How does kinetic theory remember about initial conditions?

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Introduction

- Initial conditions of kinetic theory transform into hydrodynamic information
- How is information from the pre-hydrodynamic stage kept?

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Answers in Bjorken flow described by BRSSS or Holography

Transients/Quasinormal modes/Non-hydrodynamic modes

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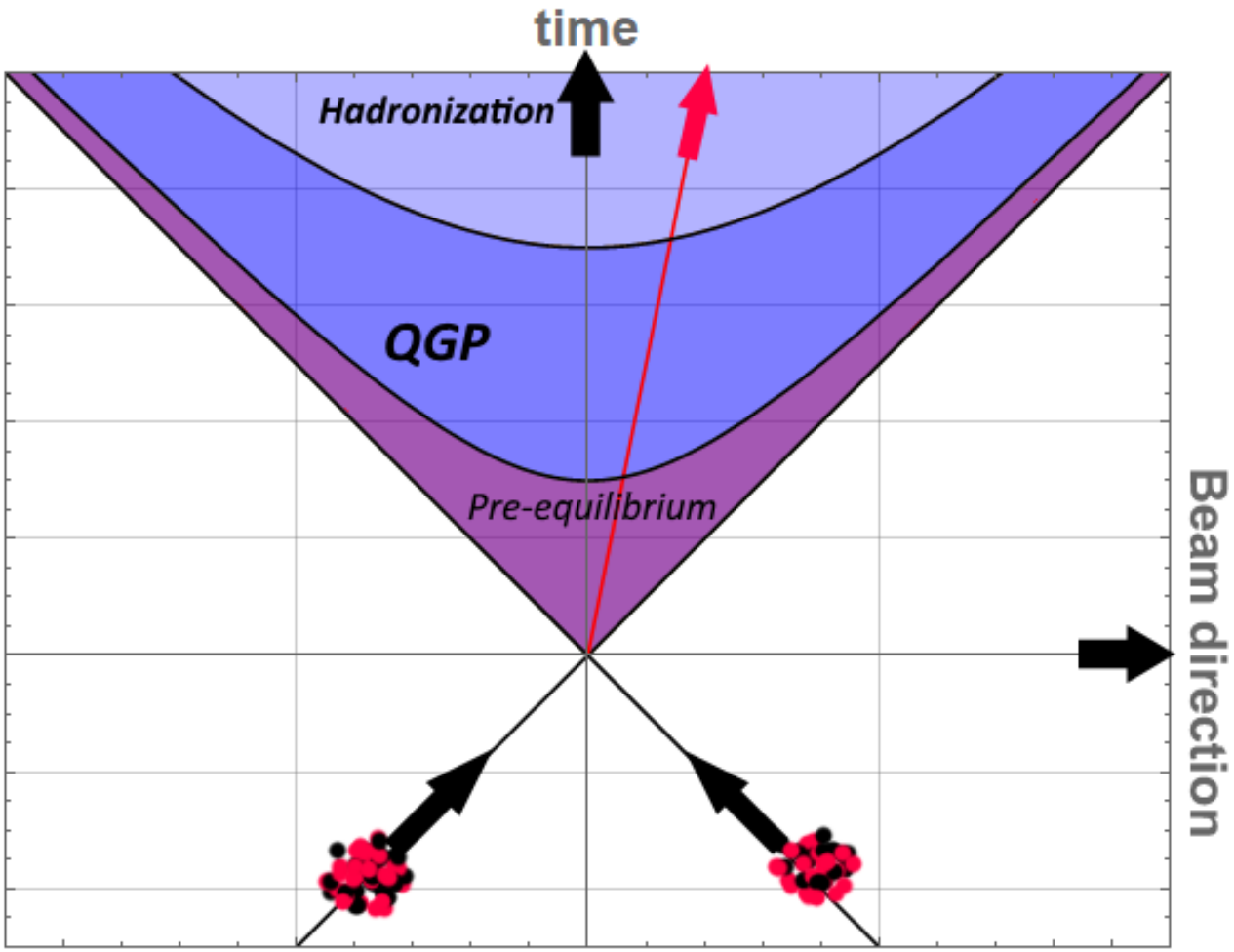
Hydrodynamization in kinetic theory?

Similiarities between hydrodynamization at strong and weak coupling $\longrightarrow$ universal lessons

Bjorken flow

Highly symmetric flow

- Homogeneous and isotropic transverse to beam
- Boost-invariant in beam direction



BRSSS in conformal Bjorken flow [Heller, Spalinski,1503.07514]

Pressure anisotropy $\mathcal{A}(w) = \frac{\mathcal{P}_T(\tau)-\mathcal{P}_L(\tau)}{\mathcal{P}(\tau)}$ as function of $w = \tau T$

- Governed by differential equation:
  $$C_{\tau_\pi} w(1+\tfrac{\mathcal{A}}{12})\mathcal{A}' + \left(\frac{C_{\tau_\pi}}{3} + \frac{C_{\lambda_1}}{8C_\eta}\right)\mathcal{A}^2 + \tfrac{3}{2}w\mathcal{A} - 12C_\eta = 0$$

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- Formal solution in gradient expansion: $\mathcal{A}(w) = f_0 + \frac{f_1}{w} + \frac{f_2}{w^2} + \ldots$

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Perturbations around the gradient expansion

$$\mathcal{A}(w) + \delta\mathcal{A}(w) \to \delta\mathcal{A}(w) \sim e^{-\frac{3}{2C_{\tau_\pi}}w} w^{\frac{C_\eta - 2C_{\lambda_1}}{C_{\tau_\pi}}}$$

The gradient expansion diverges and reveals transients

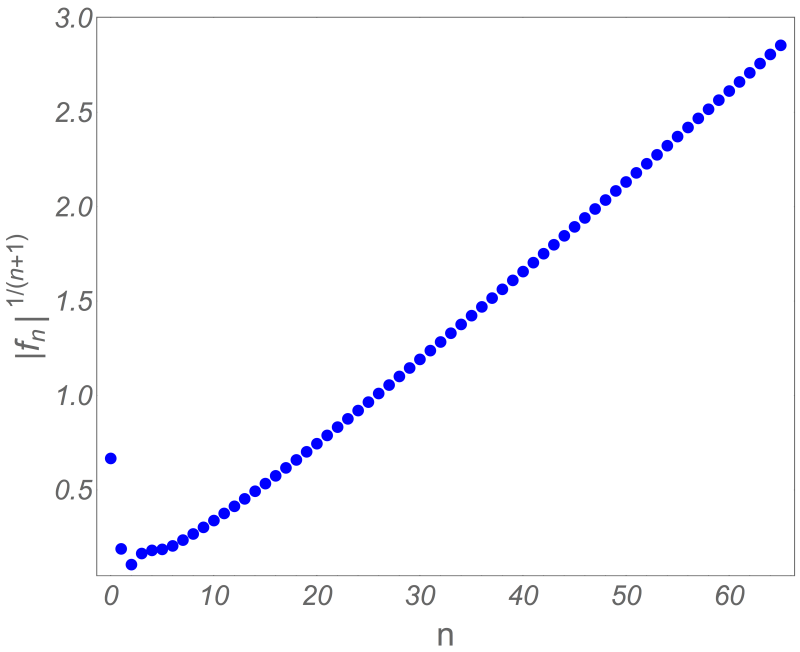

Figure 1: [Heller, Spalinski,1503.07514]

$$f_k \sim \frac{\Gamma(k+\beta)}{(-S)^k} \xrightarrow{\text{Borel transform}} \text{poles of the form } \frac{1}{(S-x)^{\beta+1}}$$

$$\xrightarrow{\text{Resummation}} \text{contributions of the form } e^{-Sw} w^{\beta}$$

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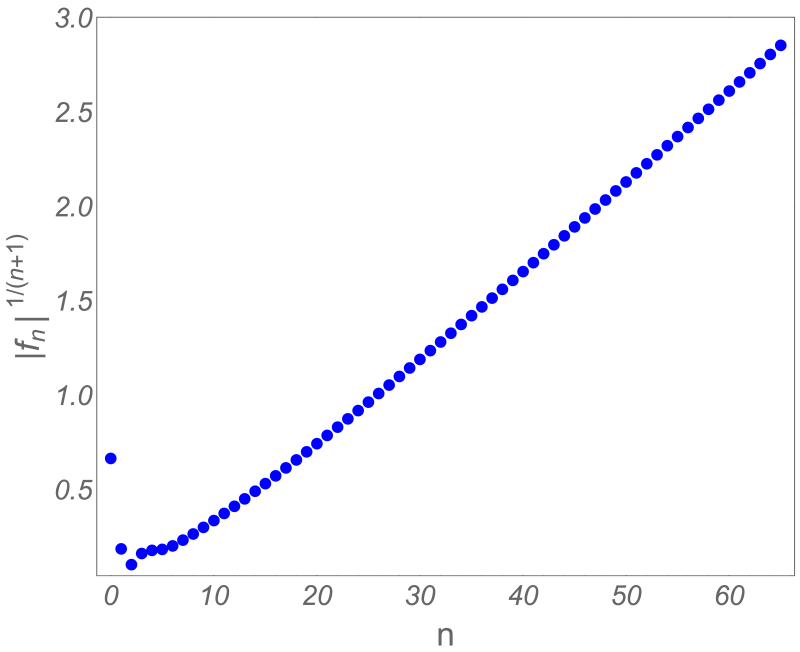

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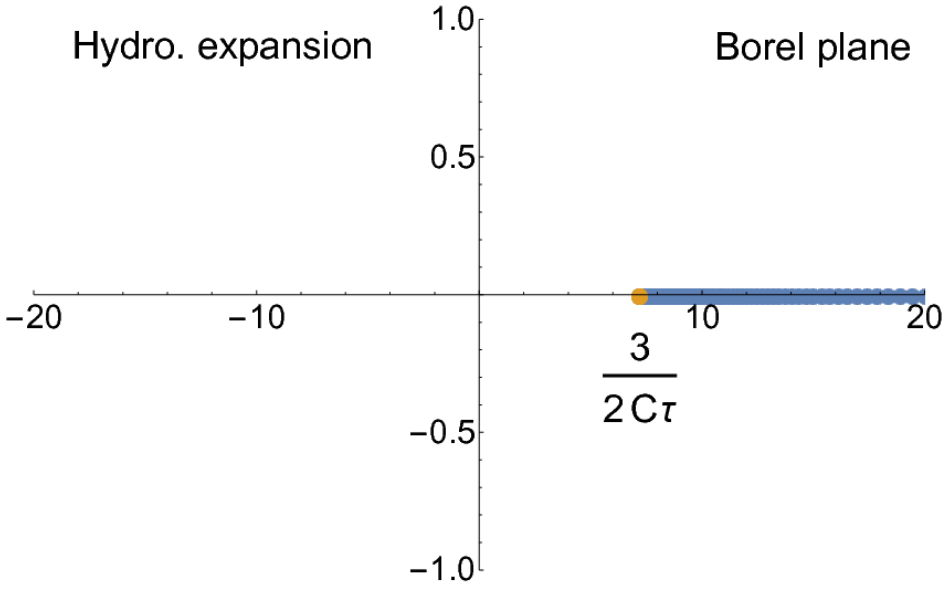

Figure 2: [Basar, Dunne, 1509.05046]

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Similar story in holography

- Gradient expansion in Bjorken flow diverges [Heller,Janik,Witaszczyk,1302.06979]
- Initial conditions: Specify metric on a timeslice

Quasinormal modes in Bjorken flow

$$\delta\mathcal{E}_i = \sigma_i \, \tau^{\alpha_i} \, e^{-s_i \tau^{2/3}}$$

Borel plane in holography

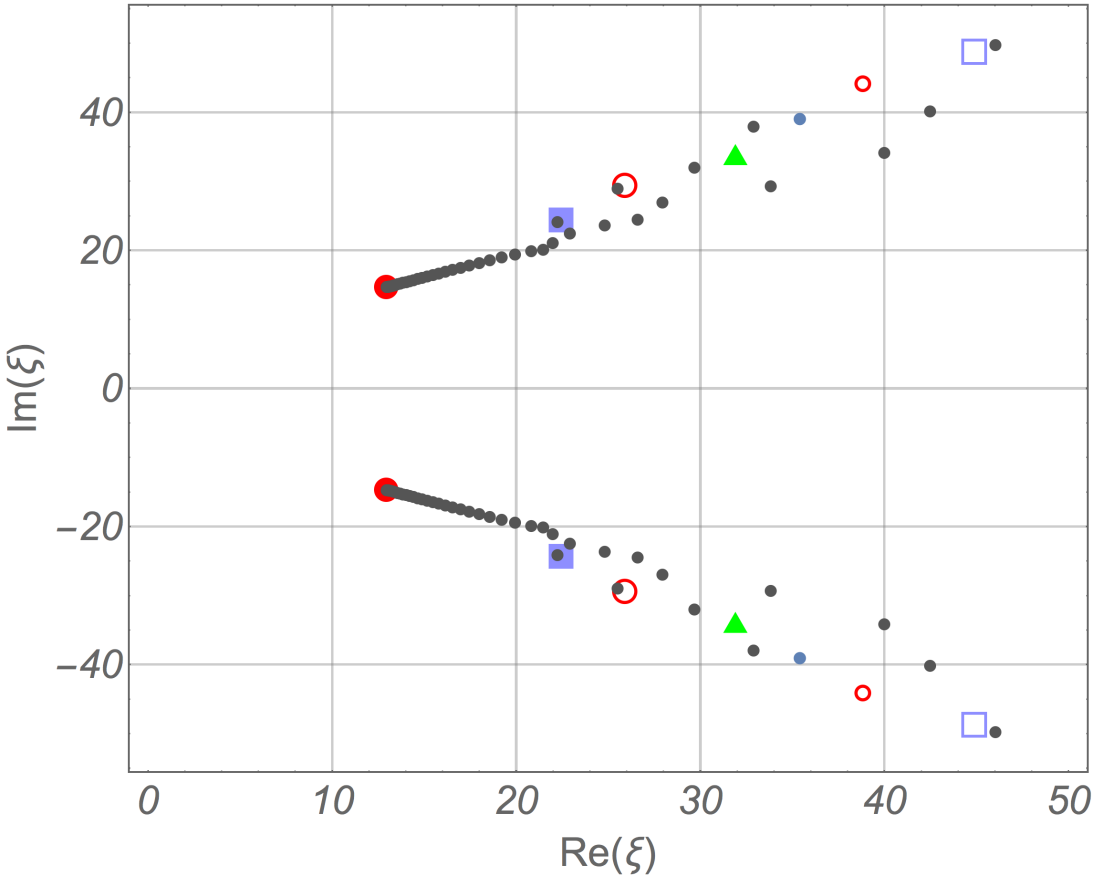

Figure 3: [1302.0697,1707.02282]

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Initial conditions, gradient expansion and non-hydrodynamic modes in BRSSS and Holography

- Divergence of gradient expansion revealed non-hydrodynamic modes
- Each mode has a free parameter set by initial conditions
- In BRSSS: one free parameter $\rightarrow$ one transient
- Holography: infinite number of transients

Kinetic theory

Distribution function

$f(x, p) =$ number of particles at position x with momentum p.

Evolution described by Boltzmann equation

Free streaming $\longrightarrow$ $p^\mu \partial_\mu f(x, p) = \mathcal{C}[f]$ $\longleftarrow$ Collision kernel

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Simplifies in Bjorken flow and Relaxation time approximation (RTA)

$$\partial_\tau f(\tau,p) = \frac{f_{eq}(\tau,p) - f(\tau,p)}{\tau_{\rm rel}}$$

$f_{eq}(\tau,p) \propto e^{-\frac{E(p)}{T(\tau)}}$ (we consider classical, massless particles)

$$\tau_{\rm rel} \propto T^{-\Delta}, \qquad \Delta < 3$$

Energy density satisfies an integral equation [Florkowski,Ryblewski,Strickland,1305.7234]

$$\mathcal{E}(\tau) = D(\tau,\tau_0)\mathcal{E}^0(\tau) + \int_{\tau_0}^{\tau} \frac{\mathrm{d}\tau'}{2\tau_{\mathrm{rel}}(\tau')} D(\tau,\tau') H\left(\frac{\tau'}{\tau}\right)\mathcal{E}(\tau'),$$

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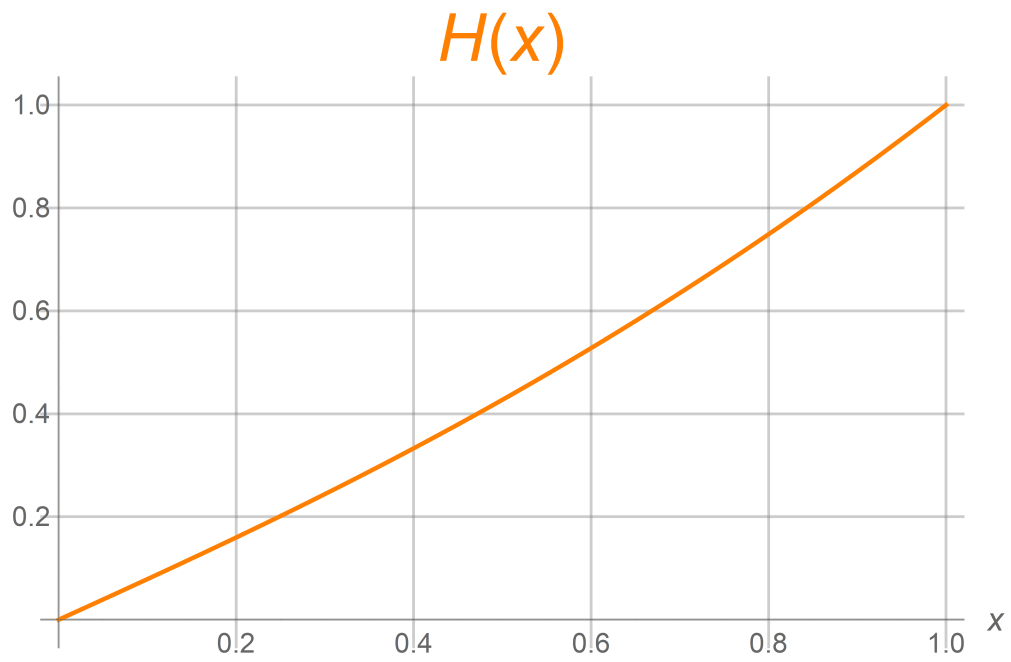


Gradient expansion in conformal RTA kinetic theory

$$\mathcal{E}(\tau) \propto \frac{1}{\tau^{4/3}} \left(1 + e_1 \frac{\tau_{\rm rel}}{\tau} + e_2 \left(\frac{\tau_{\rm rel}}{\tau} \right)^2 + \ldots \right)$$

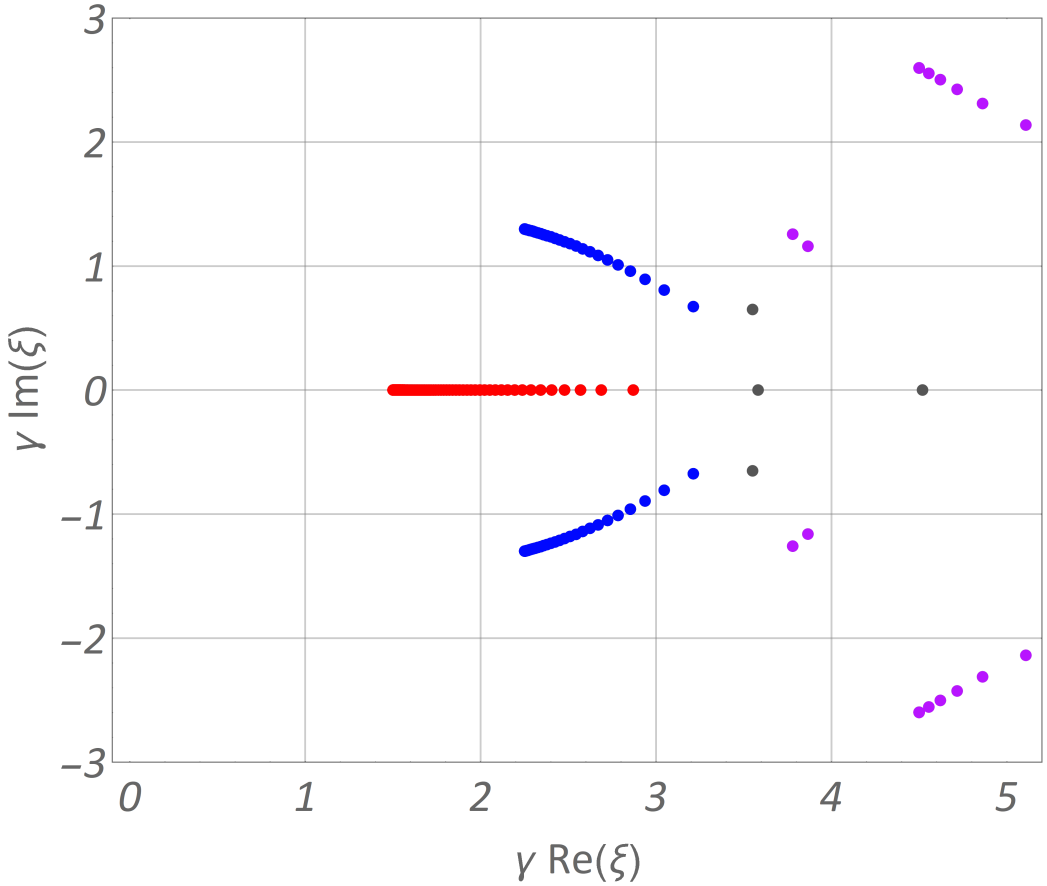
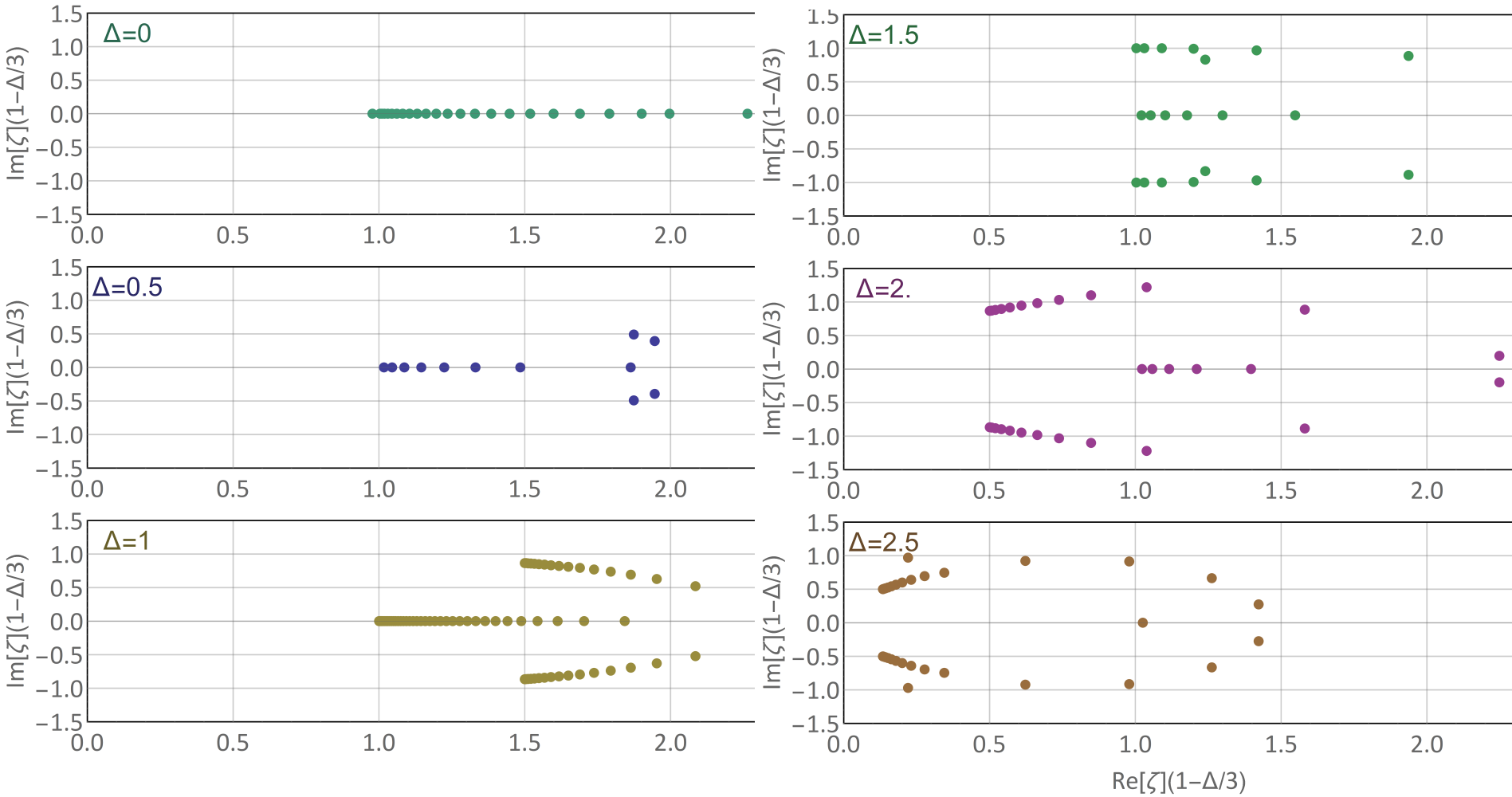


Figure 4: [Heller,Kurkela,Spalinski,VS,1609.04803]

Gradient expansion for different Δ



[Heller,VS,1802.08225]

Perturbations around gradient expansion

Use ansatz

$$\mathcal{E}(\tau) = \mathcal{E}_{ge}(\tau) + \sigma e^{-S\frac{\tau}{\tau_{\rm rel}}} \tau^{\beta} \mathcal{E}_{\beta}(\tau)$$

Exponential decay rate S

- Only one purely decaying solution: S is real.
- No off-axis singularities.

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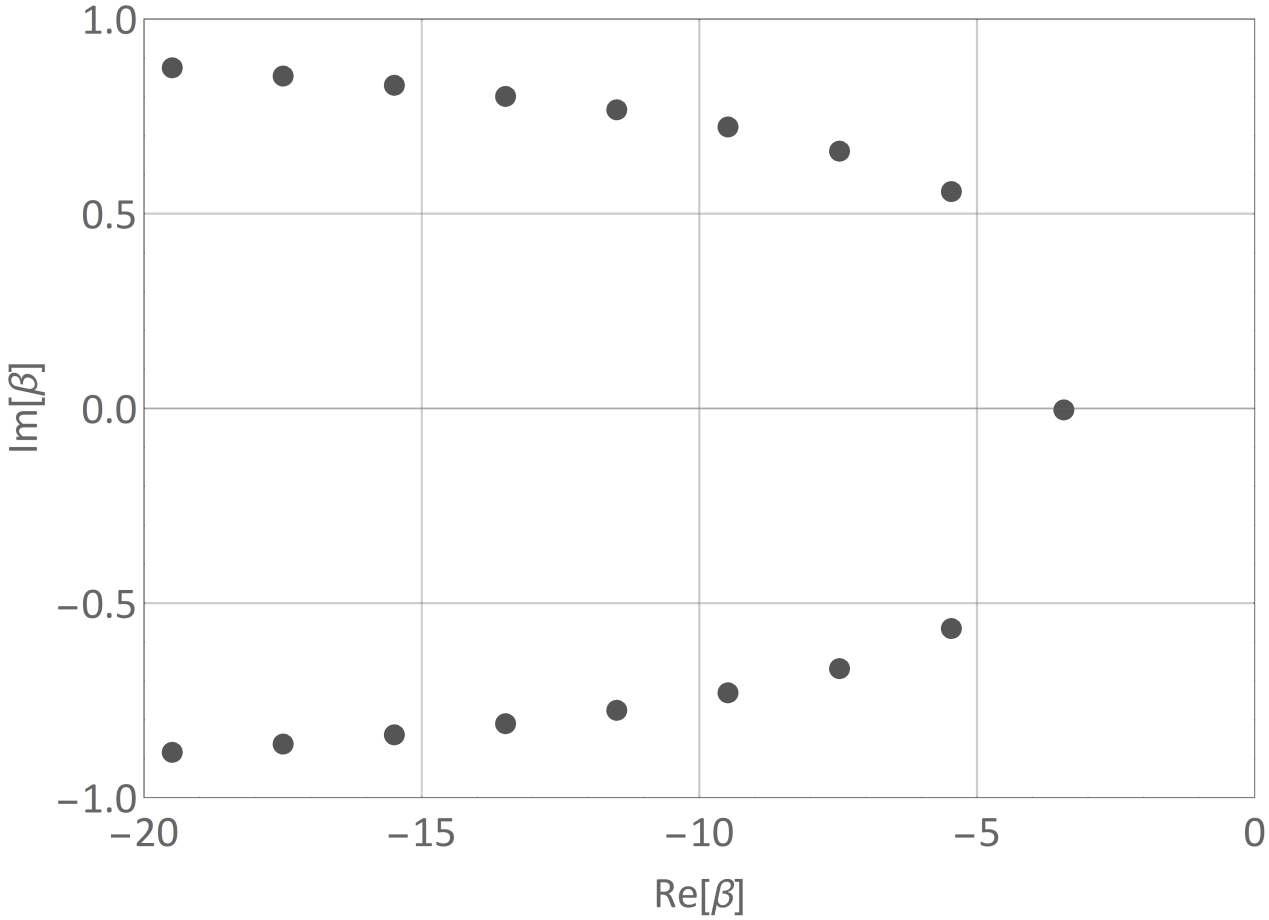
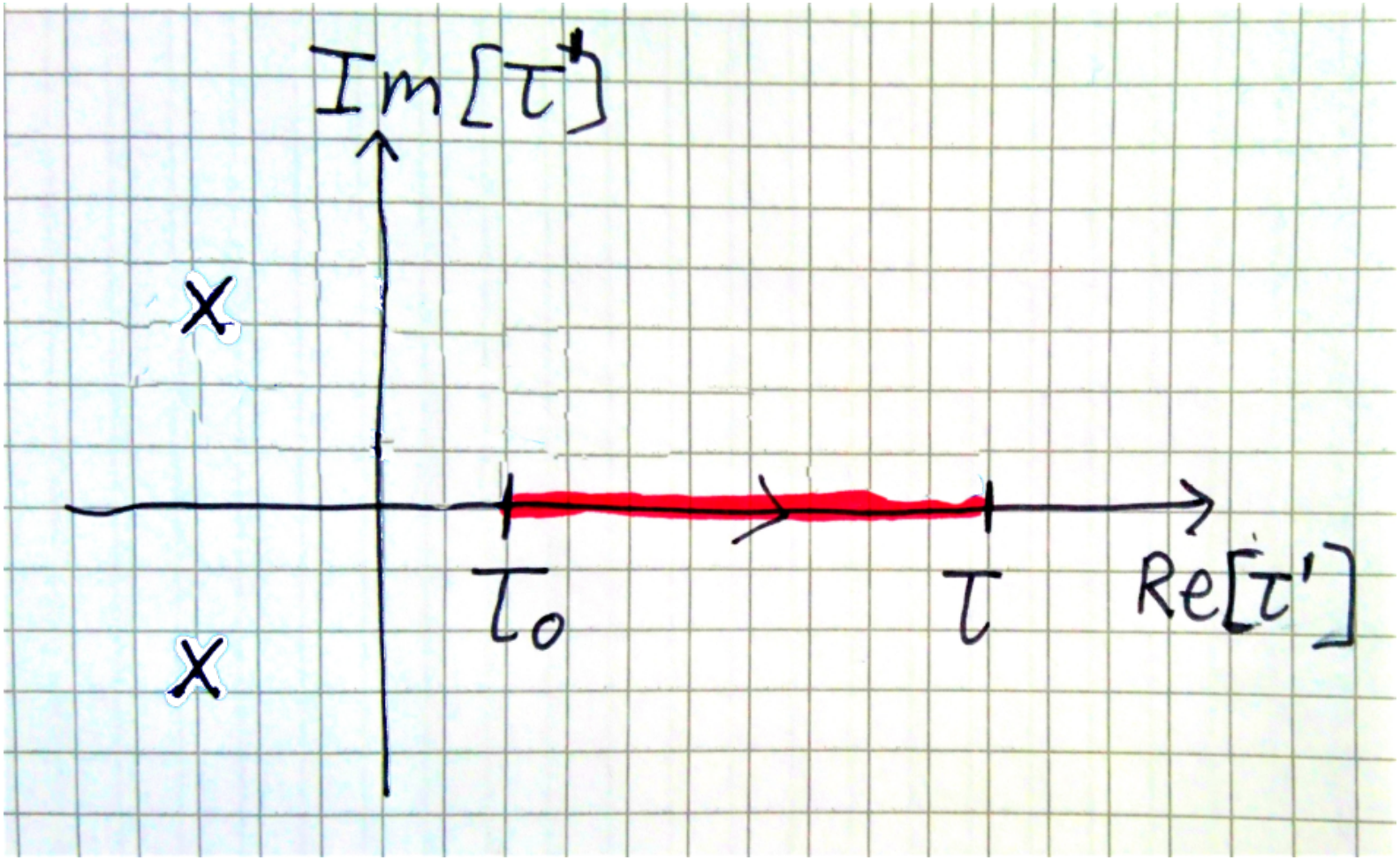
$$\frac{{}_3F_2\left(1, \frac{\beta}{2}+2, \frac{\beta}{2}+2; \frac{\beta}{2}+\frac{5}{2}, \frac{\beta}{2}+3; 1\right)}{2\,\beta^2 + 14\,\beta + 24} + \frac{1}{2(\beta+4)} = 0$$

Allowed power laws β



Leading power law: $\beta = -3.43\ldots$
Logarithmic oscillations: $\Re(\sigma\tau^{\beta}) \propto \tau^{\Re(\beta)} \cos(\theta + \Im(\beta)\log(\tau))$

Which contour to use in the integral equation?



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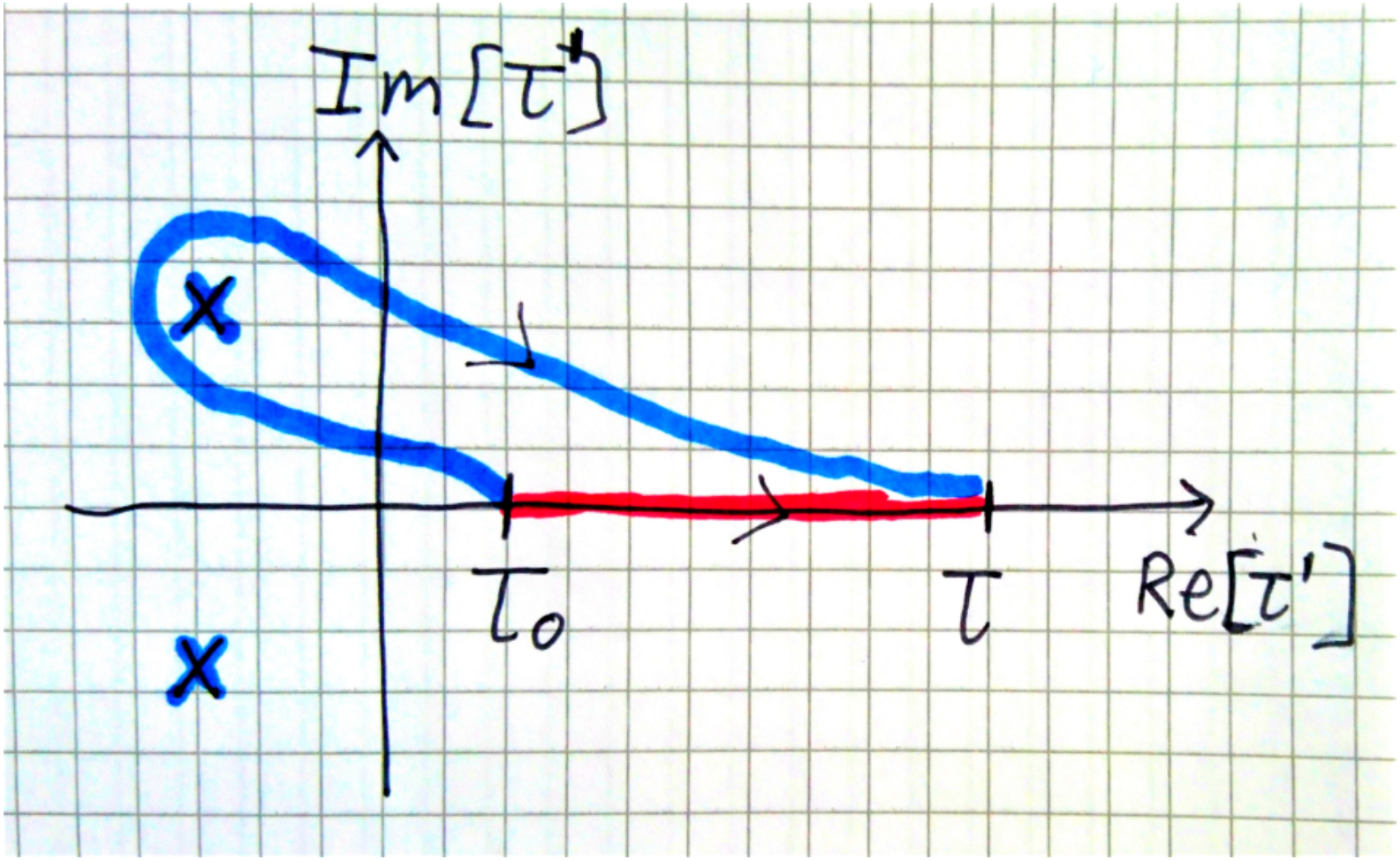

Finding transients in numerical solutions

$$\mathcal{E}(\tau) = D(\tau, \tau_0)\mathcal{E}^0(\tau) + \int_{\tau_0}^{\tau} \frac{\mathrm{d}\tau'}{2\tau_{\mathrm{rel}}} D(\tau, \tau') H\left(\frac{\tau'}{\tau}\right) \mathcal{E}(\tau')$$

Numerically solvable by iteration, starting with $\mathcal{E}^0$

[Phys.Lett. B224, 16 (1989), Banerjee, Bhalerao, Ravishankar]

[1305.7234, Florkowski, Ryblewski, Strickland]

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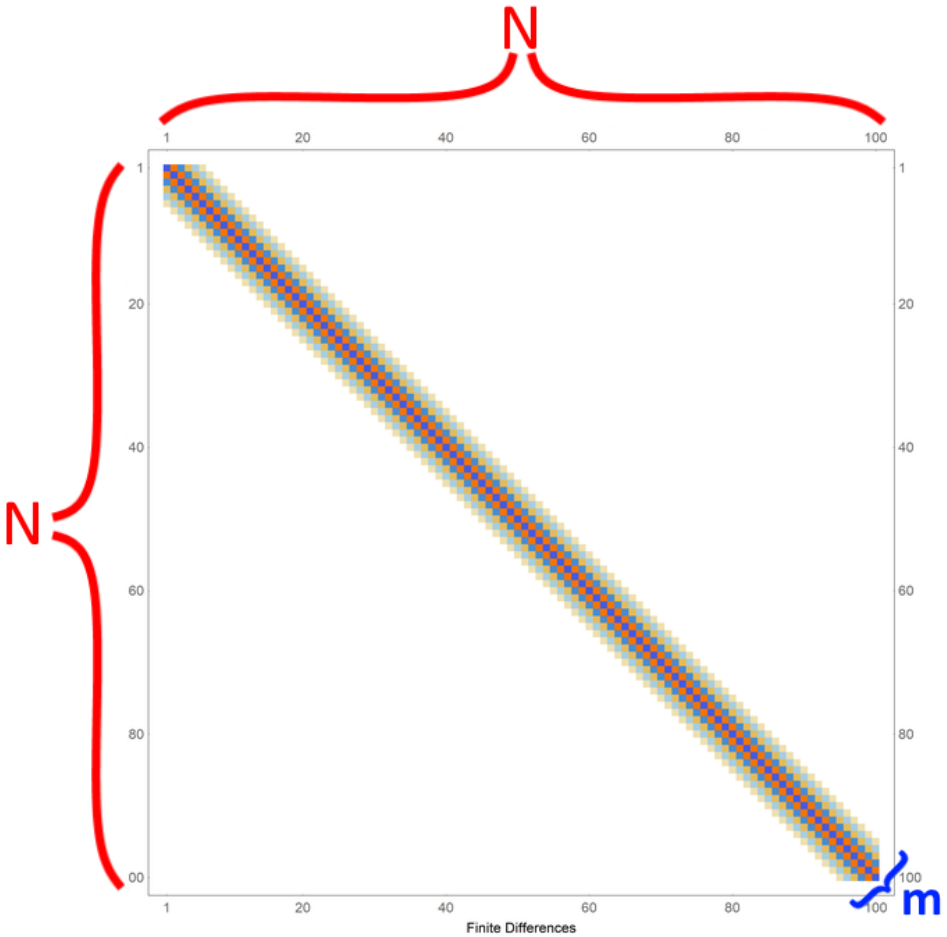
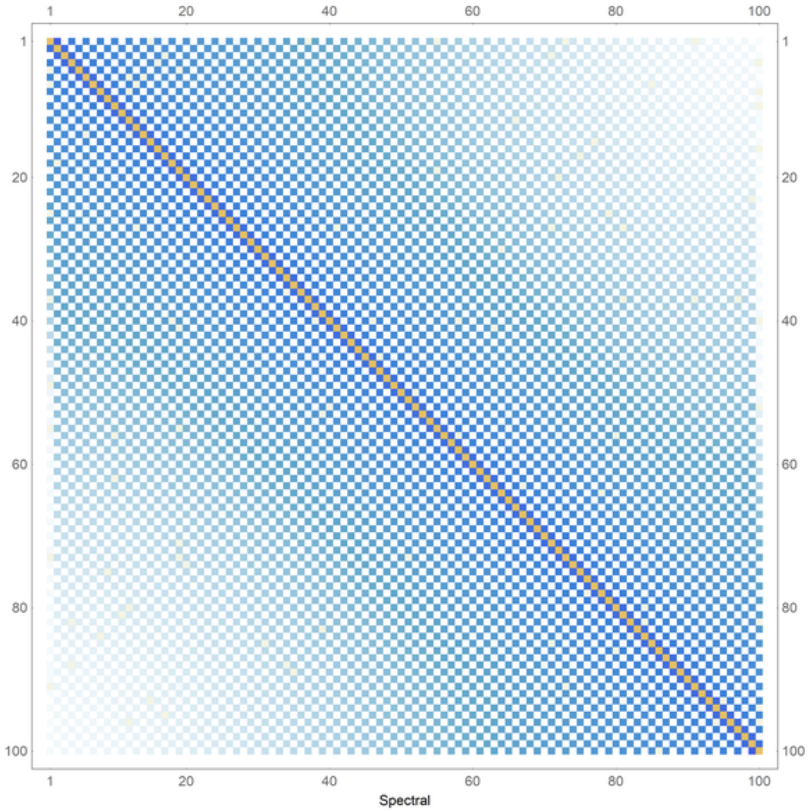
Can we study transients numerically?

Need $w \equiv \frac{\tau}{\tau_{\mathrm{rel}}} \sim 100 \Rightarrow e^{-w} \sim 10^{-50}$

Precision numerics with spectral methods





$error \propto \quad (\#\text{ grid points})^{-m}$

$e^{-\#\text{ grid points}}$

Subtracting solutions to see transients

Gradient expansion of pressure anisotropy $\mathcal{A}(w) = \frac{\mathcal{P}_T(\tau) - \mathcal{P}_L(\tau)}{\mathcal{P}(\tau)}$ is universal

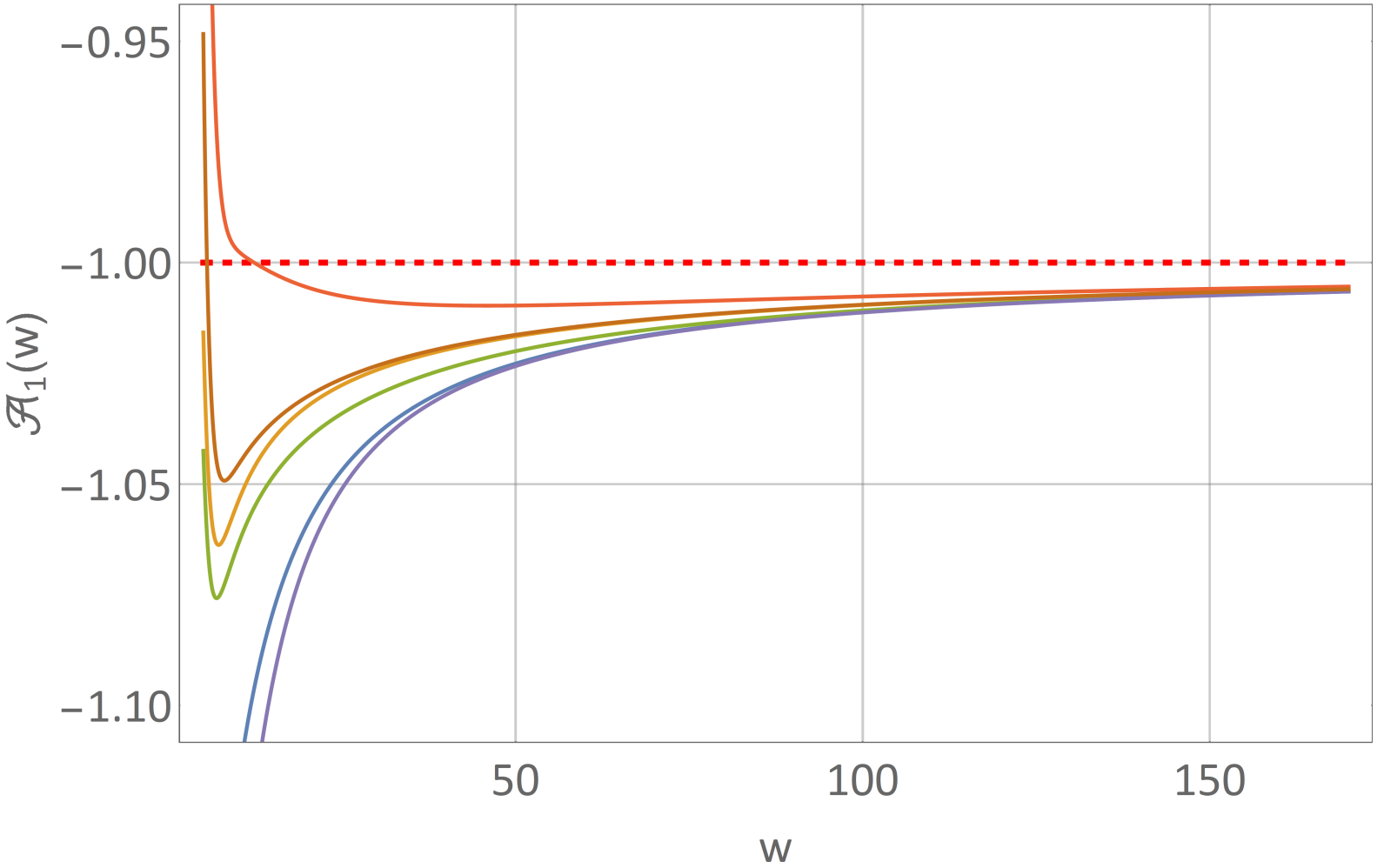
$$\text{Gradient expansion: } \mathcal{A}_0 = \mathcal{A}$$

$$\text{Leading transient: } \mathcal{A}_1 = \frac{\mathrm{d}}{\mathrm{d}w} \log\left(\mathcal{A}_0 - \mathcal{A}_0'\right)$$

$$\text{Subleading transient: } \mathcal{A}_2 = \frac{\mathrm{d}}{\mathrm{d}w} \log\left(\mathcal{A}_1 - \mathcal{A}_1'\right)$$

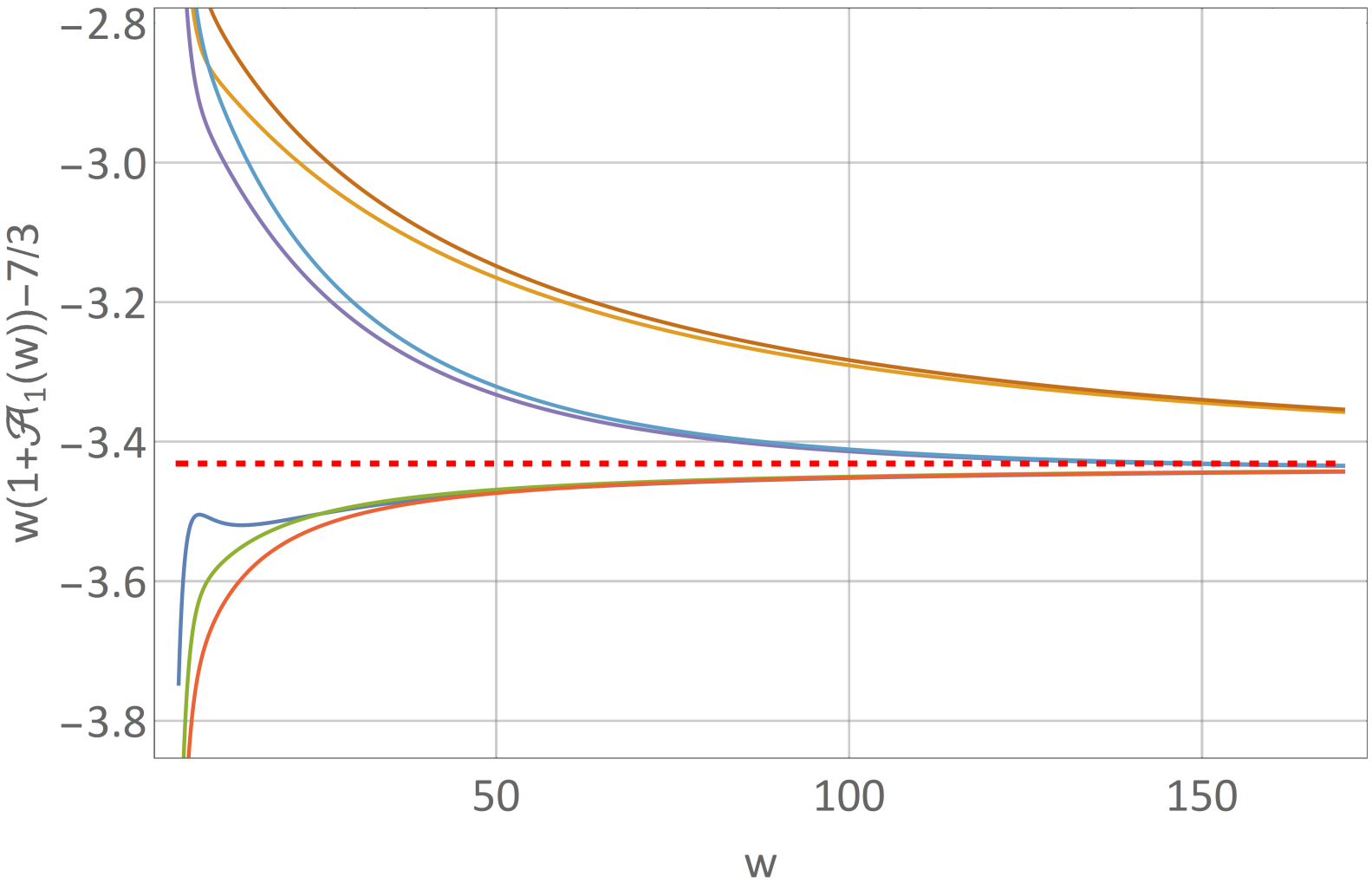
Exponential decay of leading transient

$$\mathcal{A}_1(w) = -1 + \frac{\beta_1 + 7/3}{w} + \dots$$



Power law of leading transient

$$\beta_1 + \ldots = w(\mathcal{A}_1(w) + 1) - 7/3$$



Subleading transient

$$w\mathcal{A}_2(w) \approx -3.0271 - 0.5614 \tan\left(\theta + 0.5614 \log(w)\right)$$

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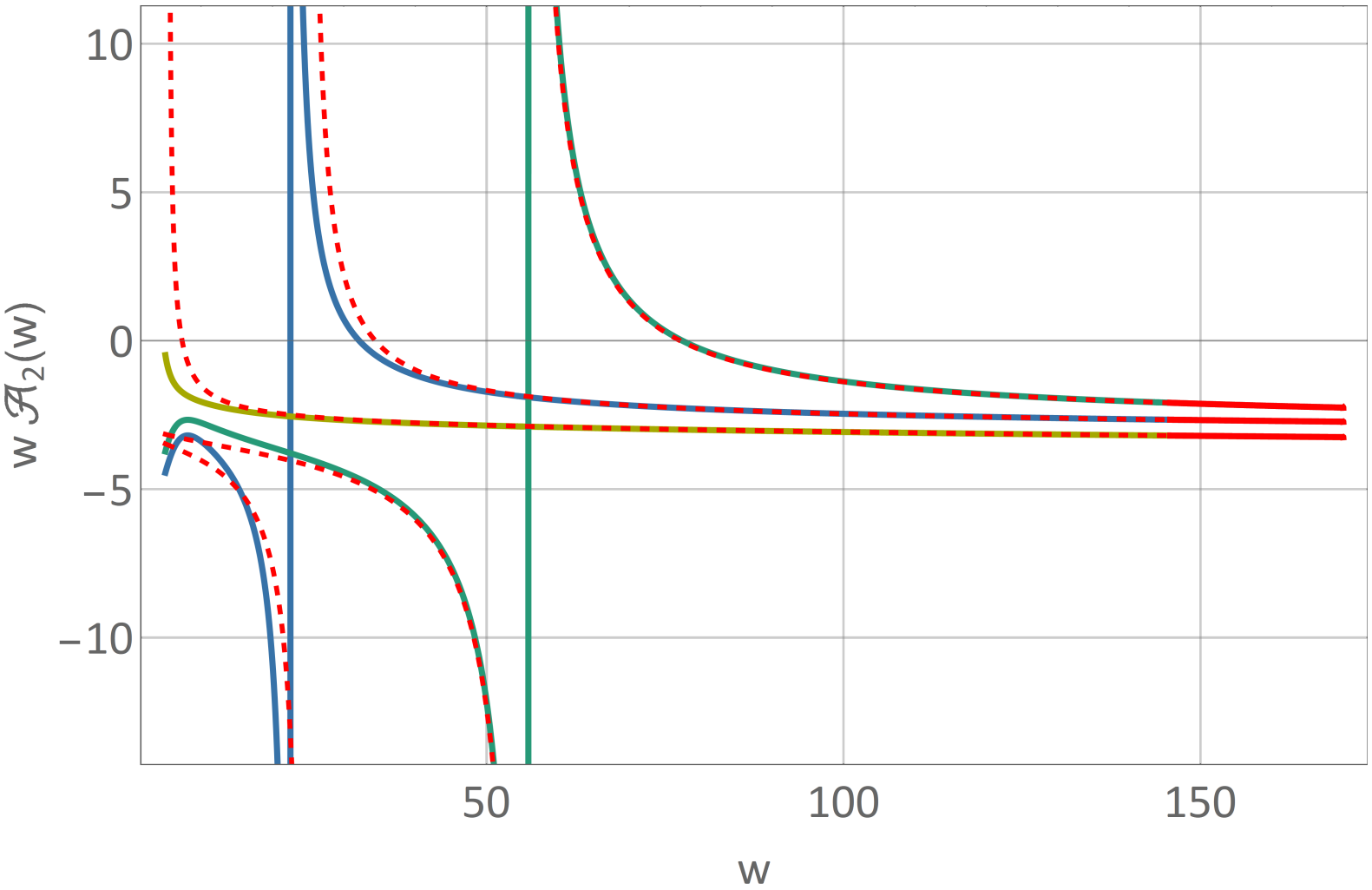
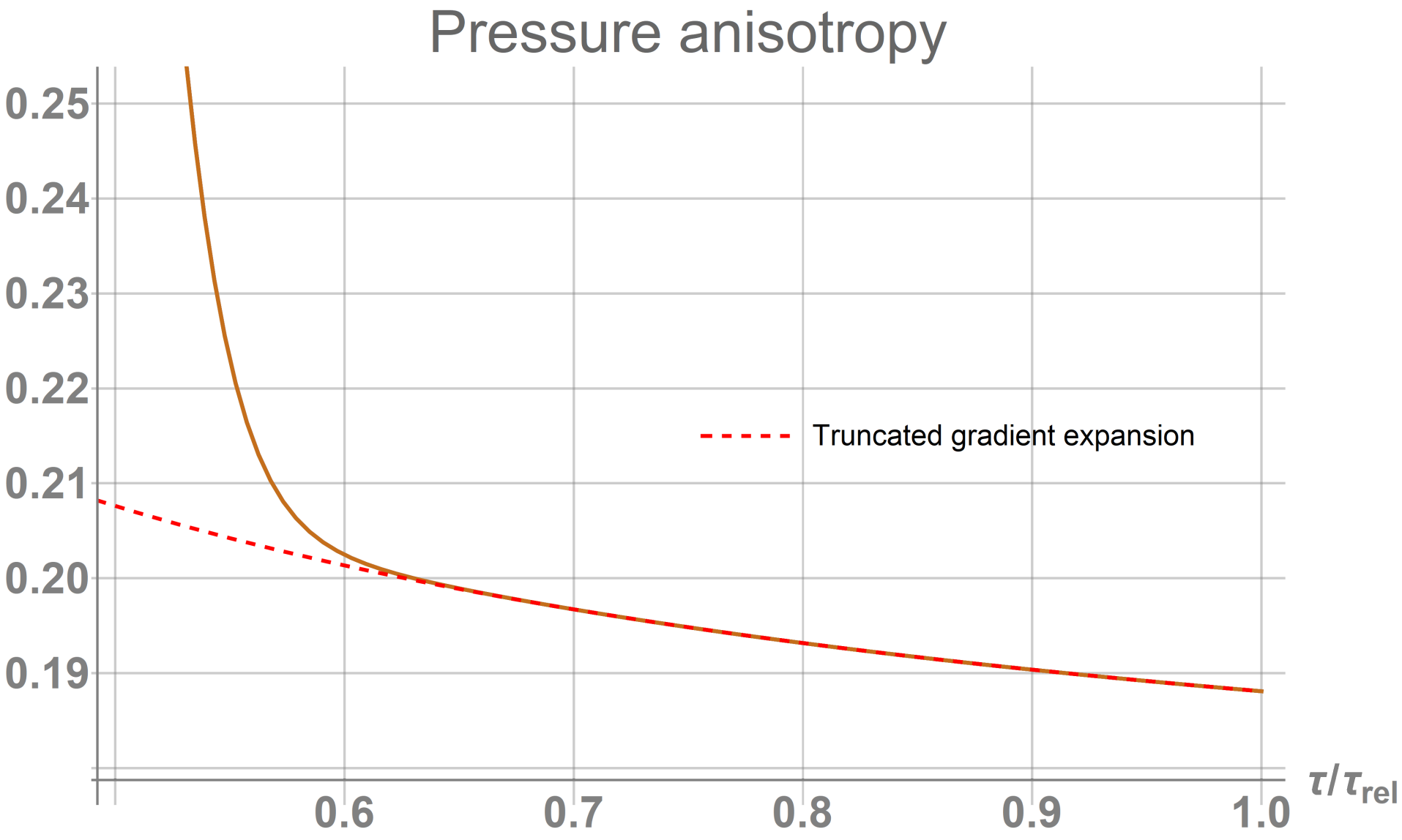


Signatures of off-axis modes at $\Delta = 2.5$



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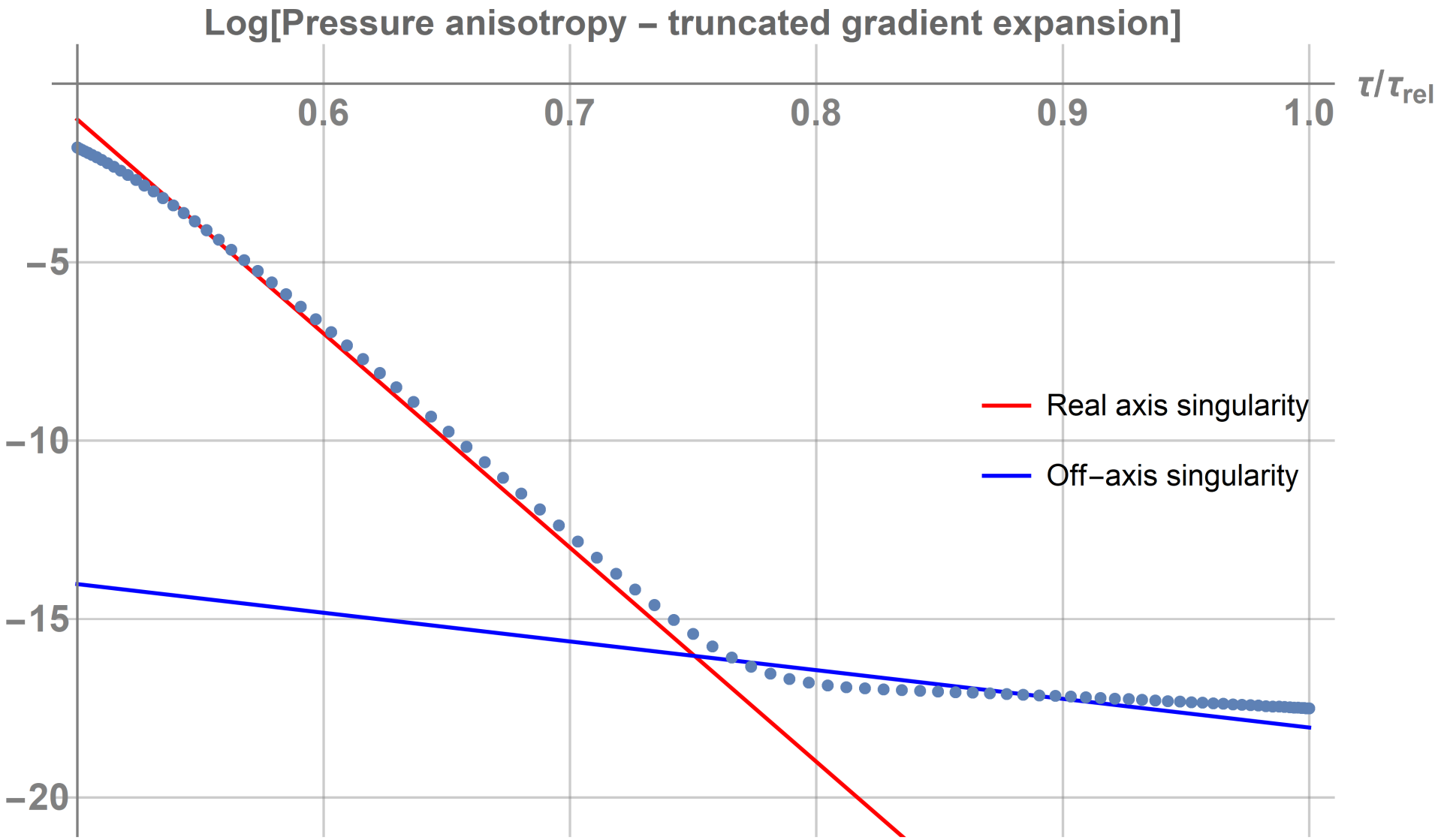
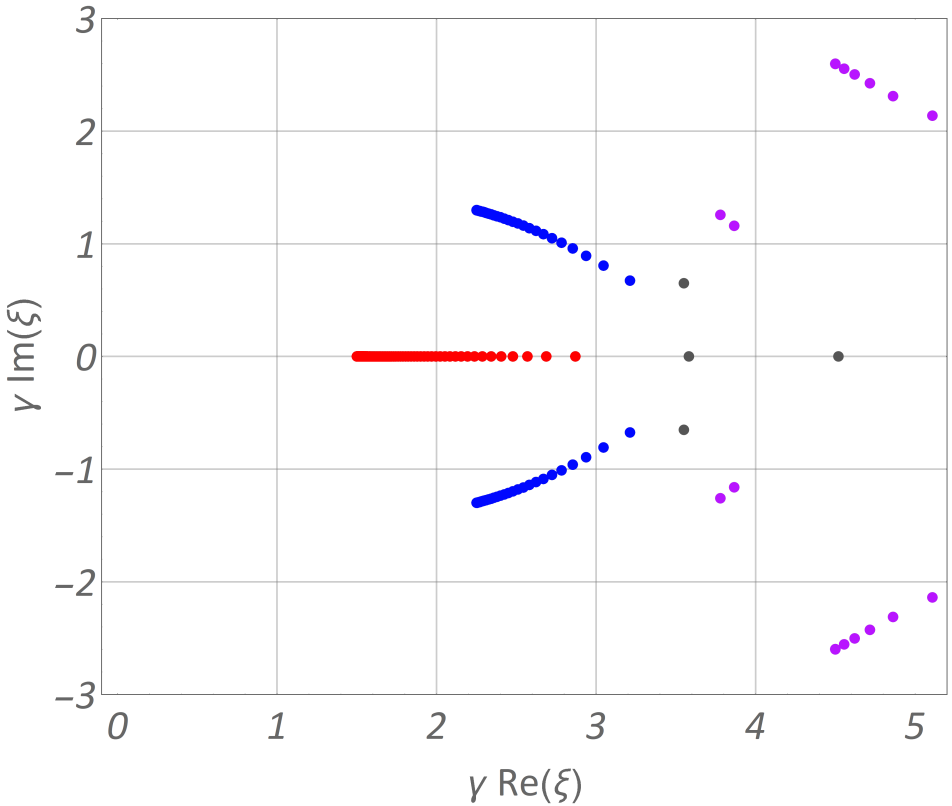


Two types of modes are extractable from the divergent behaviour of the gradient expansion



- Each σ_i is a free parameter
- Off axis modes are fixed

$$\mathcal{E}(\tau) = \mathcal{E}_{ge}(\tau) + e^{-S_{\text{real axis}}\frac{\tau}{\tau_{\text{rel}}}} \sum_i \sigma_i \tau^{\beta_i} \mathcal{E}_{\beta_i}(\tau)$$
$$+ e^{-S_{\text{off axis}}\frac{\tau}{\tau_{\text{rel}}}} \mathcal{E}_{\text{off axis}}(\tau) + \dots$$

Outlook

Three ways to study transients

- Borel transform of gradient expansion
- Perturbations around gradient expansion
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- Borel transform of gradient expansion
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Non-hydrodynamic sector in other models?

- More general relaxation time, e.g. momentum dependent...
- Other flows, e.g. Gubser...
- Beyond RTA: more realistic collision kernels, EKT for QCD...