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Exact solution for an accelerated relativistic fluid

Motivation:

Find the exact form of the stress-energy tensor at thermodynamic equilibrium

ECT*, FARH workshop
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THERMODYNAMIC EQUILIBRIUM IN SPECIAL RELATIVITY

General covariant form of the density operator

$$\hat{\rho} = \frac{e^{-\int d\Sigma_\mu \hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu}}{Z}$$

Equilibrium : $\nabla_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \zeta \hat{j}^\mu \right) = 0 \implies \boxed{\begin{array}{l} \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0 \\ \partial_\mu \zeta = 0 \end{array}}$

Flat spacetime : $\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu$

constant t.l. vector

constant antisymm. tensor

thermal vorticity

$$\Rightarrow \quad \boxed{\hat{\rho} = e^{-b_\mu \hat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\hat{J}^{\mu\nu}}}$$

10 constants
10 generators
10 independent Killing vectors

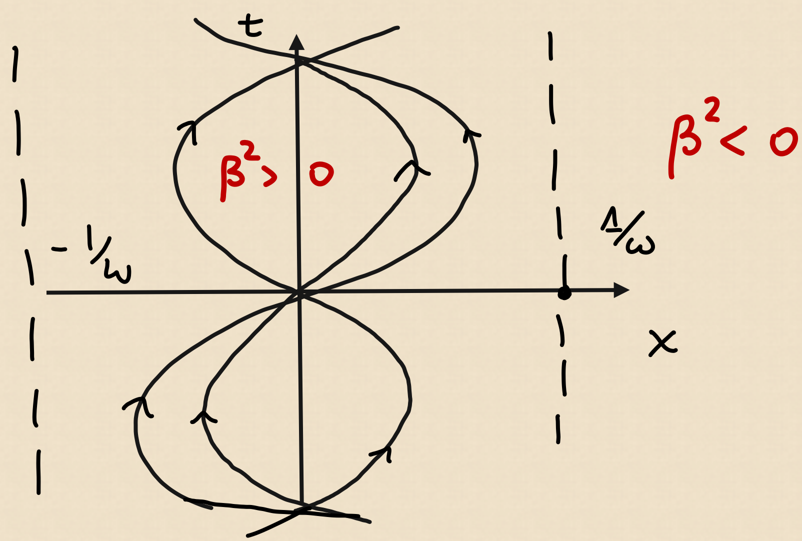
EXAMPLE: ROTATION

$$b = \frac{1}{T_0}(1, \underline{0}) \qquad \varpi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega/T_0 & 0 \\ 0 & -\omega/T_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta = \frac{1}{T_0}\left(1, \vec{\omega} \times \vec{r}\right)$$

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{H}/T_0 + \omega/T_0 \hat{J}_z}$$

$$\frac{dx^\mu}{d\tau} = \beta^\mu \qquad \text{field lines}$$



NOTE: $\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$

implies $\nabla^\perp_\mu u_\nu + \nabla^\perp_\nu u_\mu = 0$

Shear $= 0$

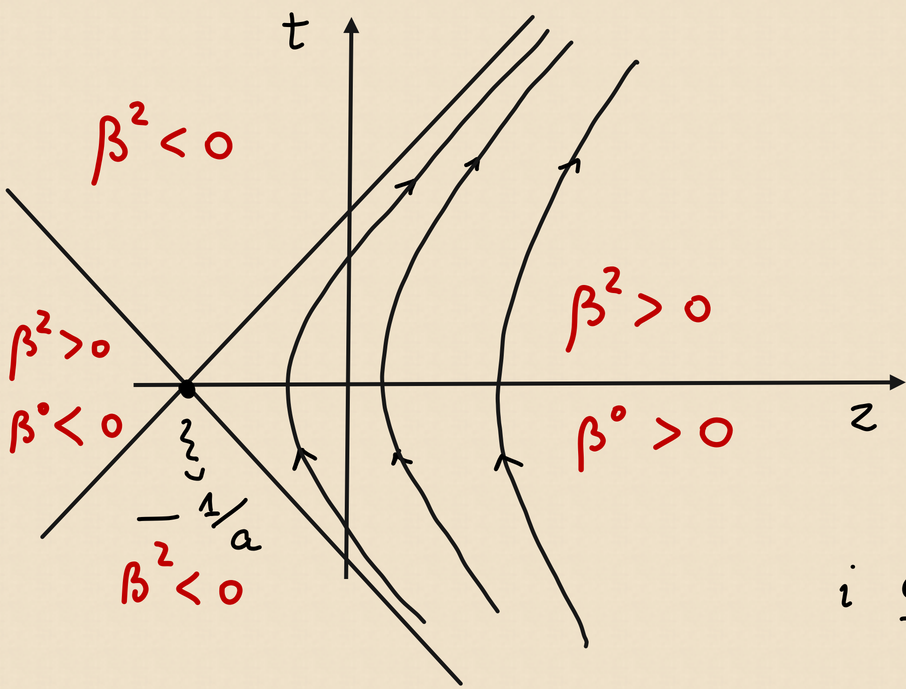
2nd EXAMPLE: ACCELERATION

$$b = \frac{1}{T_0}(1, \underline{0}) \qquad \varpi = \begin{pmatrix} 0 & 0 & 0 & a/T_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a}{T_0} & 0 & 0 & 0 \end{pmatrix}$$

$$\beta^\mu = \frac{1}{T_0}\left(1 + az, 0, 0, at\right)$$

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{H}/T_0 + a/T_0 \hat{K}_z}$$

$$T = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{ka} \qquad k = \sqrt{(1+az)^2 - t^2 a^2}$$



$$A^\mu = \frac{1}{k^2}\left(at, 0, 0, 1 + az\right)$$

$$A^2/T^2 = -\, a^2/T_0^2 = \text{cost}$$

NOTE:

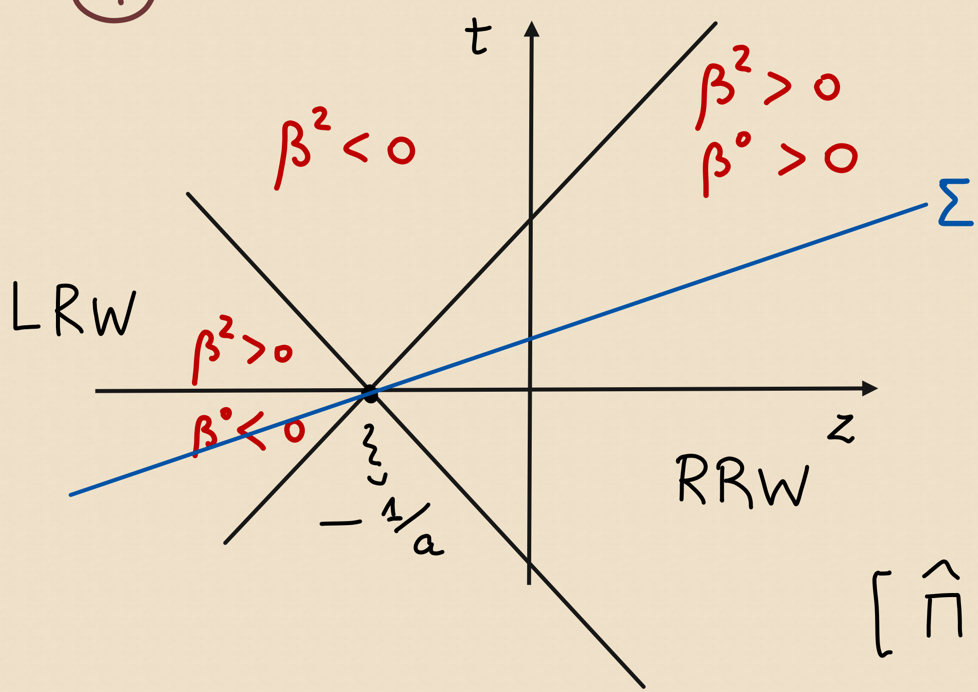
$$i\,\frac{d\hat{K}_t}{dt} = \left[\hat{K}_z, \hat{H}\right] + i\,\frac{\partial \hat{K}_t}{\partial t} = -i\,\hat{P}_z + i\,\hat{P}_t = 0$$

SOLUTION OF ACCELERATED TH. EQ.

F.B., Phys. Rev. D 97, 085013 (2018)   1712.08031

$\mathrm{tr}(\hat\rho\,\widehat{\mathrm{Anything}})$ can be calculated exactly for a free field theory in the Right Rindler Wedge

1



$$\frac{\hat H}{T_0} - \frac{a}{T_0}\hat K_z = \int d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu =$$

$$= \int_{z>-1/a} d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu + \int_{z<-1/a} d\Sigma_\mu \hat T^{\mu\nu}\beta_\nu$$

$$\quad\ \equiv \frac{\hat\Pi_R}{T_0} \qquad\qquad \equiv -\frac{\hat\Pi_L}{T_0}$$

$$[\hat\Pi_R, \hat\Pi_L] = 0$$

because $\beta = 0$ in $\begin{cases} z = -1/a \\ t = 0 \end{cases}$!

2

$$\hat{\rho} = \frac{e^{-\hat{H}/T_0 + a/T_0 \hat{K}_z}}{Z} = \hat{\rho}_R \otimes \hat{\rho}_L = \frac{e^{-\hat{\Pi}_R/T_0}}{Z_R} \otimes \frac{e^{\hat{\Pi}_L/T_0}}{Z_L}$$

Therefore: $\langle \hat{O}(x) \rangle_{RRW} = \mathrm{tr}(\hat{\rho}_R \hat{O}(x)) = \mathrm{tr}\left(\frac{e^{-\hat{\Pi}_R/T_0} \hat{O}(x)}{Z_R} \right)$

3 SOLVE KLEIN-GORDON EQUATION IN RINDLER COORDINATES

L. Crispino, A. Higuchi, G. Matsas, Rev. Mod. Phys. 80 787 (2008)

$$\hat{\psi}(RRW) = \int_0^\infty d\omega \, d^2k_T \left[u_{\omega \vec{k}_T}(x) \, a_R(\omega \vec{k}_T) + h.c. \right]$$

Real scalar field

$$\left[a_R(\omega, \vec{k}_T), a_R^\dagger(\omega', \vec{k}_T') \right] = \delta(\omega - \omega') \, \delta^2(\vec{k}_T - \vec{k}_T')$$

$$t = \frac{e^{a\xi}}{a} \sinh(a\tau)$$

$$u_{\omega \vec{k}_T}(x) = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} \, K_{\frac{i\omega}{a}}\left(\frac{m_T e^{a\xi}}{a} \right) e^{-i\omega\tau} e^{i\vec{k}_T \cdot \vec{x}_T}$$

$$z + 1/a = \frac{e^{a\xi}}{a} \cosh(a\tau)$$

NOTE: a_R, a_L can be written as linear combinations of $a(p)$, $a^\dagger(p)$; the so-called Bogoliubov relations

(4) Show that $\hat{\Pi}_{\substack{R\\L}} = \frac{1}{2}\int_0^\infty d\omega \int d^2k_T\, \omega\left(a^\dagger_{\substack{R\\L}} a_{\substack{R\\L}} + a_{\substack{R\\L}} a^\dagger_{\substack{R\\L}}\right)$

$\Downarrow$

$$tr\left(\hat{\rho}\, a^\dagger_R(\omega,\vec{k}_T)\, a_R(\omega',\vec{k}'_T)\right) = \langle a^\dagger_R(\omega,\vec{k}_T)\, a_R(\omega',\vec{k}'_T)\rangle = \frac{\delta(\omega-\omega')\,\delta^2(\vec{k}_T-\vec{k}'_T)}{e^{\omega/T_0}-1}$$

$\langle a^\dagger_L a_L \rangle = +\infty$ We don't care

Any quadratic operator in the RRW

$$\langle A\hat{\varphi}(x)\, B\hat{\varphi}(x)\rangle = \int_0^\infty d\omega \int d^2\vec{k}_T\; f_{\omega,\vec{k}_T}(x)\,\frac{1}{e^{\omega/T_0}-1} + f^*_{\omega\vec{k}_T}(x)\left(\frac{1}{e^{\omega/T_0}-1}+1\right)$$

⑤ RENORMALIZATION (FOR THE FREE FIELD)

Usual prescription for the stress-energy tensor $\langle :\hat{T}^{\mu\nu}(x): \rangle$

But $: \; :_{\text{Rindler}} \neq \; : \; :_{\text{Minkowski}} \iff |0_R\rangle \neq |0_M\rangle$

Correct prescription $\langle \hat{T}^{\mu\nu} \rangle_{\text{Ren}} = \langle \hat{T}^{\mu\nu} \rangle - \langle 0_M | \hat{T}^{\mu\nu} | 0_M \rangle$

Unruh effect : (Bisognano Wichmann 1975)

$$\langle 0_M | A \hat{\varphi} B \hat{\varphi} | 0_M \rangle = \mathrm{tr}\left(\frac{e^{-\frac{2\pi}{a}(\hat{\Pi}_R - \hat{\Pi}_L)}}{Z} A \hat{\varphi} B \hat{\varphi} \right)$$

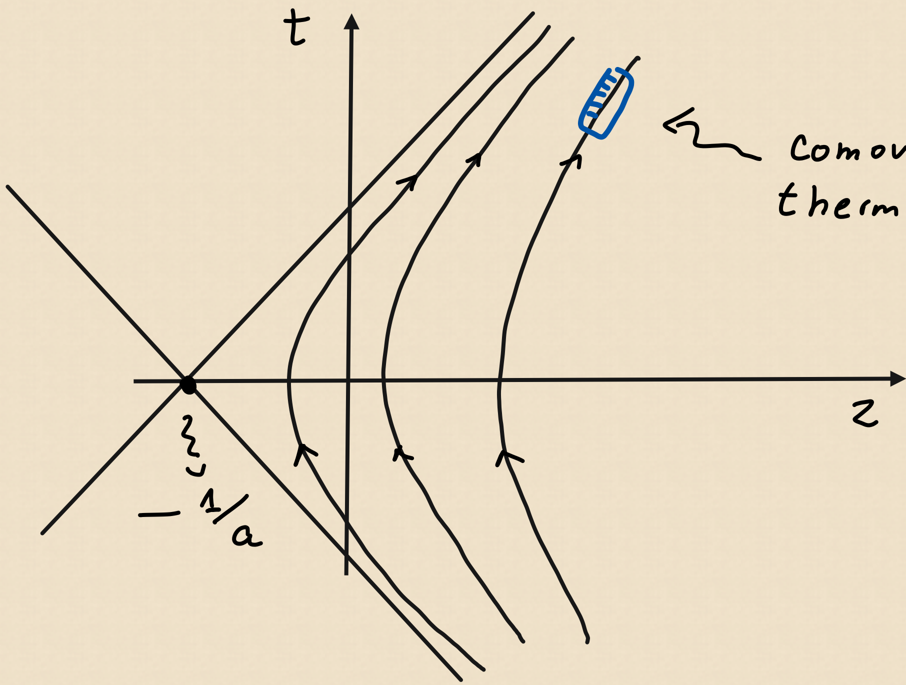
$$\langle A \hat{\varphi} B \hat{\varphi} \rangle_{\text{Ren}} = \int_0^\infty d\omega \int d^2 k_T \left[f_{\omega \vec{k}_T}(x) + f_{\omega \vec{k}_T}(x)^* \right] \left(\frac{1}{e^{\omega/T_0} - 1} - \frac{1}{e^{\omega/2\pi/a} - 1} \right)$$

!

Consequence:

$$T_0 = \frac{a}{2\pi} \implies T^{\mu\nu}_{Ren} = j^{\mu}_{Ren} = \dots = O(x)_{Ren} = 0$$

$$T_0 > \frac{a}{2\pi} = T_U \qquad \text{Since } \frac{|A^2|}{T^2} = \frac{a^2}{T_0^2} \implies T_{local} > \frac{|A|}{2\pi}$$



comoving thermometer

$T = \frac{|A|}{2\pi}$ in Vacuum

From now on

$$\alpha^{\mu} = \frac{A^{\mu}}{T} \qquad \alpha^2 = \frac{|A|^2}{T^2} = |\alpha|^2$$

THERMAL EXPECTATION VALUES

F.B., M. Buzzegoli, in preparation

$$\hat{T}^{\mu\nu}_{Can} = \partial^\mu \hat{\varphi}^+ \partial^\nu \hat{\varphi} + \partial^\nu \hat{\varphi}^+ \partial^\mu \hat{\varphi} - g_{\mu\nu} \left(\partial \hat{\varphi}^+ \cdot \partial \varphi - m^2 \hat{\varphi}^+ \hat{\varphi} \right)$$

$$= \partial^\mu \hat{\varphi}^+ \partial^\nu \hat{\varphi} + \partial^\nu \hat{\varphi}^+ \partial^\mu \hat{\varphi} - g_{\mu\nu} \frac{1}{2} \Box \left(\hat{\varphi}^+ \hat{\varphi} \right)$$

on-shell

$$u_\mu u_\nu \hat{T}^{\mu\nu} = 2 (u \cdot \partial) \hat{\varphi}^+ (u \cdot \partial) \hat{\varphi} - \frac{1}{2} \Box \left(\hat{\varphi}^+ \hat{\varphi} \right)$$

$$\Rightarrow \langle u_\mu u_\nu \hat{T}^{\mu\nu} \rangle_{Ren} \equiv \varepsilon = 2 \left\langle \left(\frac{d \hat{\varphi}}{d\tau} \right)^2 \right\rangle_{Ren} - \frac{1}{2} \Box \langle \hat{\varphi}^+ \hat{\varphi} \rangle_{Ren}$$

Can be calculated from the known field expansion exactly for $m = 0$

Result:

$$\boxed{\varepsilon = \frac{\pi^2}{15} T^4 \left(1 + \frac{5}{2\pi^2}\alpha^2 \right) - \frac{\pi^2}{15} T_U^4 \left(1 + \frac{5\,\alpha_U^2}{2\pi^2} \right)} \qquad \alpha^2 = \frac{|A|^2}{T^2}$$

Note: $\alpha_U = 2\pi \quad T_U = \frac{|A|}{2\pi} = \frac{T}{2\pi}\alpha$, the energy density can be written as:

$$\boxed{\varepsilon = \frac{\pi^2}{15} T^4 \left(1 + \frac{5\alpha^2}{2\pi^2} - \frac{11}{16}\frac{\alpha^4}{\pi^4} \right)}$$

Vanishes in the Minkowski vacuum $\alpha = 2\pi$

quantum corrections $\alpha = \frac{\hbar A}{c k T}$

The coefficient of α^2 coincides with that calculated in the quadratic expansion of $\hat{\rho}$ { F.B., E. Grossi PRD 92 (2015) 045037; M. Buzzegoli, E. Grossi, F.B. JHEP 1710 (2017) 91 }

STRESS-ENERGY TENSOR

$$\boxed{\begin{aligned} T^{\mu\nu} &= (\varepsilon + p) u^\mu u^\nu - p\, g^{\mu\nu} + A\, \hat{\alpha}^\mu \hat{\alpha}^\nu \\ \varepsilon &= \frac{\pi^2}{15} T^4 \left(1 + \frac{5\alpha^2}{2\pi^2} - \frac{11}{16} \frac{\alpha^4}{\pi^4} \right) \\ p &= \frac{\pi^2}{45} T^4 \left(1 - \frac{5\alpha^2}{\pi^2} + \frac{19\alpha^4}{16\pi^4} \right) \\ A &= T^4 \left(\frac{\alpha^2}{6} - \frac{2}{3}\pi^2 \right) \end{aligned}}$$

$$\alpha^\mu = \frac{A^\mu}{T}$$

$$\lambda_4 = \frac{A}{gT^2}$$

NOTE These coefficients are specific for the Canonical stress-energy tensor

$$\hat{T}'^{\mu\nu} = \hat{T}_c^{\mu\nu} + \partial_\alpha \partial_\rho \left(\hat{Z}^{\alpha\mu,\rho\nu} + \hat{Z}^{\alpha\nu,\rho\mu} \right) \qquad \hat{Z}^{\alpha\mu,\rho\nu} = \xi \left(g^{\rho\alpha} g^{\mu\nu} - g^{\mu\alpha} g^{\rho\nu} \right) \hat{\psi}^+ \hat{\psi}$$

CONCLUSIONS

Quantum corrections can be important

$$W(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4y\, \langle :\widehat{\psi}^\dagger(x+y/2)\widehat{\psi}(x-y/2):\rangle \mathrm{e}^{-iy\cdot k}$$

$$j^\mu(x) = i\langle :\widehat{\psi}^\dagger(x)\overleftrightarrow{\partial^\mu}\widehat{\psi}(x):\rangle$$

$$i\langle :\widehat{\psi}^\dagger(x)\overleftrightarrow{\partial^\mu}\widehat{\psi}(x):\rangle = \int \mathrm{d}^4k\, k^\mu W(x,k)$$

$$\Rightarrow \boxed{j^\mu(x) = \mathrm{Re}\int \frac{d^3p}{\varepsilon}\, p^\mu \left[f_c(x,p) - \bar{f}_c(x,p)\right]}$$

$$f_c(x,p) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathrm{p}'}{2\varepsilon'}\, \mathrm{e}^{i(p-p')\cdot x}\langle \widehat{a}^\dagger(p)\widehat{a}(p')\rangle$$

$$\boxed{\langle :\widehat{T}_C^{\mu\nu}:\rangle = \mathrm{Re}\left[\int \frac{\mathrm{d}^3p}{\varepsilon}\left(p^\mu p^\nu + \frac{1}{4}\underbrace{(ip^\mu\partial^\nu + ip^\nu\partial^\mu)}_{\hbar} + \frac{1}{4}\underbrace{(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)}_{\hbar^2}\right)(f_c(x,p) + \bar{f}_c(x,p))\right]}$$