

\mathcal{L} -moment and out-of-equilibrium hydrodynamics

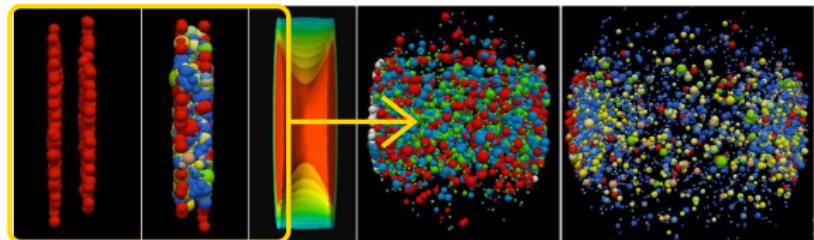
Li Yan

Department of Physics, McGill University



Foundational aspects of relativistic hydrodynamics
ECT* Trento, Italy, 2018

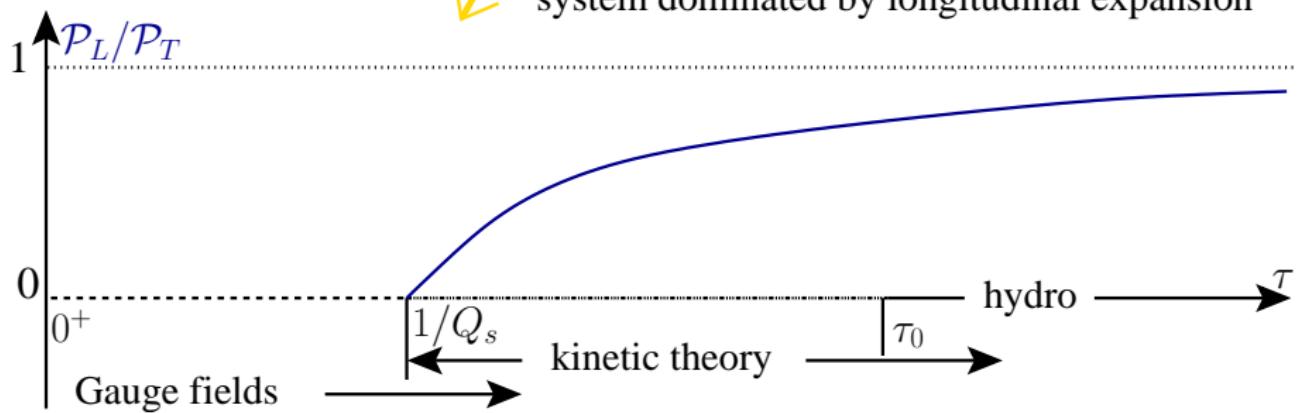
Out-of-equilibrium QGP system in heavy-ion collisions



\mathcal{P}_T : transverse pressure

\mathcal{P}_L : longitudinal pressure

system dominated by longitudinal expansion



Beyond pressure anisotropy: \mathcal{L} -moment

p^2 -moment weighted with Legendre Polynomial P_{2n} :

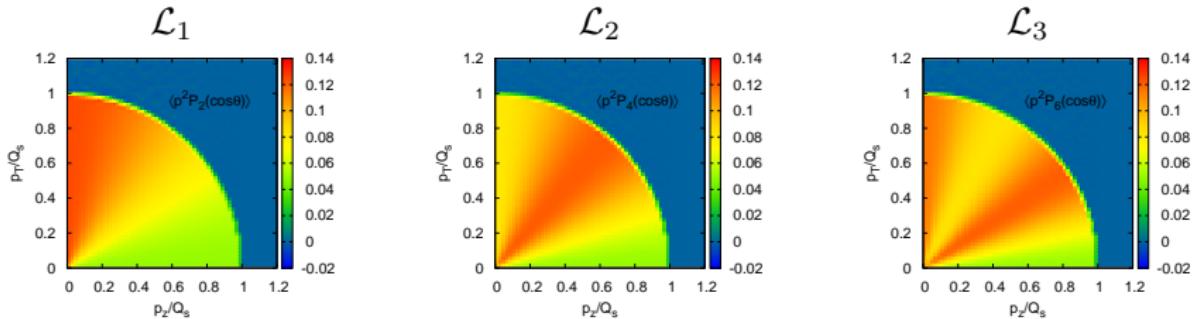
$$\mathcal{L}_n = \int \frac{d^3 p}{(2\pi)^3 p^0} \textcolor{blue}{p^2 P_{2n}(p_z/p_\perp)} f(\tau, \vec{p}_\perp, p_z),$$

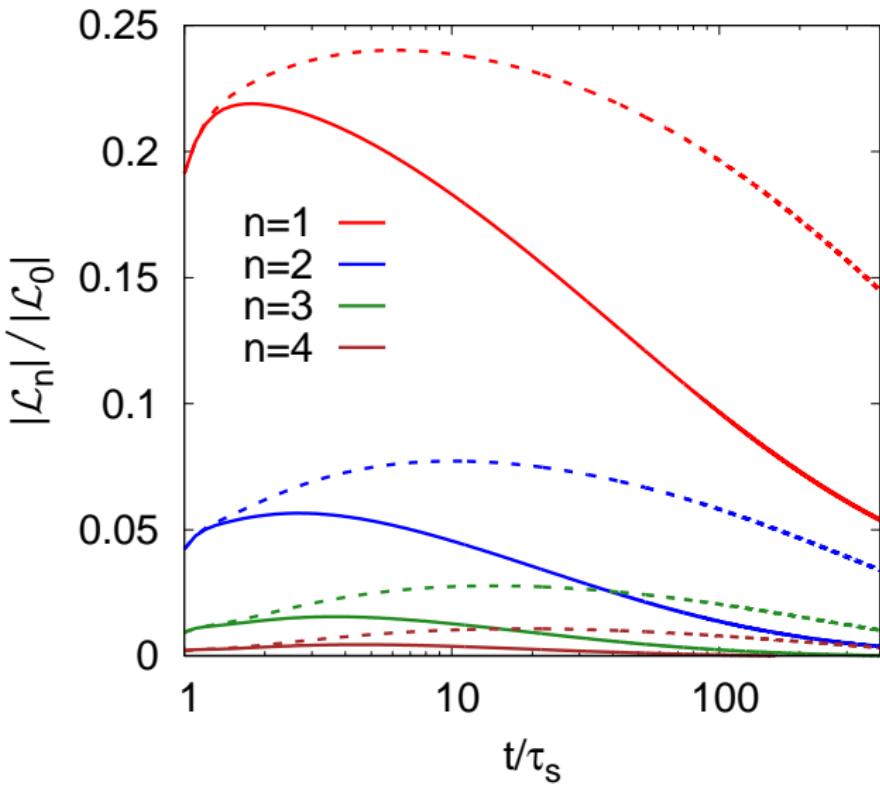
Beyond pressure anisotropy: \mathcal{L} -moment

p^2 -moment weighted with Legendre Polynomial P_{2n} :

$$\mathcal{L}_n = \int \frac{d^3p}{(2\pi)^3 p^0} p^2 P_{2n}(p_z/p_\perp) f(\tau, \vec{p}_\perp, p_z),$$

- $n = 0 \Leftrightarrow$ energy density: $\mathcal{L}_0 = \mathcal{E}$.
- $n = 1 \Leftrightarrow$ pressure anisotropy: $\mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$.
- $n \geq 2 \Leftrightarrow$ finer structure of f (or δf).





Solved by kinetic theory with respect to QCD 2-to-2 scattering.

Equation of motion for \mathcal{L}_n

Transport equation with relaxation time approximation :

$$\left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(\mathbf{p}, \tau) = - \frac{f(\mathbf{p}, \tau) - f_{\text{eq}}(p/T)}{\tau_R}, \quad \tau_R = \tau_R(T)$$

Equation of motion for \mathcal{L}_n

Transport equation with relaxation time approximation :

$$\left[\partial_\tau - \frac{p_z}{\tau} \partial_{p_z} \right] f(\mathbf{p}, \tau) = - \frac{f(\mathbf{p}, \tau) - f_{\text{eq}}(p/T)}{\tau_R}, \quad \tau_R = \tau_R(T)$$

which is equivalent to

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = - \frac{1}{\tau} [a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}] - \frac{\mathcal{L}_n}{\tau_R} (1 - \delta_{n0}), \quad n = 0, 1, \dots$$

- a_n , b_n and c_n are constant coefficients from free-streaming,

$$a_0 = \frac{4}{3}, \quad a_1 = \frac{38}{21}, \quad \dots$$

- τ_R/τ defines Knudsen number.

Truncation of the coupled equations

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau} [a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}] - \frac{\mathcal{L}_n}{\tau_R} (1 - \delta_{n0}) \quad n = 0, 1, \dots$$

Truncate at n -th order : ignore all \mathcal{L} -moments higher than n -th order

- at $n = 0$

$$\frac{\partial \mathcal{E}}{\partial \tau} + \frac{4}{3} \frac{\mathcal{E}}{\tau} = 0 \quad \rightarrow \quad \mathcal{E} \sim \tau^{-4/3} \quad \text{ideal hydro}$$

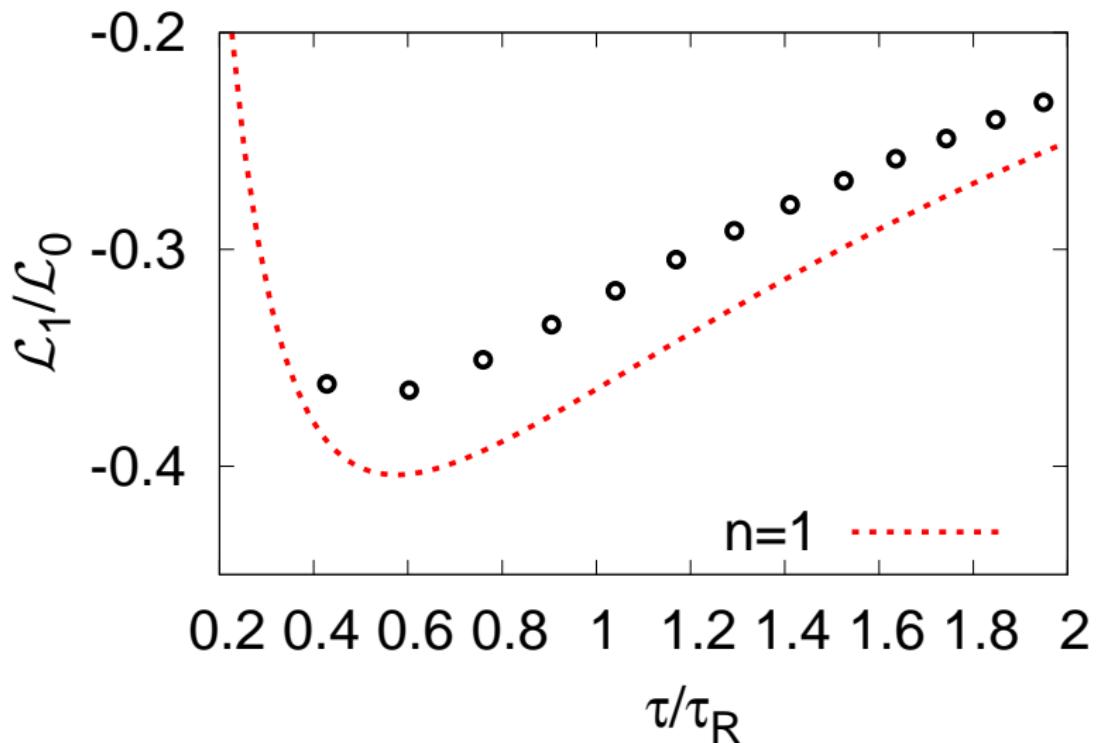
- at $n = 1$

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau} [a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1]$$

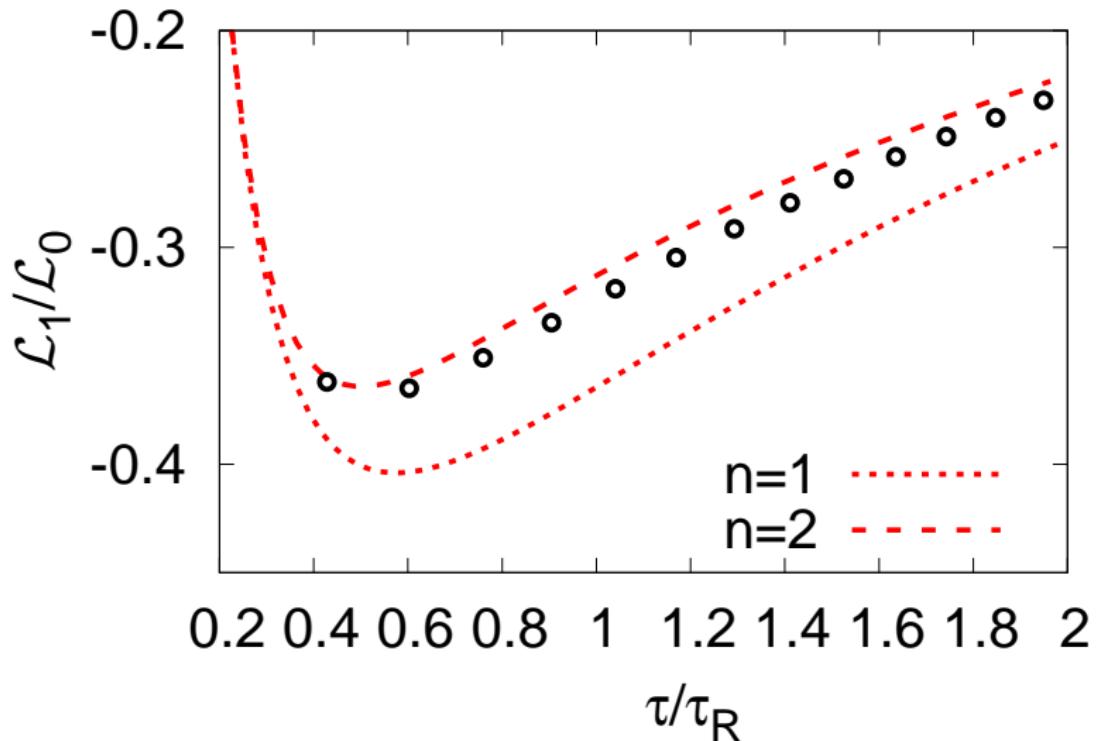
$$\frac{\partial \mathcal{L}_1}{\partial \tau} = -\frac{1}{\tau} [a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0] - \frac{\mathcal{L}_1}{\tau_R} \quad \text{2nd order viscous hydro?}$$

- ...

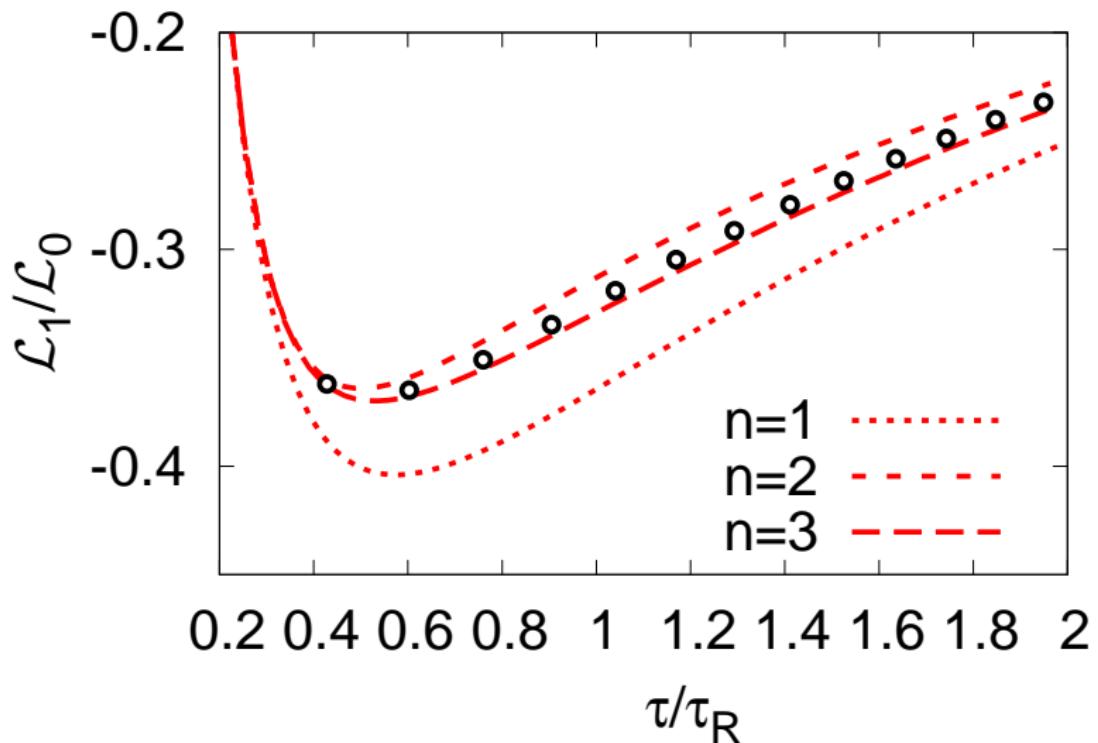
Convergence of truncation



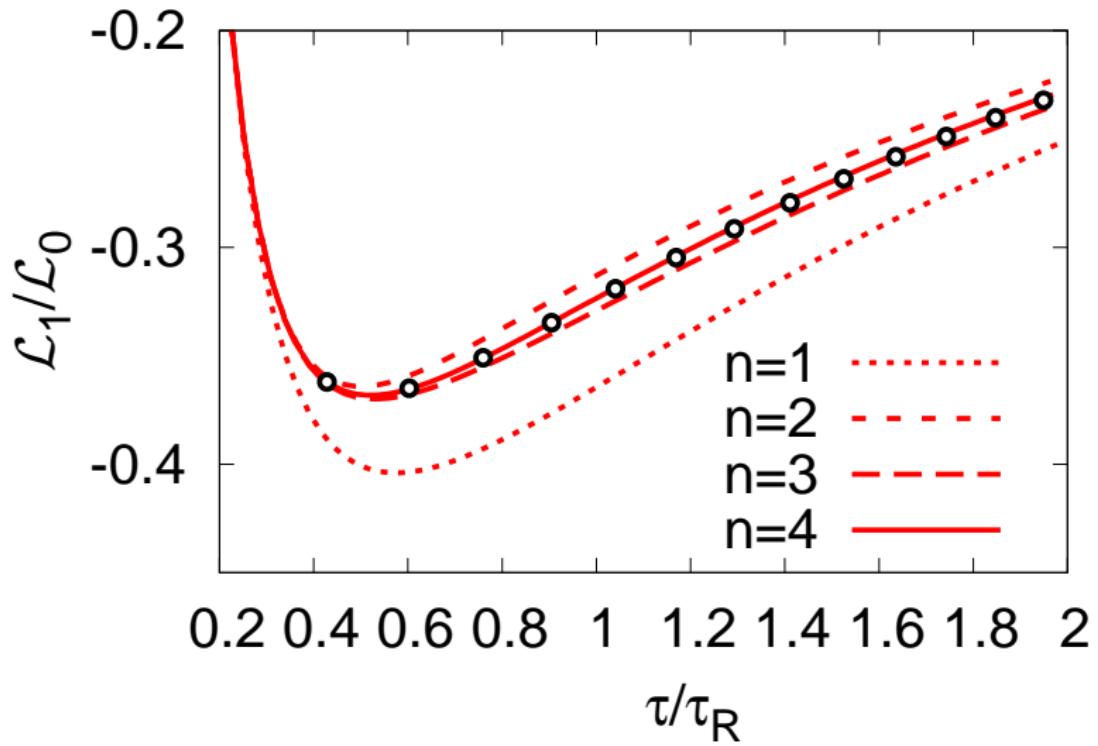
Convergence of truncation



Convergence of truncation



Convergence of truncation



The free-streaming fixed points: $\tau/\tau_R \rightarrow 0$

$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau} [a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}]$$

For infinite n :

- $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \dots$

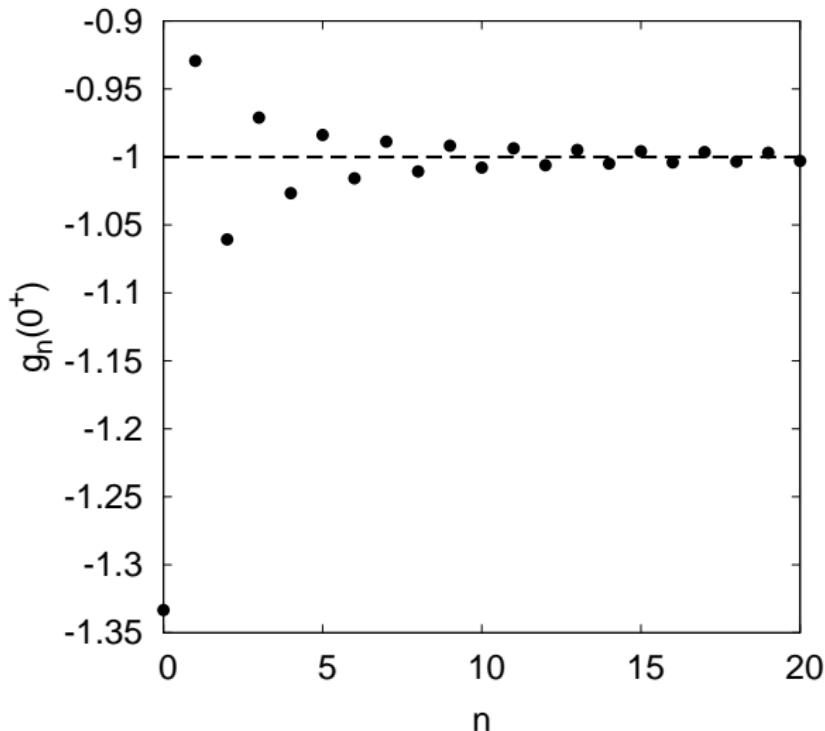
$$\Rightarrow \mathcal{L}_n = \mathcal{L}_n(\tau_0) \left(\frac{\tau_0}{\tau} \right)^2 \quad \rightarrow \tau \partial_\tau \ln \mathcal{L}_n = 2$$

- $\mathcal{L}_n(\tau) = P_{2n}(0)\mathcal{L}_0(\tau),$

$$\Rightarrow \mathcal{L}_n(\tau) = \mathcal{L}_n(\tau_0) \left(\frac{\tau_0}{\tau} \right) \quad \rightarrow \tau \partial_\tau \ln \mathcal{L}_n = 1$$

For finite n ,

$$\begin{pmatrix} a_0 & c_0 & 0 & 0 & \dots \\ b_1 & a_1 & c_1 & 0 & \dots \\ 0 & b_2 & a_2 & c_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \Rightarrow \lambda \approx 2(\text{unstable}) \text{ and } \approx 1(\text{stable})$$



- Stable fixed points in terms of $g_n = \tau \partial_\tau \ln \mathcal{L}_n$
- Note that: $g_0 \sim \mathcal{L}_1/\mathcal{L}_0 \sim \mathcal{A} = (\mathcal{P}_L - \mathcal{P}_T)/\mathcal{P}$ M.Heller, V. Svensson, ...

The hydro fixed points: $\tau/\tau_R \rightarrow \infty$

Hydro EoM via truncation order by order:

- Truncation at $n = 1$ gives 2nd order viscous hydro: ($c_0 \mathcal{L}_1 = \Pi = \Pi_\xi^\xi$)

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau} (a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1) \quad \rightarrow \quad \partial_\tau \mathcal{E} + \frac{4}{3} \frac{\mathcal{E}}{\tau} = -\frac{\Pi}{\tau}$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0) - \frac{1}{\tau_R} \mathcal{L}_1 \quad \rightarrow \quad \Pi = -\eta \sigma - \tau_R \partial_\tau \Pi - a_1 \frac{\tau_R}{\tau} \Pi.$$

* Have used conformal EoS: $\mathcal{E} = 3\mathcal{P}$ and $\tau_R \sim \eta$

* Note $a_1 = 38/21 (= \beta_{\pi\pi})$ is 2nd transport coefficient in DNMR.

Assuming homogeneous under conformal transformation, so that

$$\text{ambiguity of } \frac{1}{\tau} = \frac{3\sigma_\xi^\xi}{4} \quad \text{or} \quad \frac{1}{\tau} = \nabla \cdot u$$

is properly fixed which leads to BRSSS hydro:

$$\Pi = -\eta\sigma - \underbrace{\tau_R}_{\tau_\pi} \left[\partial_\tau \Pi + \frac{4}{3} \Pi \nabla \cdot u \right] + \tau_R \underbrace{\left(a_1 - \frac{4}{3} \right)}_{\lambda_1} \frac{3}{4} \frac{\Pi^2}{\eta},$$
$$\sim \frac{\lambda_1}{\eta\tau_\pi} = \frac{5}{7}$$

M. York and G. Moore

Assuming homogeneous under conformal transformation, so that

$$\text{ambiguity of } \frac{1}{\tau} = \frac{3\sigma_\xi^\xi}{4} \quad \text{or} \quad \frac{1}{\tau} = \nabla \cdot u$$

is properly fixed which leads to BRSSS hydro:

$$\begin{aligned} \Pi &= -\eta\sigma - \underbrace{\tau_R}_{\tau_\pi} \left[\partial_\tau \Pi + \frac{4}{3} \Pi \nabla \cdot u \right] + \tau_R \underbrace{\left(a_1 - \frac{4}{3} \right)}_{\sim \frac{\lambda_1}{\eta\tau_\pi}} \frac{3}{4} \frac{\Pi^2}{\eta}, \\ &\sim \frac{\lambda_1}{\eta\tau_\pi} = \frac{5}{7} \end{aligned}$$

M. York and G. Moore

Restore tensor indices:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_R \left[\langle D\Pi^{\mu\nu} \rangle + \frac{d}{d-1} \Pi^{\mu\nu} \nabla \cdot u \right] + \frac{\tau_R}{\eta} \left(a_1 - \frac{2}{3} \right) \frac{3}{2} \langle \Pi^{\mu\lambda} \Pi_\lambda^\nu \rangle$$

- Truncation at $n = 2$ gives 3rd order viscous hydro: $(c_0 \mathcal{L}_2 = \Sigma = \Sigma_\xi^\xi)$

$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau} (a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1) \quad \rightarrow \quad \partial_\tau \mathcal{E} + \frac{\mathcal{E} + \mathcal{P}_L}{\tau} = 0$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (a_1 \mathcal{L}_1 + b_1 \mathcal{L}_0 + \textcolor{red}{c_1 \mathcal{L}_2}) - \frac{1}{\tau_R} \mathcal{L}_1 \quad \rightarrow \quad \text{Eq (a): see next slide}$$

$$\partial_\tau \mathcal{L}_2 = -\frac{1}{\tau} (a_2 \mathcal{L}_2 + b_2 \mathcal{L}_1) - \frac{1}{\tau_R} \mathcal{L}_2 \quad \rightarrow \quad \text{Eq (b): see next slide}$$

Assuming Σ to be homogeneous under conformal transformation,

$$\text{Eq (a): } \Pi = -\eta\sigma - \tau_\pi \left[\partial_\tau \Pi + \frac{4}{3} \Pi \nabla \cdot u \right] + \underbrace{\tau_\pi \left(a_1 - \frac{4}{3} \right) \frac{3}{4} \frac{\Pi^2}{\eta}}_{\frac{\lambda_1}{\eta \tau_\pi}} + \underbrace{\frac{3c_1 \tau_\pi}{4}}_{\sim 3\text{rd order transport}} \frac{\Sigma \Pi}{\eta}$$

A. Jaiswal (2013)

Assuming Σ to be homogeneous under conformal transformation,

$$\text{Eq (a): } \Pi = -\eta\sigma - \tau_\pi \left[\partial_\tau \Pi + \frac{4}{3} \Pi \nabla \cdot u \right] + \underbrace{\tau_\pi \left(a_1 - \frac{4}{3} \right) \frac{3}{4} \frac{\Pi^2}{\eta}}_{\frac{\lambda_1}{\eta \tau_\pi}} + \underbrace{\frac{3c_1 \tau_\pi}{4}}_{\sim \text{3rd order transport}} \frac{\Sigma \Pi}{\eta}$$

A. Jaiswal (2013)

$$\text{Eq (b): } \Sigma = \frac{\lambda_1 + \eta \tau_\pi}{2\eta^2} \Pi^2 - \tau_\pi \left(\partial_\tau \Sigma + \frac{4}{3} \Sigma \nabla \cdot u \right)$$

$$+ \underbrace{\tau_\pi \left(a_2 - \frac{4}{3} \right) \frac{3}{4}}_{\sim \text{3rd order transport}} \frac{\Sigma \Pi}{\eta}$$

Restore tensor indices of Σ :

$$\begin{aligned}\Sigma^{\mu\nu} = & \frac{\lambda_1 + \eta\tau_\pi}{\eta^2} \langle \Pi^{\mu\lambda} \Pi^\nu_{\lambda} \rangle - \tau_\pi \left[\langle D\Sigma^{\mu\nu} \rangle + \frac{d}{d-1} \Sigma^{\mu\nu} \nabla \cdot u \right] \\ & + \frac{\tau_\pi}{\eta} \left(a_2 - \frac{2}{3} \right) \frac{3}{2} \langle \Sigma^{\mu\lambda} \Pi^\nu_{\lambda} \rangle\end{aligned}$$

Restore tensor indices of Σ :

$$\begin{aligned}\Sigma^{\mu\nu} = & \frac{\lambda_1 + \eta\tau_\pi}{\eta^2} \langle \Pi^{\mu\lambda} \Pi^\nu{}_\lambda \rangle - \tau_\pi \left[\langle D\Sigma^{\mu\nu} \rangle + \frac{d}{d-1} \Sigma^{\mu\nu} \nabla \cdot u \right] \\ & + \frac{\tau_\pi}{\eta} \left(a_2 - \frac{2}{3} \right) \frac{3}{2} \langle \Sigma^{\mu\lambda} \Pi^\nu{}_\lambda \rangle\end{aligned}$$

Remarks:

- A systematic scheme to derive (arbitrarily) higher order viscous hydro !
- Under Bjorken and conformal symmetry: no vorticity terms $\Omega^{\mu\nu}$, etc.
- $\mathcal{L}_n \Leftrightarrow$ n-th order viscous correction of hydro.

The hydro fixed points: $\tau/\tau_R \rightarrow \infty$

Ansatz gradient expansion form of \mathcal{L}_n in hydro regime,

$$\mathcal{L}_n = \sum_{m=0} \frac{\alpha_m^{(n)}}{\tau^n}, \quad \alpha_m^{(n)} \rightarrow \text{transport coefficient}$$

asymptotic decay rate determined by the leading term: $\mathcal{L}_n \sim \alpha_n^{(n)}/\tau$

$$\Rightarrow \tau \partial_\tau \ln \mathcal{L}_n = -\frac{4+2n}{3} \quad (\tau_R \propto 1/T)$$

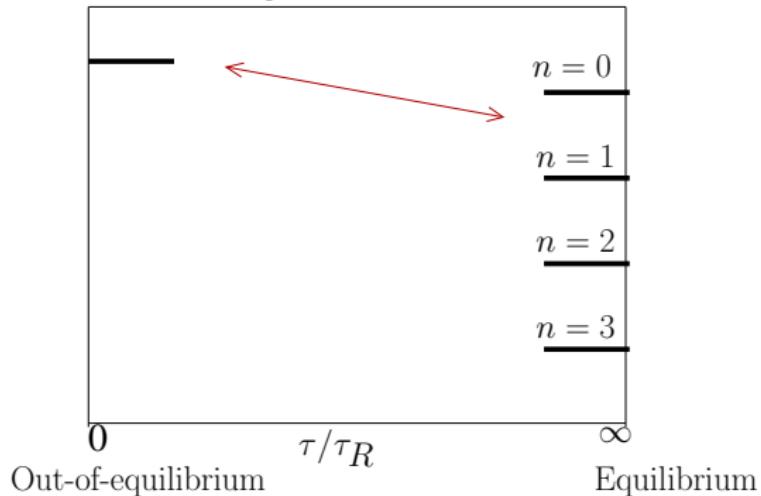
$$\Rightarrow \tau \partial_\tau \ln \mathcal{L}_n = -\frac{4+3n}{3} \quad (\tau_R \text{ constant})$$

* These are stable fixed points in the hydro regime.

* $\tau \partial_\tau \ln \mathcal{L}_0 = -4/3$

A simple summary on the fixed points

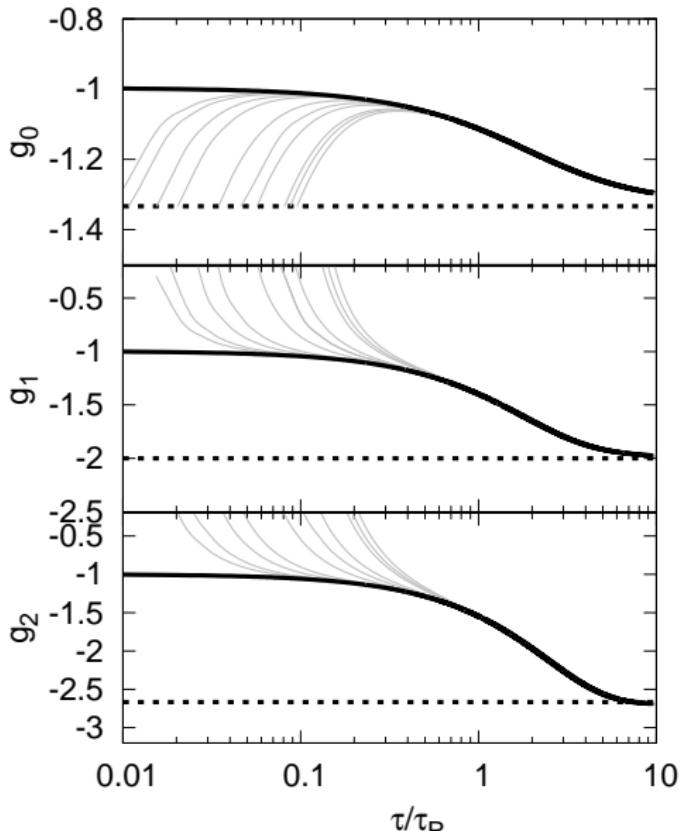
$$g_n = \tau \partial_\tau \ln \mathcal{L}_n$$



- Degenerated fixed point of all $g_n \approx -1$ in the free-streaming limit.
- Hydro fixed points of g_n split according to n .
- System evolves between there two types of fixed points – attractor.

Ideal hydro is a trivial attractor solution: $g_0 = \text{const.} = -4/3$

Attractor solution



- Attractor solution exists, with or without conformal symmetry, beyond Bjorken symmetry.
P. Romatschke, M. Martinez, M. Strickland, G. Denicol, ...

- Non-hydro modes decay exponentially, w.r.t. attractor solutions.

P. Romatschke, A. Kurkela, U. Wiedemann, ...

- Attractor corresponds to Borel-summation of hydro gradient expansion.

M. Heller, M. Spalinski, R. Janik, P. Witaszczyk, G. Basar, G. Dunne, ...

Renormalization of η/s

Effects from higher order moments/viscous hydro (leading order):

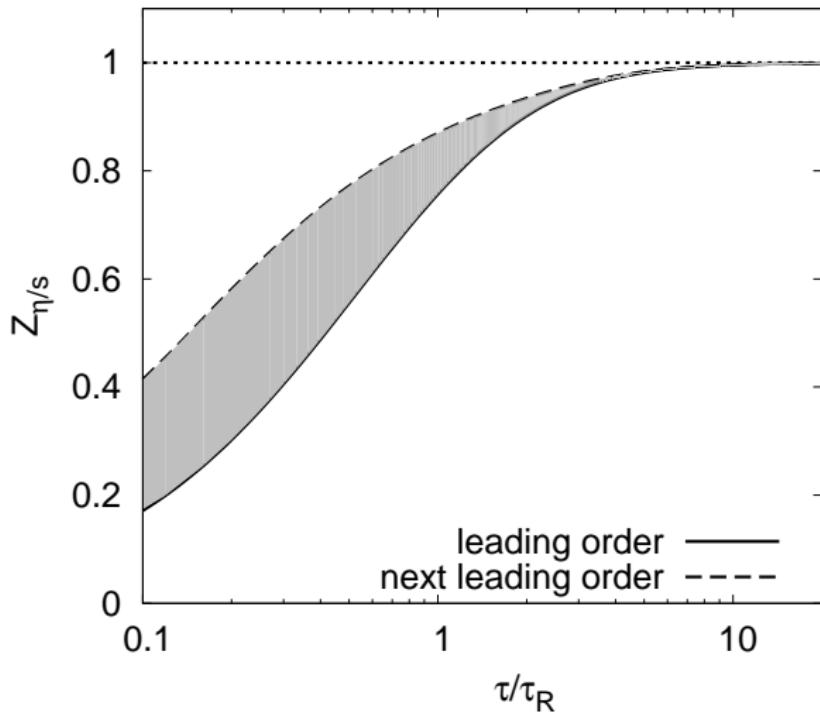
$$\partial_\tau \mathcal{L}_0 = -\frac{1}{\tau} (a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1),$$

$$\partial_\tau \mathcal{L}_1 = -\frac{1}{\tau} (a_1 \mathcal{L}_1 + b_0 \mathcal{L}_0) - \underbrace{\left[1 + \frac{c_1 \tau_R}{\tau} \frac{\mathcal{L}_2}{\mathcal{L}_1} \right]}_{Z_{\eta/s}^{-1}} \frac{\mathcal{L}_1}{\tau_R} \quad (\mathcal{L}_2 \text{ in 2nd hydro}),$$

$$g_2(\tau/\tau_R) = -a_2 - b_2 \frac{\mathcal{L}_2}{\mathcal{L}_1} - \frac{\tau}{\tau_R}.$$

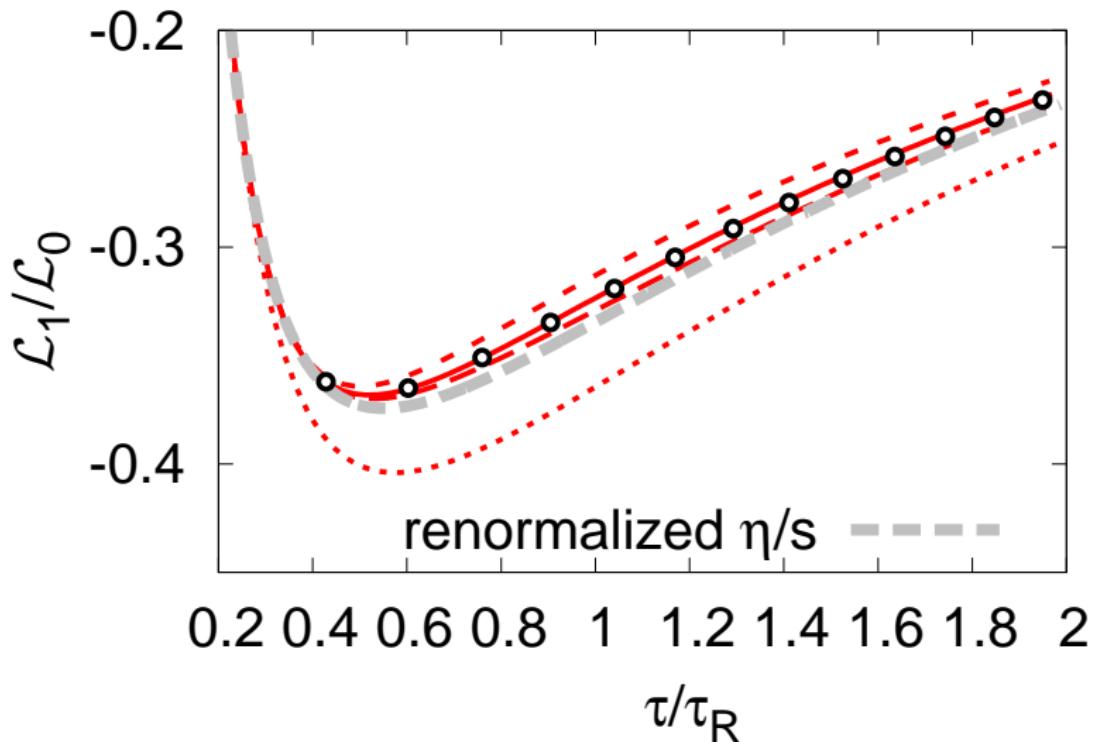
- Taking attractor solution for g_2 : resummed of gradients.
- Effectively, for 2nd order hydro, η/s is renormalized.

Attractor solution



Out-of-equilibrium physics can be effectively absorbed into a reduced η/s .

E. Shuryak, M. Lublinsky, P. Romatschke



2nd order viscous hydro + renormalized η/s

Summary

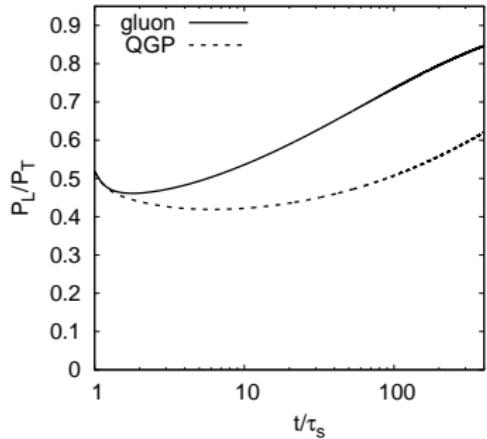
- \mathcal{L} -moments are proposed to quantify system thermalization.
- Coupled equations for \mathcal{L}_n are derived, and valid truncations.
 - ⇒ fluid dynamics for out-of-equilibrium system
- Attractor solution smoothly connect fixed points of \mathcal{L}_n in two limits.
- The systems in heavy-ion collisions may be out-of-equilibrium:
 - Actual value of η/s could be larger than phenomenological expectation.

Summary

- \mathcal{L} -moments are proposed to quantify system thermalization.
- Coupled equations for \mathcal{L}_n are derived, and valid truncations.
 - ⇒ fluid dynamics for out-of-equilibrium system
- Attractor solution smoothly connect fixed points of \mathcal{L}_n in two limits.
- The systems in heavy-ion collisions may be out-of-equilibrium:
 - Actual value of η/s could be larger than phenomenological expectation.
 - ⇒ strongly coupled QGP?

Back-up slides

Kinetic theory description

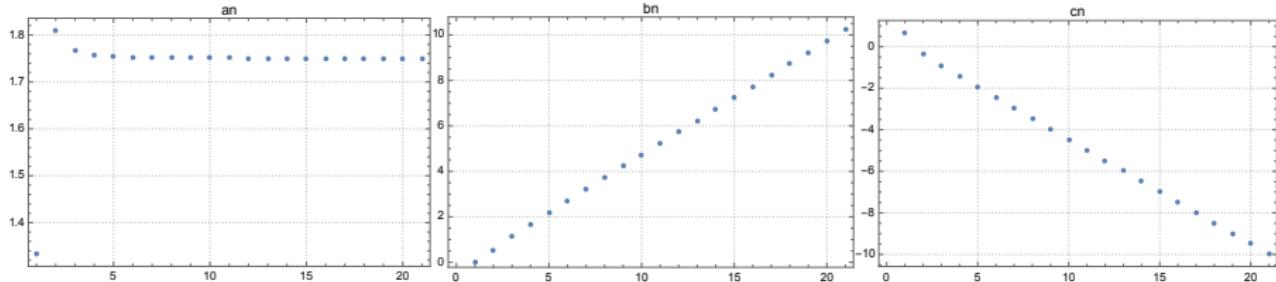


- In kinetic theory: $p^\mu \partial_\mu f(\tau, p_\perp, p_z) = -\mathcal{C}[f]$

$$\mathcal{P}_L = \int \frac{d^3 p}{(2\pi)^3 p^0} p_z^2 f(\tau, p_\perp, p_z), \quad \mathcal{P}_T = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3 p^0} (p_x^2 + p_y^2) f(\tau, p_\perp, p_z)$$

$$\mathcal{E} = \mathcal{P}_L + 2\mathcal{P}_T = \int \frac{d^3 p}{(2\pi)^3 p^0} p^2 f(\tau, p_\perp, p_z)$$

Bjorken symmetry: boost invariance along ξ , translation invariance in \vec{x}_\perp .



$$a_n = \frac{2(14n^2 + 7n - 2)}{(4n-1)(4n+3)}, \quad b_n = \frac{(2n-1)2n(2n+2)}{(4n-1)(4n+1)},$$

$$c_n = \frac{(1-2n)(2n+1)(2n+2)}{(4n+1)(4n+3)},$$