Comments on thermodynamics in external fields

Pavel Kovtun
University of Victoria

Foundational aspects of relativistic hydrodynamics
ECT* Trento - May 8, 2018

This workshop is about hydrodynamics

To understand hydrodynamics, first understand thermodynamics

Thermodynamics

System in external time-independent $g_{\mu\nu}$, A_μ

Compute $W = -i \ln Z[g_{\mu\nu}, A_\mu]$

Local correlations $\Longrightarrow$ $W[g,A] = \int d^{d+1}x \, \sqrt{-g} \, \mathcal{F}(g,A)$

Near-uniform fields $\Longrightarrow$ expand $\mathcal{F}(g,A)$ in derivatives of g,A

Leading order $\Longrightarrow$ $\mathcal{F}(g,A) = P + O(\partial)$

BBBJMS arXiv:1203.3544
JKKMRY arXiv:1203.3556

Thermodynamic variables

Timelike Killing vector V^μ, e.g. $V^\mu = (1,\mathbf{0})$ for matter "at rest"

$$T = \frac{1}{\beta_0\sqrt{-V^2}}\,, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}\,, \qquad \mu = \frac{V^\mu A_\mu + \Lambda_V}{\sqrt{-V^2}}$$

JLY arXiv:1310.7024

Definition of electric and magnetic fields:

$$F_{\mu\nu} = u_\mu E_\nu - u_\nu E_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma$$

Equilibrium relations

$$u^\lambda \partial_\lambda T = 0\,, \qquad u^\lambda \partial_\lambda \mu = 0$$

things don't depend on time

$$a_\lambda = -\partial_\lambda T / T$$

gravitational potential induces temperature gradient

$$E^\alpha - T\Delta^{\alpha\beta}\partial_\beta\left(\frac{\mu}{T}\right) = 0$$

electric field induces charge gradient: this is electric screening

$$\nabla_\mu u_\nu = -u_\mu a_\nu - \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\, u^\alpha \Omega^\beta$$

shear and expansion vanish in equilibrium

$$a^\mu \equiv u^\lambda \nabla_\lambda u^\mu$$
$$\Omega^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha u_\beta$$

Bound charges and bound currents

$$\delta_{A,F} W = \int d^{d+1}x \sqrt{-g} \left[J_{\rm f}^{\mu} \delta A_{\mu} + \tfrac{1}{2} M^{\mu\nu} \delta F_{\mu\nu} \right]$$

The separation of J_f and M is ambiguous.
But the total current is not:

$$J^{\mu} = J_{\rm f}^{\mu} - \nabla_{\lambda} M^{\lambda\mu}$$

"free current" "bound current"

Can fix the ambiguity by trading $\partial_\alpha\mu$ for E_α.

Then $J_{\rm f}^{\mu} = \rho u^{\mu}$ where $\rho \equiv \partial \mathcal{F} / \partial \mu$

Bound charges and bound currents

Define charge density and spatial current:

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

charge density

spatial current, orthogonal to u_μ

Polarization vectors:

$$M_{\mu\nu} = p_\mu u_\nu - p_\nu u_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho m^\sigma$$

$$\boxed{\begin{aligned} \mathcal{N} &= \rho - \nabla_\mu p^\mu + p^\mu a_\mu - m_\mu \Omega^\mu \\ \mathcal{J}^\mu &= \epsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma} u_\nu a_\rho m_\sigma \end{aligned}}$$

a_μ = acceleration
Ω_μ = vorticity

Bound charges and bound currents

Define charge density and spatial current:

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

charge density

spatial current, orthogonal to u_μ

Polarization vectors:

$$M_{\mu\nu} = p_\mu u_\nu - p_\nu u_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho m^\sigma$$

$$\boxed{\begin{aligned} n &= \rho - \boldsymbol{\nabla}\cdot\mathbf{p} - \mathbf{p}\cdot\boldsymbol{\nabla}T/T - 2\mathbf{m}\cdot\boldsymbol{\omega} \\ \mathbf{J} &= \boldsymbol{\nabla}\times\mathbf{m} + \mathbf{m}\times\boldsymbol{\nabla}T/T \end{aligned}}$$

These were equilibrium charges and currents.

Now need to find equilibrium $T^{\mu\nu}$.

For that, need the derivative expansion.

Derivative expansion

$$W[g,A] = \int \sqrt{-g}\, p + O(\partial)$$

How do we count derivatives?

Clearly, $g_{\mu\nu}$,T~O(1)

In equilibrium, $E^\alpha - T\Delta^{\alpha\beta}\partial_\beta\left(\frac{\mu}{T}\right) = 0$

So if μ~O(1), then E~O(∂). This is screening.

No similar constraint on B, can take B~O(∂) or B~O(1)

Derivative expansion

$$W[g, A] = \int \sqrt{-g}\, p + O(\partial)$$

Weak E, B: p=p(T, μ)

Insulator in strong E, B fields: $p=p(T, E^2, B^2, E\cdot B)$

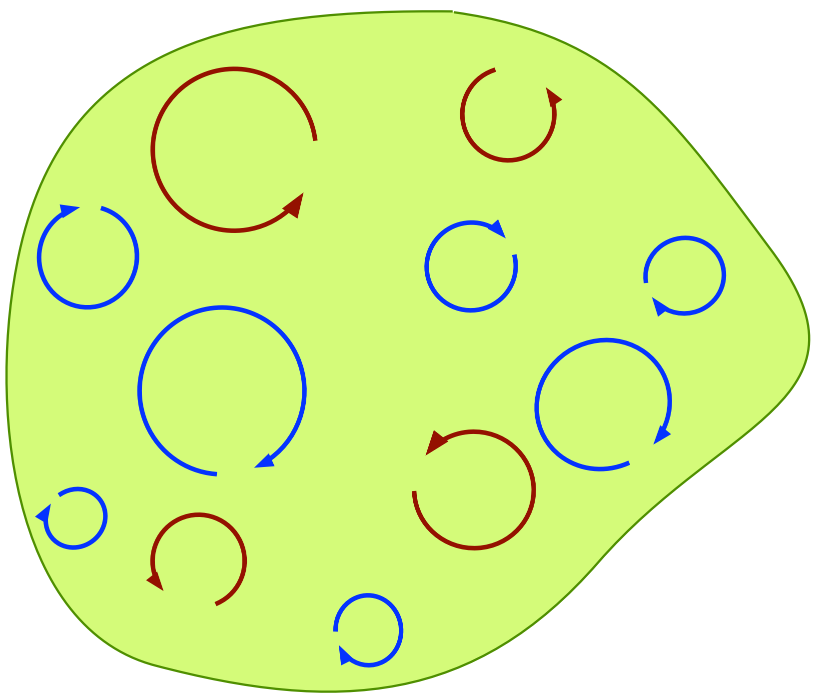
Conductor in strong B-field: $p=p(T, \mu, B^2)$

Example: P-invariant conductor in strong B field

Free energy: $\mathcal{F}(g,A) = p(T,\mu,B^2) + M_\Omega(T,\mu,B^2)\, B\cdot\Omega + O(\partial^2)$

Vary $W[g,A] = \int d^{d+1}x\,\sqrt{-g}\,\mathcal{F}(g,A)$ to find $T^{\mu\nu}$, J^μ

In constant B-field: $T_s^{\mu\nu} = Q_s^\mu u^\nu + Q_s^\nu u^\mu$, $Q_s^\alpha = M_\Omega \epsilon^{\alpha\mu\nu\rho} u_\mu B_\nu n_\rho$



Angular momentum:

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B}$$

Example: P-invariant conductor in strong B field

System at rest
in flat space,
constant B-field:

$$\frac{\mathbf{L}}{V} = 2M_\Omega \mathbf{B}$$

System rotating
in flat space,
no B-field:

$$\mathbf{m} = 2M_\Omega \, \boldsymbol{\omega}$$

Fluid with a global U(1)

$$W[g,A] = \int d^4x\,\sqrt{-g}\left[p(T,\mu) + \sum_n f_n(T,\mu)s_n^{(2)}\right] + \ldots$$

n	1	2	3	4	5	6	7	8	9
$s_n^{(2)}$	R	a^2	Ω^2	B^2	$B\cdot\Omega$	E^2	$E\cdot a$	$B\cdot E$	$B\cdot a$
P	+	+	+	+	+	+	+	−	−
C	+	+	+	+	−	+	−	+	−
T	+	+	+	+	+	+	+	−	−
W	n/a	n/a	2	4	3	4	n/a	4	n/a

Nine thermodynamic susceptibilities $f_n(T,\mu)$, have to be computed from the microscopics, just like $p(T,\mu)$

Fluid with a global U(1)

$$W[g,A] = \int d^4x\,\sqrt{-g}\left[p(T,\mu) + \sum_n f_n(T,\mu) s_n^{(2)}\right] + \dots$$

f_1 : T- and μ-dependent Newton's constant

f_2 : pressure response to $(\boldsymbol{\nabla}T)^2$ [talk by FB earlier today]

f_3 : pressure response to $(\text{vorticity})^2$

$f_{4,6,8}$: magnetic, electric, and magneto-electric suseptibilities

f_5 : magneto-vortical susceptibility, determines **L**~**B**, **m**~**ω**

$f_{7,9}$: pressure response to $\mathbf{E}\cdot\boldsymbol{\nabla}T$, $\mathbf{B}\cdot\boldsymbol{\nabla}T$

Example: no external E,B fields

QCD with $\mu_B \neq 0$: vary W[g,A], get $T^{\mu\nu}$ and J^{μ} in terms of five susceptibilities $f_n(T,\mu)$, n=1,2,3,5,7 besides the pressure $p(T,\mu)$

CFT with $\mu \neq 0$: vary W[g,A], get $T^{\mu\nu}$ and J^{μ} in terms of three susceptibilities $f_n(T,\mu)$, n=1,3,5 besides the pressure $p(T,\mu)$

Various combinations of $f_n(T,\mu)$ and their derivatives in $T^{\mu\nu}$, J^{μ} are often called "thermodynamic transport coefficients".

Can be computed perturbatively, on the lattice, or in AdS/CFT

BRSSS 0712.2451, Romatschke, Son 0903.3946, Moore, Sohrabi 1007.5333, 1210.3340, Arnold, Vaman, Wu, Xiao 1105.4645, Philipsen, Schäfer 1311.6618, Megias, Valle 1408.0165, Finazzo, Rougemont, Marrochio, Noronha 1412.2968, Buzzegoli, Grossi, Becattini 1704.02808

If you really want to see the expressions

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}$$

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\mathcal{E} = \epsilon + (f_1' - f_1)R + (4f_1' + 2f_1'' - f_2 - f_2')a^2 + (f_1' - f_2 - 3f_3 + f_3')\,\Omega^2 - 2(f_1 + f_1' - f_2)\,u^\alpha R_{\alpha\beta} u^\beta\,,$$

$$\mathcal{P} = p + \tfrac{1}{3} f_1 R - \tfrac{1}{3}(2f_1' + f_3)\,\Omega^2 - \tfrac{1}{3}(2f_1' + 4f_1'' - f_2)a^2 + \tfrac{2}{3}(2f_1' - f_1)\,u^\alpha R_{\alpha\beta} u^\beta\,,$$

$$\mathcal{Q}_\mu = (f_1' + 2f_3')\,\epsilon_{\mu\lambda\rho\sigma} a^\lambda u^\rho \Omega^\sigma + (2f_1 + 4f_3)\Delta_\mu^\rho R_{\rho\sigma} u^\sigma\,,$$

$$\mathcal{T}_{\mu\nu} = (4f_1' + 2f_1'' - 2f_2)a_{\langle\mu}a_{\nu\rangle} - \tfrac{1}{2}(f_1' - 4f_3)\,\Omega_{\langle\mu}\Omega_{\nu\rangle} + 2f_1'\,u^\alpha R_{\alpha\langle\mu\nu\rangle\beta} u^\beta - 2f_1 R_{\langle\mu\nu\rangle}\,.$$

$$\mathcal{N} = n + f_{1,\mu} R + (f_{2,\mu} + f_7 + f_7')a^2 + \left(f_{3,\mu} - f_5 + \tfrac{1}{2} f_7\right)\Omega^2 - f_7\,u^\alpha R_{\alpha\beta} u^\beta\,,$$

$$\mathcal{J}^\mu = -(f_5 + f_5')\epsilon^{\mu\nu\rho\sigma} u_\nu a_\rho \Omega_\sigma + 2f_5 \Delta^{\mu\rho} R_{\rho\lambda} u^\lambda\,,$$

$f_n' \equiv T f_{n,T} + \mu f_{n,\mu}$, $f_n'' \equiv T^2 f_{n,T,T} + 2\mu T f_{n,T,\mu} + \mu^2 f_{n,\mu,\mu}$

Kubo formulas

Makes sense to write Kubo formulas for susceptibilities $f_n(T,\mu)$ rather than for thermodynamic transport coefficients

The above $T^{\mu\nu}$ and J^{μ} in terms of $f_n(T,\mu)$ are *automatically* conserved in equilibrium, no need to solve any hydro eqs

Response to a time-independent source gives zero-frequency correlation functions

Know $T^{\mu\nu}[g,A]$ and $J^{\mu}[g,A]$, take the variations w.r.t. $g_{\alpha\beta}$, A_{α}, get Kubo formulas

Kubo formulas

All seven parity-even susceptibilities are given by 2-point functions of $T^{\mu\nu}$ and J^{μ}.

Only need 3-point functions in parity-breaking theories.

Calculate all parity-even susceptibilities on the lattice or in holography?

If you really want to see the expressions

$$f_1 = -\tfrac{1}{2} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} G_{T^{xy}T^{xy}}$$

$$f_2 = \tfrac{1}{4} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} \left(G_{T^{tt}T^{tt}} + 2G_{T^{tt}T^{xx}} - 4G_{T^{xy}T^{xy}} \right)$$

$$f_3 = \tfrac{1}{4} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} \left(G_{T^{tx}T^{tx}} + G_{T^{xy}T^{xy}} \right)$$

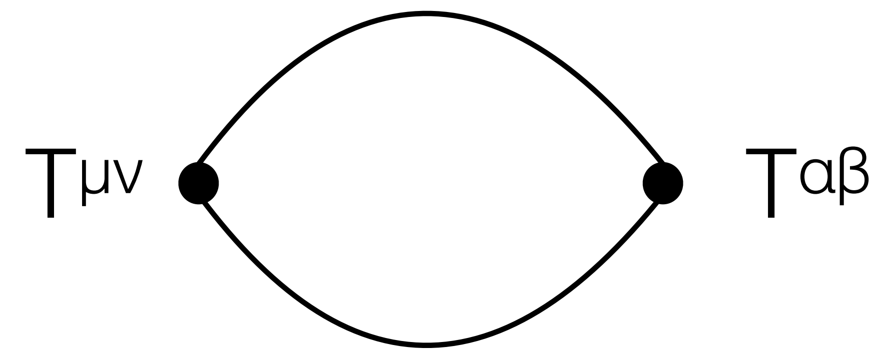
$$f_4 = \tfrac{1}{4} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} G_{J^x J^x} \qquad f_6 = \tfrac{1}{4} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} G_{J^t J^t}$$

$$f_5 = - \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z \partial k_x} G_{J^z T^{tx}} \qquad f_7 = -\tfrac{1}{2} \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial k_z^2} (G_{J^t T^{tt}} + G_{J^t T^{xx}})$$

$$f_8' + f_{9,\mu} = \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial p_y \partial k_z} G_{J^t J^x T^{tt}}(p,k) \qquad f_9 = - \lim_{\mathbf{k}\to 0} \frac{\partial^2}{\partial p_y \partial k_z} G_{J^t T^{tx} T^{tt}}(p,k)$$

Example: free fields

Evaluate the one-loop diagram:



Free massless real scalar:

$$f_1 = \frac{T^2}{144}(1-6\xi)\ , \qquad f_2 = 0\,, \qquad f_3 = -\frac{T^2}{144}\,.$$

Free massless Dirac fermion at μ=0:

$$f_1 = -\frac{T^2}{144}\,, \qquad f_2 = -\frac{T^2}{24}\,, \qquad f_3 = -\frac{T^2}{288}\,.$$

WIP w/ Ashish Shukla

Application: Einstein equations in matter

Equilibrium generating functional $W[g_{\mu\nu}]$ =

Equilibrium effective action $S[g_{\mu\nu}]$

Usual case:

$$S_{\text{eff}}[g] = \int d^{d+1}x \, \sqrt{-g} \left[p(T,\mu) + \frac{1}{16\pi G} R \right] \qquad T = \frac{T_0}{\sqrt{-g_{00}}} \, , \quad \mu = \frac{\mu_0}{\sqrt{-g_{00}}}$$

$\delta_g S_{\text{eff}} = 0 \Longrightarrow$ Einstein equations: $T^{\mu\nu}=0$. Get e.g. Tolman-Oppenheimer-Volkoff equations

Application: Einstein equations in matter

Equilibrium generating functional $W[g_{\mu\nu}]$ =

Equilibrium effective action $S[g_{\mu\nu}]$

Actually have:

$$S_{\text{eff}}[g] = \int d^{d+1}x \,\sqrt{-g}\,\left[p(T,\mu) + f_1(T,\mu)R + f_2(T,\mu)a^2 + f_3(T,\mu)\Omega^2\right]$$

$$f_1 = \frac{1}{16\pi G} + O(T^2, \mu^2)$$

$\delta_g S_{\text{eff}} = 0 \Longrightarrow$ Einstein equations: $T^{\mu\nu}=0$. Get e.g. modified Tolman-Oppenheimer-Volkoff equations due to the pressure response to curvature, modified mass-radius relation

Application: hydro with O(1) external magnetic field

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$ ← diffeomorphism invariance

$$\nabla_\mu J^\mu = 0$$ ← gauge invariance

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + T^{\mu\nu}_{non\text{-}eq}\,, \qquad J^\mu = J^\mu_{eq} + J^\mu_{non\text{-}eq}$$

get from equilibrium $W[g,A]=\int p + O(\partial)$

e.g. $J^\mu_{eq} = \rho u^\mu - \nabla_\lambda M^{\lambda\mu}$

Application: hydro with O(1) external magnetic field

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda$$ ← diffeomorphism invariance

$$\nabla_\mu J^\mu = 0$$ ← gauge invariance

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + T^{\mu\nu}_{non\text{-}eq}\,, \qquad J^\mu = J^\mu_{eq} + J^\mu_{non\text{-}eq}$$

vanish in equilibrium, depend on ∂_μ, B_μ, E_μ, η, ζ, …

Application: hydro with O(1) external magnetic field

For P-invariant conducting fluid in 3+1dim:

- one thermodynamic susceptibility $M_\Omega = f_5$
- two shear viscosities ($\perp$ and || to B)
- three bulk viscosities
- two electrical conductivities ($\perp$ and || to B)
- two Hall viscosities ($\perp$ and || to B)
- one Hall conductivity

Eleven coefficients total:

1 thermodynamic, non-dissipative
3 non-equilibrium, non-dissipative
7 non-equilibrium, dissipative

Application: hydro with O(1) external magnetic field

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \mathcal{T}^{\mu\nu}\,, \quad J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu$$

$$\begin{aligned}
\mathcal{E} &= -p + T\,p_{,T} + \mu\,p_{,\mu} + \left(TM_{\Omega,T} + \mu M_{\Omega,\mu} - 2M_\Omega\right) B{\cdot}\Omega\,,\\
\mathcal{P} &= p - \tfrac{4}{3}\,p_{,B^2}B^2 - \tfrac{1}{3}(M_\Omega + 4M_{\Omega,B^2}B^2)B{\cdot}\Omega - \zeta_1\nabla{\cdot}u - \zeta_2 b^\mu b^\nu \nabla_\mu u_\nu\,,\\
\mathcal{Q}^\mu &= -M_\Omega \epsilon^{\mu\nu\rho\sigma}u_\nu \partial_\sigma B_\rho + (2M_\Omega - TM_{\Omega,T} - \mu M_{\Omega,\mu})\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma T/T\\
&\quad - M_{\Omega,B^2}\epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho \partial_\sigma B^2 + (-2p_{,B^2} + M_{\Omega,\mu} - 2M_{\Omega,B^2}B{\cdot}\Omega)\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma\\
&\quad + M_\Omega \epsilon^{\mu\nu\rho\sigma}\Omega_\nu E_\rho u_\sigma\,,\\
\mathcal{T}^{\mu\nu} &= 2p_{,B^2}\left(B^\mu B^\nu - \tfrac{1}{3}\Delta^{\mu\nu}B^2\right) + M_{\Omega,B^2}B^{\langle\mu}B^{\nu\rangle}B{\cdot}\Omega + M_\Omega B^{\langle\mu}\Omega^{\nu\rangle}\\
&\quad - \eta_\perp \sigma_\perp^{\mu\nu} - \eta_\parallel(b^\mu\Sigma^\nu + b^\nu\Sigma^\mu) - b^{\langle\mu}b^{\nu\rangle}\left(\eta_1\nabla{\cdot}u + \eta_2 b^\alpha b^\beta \nabla_\alpha u_\beta\right)\\
&\quad - \tilde\eta_\perp \tilde\sigma_\perp^{\mu\nu} - \tilde\eta_\parallel(b^\mu\tilde\Sigma^\nu + b^\nu\tilde\Sigma^\mu)\,,\\
\mathcal{N} &= p_{,\mu} + M_{\Omega,\mu}B{\cdot}\Omega - m{\cdot}\Omega\,,\\
\mathcal{J}^\mu &= \epsilon^{\mu\nu\rho\sigma}u_\nu \nabla_\rho m_\sigma + \epsilon^{\mu\nu\rho\sigma}u_\nu a_\rho m_\sigma + \left(\sigma_\perp \mathbb{B}^{\mu\nu} + \sigma_\parallel \frac{B^\mu B^\nu}{B^2}\right)V_\nu + \tilde\sigma\,\tilde V^\mu
\end{aligned}$$

$$\begin{aligned}
\Delta^{\mu\nu} &\equiv g^{\mu\nu} + u^\mu u^\nu \qquad b^\mu \equiv B^\mu/B\\
\sigma^{\mu\nu} &\equiv \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \tfrac{2}{3}\Delta_{\alpha\beta}\nabla{\cdot}u)\\
\tilde\sigma^{\mu\nu} &\equiv \frac{1}{2B}\left(\epsilon^{\mu\lambda\alpha\beta}u_\lambda B_\alpha \sigma_\beta{}^\nu + \epsilon^{\nu\lambda\alpha\beta}u_\lambda B_\alpha \sigma_\beta{}^\mu\right)\\
\mathbb{B}^{\mu\nu} &\equiv \Delta^{\mu\nu} - b^\mu b^\nu \qquad \Sigma^\mu \equiv \mathbb{B}^{\mu\lambda}\sigma_{\lambda\rho}b^\rho\\
V^\mu &\equiv E^\mu - T\Delta^{\mu\nu}\partial_\nu(\mu/T)\\
\tilde v^\mu &\equiv \epsilon^{\mu\nu\rho\sigma}u_\nu B_\rho v_\sigma/B\\
m^\mu &= \left(2\,p_{,B^2} + 2M_{\Omega,B^2}B{\cdot}\Omega\right)B^\mu + M_\Omega \Omega^\mu
\end{aligned}$$

* In thermodynamic frame, up to O(∂)

Application: hydro with O(1) external magnetic field

Inequality constraints on η's, ζ's, σ's from 2-nd law

Equality constraints on η's, ζ's, σ's from Onsager relations

Eigenmodes: collective cyclotron modes, sound, diffusion,…

Express η's, ζ's, σ's in terms of $\langle T_{\mu\nu} T_{\alpha\beta} \rangle$, $\langle T_{\mu\nu} J_{\alpha} \rangle$, $\langle J_{\mu} J_{\alpha} \rangle$

Transport coefficients for P-violating fluids

Hernandez, PK 1703.08757

Huang, Sedrakian, Rischke 1108.0602

Finazzo, Rougemont, Marrochio, Noronha 1412.2968

Application: Maxwell equations in matter

Equilibrium generating functional $W[g_{\mu\nu}, A_\mu]$ =

Equilibrium effective action $S[g_{\mu\nu}, A_\mu]$

In the vacuum:

$$S_{\text{eff}}[g, A] = \int d^{d+1}x \, \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu}^2 \right]$$

$\delta_A S_{\text{eff}} = 0 \Longrightarrow$ Maxwell equations: $J^\mu=0$, or $\nabla_\nu F^{\mu\nu} = 0$.

Application: Maxwell equations in matter

Equilibrium generating functional $W[g_{\mu\nu}, A_\mu]$ =

Equilibrium effective action $S[g_{\mu\nu}, A_\mu]$

In matter:

$$S_{\text{eff}}[g, A] = \int d^{d+1}x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu}^2 + \mathcal{F}_{\text{m}}[T, \mu, E^2, B^2, B\cdot\Omega, \dots] \right]$$

$\delta_A S_{\text{eff}} = 0 \Longrightarrow$ Maxwell equations: $J^\mu = 0$, or $\nabla_\nu H^{\mu\nu} = n u^\mu$.

$$H^{\mu\nu} \equiv F^{\mu\nu} - M_{\text{m}}^{\mu\nu}$$

$$n \equiv \partial \mathcal{F}_{\text{m}} / \partial \mu$$

Application: Maxwell equations in matter

Equilibrium generating functional $W[g_{\mu\nu}, A_\mu]$ =

Equilibrium effective action $S[g_{\mu\nu}, A_\mu]$

Equations to solve:

$$\nabla_\mu T^{\mu\nu} = F^{\lambda\nu} J_{\mathrm{ext}\,\lambda}\,,$$

$$J^\mu + J^\mu_{\mathrm{ext}} = 0\,,$$

$$\epsilon^{\mu\nu\alpha\beta}\nabla_\nu F_{\alpha\beta} = 0\,.$$

This is relativistic MHD, with 11 transport coefficients

MHD vs hydro in external B-field

- MHD has the same 11 transport coef-s (7 are dissipative)

- MHD has the same entropy current

- MHD has the same Kubo formulas for viscosities

- MHD has *different* Kubo formulas for conductivities

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\mathrm{ret.}}_{E_z E_z}(\omega,\mathbf{k}{=}0) = \rho_{\|}$$

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\mathrm{ret.}}_{E_x E_x}(\omega,\mathbf{k}{=}0) = \rho_{\perp}$$

$$\frac{1}{\omega}\,\mathrm{Im}\,G^{\mathrm{ret.}}_{E_x E_y}(\omega,\mathbf{k}{=}0) = -\tilde{\rho}_{\perp}\mathrm{sign}(B_0)$$

$$\sigma_{ab} \equiv \sigma_{\perp}\delta_{ab} + \tilde{\sigma}\epsilon_{ab}$$

$$(\sigma^{-1})_{ab} = \rho_{\perp}\delta_{ab} + \tilde{\rho}_{\perp}\epsilon_{ab}$$

$$\rho_{\|} \equiv 1/\sigma_{\|}$$

- MHD has *different* eigenmodes (e.g. Alfven waves)

Hernandez, PK arXiv:1703.08757

Questions

Can get all 2-nd order thermodynamic transport coef-s in QCD from Euclidean 2-point functions. Lattice & AdS?

Well-posedness of MHD a la Israel-Stewart? [See Dirk's talk on Friday.] DHMMNNRW 1804.05210

Transport coef-s in B-field at weak vs strong coupling? Physical implications?

Statistical fluctuations, aggravated by the B-field?

There is a "dual" formulation of MHD in terms of the magnetic flux. Relation to "conventional" MHD underexplored.

Grozdanov, Hofman, Iqbal 1610.07392

Thank you!