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Heavy-quark potential from classical statistical real-time lattice simulations

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Some selected references:
A. Lehmann, A.R. JHEP **07** (2021) 067
A.R. "Heavy Quarkonium in Extreme Conditions" Phys.Rept. **858** (2020)

Norwegian Particle, Astroparticle & Cosmology Theory network

Open theory questions

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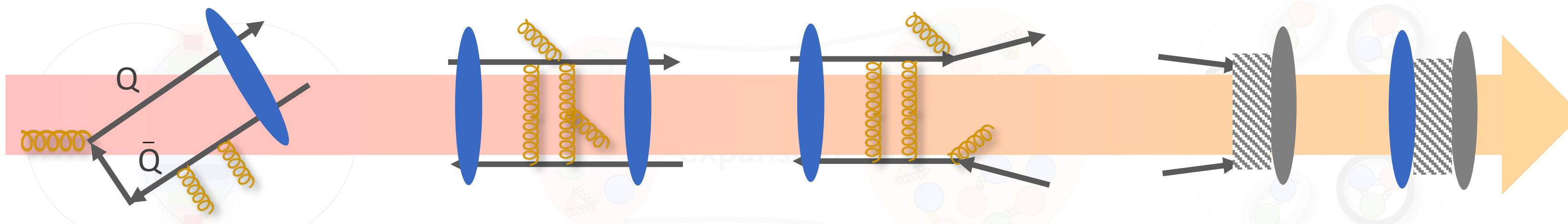
bulk: pre-thermalization   Quark-Gluon-Plasma   hadronization



$Q\bar{Q}$: production /formation   medium interaction   freeze-out

$Q\bar{Q}$ realtime evol. in the initial stages

Formation: probe of strong field QCD and non-equilibrium dynamics

Towards real-time lattice simulations for gluons & heavy quarks

A.Lehmann, A.R. JHEP 07 (2021) 067
R. Larsen, A.R. in progress

Real-time $Q\bar{Q}$ evol. in local thermal equilibrium

Beyond Matsui&Satz:

Open-quantum-systems real-time evolution

see e.g. talk by Peter van der Griend

and also O. Ålund, Y. Akamatsu, F. Laurén, T. Miura, J. Nordström, A.R. JCP 425 (2021) 109917, T. Miura, Y. Akamatsu, M. Asakawa, A.R. PRD 101 (2020) 034011

Properties of equilibrium $Q\bar{Q}$

First principles extraction of **the heavy quark potential**

Extraction of **thermal spectral properties** on the lattice

see talk by Johannes Weber

The two faces of lattice QCD

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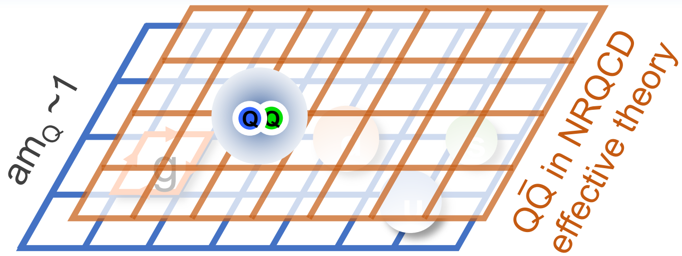
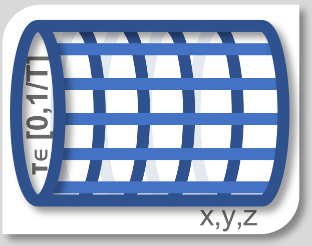


Lattice discretization

- Gauge fields as links: $U_\mu(x)=\exp[\, i\, g\, a_\mu A_\mu(x)\,]$
- Discretized N_f flavors of light fermions on the nodes
- Heavy quarks as relativistic fermion fields OR non-relativistic EFT fields (“without loops”)

Euclidean quantum



Formulated in compact **imaginary time** for MC

Gattringer, Lang, QCD on a lattice
10.1007/978-3-642-01850-3

$$\langle O(\tau)\rangle = \int \mathcal{D}U O(U) e^{-S_E^{QCD}[U]}$$

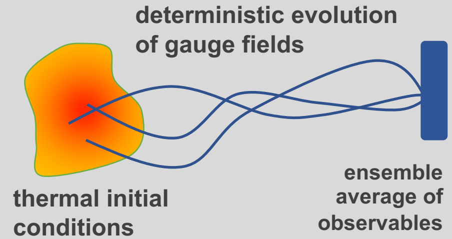
ab-initio sim. of a quantum path integral

$$P[U] \propto e^{-S_E[U,\psi,\bar{\psi}]}$$

$$\langle O\rangle = \frac{1}{N} \lim_{N\to\infty} \sum_{k=1}^{N} O(U^k)$$

return to real-time very costly

Real-time classical statistical



Formulated in **Minkowski time** directly

V. Kasper et. al.
PRD 90, 025016 (2014)

$$\langle O(t)\rangle = \int dE_0 dU_0\, P[U_0, E_0] O(U(t), E(t))$$

valid at highly occupancy: glasma
or deep in the IR: sphaleron transitions

$$P[U_i, E_i]|_{t=0} \sim e^{-H/T}$$

$$\partial_\mu F^a_{\mu\nu}[U,E] = j^a_\nu[\psi]$$

continuum limit intricate & no confinement

Static quark potential at T>0

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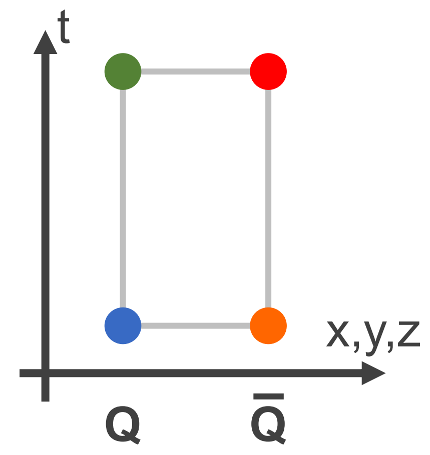
- Simplest model system: **infinitely heavy** color sources

$$\langle (\bar{Q}Q)(\bar{Q}Q)^\dagger \rangle \overset{m_Q \to \infty}{=} W_\square(r,t) = exp\Big[ig \int_\square dz^\mu A_\mu\Big]$$



Central question: can its real-time evolution be understood via a **Schrödinger equation** i.e. by a **potential**?

$$i\partial_t W_\square(r,t) \overset{?}{=} V(r) W_\square(r,t)$$

In general, we know that this is not true (pNRQCD non-potential effects). But with time coarse graining (late times) we may have a chance.

For weak coupling results see
M. Laine et.al. JHEP 03 (2007) 054,
N. Brambilla et.al. PRD 78 (2008) 014017

- If V(r) exists, how to extract it from lattice simulations: **spectral functions**

A.R., T. Hatsuda, S. Sasaki PRL 108 (2012) 162001

$$V(r) = \lim_{t\to\infty} \frac{i\partial_t W_\square(r,t)}{W_\square(r,t)} = \lim_{t\to\infty} \frac{\int d\omega\omega\rho_\square(r,\omega)e^{-i\omega t}}{\int d\omega\rho_\square(r,\omega)e^{-i\omega t}} \qquad W_\square(r,t) = \int_{-\infty}^{\infty} e^{-i\omega t}\rho_\square(r,\omega)d\omega$$

Instead of having to compute the RHS, can we directly read off the values of the potential from ϱ(r,ω) ?

Identifying a static interquark potential (II)

- Starting point: non-relativistic e.o.m. for the Wilson loop (non-perturbative)

argument first presented in Y.Burnier, A.R. Phys.Rev.D 86 (2012) 051503

$$i\partial_t W_\square(r,t) = \underbrace{\Phi(r,t)}_{\in\mathbb{C}}\, W_\square(r,t) \qquad W_\square(r,-t) = W^*_\square(r,t)$$

e.o.m. from QCD: see e.g.
Brambilla et.al. Rev. Mod. Phys. 77, 1423 (2005)

- If a potential description is applicable after coarse graining over $\tau_{\rm pot}$

$$\Phi(r,t) \xrightarrow{t\gg\tau_{pot}} V(r) \qquad i\partial_t W_\square(r,t) = \big\{V(r) + \varphi(r,t)\big\} W_\square(r,t)$$

one can always decompose Φ = Φ - V(r) + V(r) with φ(r,t)= Φ-V(r)

- It is possible to formally solve such an e.o.m. via exponentiation

$$W_\square(r,t) = exp\Big[-i\Big(Re[V](r)\cdot t + Re[\sigma](r,t)\Big) - |Im[V](r)|t + Im[\sigma](r,t)\Big]$$

$$\sigma(r,t) = \int_0^t \varphi(r,t')dt' \quad \sigma_\infty(r) = \sigma(r,|t|>\tau_{pot}) = \int_0^\infty \varphi(r,t')dt'$$

Wilson loop contains both potential (t>$\tau_{\rm pot}$) and non-potential (t<$\tau_{\rm pot}$) physics

Identifying a static interquark potential (III)

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- Spectral function: disentangling of physics from different timescales

$$W_\square(r,t) = exp\Big[-i\Big(Re[V](r)\cdot t + Re[\sigma](r,t)\Big) - |Im[V](r)|t + Im[\sigma](r,t)\Big]$$

real-time Wilson loop and spectral function related via Fourier transform (non-perturbative)

$$\rho_\square(r,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, exp\Big[i\Big(\omega - Re[V](r)\Big)t - iRe[\sigma](r,|t|)sign(t) - |Im[V](r)||t| + Im[\sigma](r,|t|)\Big]$$

- Exploit that σ becomes time independent for |t|> τ_{pot} , its value being σ_∞

$$\rho_\square(r,\omega) = \frac{1}{2\pi}e^{Im[\sigma_\infty](r)}\int_{-\infty}^{\infty} dt\, exp\Big[i\Big(\omega - Re[V](r)\Big)t - |Im[V](r)||t| - iRe[\sigma_\infty](r)sign(t)\Big]$$
$$+\frac{1}{2\pi}\int_{-\tau_{pot}}^{\tau_{pot}} dt\, e^{i\left(\omega - Re[V](r)\right)t - |Im[V](r)||t|}\Big(e^{-iRe[\sigma](r,|t|)sign(t)+Im[\sigma](r,|t|)} - e^{-iRe[\sigma_\infty](r)sign(t)+Im[\sigma_\infty](r)}\Big)$$

First term gives skewed Breit-Wigner around dominant peak

Second contribution: background in which potential peak is embedded

$$\rho_\square(r,\omega) = \frac{1}{\pi}e^{Im[\sigma_\infty](r)}\frac{|Im[V](r)|cos\big[Re[\sigma_\infty](r)\big] - (Re[V]^2(r)-\omega)sin\big[Re[\sigma_\infty](r)\big]}{Im[V]^2(r) + (Re[V]^2(r)-\omega)^2}$$
$$+ c_0(r) + c_1(r)\tau_{pot}(Re[V]^2(r)-\omega) + c_2(r)\tau_{pot}^2(Re[V]^2(r)-\omega)^2 + \ldots$$

Identifying a static interquark potential (IV)

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- Assume potential description viable: dominant skewed Breit Wigner peak

$$\rho_\Box(r,\omega) = \frac{1}{\pi} e^{Im[\sigma_\infty](r)} \frac{|Im[V](r)|cos\big[Re[\sigma_\infty](r)\big] - (Re[V]^2(r)-\omega)sin\big[Re[\sigma_\infty](r)\big]}{Im[V]^2(r) + (Re[V]^2(r)-\omega)^2}$$
$$+ c_0(r) + c_1(r)\tau_{pot}(Re[V]^2(r)-\omega) + c_2(r)\tau_{pot}^2(Re[V]^2(r)-\omega)^2 + \ldots$$

- Inserting our result into the spectral decomposition: only skew-BW relevant

$$V(r) = \lim_{t\to\infty} \frac{i\partial_t W_\Box(r,t)}{W_\Box(r,t)} = \lim_{t\to\infty} \frac{\int d\omega\omega\rho(r,\omega)e^{-i\omega t}}{\int d\omega\rho(r,\omega)e^{-i\omega t}}$$

Position of skew-BW peak: **Re[V](r)**
Width of skew-BW peak: **Im[V](r)**

- Residual non-pot effects important when fitting V(r) from ϱ(r,ω)

mock data tests in HTL perturbation theory: **if skewing is neglected Re[V] overestimated**

shown in detail in Y.Burnier, A.R. Phys.Rev.D 87 (2013) 114019

- Different spectral shapes do not lead to time-independent potential

$$\rho(\omega) \sim A exp\big[-(\omega-\omega_0)^2/\Gamma^2\big] \qquad Re[V](r) = \omega_0 \qquad Im[V](r) = -\Gamma^2 t^2$$

so far skew-BW only derived fit form from Wilson loop e.o.m.

- In absence of any low-frequency poles hiding in the spectral function:
if dominant low-lying skew-BW exists in spectrum, potential picture valid

A potential puzzle

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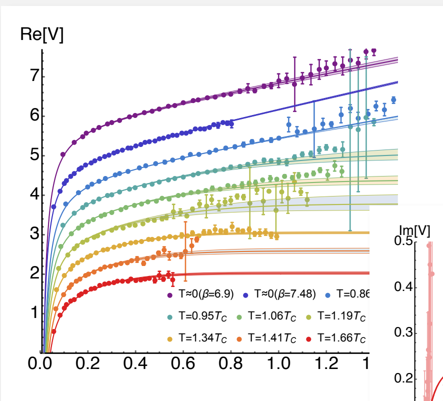
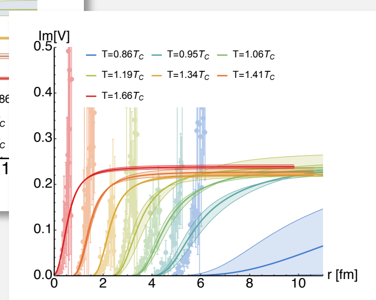
Quantum computation at T>0

- HTL perturbation theory predicts existence of potential with ReV & ImV

A. Beraudo et.al. NPA 806 (2008) 312
M. Laine et.al. JHEP 03 (2007) 054,
N. Brambilla et.al. PRD 78 (2008) 014017





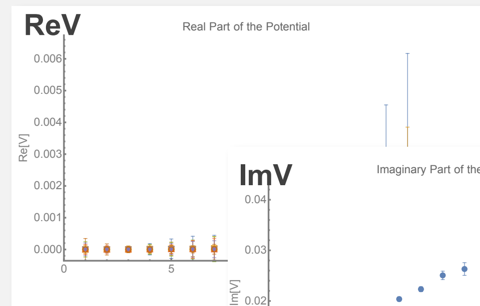
see e.g. Y. Burnier, O. Kaczmarek, A.R. JHEP 12 (2015) 101

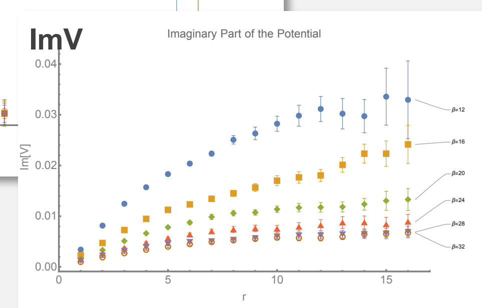
- Multiple lattice evaluations of Wilson correlators observe screened ReV & presence of ImV

Classical statistical computation

- Taking naïve classical limit ħ ⇨ 0 in HTL leads to a vanishing ReV

see Laine, Philipsen, Tassler, JHEP 09 (2007) 066





- Repeating Euclidean recipe for V(r) step by step in classical limit: ReV=0

- Central question: where is the real-part of the classical potential hiding?

Static sources in the path integral

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- Introducing static quarks in the quantum path integral via Wilson loops

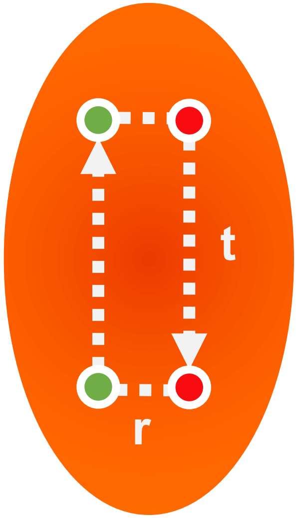
$$\langle W(t, r = |\mathbf{x}_0 - \mathbf{x}_1|)\rangle$$

$$= \frac{1}{Z_{\text{no src}}} \int \mathcal{D}[A] \mathrm{Tr}\Big[\exp\Big[ig \int dt A_0(\mathbf{x}_0, t)\Big] \exp\Big[-ig \int dt A_0(\mathbf{x}_1, t)\Big]\Big] e^{\,\mathrm{i} S_{\text{gluon}}[A]}$$

$$= \frac{1}{Z_{\text{no src}}} \int \mathcal{D}[A] \exp\Big[\ \mathrm{i}\, S_{\text{gluon}}[A] - \mathrm{i}\, g \int d^4x\, \mathrm{ReTr}\big[A_0(x) j_0(x, \mathbf{x}_0, \mathbf{x}_1)\big]\Big]$$

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x} - \mathbf{x}_0) - \delta^{(3)}(\mathbf{x} - \mathbf{x}_1)\Big)$$

$$= \frac{Z_{\text{src}}(t, r)}{Z_{\text{no src}}}.$$



Evaluating W(t,r) amounts to reweighting to a theory with static sources!

In the naïve classical limit

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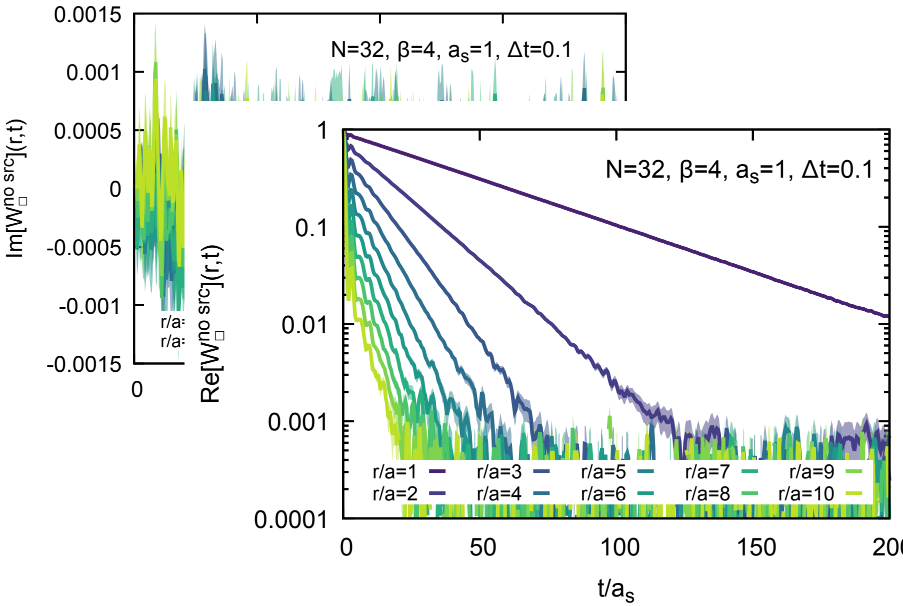
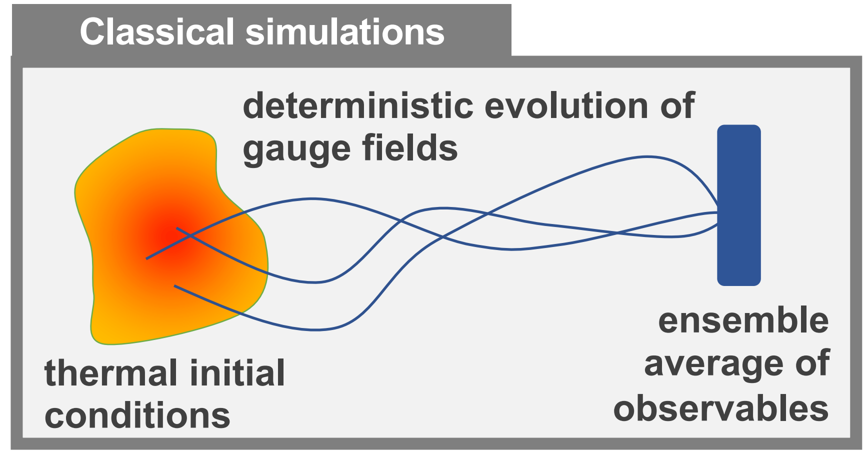
- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

- In the literature: compute Wilson loop in a thermal ensemble of gauge fields

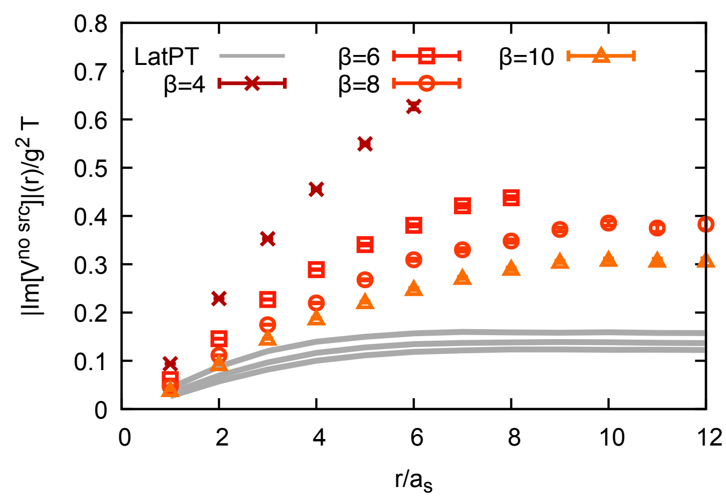
first in Laine, Philipsen, Tassler, JHEP 09 (2007) 066



A. Lehmann, A.R. arXiv:2012.10089



$$\langle W_\square\rangle(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-tImV^{(cl)}(r)]$$



A. Lehmann, A.R. JHEP 07 (2021) 067

- ImV increases with distance & temperature, qualitatively similar to quantum result

Static sources in classical LGT

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- Where do the static sources enter in the classical theory?

see e.g. Kasper, Hebenstreit, Berges, PRD 90 (2014) 025016

$$Z_{\rm src} = \int \mathcal{D}A(t=0) \int \mathcal{D}\Pi(t=0)\rho(A,\Pi,t=0)\delta\Big(D_\mu F^{\mu\nu}[A] - j^\nu\Big)$$

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) = \rho(\mathbf{x};\mathbf{x}_0,\mathbf{x}_1)$$

- Evaluating the Wilson loop in the classical simulation does NOT introduce the sources but instead we need to enforce a modified Gauss-Law by hand:

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \Big[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t)E_i(\mathbf{x}-i,t)U^\dagger_{-i}(\mathbf{x},t)\Big] - \rho(\mathbf{x};\mathbf{x}_0,\mathbf{x}_1) = 0$$

- Gauss-Law plays a central role in drawing the stochastic initial conditions

Initial set of U and E: $\exp[-\beta_G H_{cl}] \prod_{\mathbf{x}} \delta(G(\mathbf{x}))$ → Evolve e.o.m. → Compute Wilson loop

- Project onto **G[ρ]=0** surface
- Symplectic leap-frog scheme for E and U

Proper Gauss law & ReV

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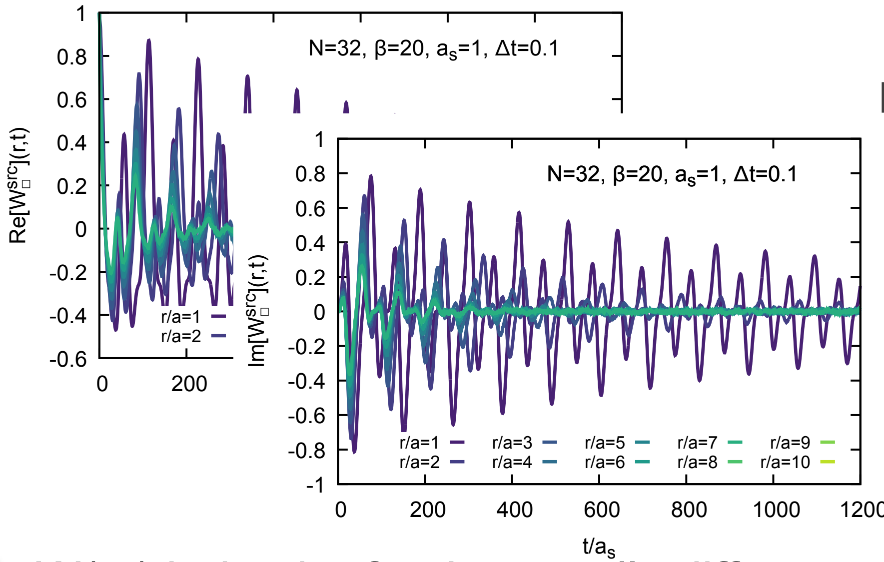
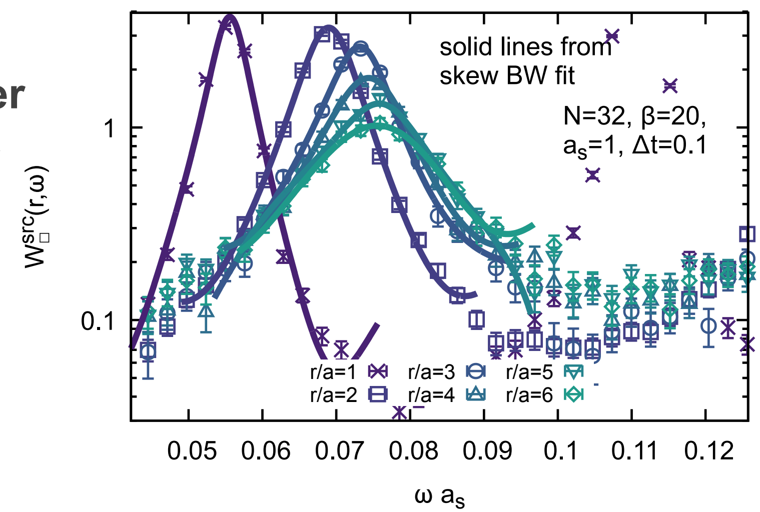
- Solving the Yang-Mills equation of motion with the proper Gauss-Law.

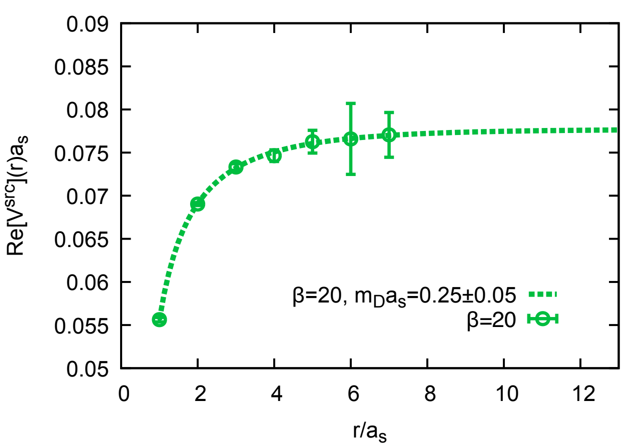
A. Lehmann, A.R. JHEP 07 (2021) 067



Fourier



- W(r,t) behavior fundamentally different, oscillations indicate presence of ReV

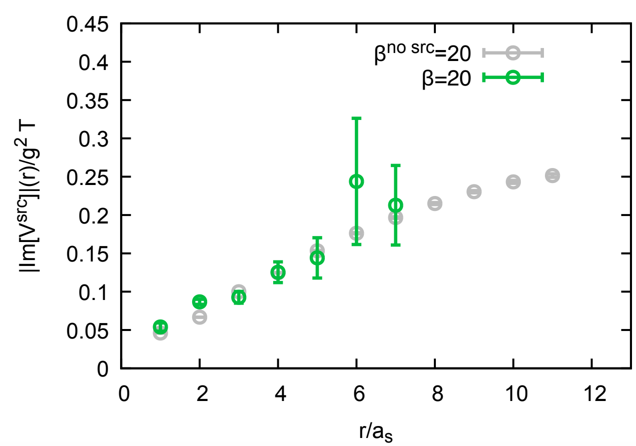


Debye screened ReV

$$ReV \sim exp[-m_D r]/r$$

$$m_D^{PT} a = 0.275$$

$$m_D^{CL} a = 0.25 \pm 0.05$$



Next steps: finite mass quarks

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- Non-relativistic heavy quarks in terms of Pauli spinors (NRQCD: expansion in v)
- Hamiltonian to order $O(v^3)$ with leading order Wilson coefficients

$$H^\psi = -\frac{\mathbf{D}^2}{2M} - c_1 \frac{g}{2M}\sigma\cdot\mathbf{B} - c_2\frac{g}{8M^2}\mathbf{D}\cdot\mathbf{E} - c_3\frac{ig}{8M^2}\sigma\cdot\left(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D}\right)$$

- Retarded real-time correlator simplifies: absence spontaneous quark pair creation

$$D_V^R(t) = \int DU G^\psi \sigma^i G^{\chi,\dagger}\sigma_i e^{iS[U]}$$

- Need to realiably compute heavy (anti-)quark motion via Crank-Nicholson

$$G(t+a_t) = \left(1+\frac{ia_t}{2}H[U(t)]\right)^{-1}\cdot\left(1-\frac{ia_t}{2}H[U(t)]\right)G(t)$$

- Spectral function via Wigner transform in relative time coordinate $s=t_1-t_2$

$$\rho_V(t=(t_1+t_2)/2,\omega,\mathbf{p}=0) = 2Im[\int_0^{s_{max}} D_V^R(t+\frac{s}{2},t-\frac{s}{2},\mathbf{p}=0)e^{-i\omega s}ds$$

1st look at NRQCD spectral funtions

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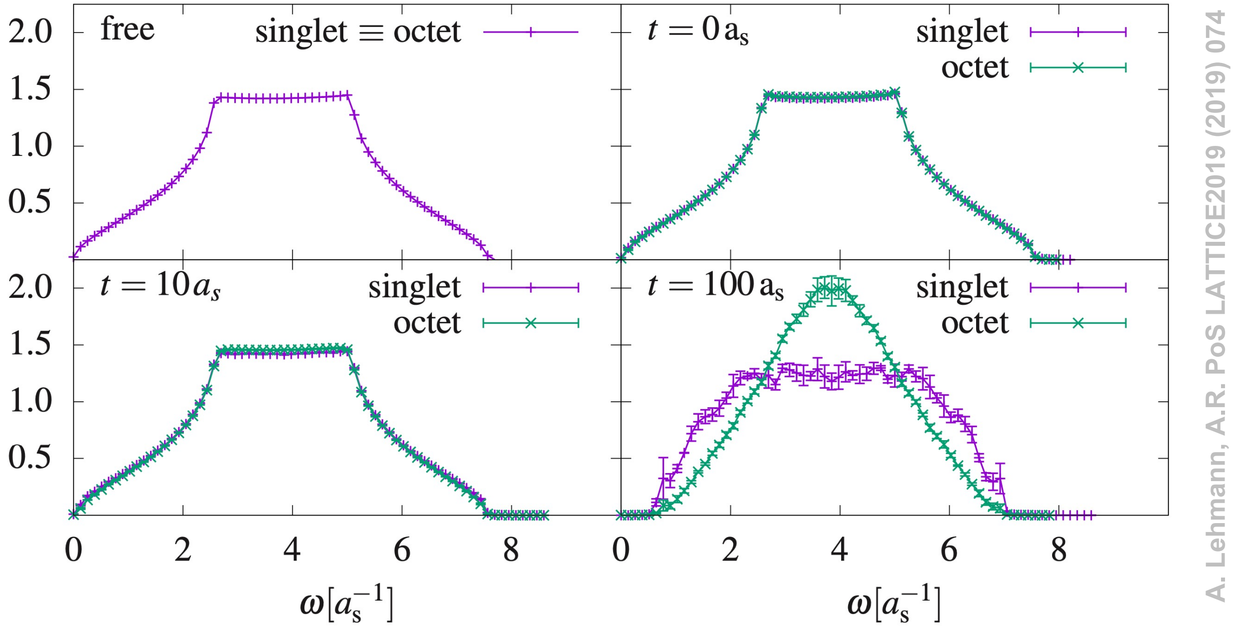
- In the absence of back-reaction: no indications for binding expected
- At initial time no interactions yet: spectrum appears as free $Q\bar{Q}$ pair



- Over time, interactions with surrounding thermal medium reduce spectral weight for singlet and increase that of octet: gluon absorption at work!
- Next step: implement the back reaction of the $Q\bar{Q}$ pair on gauge fields (in progress)

Summary and outlook

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- Based on **generic properties** of the Wilson loop we can establish **non-perturbatively** that its spectral function exhibits **skewed BW peak** if potential description is viable at late times.

- Treatment of **static sources** in the path integral and in classical statistical approximation different: **reweighting** vs. explicitly **modified Gauss-Law**

- Incorporation of proper Gauss-law in **initial conditions** changes Wilson loops: ensuing oscillatory behavior recovers **screened ReV**

- How to implement finite mass heavy quarks?
  - Relevant theory tool: real-time NRQCD on the lattice
  - Together with Rasmus Larsen: implement the backreaction of NRQCD heavy quarks onto the classical statistical gauge field simulations first steps without backreaction: A. Lehmann, A.R. PoS LATTICE2019 (2019) 074