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Asymptotic solutions of IS theory and the validity of hydrodynamics

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GSD and J. Noronha, arXiv:1711.01657, arXiv:1804.04771

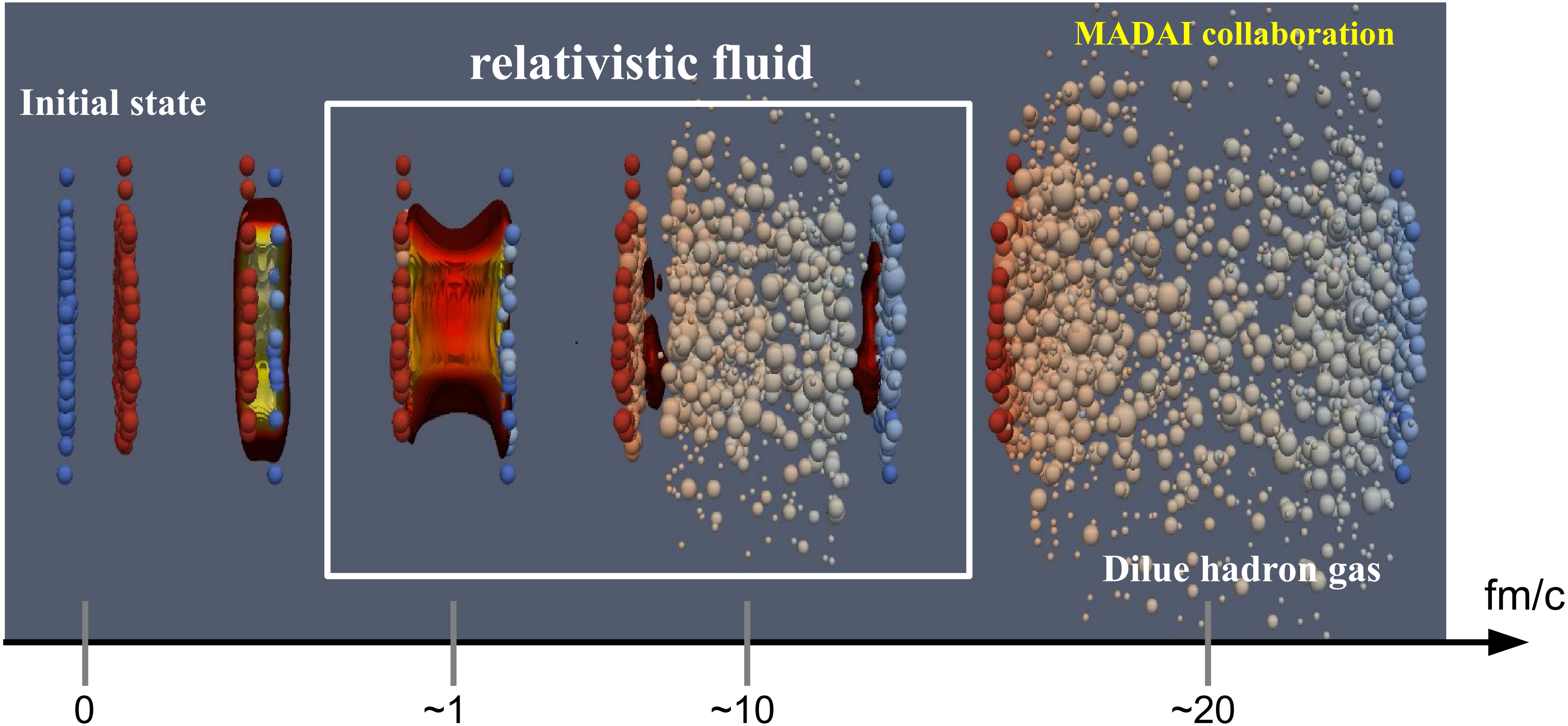
ECT*
Foundational aspects of relativistic hydrodynamics

Summary

- Introduction & motivation
- Divergence of the gradient expansion
- Asymptotic sol. of Israel-Stewart in Bjorken flow
- Asymptotic sol. of Israel-Stewart in Gubser flow

Theoretical description of HIC

Empirical: Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies



Main assumption: fluid dynamics can be applied at the very early stages – *Why?*

Validity of fluid dynamics

- **“proximity” to (local) equilibrium**
- **“small” gradients**

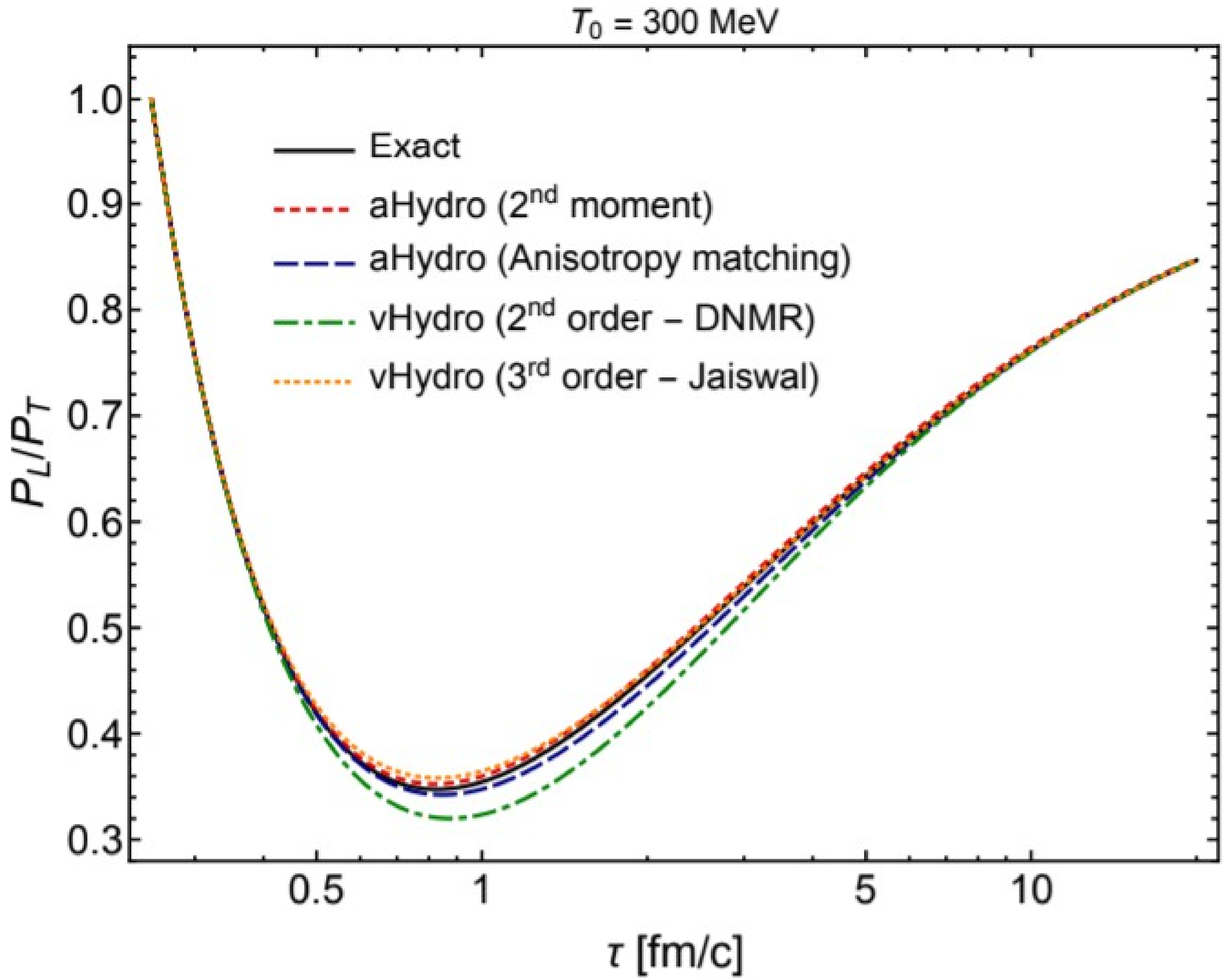
Separation of scales $\longrightarrow$ **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \dfrac{\ell}{L} \ll 1$

These things do not occur at the early stages of HIC

Simple example: Bjorken scalling

Alqahtani et al, arXiv:1712.03282v1



second-order fluid dynamics displays surprising accuracy

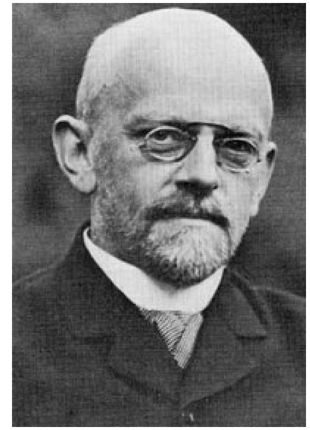
Validity of fluid dynamics

???

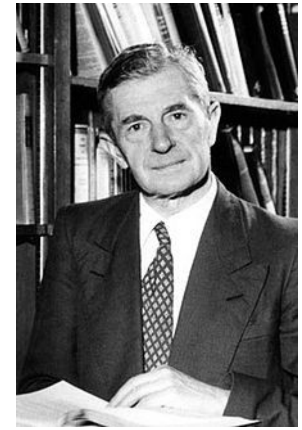
Proximity to equilibrium, small gradients

Intuition about the validity of
hydro comes mostly from the gradient expansion

Gradient Expansion



Hilbert



Chapman

Enskog

$$k^\mu \partial_\mu f_{\mathbf{k}} = \frac{1}{\epsilon} C\left[f_{\mathbf{k}}\right]$$

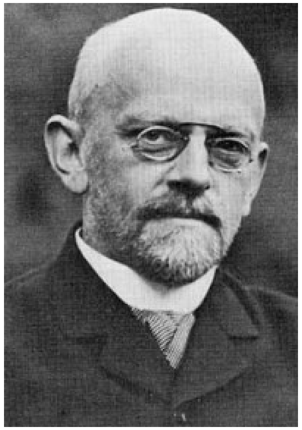
Knudsen number

Perturbative expansion

$$f_{\mathbf{k}} = f_{\mathbf{k}}^{(0)} + \epsilon f_{\mathbf{k}}^{(1)} + \epsilon^2 f_{\mathbf{k}}^{(2)} + \ldots$$

Result is a gradient expansion – more general than kinetic theory

Gradient Expansion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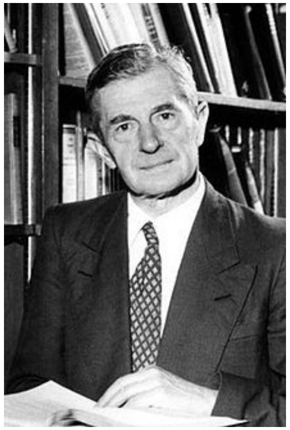

Chapman


Enskog

1st order truncation: Navier-Stokes theory

$$\boxed{\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}} \qquad \sigma^{\mu\nu} \equiv \frac{1}{2}(\nabla^{\mu}u^{\nu} + \nabla^{\nu}u^{\mu}) - \frac{1}{3}\Delta^{\mu\nu}\theta.$$

2nd order truncation: Burnett theory

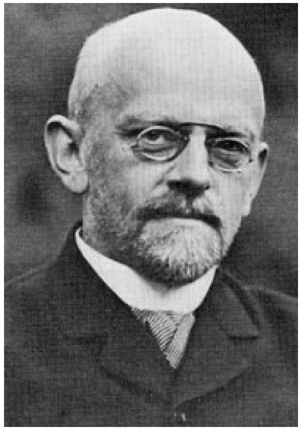
$$\boxed{\begin{aligned}\pi^{\mu\nu} = {} & 2\eta\sigma^{\mu\nu} + \eta_1\omega_{\lambda}^{\ \langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma_{\lambda}^{\ \nu\rangle} + \eta_4\sigma_{\lambda}^{\ \langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\nu\rangle} \\ & + \eta_6 J^{\langle\mu}J^{\nu\rangle} + \eta_7 I^{\langle\mu}J^{\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\nu\rangle} + \eta_9\nabla^{\langle\mu}J^{\nu\rangle}.\end{aligned}}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^{\mu}u^{\nu} - \nabla^{\nu}u^{\mu}).$$

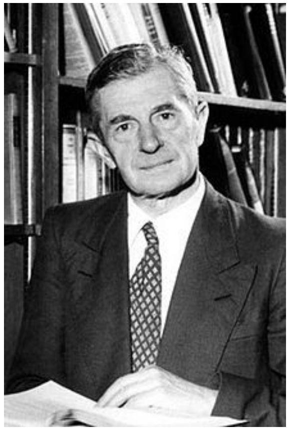
$$\theta = \nabla_{\mu}u^{\mu},$$

$$I^{\mu} \equiv \nabla^{\mu}\alpha_0, \quad J^{\mu} \equiv \nabla^{\mu}\beta_0,$$

Gradient Expansion



Hilbert



Chapman

Enskog

Second-order truncation: Burnett theory

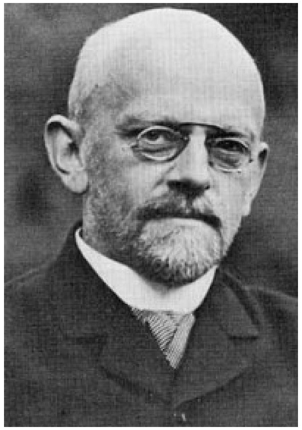
$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1\omega_{\lambda}^{\ \langle\mu}\omega^{\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\sigma_{\lambda}^{\ \nu\rangle} + \eta_4\sigma_{\lambda}^{\ \langle\mu}\omega^{\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\nu\rangle} + \eta_6 J^{\langle\mu}J^{\nu\rangle} + \eta_7 I^{\langle\mu}J^{\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\nu\rangle} + \eta_9\nabla^{\langle\mu}J^{\nu\rangle}.$$

Hydrodynamical constitutive equations are usually derived by *truncating* this series.

Effective theory: can be systematically corrected

Convergence is assumed!

Gradient Expansion Diverges (?)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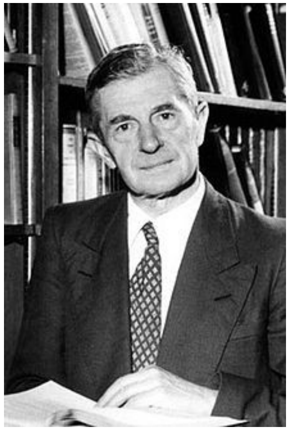

Chapman


Enskog

H. Grad: CE is an asymptotic series, Physics of Fluids 6, 147 (1963).

First example of divergence: Couette flow problem (RTA), Santos *et al*, PRL 56, 1571 (1986).

Heller *et al*: Holography+Bjorken scaling, PRL 110, 211602 (2013) -- first time for an expanding system

Not necessarily a problem; but applicability and improvability not clear.

IS theory is not "traditional" fluid dynamics

Causality: constitutive relations for the shear stress tensor cannot be imposed

Dynamical equation, e.g. Israel-Stewart theory

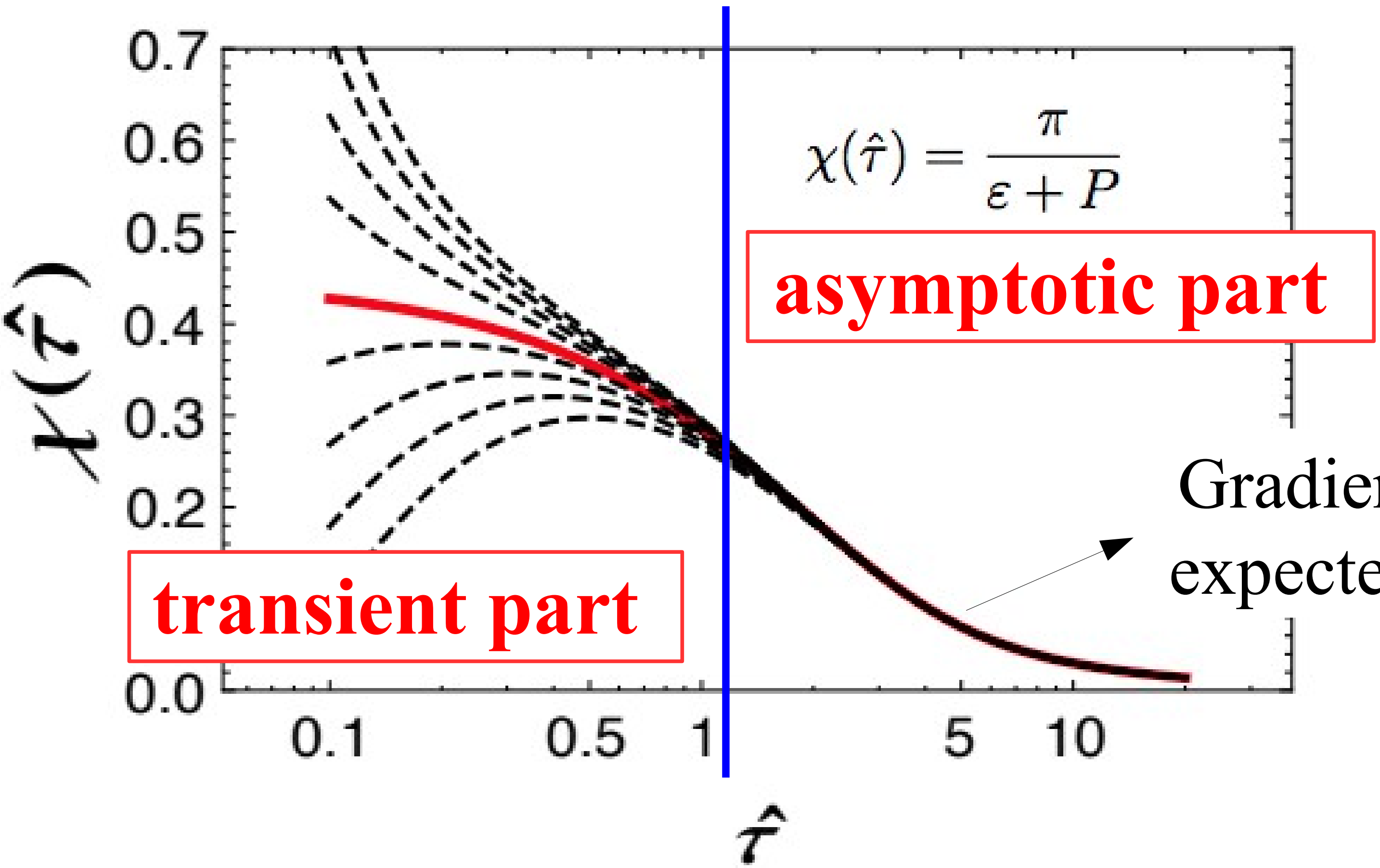
$$\Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\Theta + \frac{\pi^{\mu\nu}}{\tau_R} = 2\frac{\eta}{\tau_R}\sigma^{\mu\nu}$$

Beyond hydrodynamics: does not only describe slow processes, but fast ones as well

non-hydrodynamic modes ...

“Hydrodynamic regime” of Israel-Stewart theory

$$\boxed{\tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi}\,\theta\pi^{\mu\nu} + \tau_{\pi\pi}\,\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^{\beta}_{\lambda} - 2\,\tau_R \Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha}_{\lambda}\omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},}$$



arXiv:1711.01657

Gradient expansion was expected to describe this

Our strategy: we study this problem in Israel-Stewart theory

- Simple proxy for a microscopic theory; may help us gain intuition

- In particular, we want to understand what happens for other flow profiles

Israel-Stewart theory and the gradient expansion

- **Bjorken flow**
- **Ideal EoS**
- **Constant relaxation**

Already studied in Heller & Spalinski, PRL 115, no. 7, 072501 (2015)

Analytical Solution for constant relaxation times (Bjorken scalling)

constant

$$\partial_{\hat\tau}\chi + \chi + \frac{4}{3\hat\tau}\chi^2 = \frac{3a}{4\hat\tau},$$

$$\chi(\hat\tau) = \frac{\pi}{\varepsilon + P} \qquad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

Solution:

decays exp. **grows exp.**

$$\chi(\hat\tau) = \frac{3\sqrt{a}}{4}\left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) + K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) - K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}\right]$$

initial cond.

Analytical Solution for constant relaxation times (Bjorken scalling)

$$\partial_{\hat\tau}\chi + \chi + \frac{4}{3\hat\tau}\chi^2 = \frac{3a}{4\hat\tau},$$

$$\chi(\hat\tau) = \frac{\pi}{\varepsilon + P} \qquad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

Solution:

decays exp.

grows exp.

$$\chi(\hat\tau) = \frac{3\sqrt{a}}{4}\left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) + K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) - K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}\right]$$

initial cond.

At late times:

$$\chi(\hat\tau) \to \chi_{att}(\hat\tau) = \frac{3\sqrt{a}}{4}\left[\frac{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat\tau}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat\tau}{2}\right)}\right]$$

Analytical Solution for constant relaxation times (Bjorken scalling)

$$\partial_{\hat{\tau}}\chi + \chi + \frac{4}{3\hat{\tau}}\chi^2 = \frac{3a}{4\hat{\tau}},$$

$$\chi(\hat{\tau}) = \frac{\pi}{\varepsilon + P} \qquad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

For "physical" case, a = 16/45 gradient expansion does not converge

Curiosity: solutions becomes very simply for a=1

$$\chi(\hat{\tau})\Big|_{a=1} = \frac{3}{4}\left[\frac{\frac{1}{\hat{\tau}} + \alpha\left(1 + \frac{1}{\hat{\tau}}\right)e^{-\hat{\tau}}}{1 - \frac{1}{\hat{\tau}} + \frac{\alpha}{\hat{\tau}}e^{-\hat{\tau}}}\right]$$

Attractor becomes very simple (does not diverge)

$$\chi(\hat{\tau})\Big|_{a=1,att} = \frac{3}{4\left(\hat{\tau} - 1\right)},$$

Analytical Solution for constant relaxation times (Bjorken scalling)

$$\partial_{\hat{\tau}}\chi + \chi + \frac{4}{3\hat{\tau}}\chi^2 = \frac{3a}{4\hat{\tau}},$$

$$\chi(\hat{\tau}) = \frac{\pi}{\varepsilon + P} \quad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

Trans-series can be easily generated (a=1)

$$\chi(\hat{\tau}) = \frac{3}{4}\sum_{n=0}^{\infty}\frac{1}{\hat{\tau}^{n+1}} - \frac{3}{4}\alpha\, e^{-\hat{\tau}}\sum_{n=0}^{\infty}\frac{1}{\hat{\tau}^{n}}\left(1 + \frac{1}{\hat{\tau}} + \frac{n+1}{\hat{\tau}^2}\right) + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}})$$

$$= \frac{3}{4}\frac{1}{(\hat{\tau}-1)} - \frac{3}{4}\alpha\, e^{-\hat{\tau}}\frac{\hat{\tau}^2}{(\hat{\tau}-1)^2} + \mathcal{O}(\alpha^2 e^{-2\hat{\tau}}),$$

non-perturbative

Ressumed Gradient expansion
(*finite radius of convergence*)

Another option: slow-roll expansion

$$\boxed{\epsilon\,\hat{\tau}\frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0} \qquad a = \frac{16}{9(\tau_R T)}\frac{\eta}{s}.$$

Slow regime: only derivative
Is assumed to be small

perturbative solution

$$\chi(\hat{\tau};\epsilon) = \sum_{n=0}^{\infty}\chi_n(\hat{\tau})\epsilon^n$$

Coefficients satisfy recurrence relations:

$$\chi_0(\hat{\tau}) = \frac{3}{8}\left(\sqrt{\hat{\tau}^2 + 4a} - \hat{\tau}\right)$$

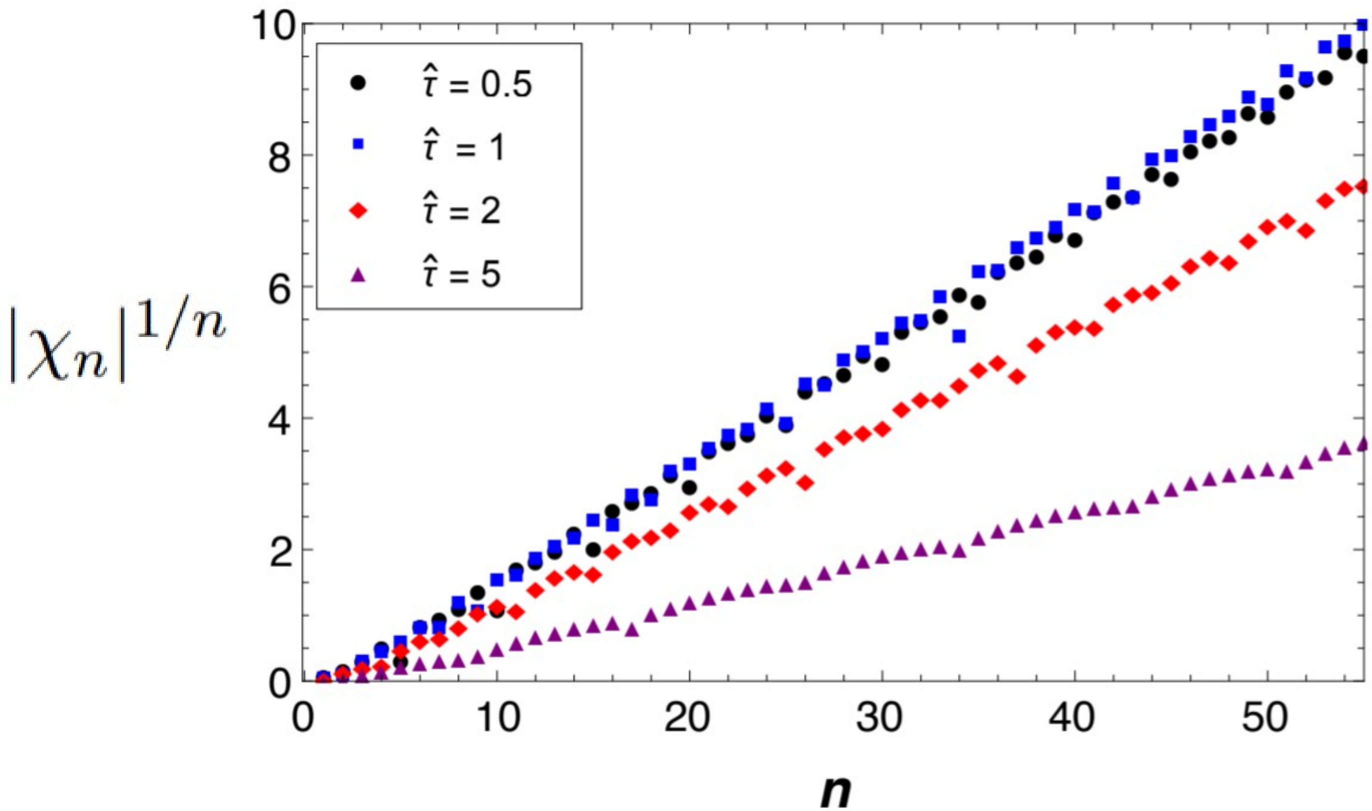
(Heller et al, Romatschke ...)

$$\chi_n(\hat{\tau}) = -\frac{1}{\sqrt{\hat{\tau}^2 + 4a}}\left(\hat{\tau}\frac{d\chi_{n-1}}{d\hat{\tau}} + \frac{4}{3}\sum_{m=1}^{n-1}\chi_{n-m}\chi_m\right)$$

Another option: slow-roll expansion

$$\epsilon\,\hat{\tau}\frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0$$

$$\chi(\hat{\tau};\epsilon) = \sum_{n=0}^{\infty} \chi_n(\hat{\tau})\epsilon^n$$

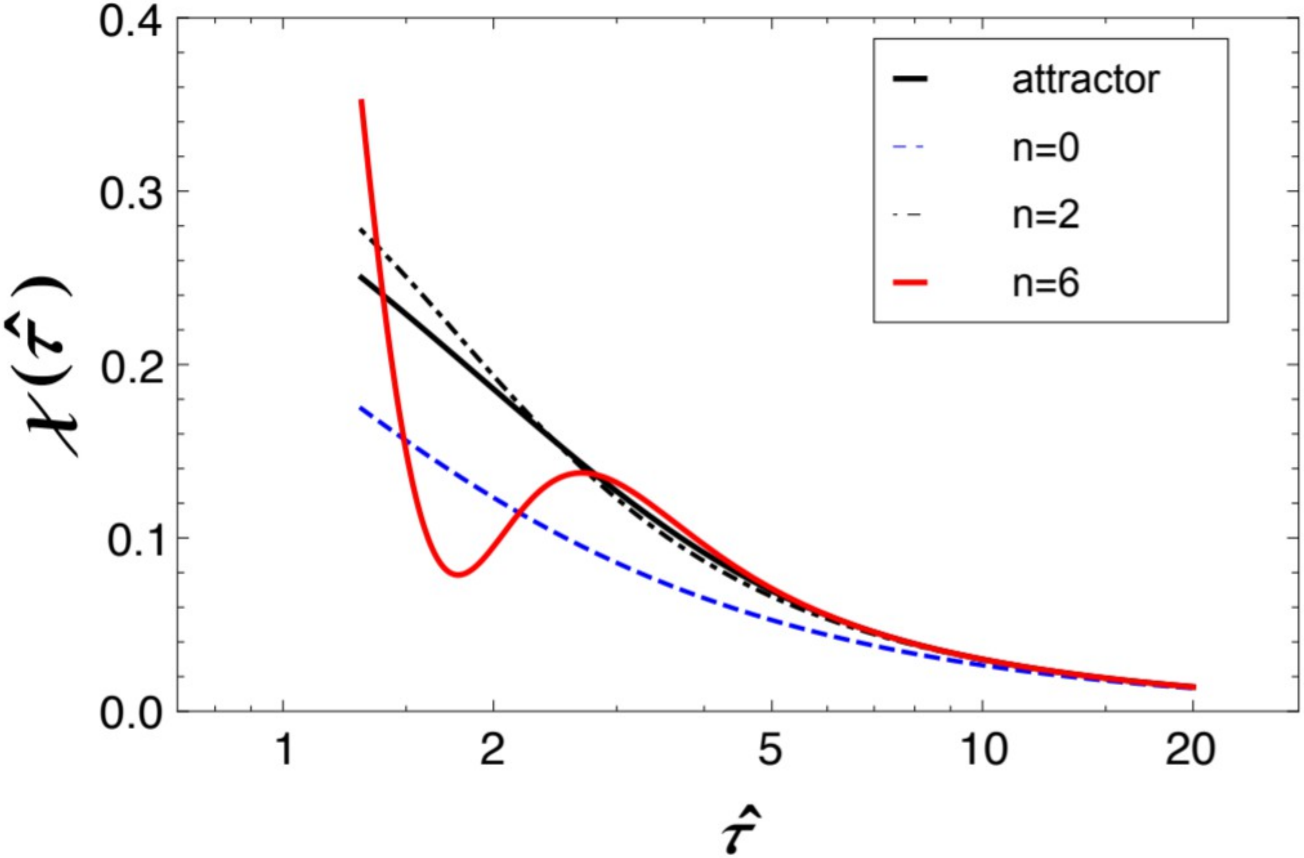


expansion also diverges!

Another option: slow-roll expansion

$$\boxed{\epsilon\,\hat{\tau}\frac{d\chi}{d\hat{\tau}} + \frac{4}{3}\chi^2 + \hat{\tau}\chi - \frac{3a}{4} = 0}$$

$$\boxed{\chi(\hat{\tau};\epsilon) = \sum_{n=0}^{\infty} \chi_n(\hat{\tau})\epsilon^n}$$



Truncated slow-roll is better than GE (improvability)

Israel-Stewart theory and the gradient expansion

- Gubser flow
- Conformal fluid

Asymptotic Solutions in Gubser flow

Homogeneous fluid expanding according to:

$$\frac{ds^2}{\tau^2} = d\rho^2 - \left(\cosh^2\rho d\theta^2 + \cosh^2\rho \sin^2\theta d\phi^2 + d\eta^2\right)$$

Gubser coordinates (includes radial expansion)

$$\sinh\rho \equiv -\frac{1-(q\tau)^2+(qr)^2}{2q\tau}, \ \tan\theta \equiv \frac{2qr}{1+(q\tau)^2-(qr)^2},$$

Israel-Stewart equations (exp. rate: $\theta = 2\tanh\rho$)

$$\frac{1}{T}\frac{dT}{d\rho} + \frac{2}{3}\tanh\rho - \frac{1}{3}\pi\tanh\rho = 0,$$

$$\frac{d\pi}{d\rho} + \frac{\pi}{\tau_R} + \frac{4}{3}\pi^2\tanh\rho = \frac{4}{15}\tanh\rho.$$

Asymptotic Solutions in Gubser flow

Convert everything into a perturbative problem

Gradient expansion

$$\epsilon\frac{d\pi}{d\rho}+\frac{\pi T}{c}+\frac{4}{3}\pi^2\epsilon\tanh\rho=\frac{4}{15}\epsilon\tanh\rho,$$

Slow-roll expansion

$$\epsilon\, c\frac{d\pi}{d\rho}+T\pi+\frac{4}{3}c\pi^2\tanh\rho=\frac{4}{15}c\tanh\rho,$$

$$\pi\equiv-\frac{\pi_\eta^\eta}{\varepsilon+P}.$$

$$\boxed{\pi\sim\sum_{n=0}^{\infty}\pi_n\left(\rho\right)\epsilon^n}$$

Method itself will produce the terms

Difficulty: resumming contributions in temperature derivatives

Asymptotic Solutions in Gubser flow

Gradient expansion

$$\pi = \frac{4}{15}\epsilon\tau_R\tanh\rho - \frac{4}{15}(\epsilon\tau_R)^2\left(1-\frac{1}{3}\tanh^2\rho\right)$$

NS

$$+\frac{8}{45}(\epsilon\tau_R)^3\left(\tanh\rho - \frac{1}{15}\tanh^3\rho\right) + \mathcal{O}(\epsilon^4).$$

$$\pi \equiv -\frac{\pi^\eta_\eta}{\varepsilon + P}.$$

Slow-roll expansion

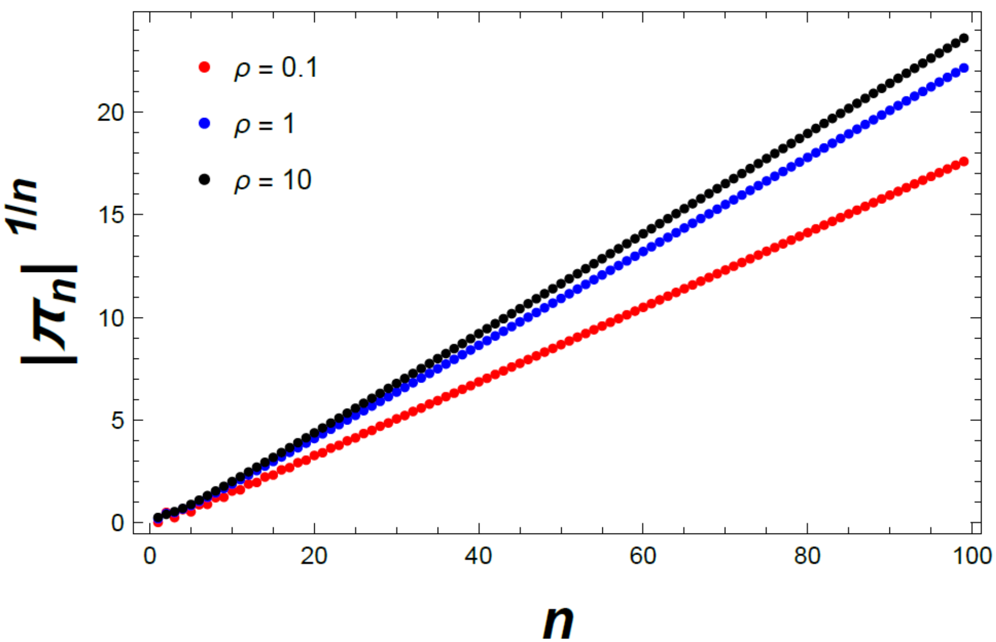
$$\bar\pi_0^\pm(\rho) = \frac{-3 \pm \sqrt{9 + \frac{4}{5}(4\tau_R\tanh\rho)^2}}{8\tau_R\tanh\rho}$$

$$\bar\pi_1(\rho) = \frac{15\bar\pi_0^+}{\tau_R\tanh\rho}\left[\frac{\left(1+\bar\pi_0^+\right)(\tau_R\tanh\rho)^2 - 3\tau_R^2}{45 + 64(\tau_R\tanh\rho)^2}\right]$$

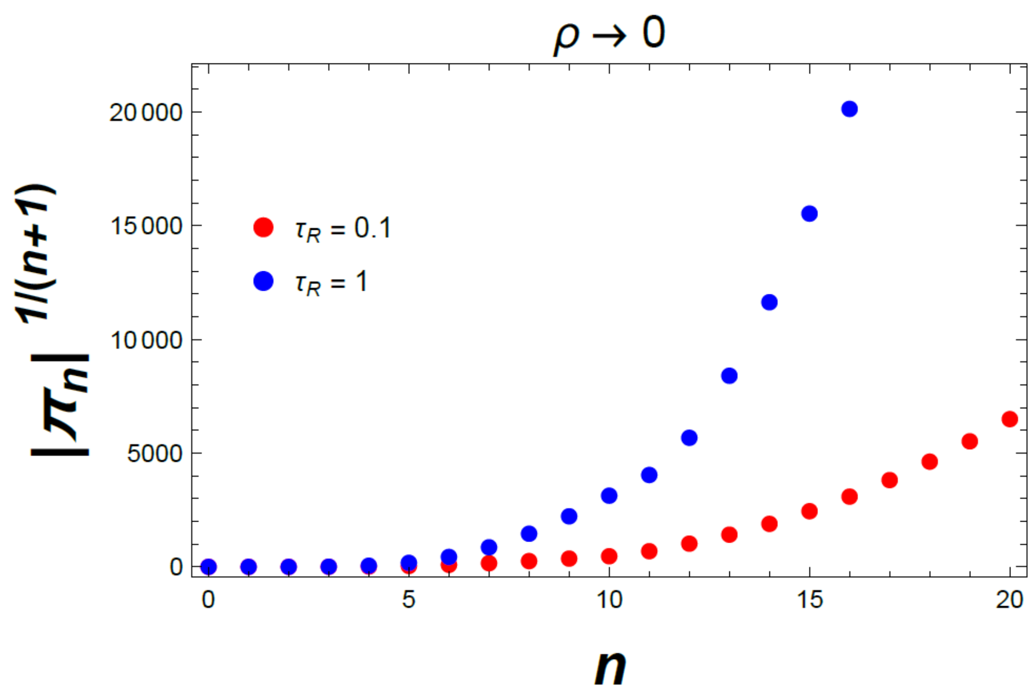
Unlike in Bjorken, *not* a series in $\tau_R\tanh\rho$.

Asymptotic Solutions in Gubser flow

Higher order behavior: both series diverge



gradient expansion

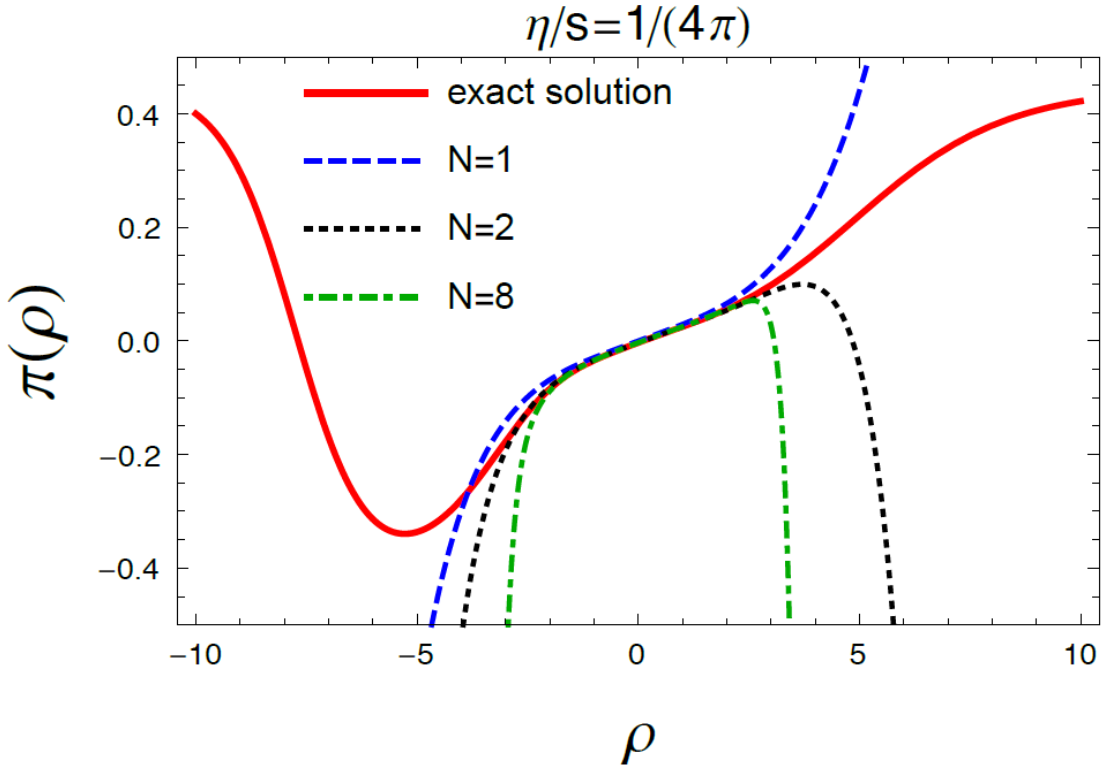


slow-roll expansion

exception: slow-roll converges at $|\rho| \to \infty$

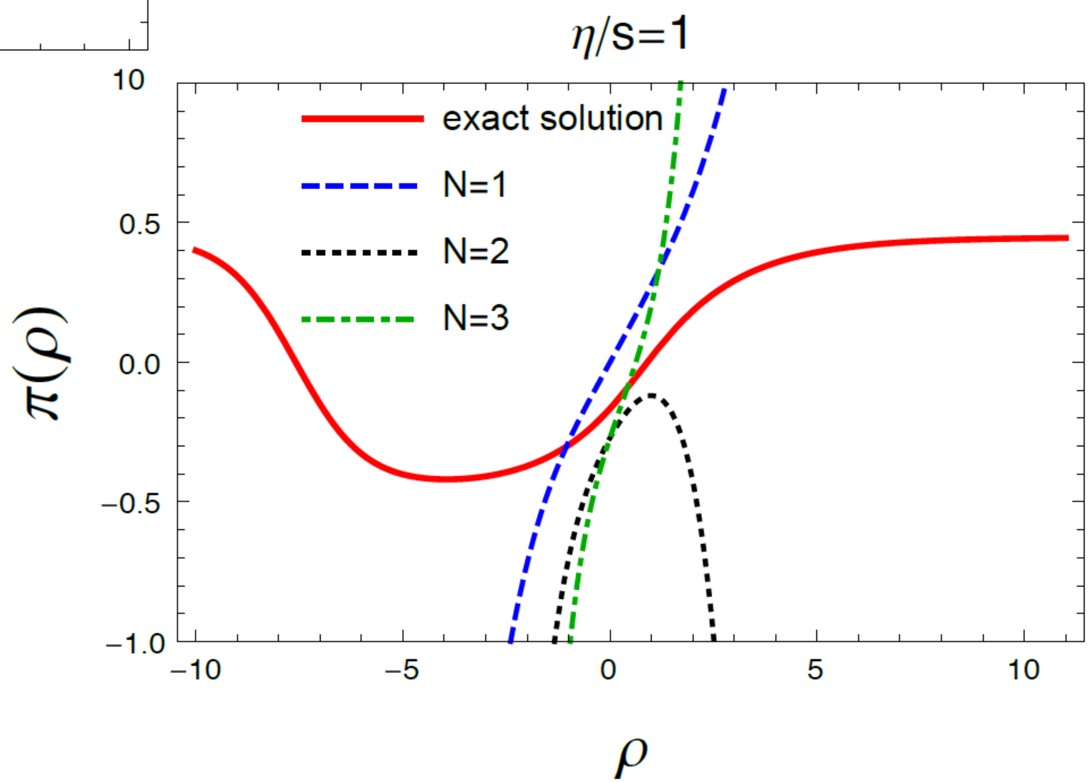
$$\bar{\pi}_0^{\pm} \to \pm \mathrm{sign}\, \rho / \sqrt{5}$$

Asymptotic Solutions in Gubser flow

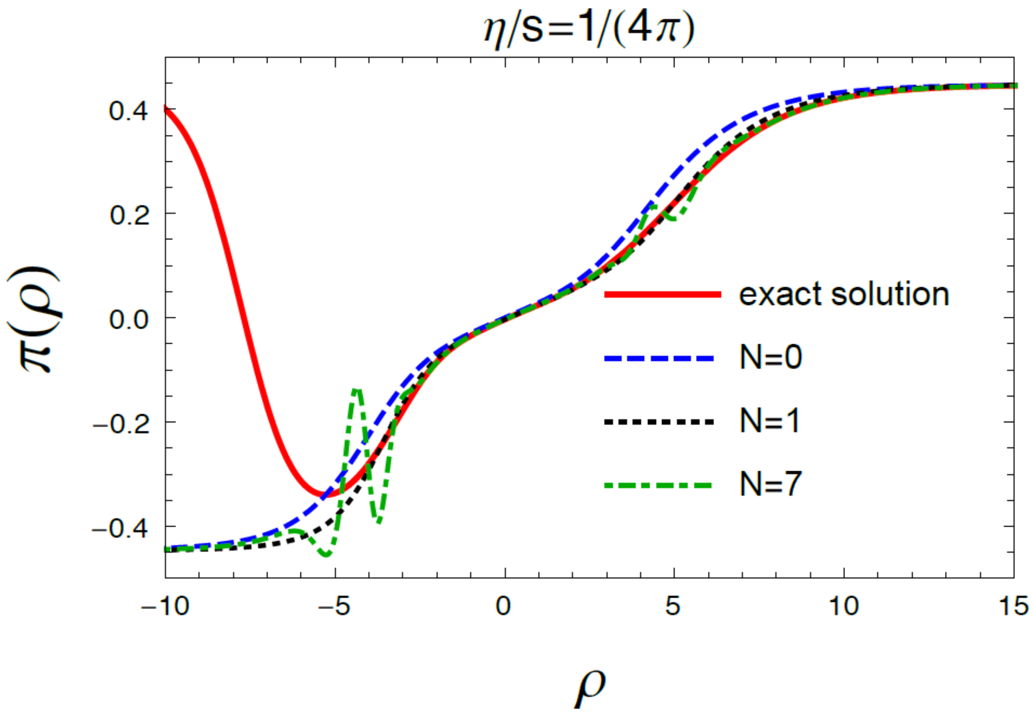


For small viscosities: region around ρ=0 is well described.

larger viscosities: no region is well described

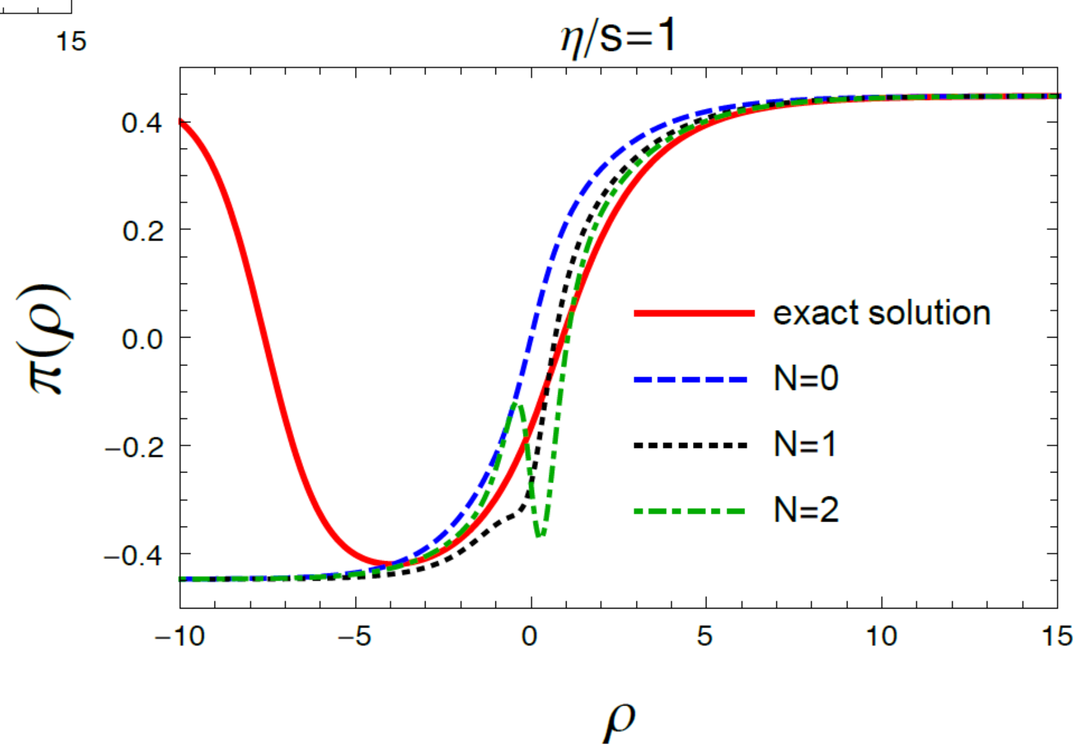


Asymptotic Solutions in Gubser flow



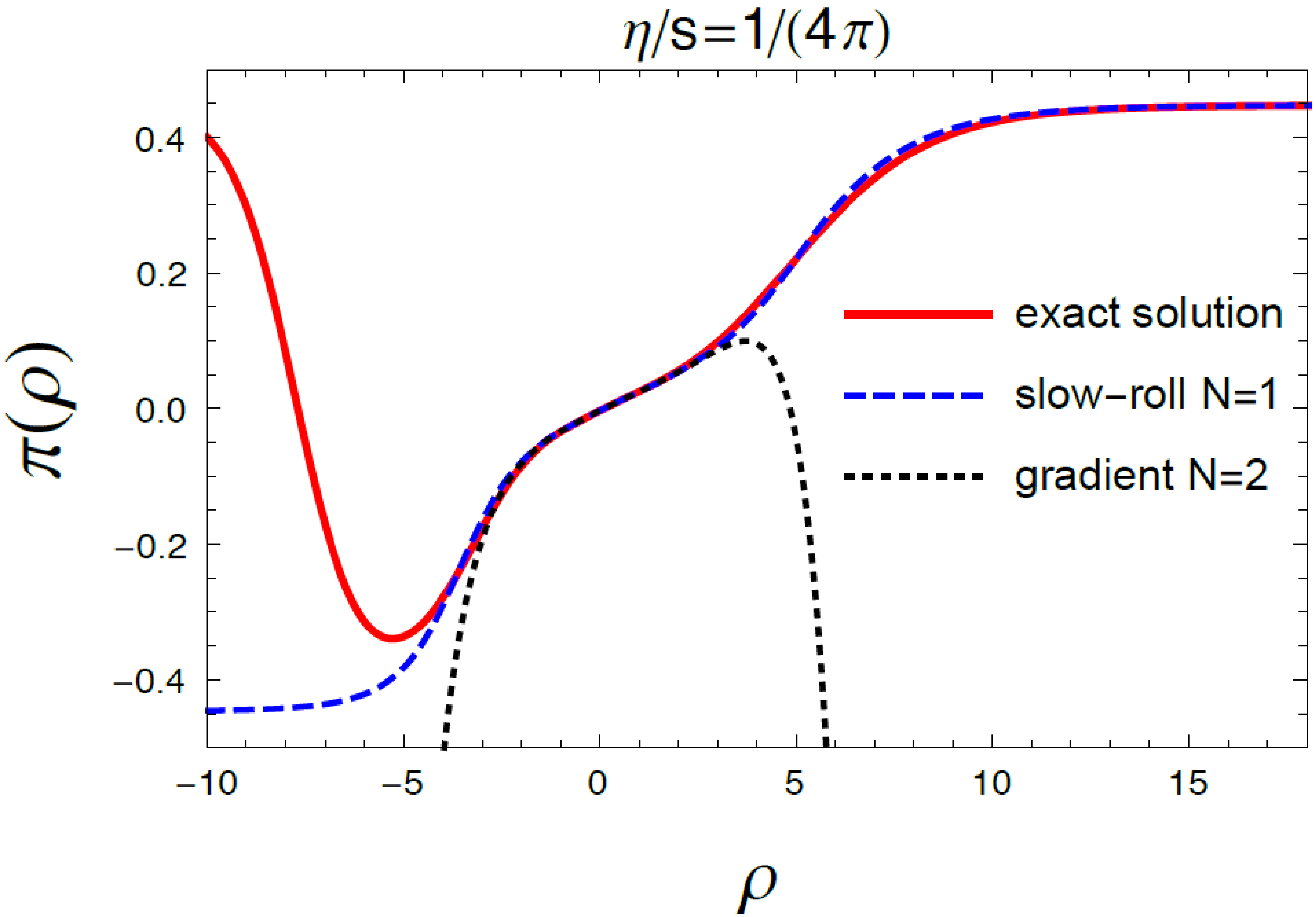
For small viscosities:
region ρ>0 is well
described.

Larger viscosities:
Only large ρ is well
described. But qualitative
description is good.



Asymptotic Solutions in Gubser flow

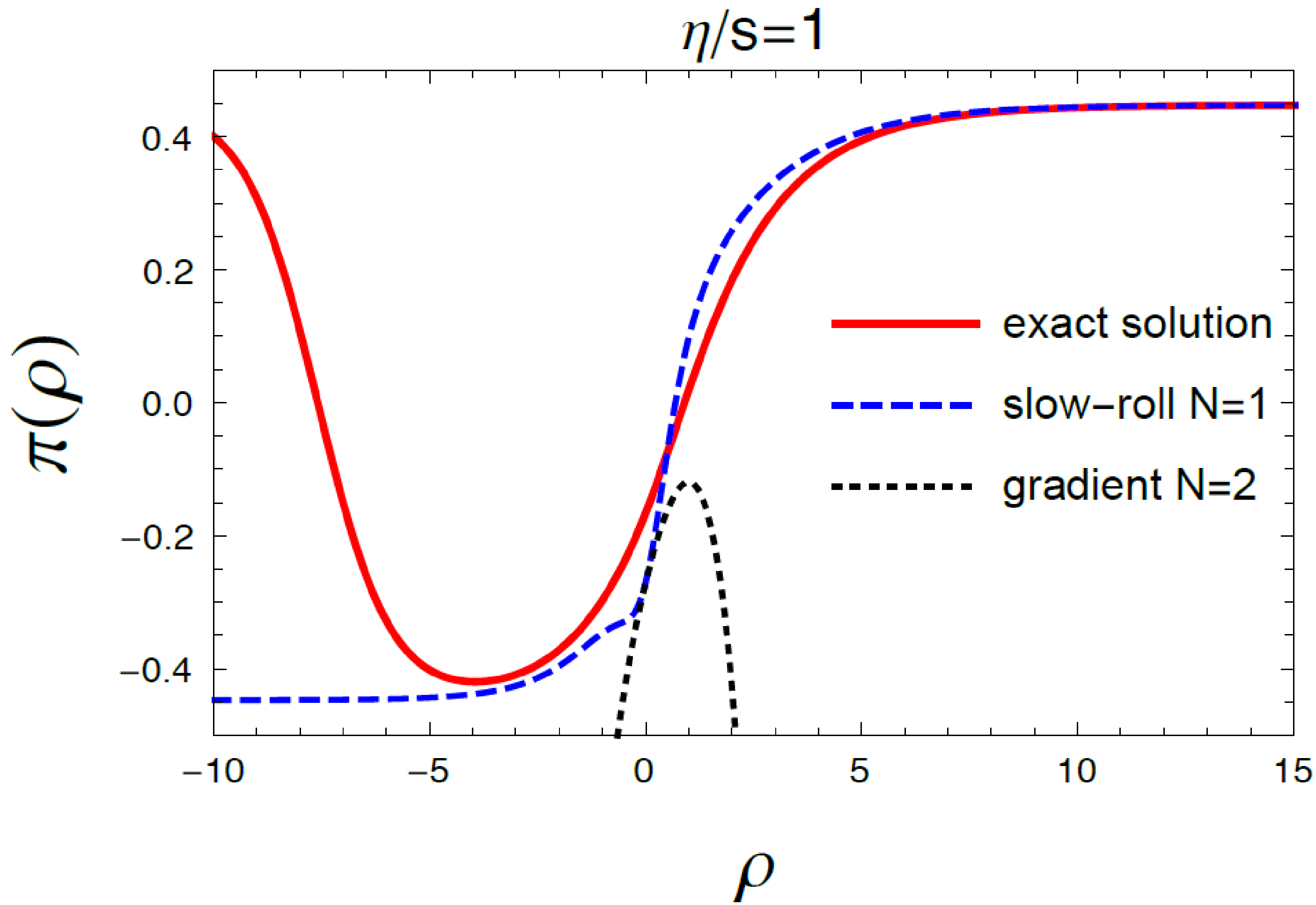
Comparison of optimal truncations



Both series diverge, but the truncated slow-roll Expansion still offers a better description

Asymptotic Solutions in Gubser flow

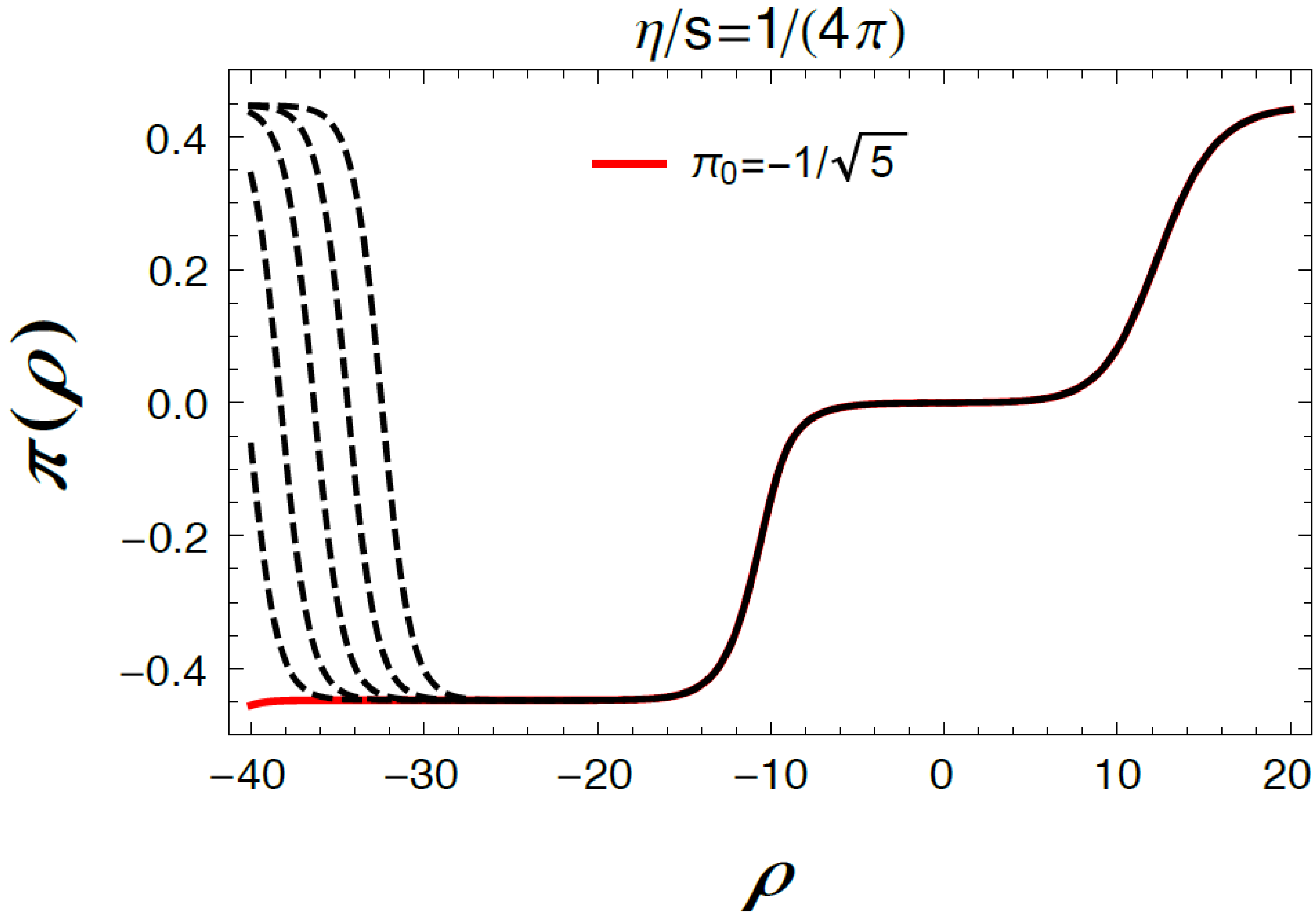
Comparison of optimal truncations



Large viscosity: the truncated slow-roll is only qualitatively good

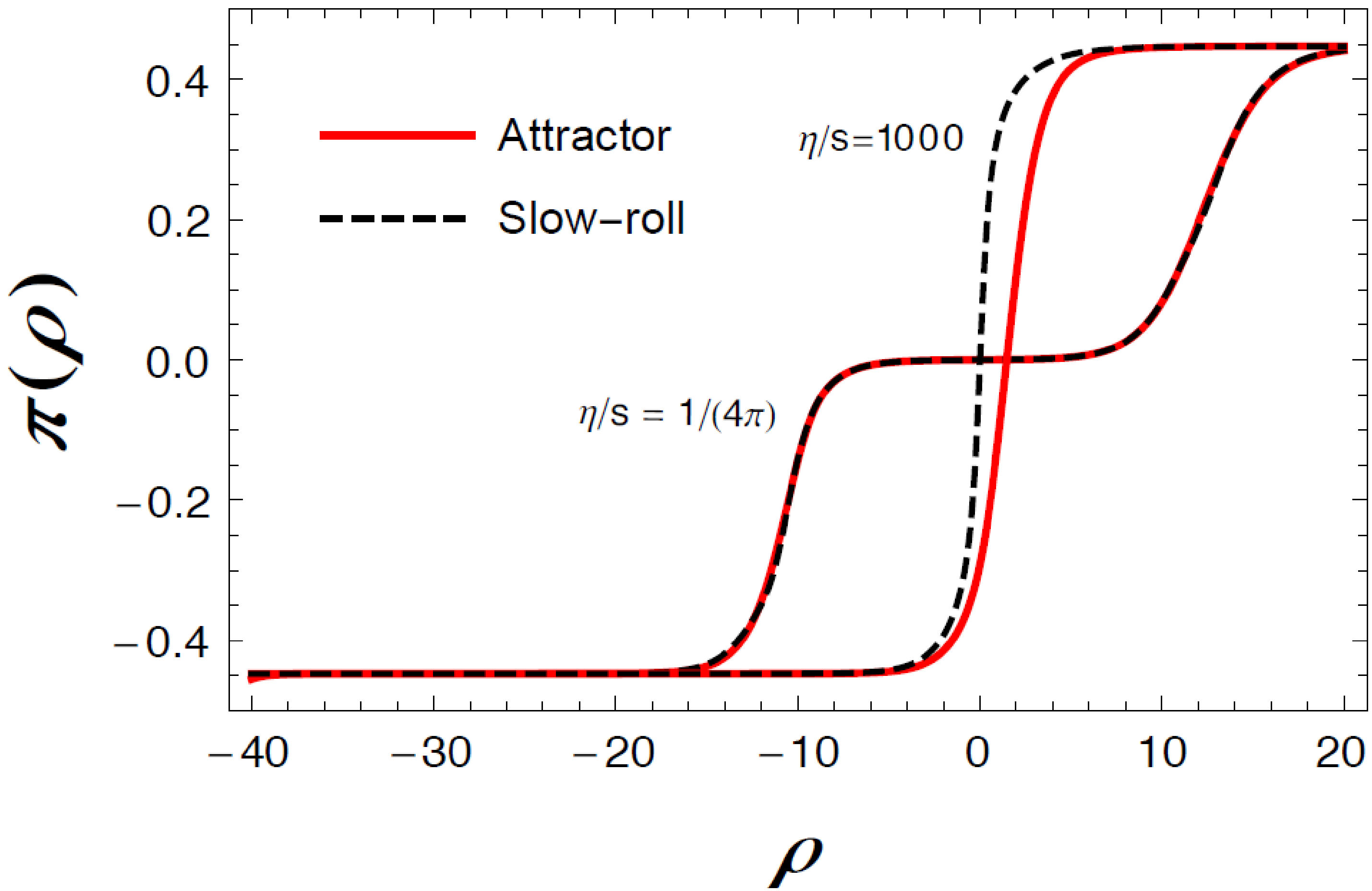
Asymptotic Solutions in Gubser flow

Since the slow expansion converges at |ρ| infinity, we can solve it numerically $\bar{\pi}_0^{\pm} \to \pm\mathrm{sign}\,\rho/\sqrt{5}$



Asymptotic Solutions in Gubser flow

Comparison to truncated (n=0) slow roll

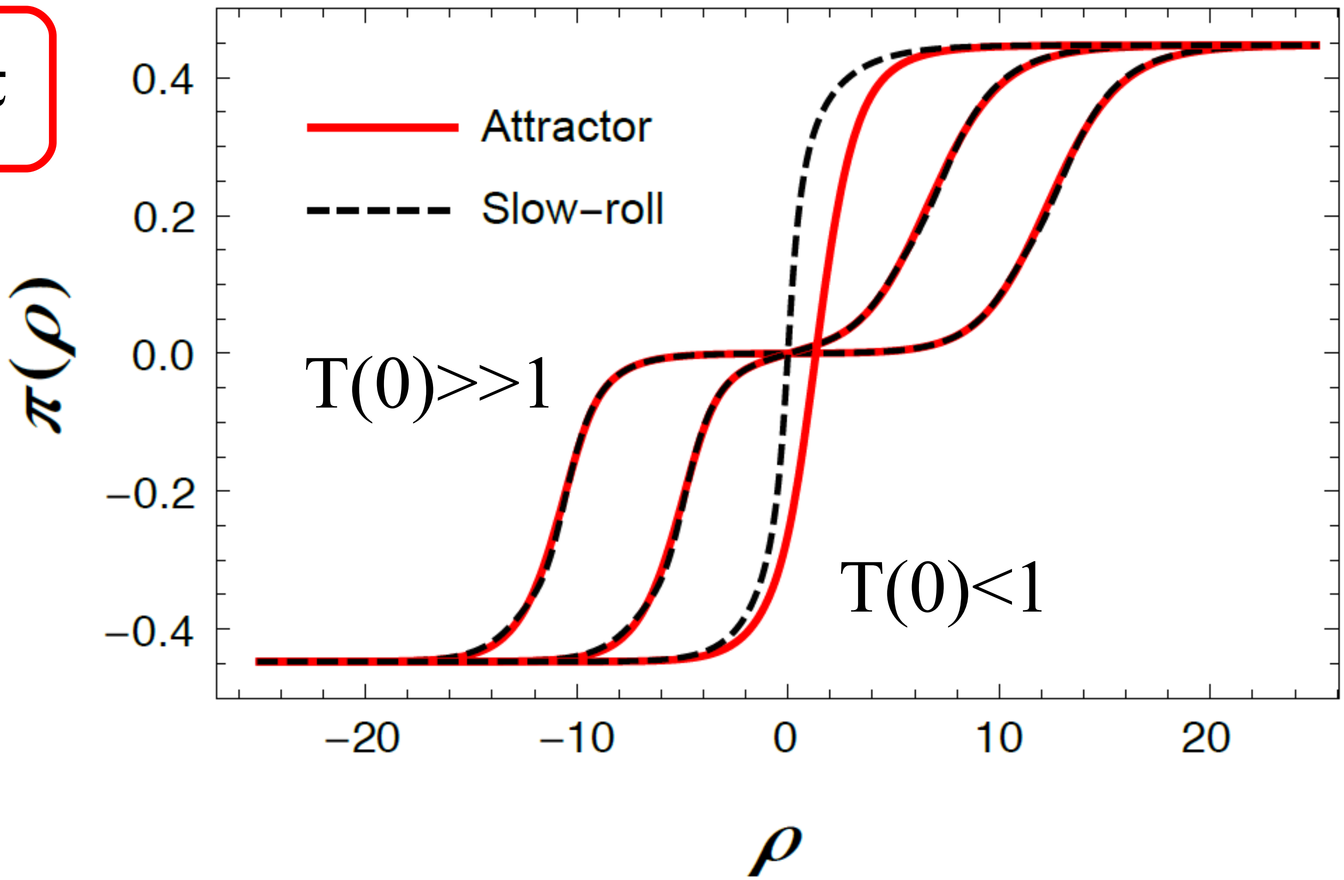


Not that bad ...

Asymptotic Solutions in Gubser flow

Comparison to truncated (n=0) slow roll: fixed viscosity and different initial temperatures

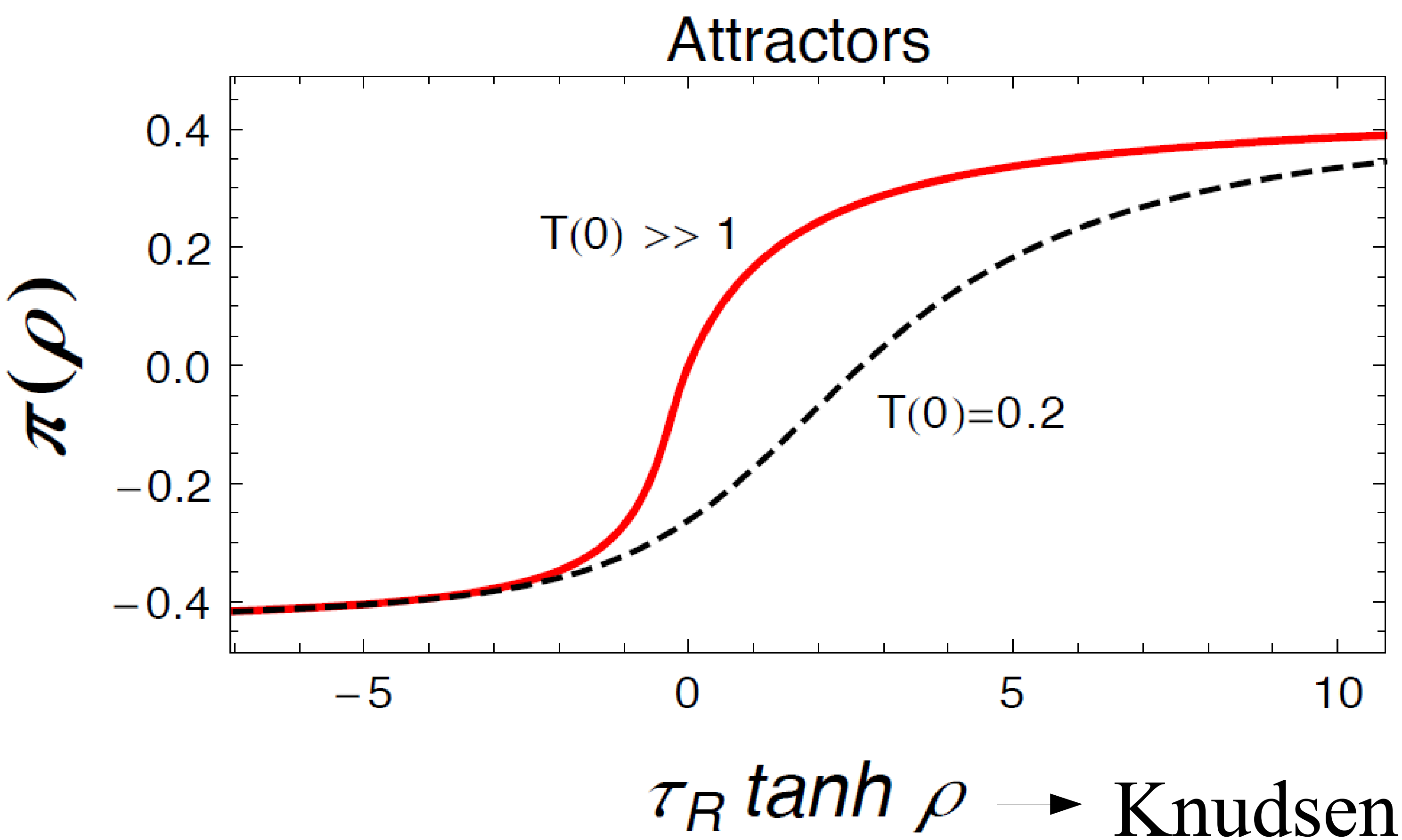
η/s=1/4π



Varying the temperature also changes the attractor

Asymptotic Solutions in Gubser flow

Attractor is no longer univeral, but a set of solutions that depend either on temperature or viscosity



$\rightarrow$ Knudsen number

similar to a constitutive relation

Conclusions

We studied asymptotic solutions of IS theory under Bjorken and Gubser flow

- ➔ CE series diverges both in Bjorken and Gubser flow.
- ➔ Slow roll expansion diverges as well
- ➔ Attractor in Gubser flow is no longer a universal function of the Knudsen number
- ➔ Slow-roll expansion is an improvement over gradient expansion