Stable (sink)}$$

$$B : \left\{-2, : \frac{8}{3\sqrt{5}}, -\frac{2}{3} - \frac{1}{3\sqrt{5}}\right\}$$

$$C : \left\{-2, \frac{7}{5} - \frac{\sqrt{821}}{15}, -\frac{7}{5} + \frac{\sqrt{821}}{15}\right\}$$

B and C $\longrightarrow$ *Unstable (saddle)*

Lyapunov exponents of IS theory



Lyapunov exponents of the attractor are read off from the eigenvalues of the matrix

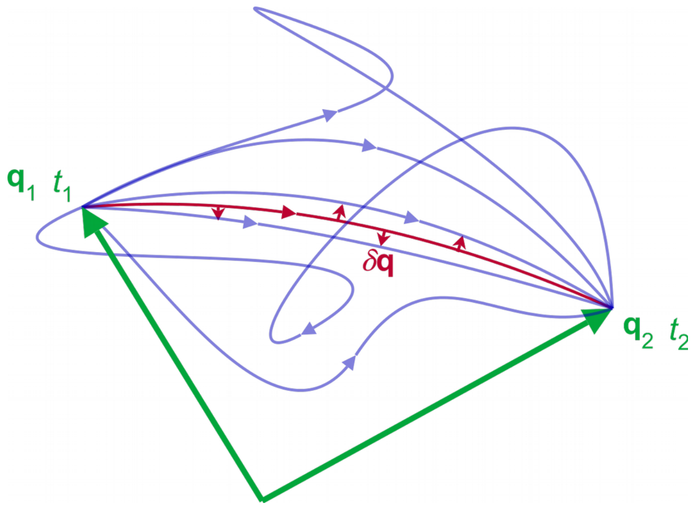
$$A : \left\{-2, -\frac{8}{3\sqrt{5}}, -\frac{2}{3} + \frac{1}{3\sqrt{5}}\right\} \longrightarrow \textit{Stable (sink)}$$

Attractor: $\mathcal{A} \sim \hat{T}_0\, e^{\lambda_{\hat{T}}\rho}\mathbf{u_1} + (\frac{1}{\sqrt{5}} - \bar{\pi}_0\, e^{\lambda_{\bar{\pi}}\rho})\mathbf{u_2} + \mathbf{u_3},$

$$\lambda_{\hat{T}} = -\frac{2}{3} + \frac{1}{3\sqrt{5}}, \quad \lambda_{\bar{\pi}} = -\frac{8}{3\sqrt{5}}, \quad \lambda_{\tau} = -2.$$

Why is the basin of attraction so interesting?

$$Z = \int_M D\phi \, e^{-S[\phi]}$$



- M *defines the space of fields or paths over which the integral is evaluated*

- *Saddle points (classical path) are determined from the action principle*

$$\frac{\delta S[\phi]}{\delta \phi} = 0$$

$\Rightarrow$ *M is a stable manifold of integration shaped by the solutions to the saddle point approximation*

Why is the basin of attraction so interesting?

Using this analogy the partition function for hydrodynamics

$$Z_{\rm eff}(c) = \int_M D\hat{T} D\bar{\pi} Dt \; e^{-\int d\rho \left(\left(\frac{d\mathbf{x}}{d\rho}\right)^2 - \mathcal{V}(\mathbf{x},c)\right)}.$$

$\mathcal{V}$ is the Lyapunov function which due to stability has to satisfy

$$\frac{d\mathcal{V}}{d\rho} \leq 0,$$

Thus M is the manifold whose paths are determined by the basin of attraction of the hydrodynamical equations!!!!

For the Gubser flow and IS theory local Lyapunov function was obtained

see arXiv:1711.01745

Determining attractors I

- *IS, DNMR and anisotropic hydro equations can be recombined into a unique equation*

$$3w\left(\coth^2\rho - 1 - \mathcal{A}(w)\right)\frac{d\mathcal{A}(w)}{dw} + H(\mathcal{A}(w), w) = 0 \qquad (1)$$

Remember, we evaluate the asymptotic attractor $\coth^2\rho \longrightarrow 1$

- *The function H depends on the hydro model*

$$H_{\mathrm{IS}} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \tfrac{3\mathcal{A}(w)+2}{cw} - \tfrac{4}{15}\,,$$

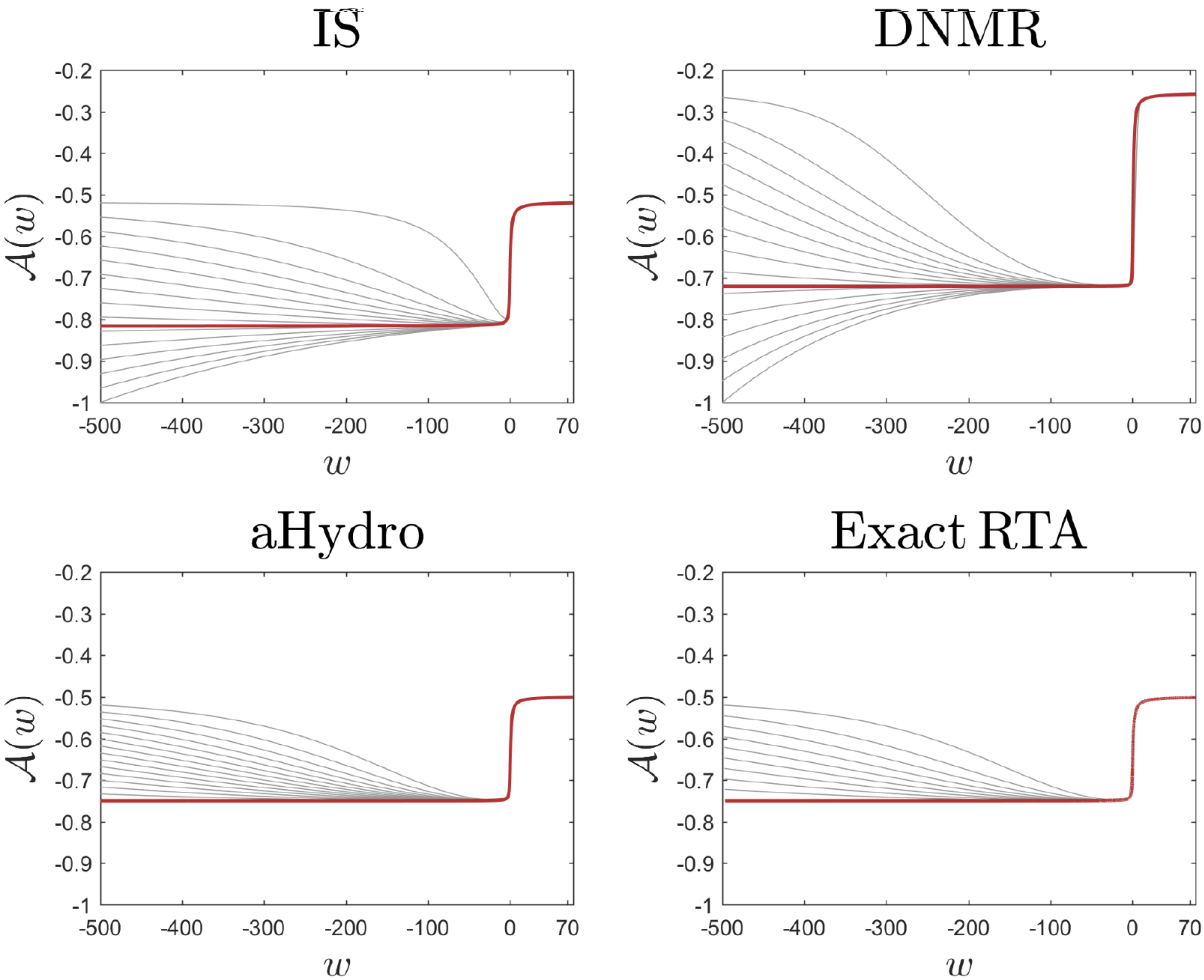
$$H_{\mathrm{DNMR}} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{c\,w} - \tfrac{10}{7}\right] - \tfrac{4}{15}\,,$$

$$H_{\mathrm{aHydro}} = \tfrac{4}{3}\left(3\mathcal{A}(w)+2\right)^2 + \left(3\mathcal{A}(w)+2\right)\left[\tfrac{1}{c\,w} - \tfrac{4}{3}\right] - \\ - \tfrac{5}{12} + \tfrac{3}{4}\mathcal{F}\left(3\mathcal{A}(w)+2\right)\,.$$

Attractors are found by a two-step process:

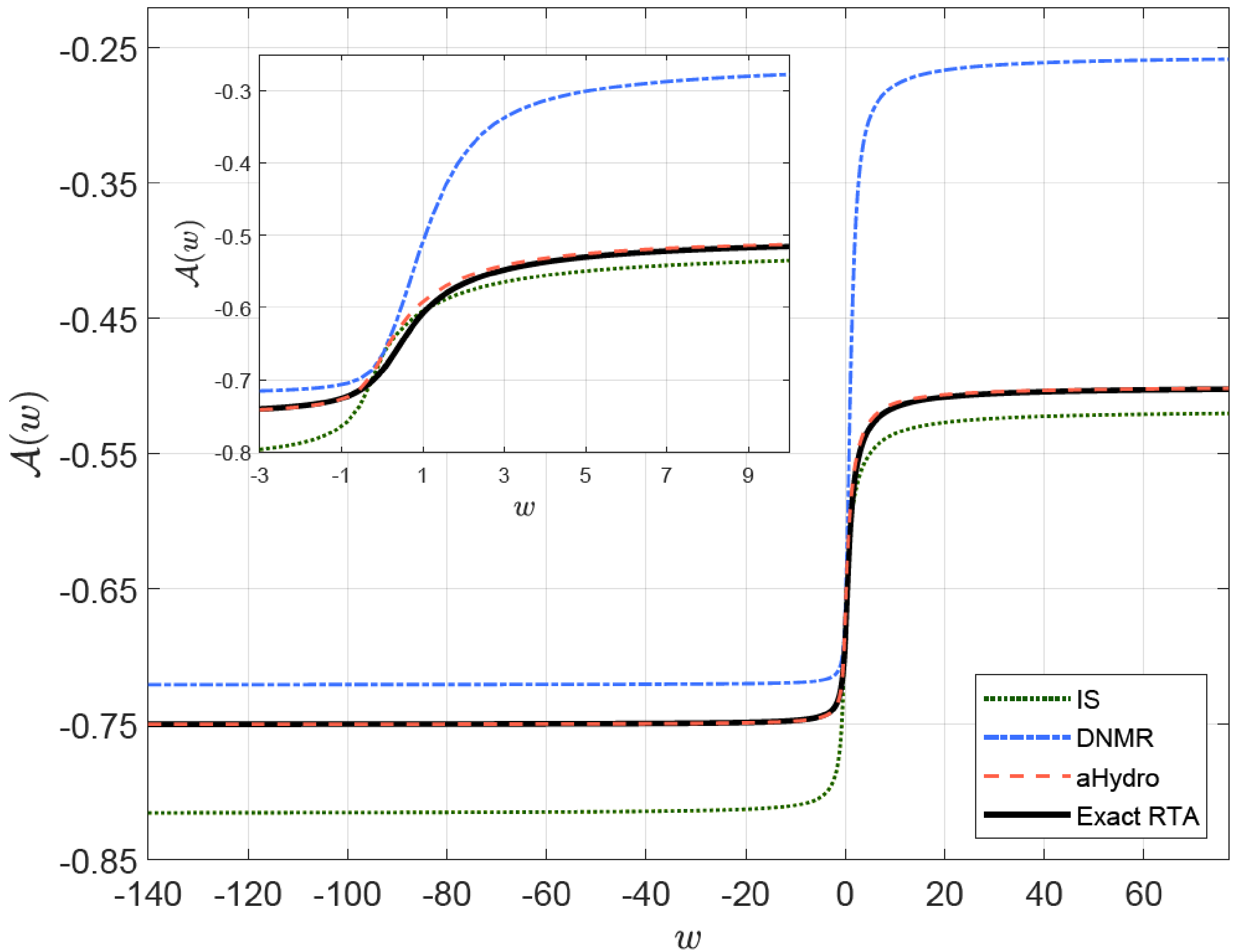
- *Finding null-lines with slow-roll down approx.* $d\mathcal{A}/dw=0$
- *The initial condition for solving* (1) *is obtained from the stable solution of the null-line* $\mathcal{A}_i=\mathcal{A}_+(w\longrightarrow-\infty)$

Universal attractors for Gubser flow



$$c = \frac{15}{4\pi}$$

Comparing attractors



$$c = \frac{15}{4\pi}$$

Anisotropic hydrodynamics matches almost exactly the exact attractor