Scattering amplitudes from spectral functions (and related ideas)

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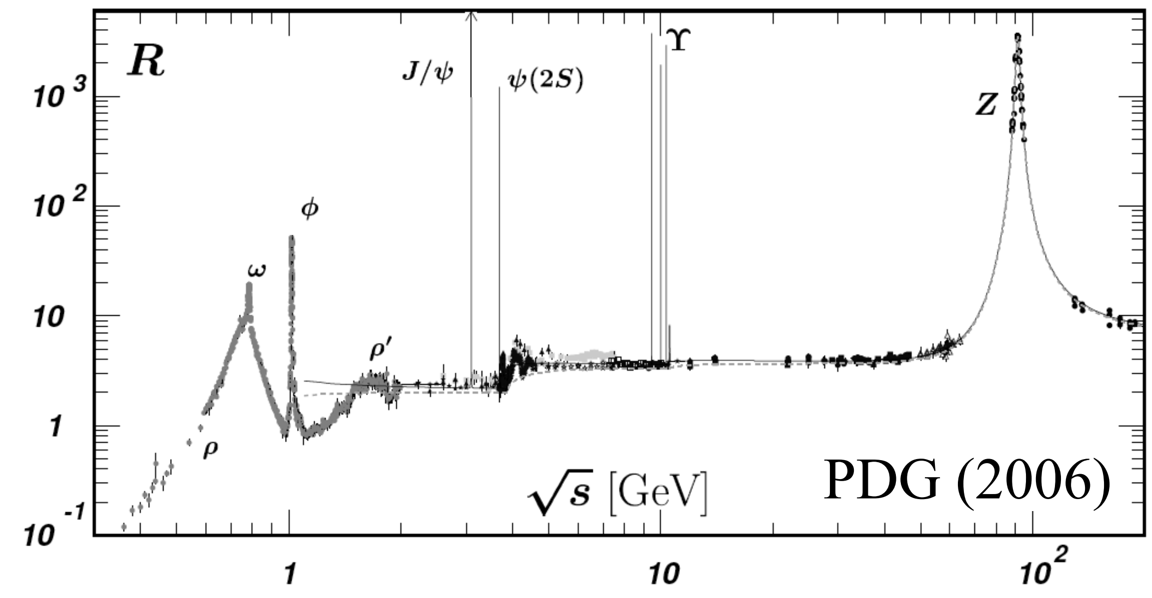
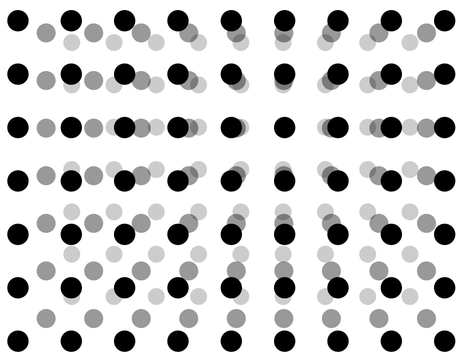
Perspectives on Euclidean

- R-ratio based (g-2) determination

$$a^{\mathrm{HVP,LO}}(m_\mu) = \int_{4M_\pi^2}^{\infty} ds\, K(m_\mu, s)\, R(s)$$



- Integral can be Wick rotated to Euclidean signature



$$G(\tau) = \int_{4M_\pi^2}^{\infty} ds\, g(\tau, s)\, R(s) \qquad g(\tau, s) \propto \sqrt{s}e^{-\sqrt{s}\tau}$$

- Can we find a function $\mathcal{K}(m_\mu, \tau)$? such that…

$$\sum_\tau \mathcal{K}(m_\mu, \tau)\, g(\tau, s) = K(m_\mu, s) \longrightarrow a^{\mathrm{HVP,LO}}(m_\mu) = \sum_\tau \mathcal{K}(m_\mu, \tau)\, G(\tau)$$

This defines the standard kernel

Reconstruction methods

- ***Linear, model-independent reconstruction*** (e.g. Backus Gilbert)

$$\sum_\tau \mathcal{K}(\bar{\omega},\tau)G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega},\tau)\int d\omega\, e^{-\omega\tau}\,\rho(\omega)$$
$$= \int d\omega\,\Big[\sum_\tau \mathcal{K}(\bar{\omega},\tau)\,e^{-\omega\tau}\Big]\,\rho(\omega)$$
$$= \int d\omega\,\widehat{\delta}_\Delta(\bar{\omega},\omega)\,\rho(\omega)$$

δ is exactly known

- Maximum Entropy Method (MEM)
- Direct fits
- Neural networks

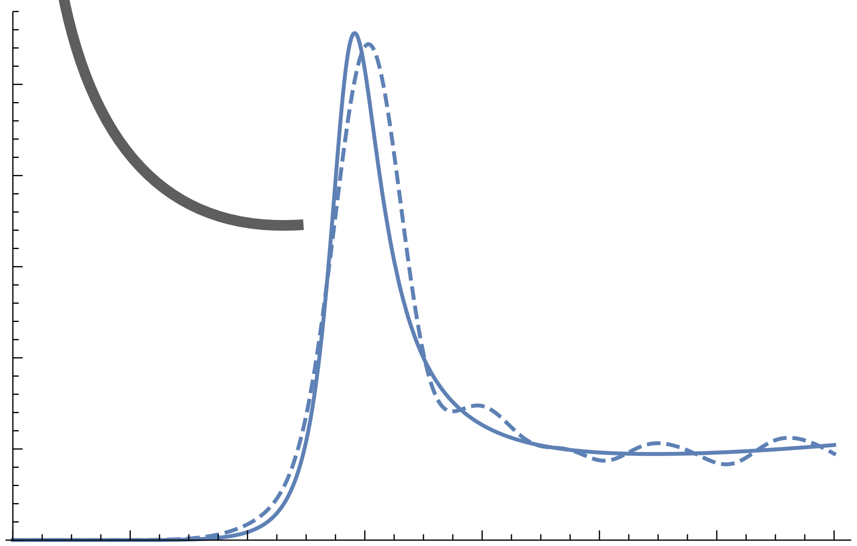
← Not discussed here…

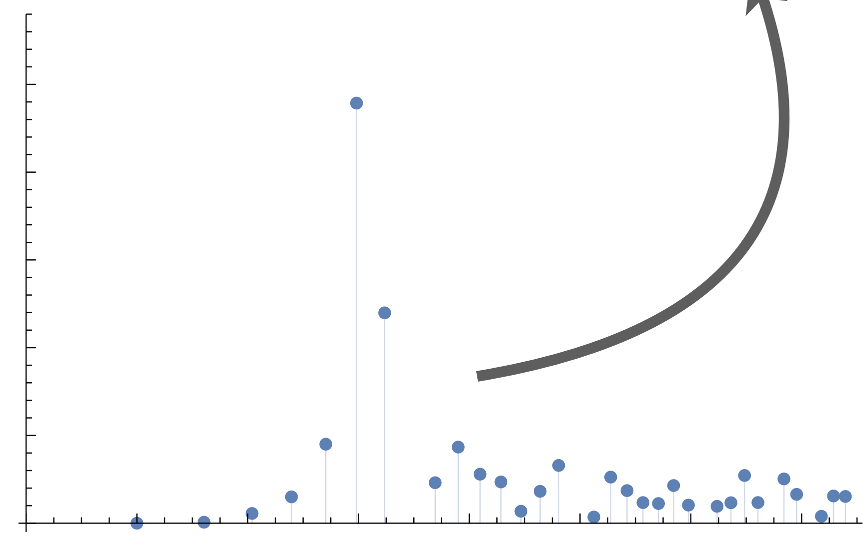
- Key idea here… we aim only to construct

$$\widehat{\rho}(\bar{\omega}) \equiv \int_0^\infty d\omega\,\widehat{\delta}_\Delta(\bar{\omega},\omega)\,\rho(\omega)$$

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega) \qquad G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- ☐ Any reconstructed spectral function that ≠ forest of deltas…

  contains implicit smearing (or else $L \to \infty$)

We require…

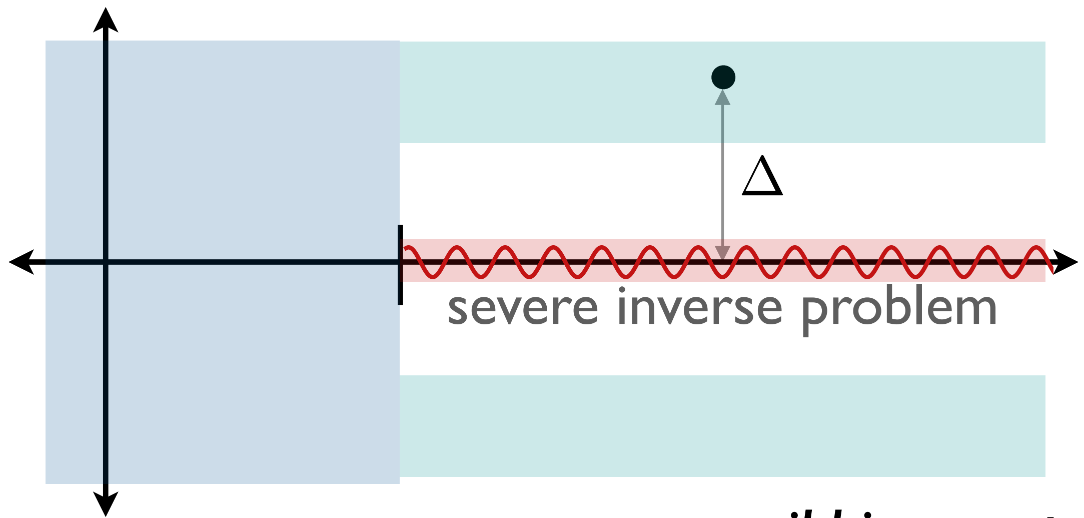
$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

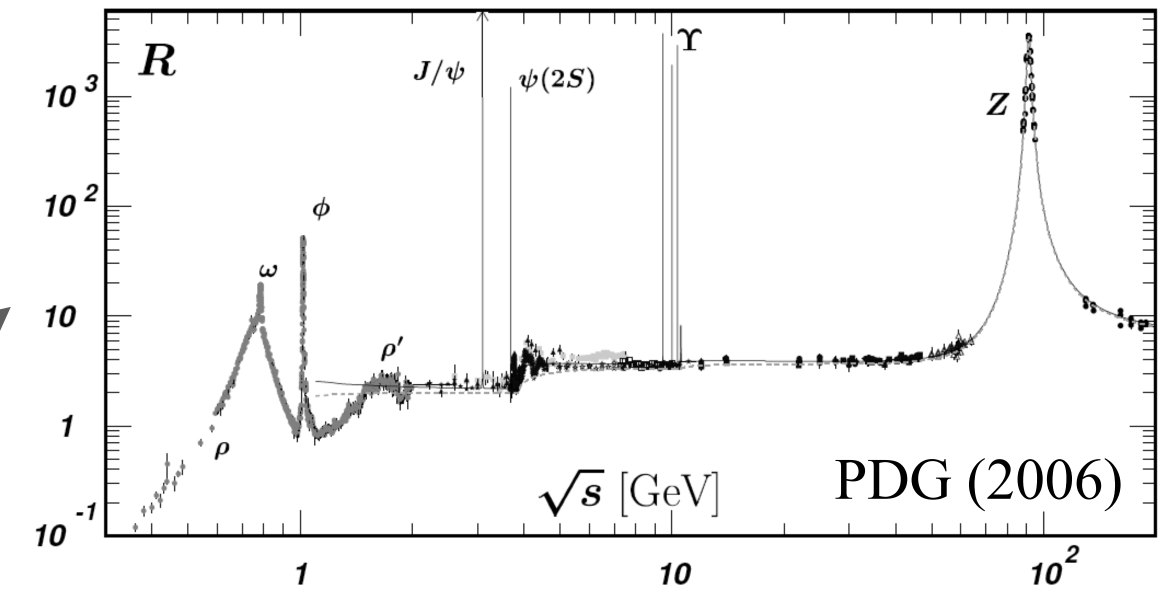
R ratio

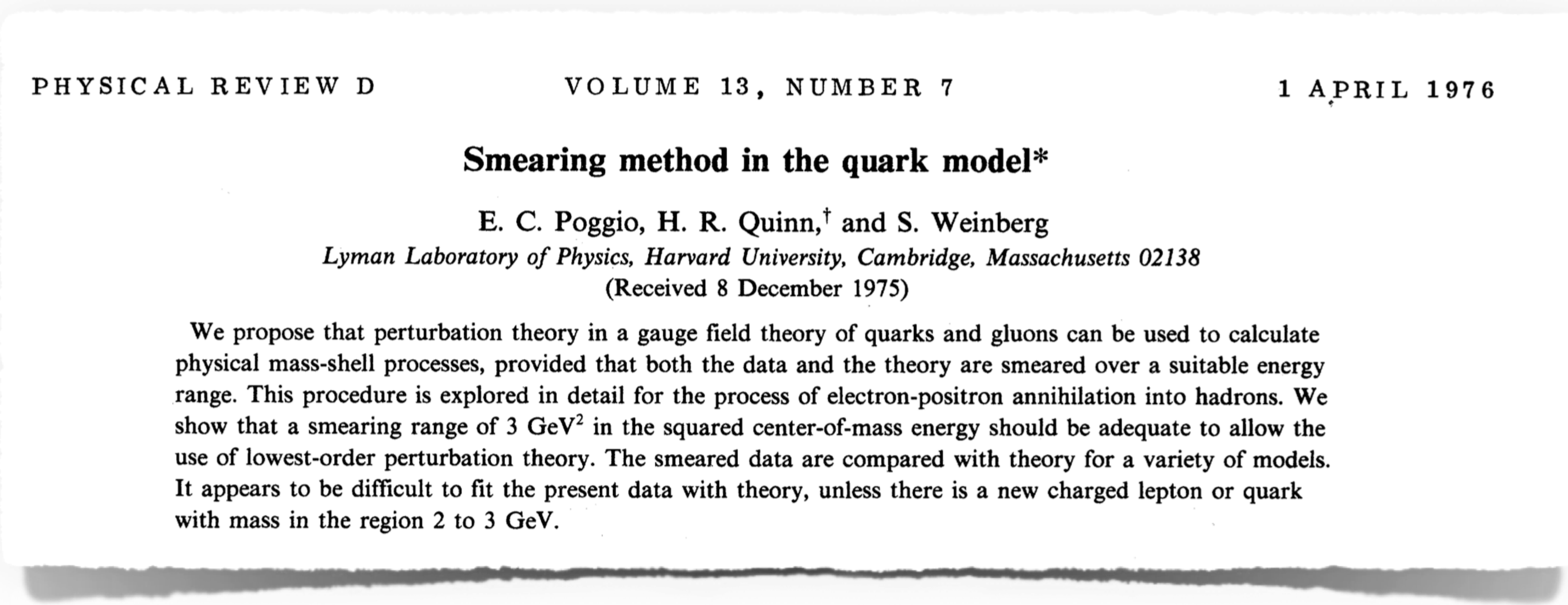


no inverse problem

Hashimoto, 2019 $\frac{1}{\bar{\omega} - \omega}$

mild inverse problem?





PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

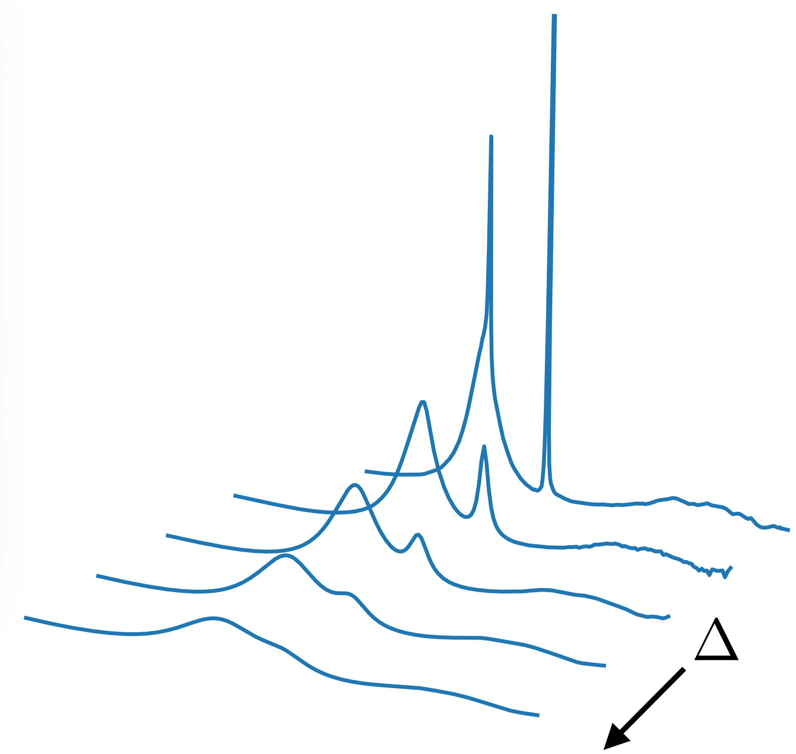
Smearing method in the quark model*

E. C. Poggio, H. R. Quinn,† and S. Weinberg
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 GeV^2 in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

$$\widehat{R}_\Delta(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \; \frac{R(s')}{(s'-s)^2 + \Delta^2}$$

into the complex plane!

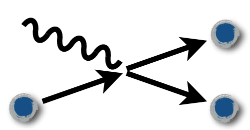
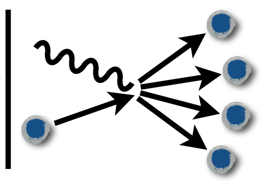
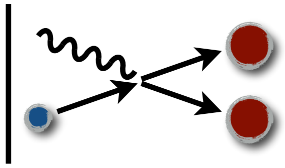


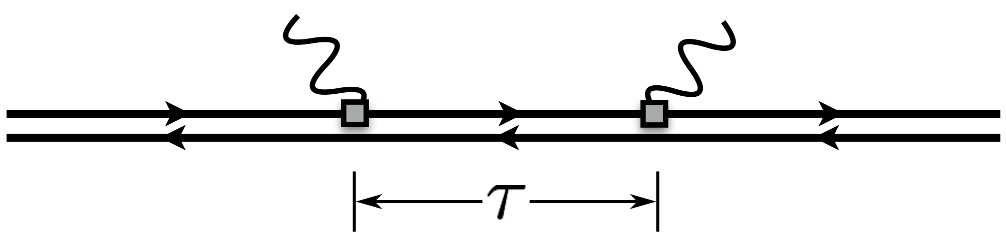
courtesy of M. Bruno
(using F. Jegerlehner's ***alphaQED***)

Total rate based applications

- Hadronic tensor, heavy flavor lifetimes

$$W_{\mu\nu}(p,q) \equiv \int d^4x \; e^{iqx} \langle \pi, p | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, p \rangle$$

$\propto \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$



$$= \int_0^\infty d\omega \, e^{-\omega\tau} \, W_{\mu\nu}(p,q)_{\omega = p^0 + q^0}$$

$$W_{\mu\nu} = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{W}_{\mu\nu;\Delta,L}$$

- What about ***scattering*** and ***transition amplitudes?***

MTH, Meyer, Robaina (2017)

Amplitudes from spectral functions

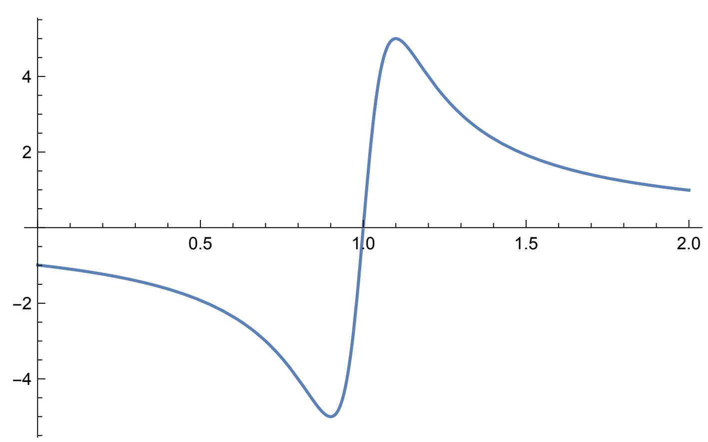
- First define the Euclidean correlator

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L = \int_0^\infty dE_3 \, \rho(E_3) e^{-E_3 \tau_3}$$

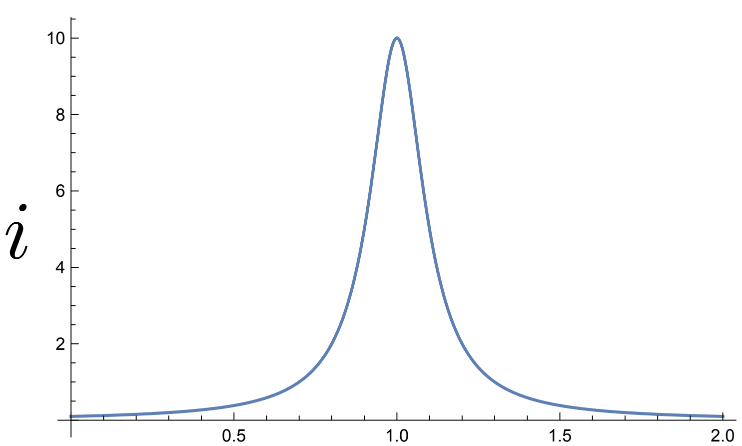
and the smeared spectral function

$$\widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(q_3) = \int_0^\infty dE_3 \, \frac{1}{q_3^0 - E_3 + i\epsilon} \, \rho(E_3) = \int_0^\infty dE_3 \, \widehat{\delta}_\epsilon(q_3^0, E_2) \, \rho(E_3)$$

$$\frac{1}{q_3^0 - E_3 + i\epsilon} =$$



$$-i$$



Bulava, MTH (2019)

Amplitudes from spectral functions

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- Next project *on shell* at finite ϵ

$$\mathcal{M}_{\rm c}^{L,\epsilon}(p_4 p_3 | p_2 p_1) \equiv \frac{2E(\boldsymbol{p}_3)}{Z^{1/2}(\boldsymbol{p}_3)} \frac{2E(\boldsymbol{p}_2)}{Z^{1/2}(\boldsymbol{p}_2)} \epsilon^2 \, \widehat{\rho}^{L,\epsilon}_{\boldsymbol{p}_4 \boldsymbol{p}_1}(E(\boldsymbol{p}_3), \boldsymbol{p}_3)$$

- Finally project out the ***scattering amplitude***

$$\mathcal{M}_{\rm c}(p_4 p_3 | p_2 p_1) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \mathcal{M}_{\rm c}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

Bulava, MTH (2019)

Some comments

- Derivation based in modified **LSZ** + *signature-independence* of ρ(E)
- Holds when LSZ holds

$$\langle m, \mathrm{out} | n, \mathrm{in} \rangle \qquad \langle m, \mathrm{out} | \mathcal{J}(0) | n, \mathrm{in} \rangle$$

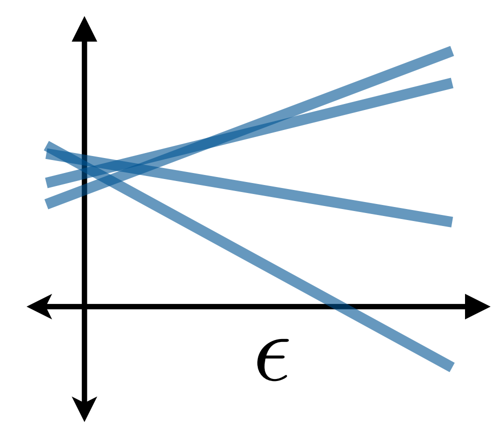
- Very challenging… but systematic

  for some (unknown) volume + correlator quality, we can get $D \to \pi\pi, K\overline{K}$

- Some nice features…

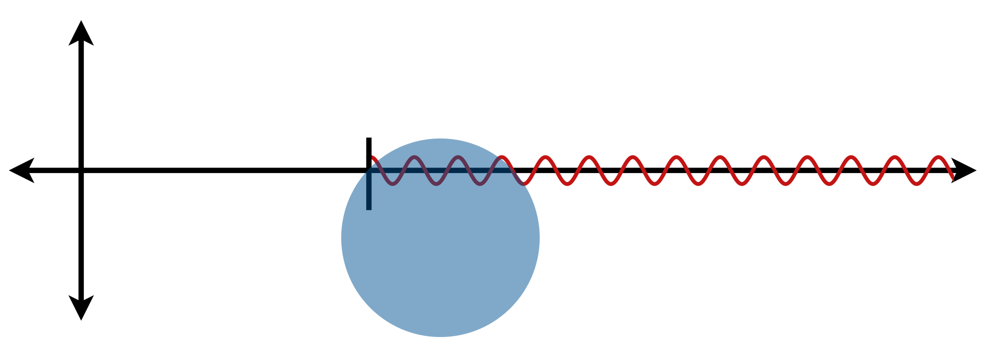
  GEVP-like operator freedom

$$G_L^{[ab]}(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi^a(\tau_3, \boldsymbol{p}_3) \pi^b(0) | \pi_{\boldsymbol{p}_1} \rangle_L \longrightarrow \mathcal{M}^{[ab],\epsilon,L}$$



Finite range of analyticity in ϵ

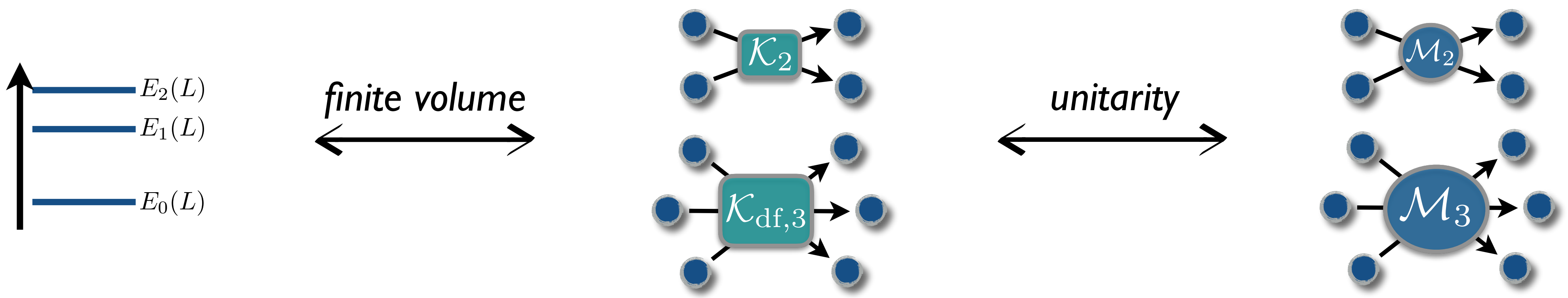
Uses overlaps



Bulava, MTH (2019)

Interlude…

- This is an alternative to finite-volume methods



- Proven very powerful and are here to stay

- Spectral function methods will be most competitive in multi-channel regime

- Biggest impact may be lower-precision long distance effects
  e.g. QED corrections of D decays with multi-hadron intermediate states

- Spectral function methods use ***matrix elements*** and ***energies***

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014) • Briceño, Davoudi (2015) • Mai, Döring • Hansen, Sharpe

Perturbative study…

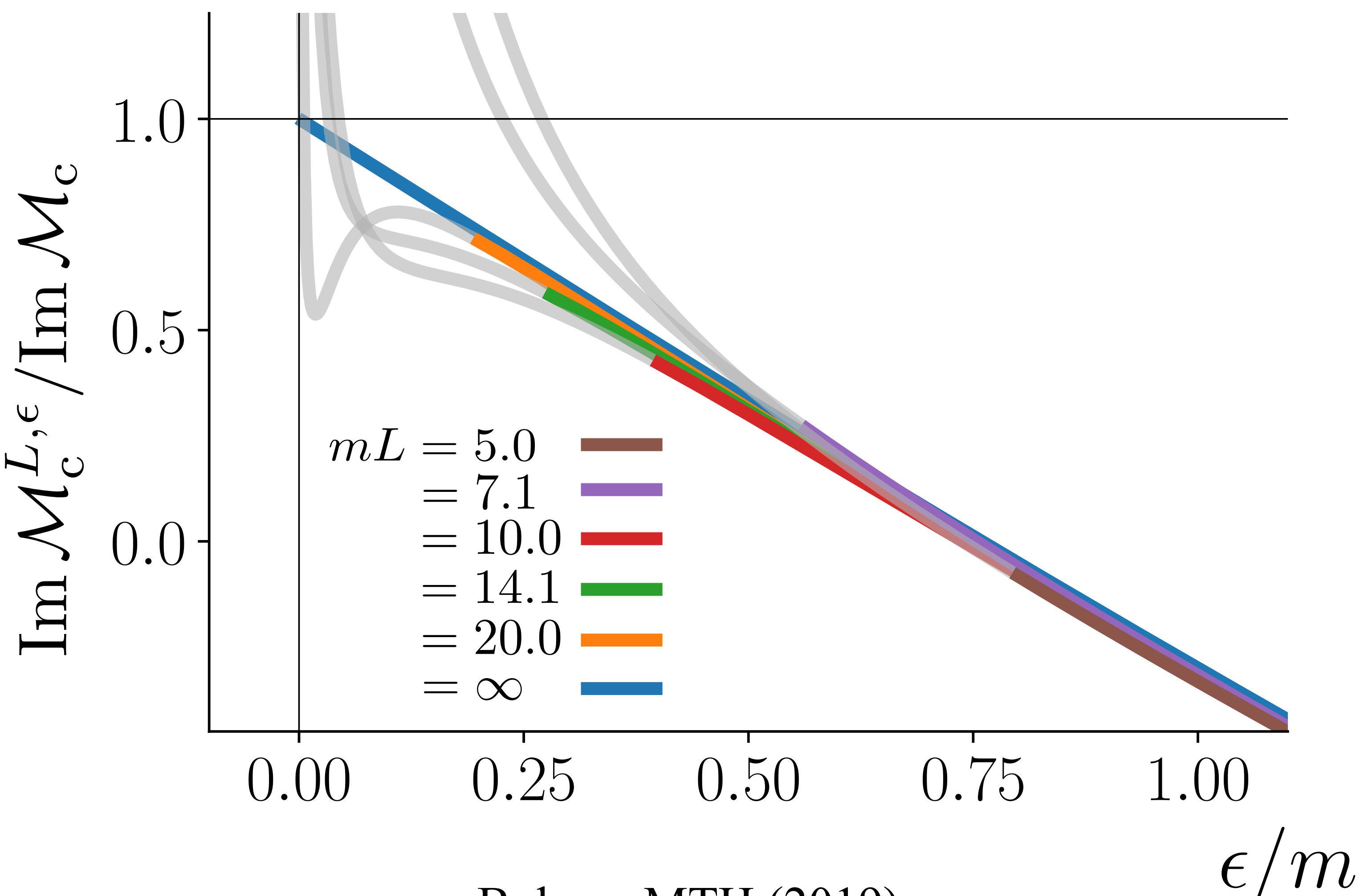
Calculate in PT

$$G_L(\tau) = \langle \pi_{\boldsymbol{p}_4} | \pi(\tau_3, \boldsymbol{p}_3) \pi(0) | \pi_{\boldsymbol{p}_1} \rangle_L$$

Convert to this

$$\mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1)$$

$$\mathrm{Im}\, \mathcal{M}_{\mathrm{c}}^{L,\epsilon}(p_4 p_3 | p_2 p_1) = \frac{\lambda^2}{2} \frac{1}{L^3} \sum_{\boldsymbol{k}'}^{\Lambda} \frac{1}{(2E(\boldsymbol{k}'))^2} \,\mathrm{Im} \left\{ \frac{1}{(E_{\mathrm{cm}} - 2E(\boldsymbol{k}') + i\epsilon)} \left[1 - \frac{\epsilon^2}{4E(\boldsymbol{k}')^2} - \frac{\epsilon(\epsilon + 2iE(\boldsymbol{k}'))}{E_{\mathrm{cm}} E(\boldsymbol{k}')} \right] \right\}$$



Bulava, MTH (2019)

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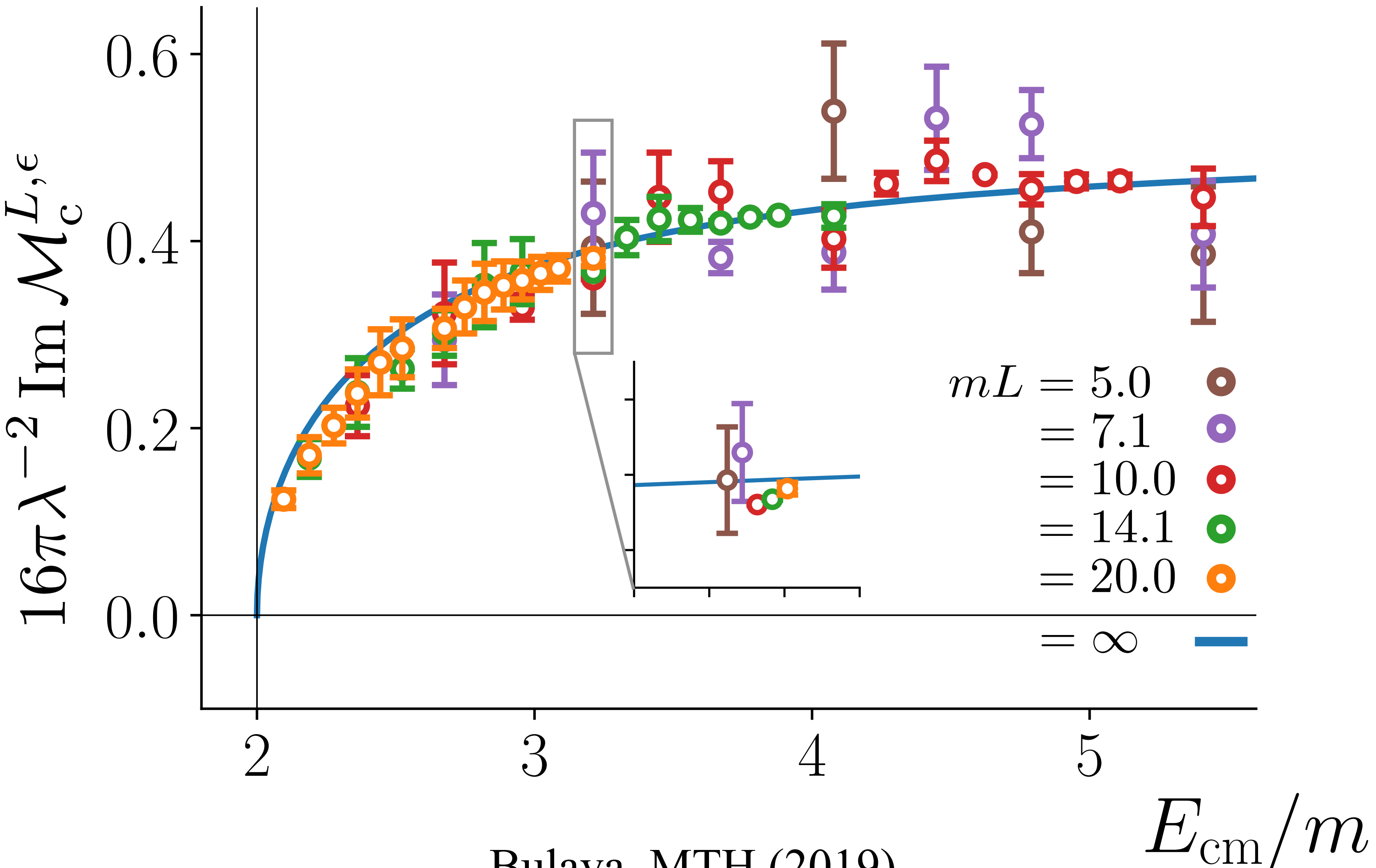


Bulava, MTH (2019)

Connection to Maiani-Testa

- Maiani and Testa considered correlators of the form

$$G_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau) J(0) | 0 \rangle$$

- And showed two key points

$$G_{[\mathbf{0}]}(\tau) = \frac{\sqrt{Z_\pi}}{2M_\pi} e^{-M_\pi \tau} f(4M_\pi^2) \left[1 - a_{\pi\pi} \sqrt{\frac{m_\pi}{\pi \tau}} + O(\tau^{-3/2}) \right]$$

$G_{[\boldsymbol{p} \neq \mathbf{0}]}(\tau)$ is plagued by un-physical (operator dependent) contributions

- Recent work with M. Bruno connects this story to spectral function amplitudes

Variations on the Maiani-Testa approach and the inverse problem

M. Bruno[a] and M. T. Hansen[b] *arXiv: 2012.11488*

G is a smeared spectral function

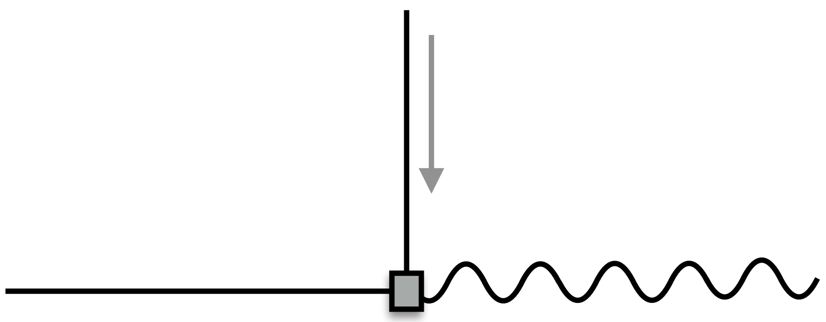
- Correlator can be viewed as a smeared spectral function

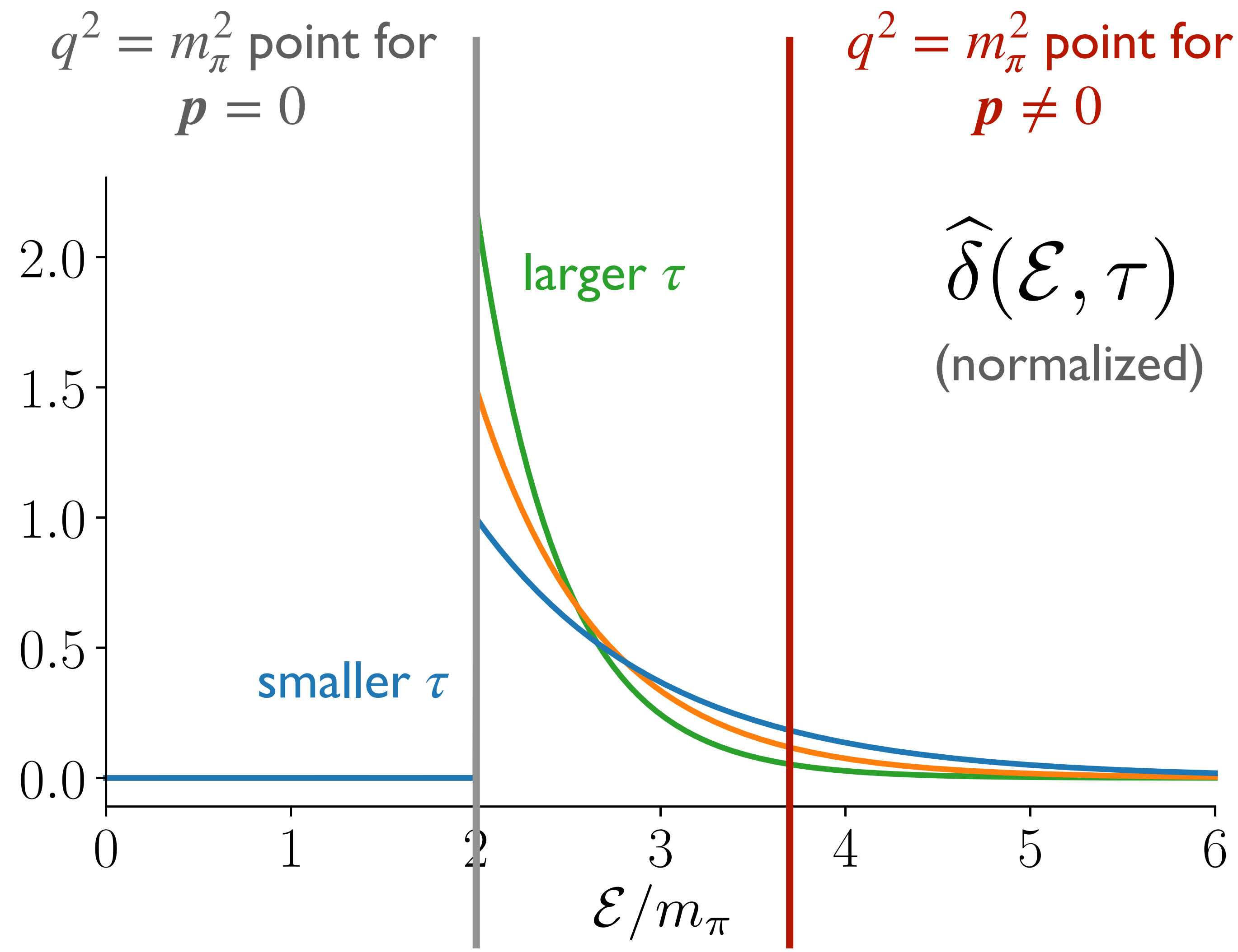
$$G_{[\boldsymbol{p}]}(\tau) = \int_0^\infty d\mathcal{E}\, \rho(\mathcal{E})\, \widehat{\delta}(\mathcal{E}, \tau)$$

$$\widehat{\delta}(\mathcal{E}, \tau) = \theta(\mathcal{E} - 2m_\pi)\exp[-\mathcal{E}\tau]$$

$$\rho(\mathcal{E})$$

$$q^2 = (\mathcal{E} - \omega_{\boldsymbol{p}})^2 - \boldsymbol{p}^2$$

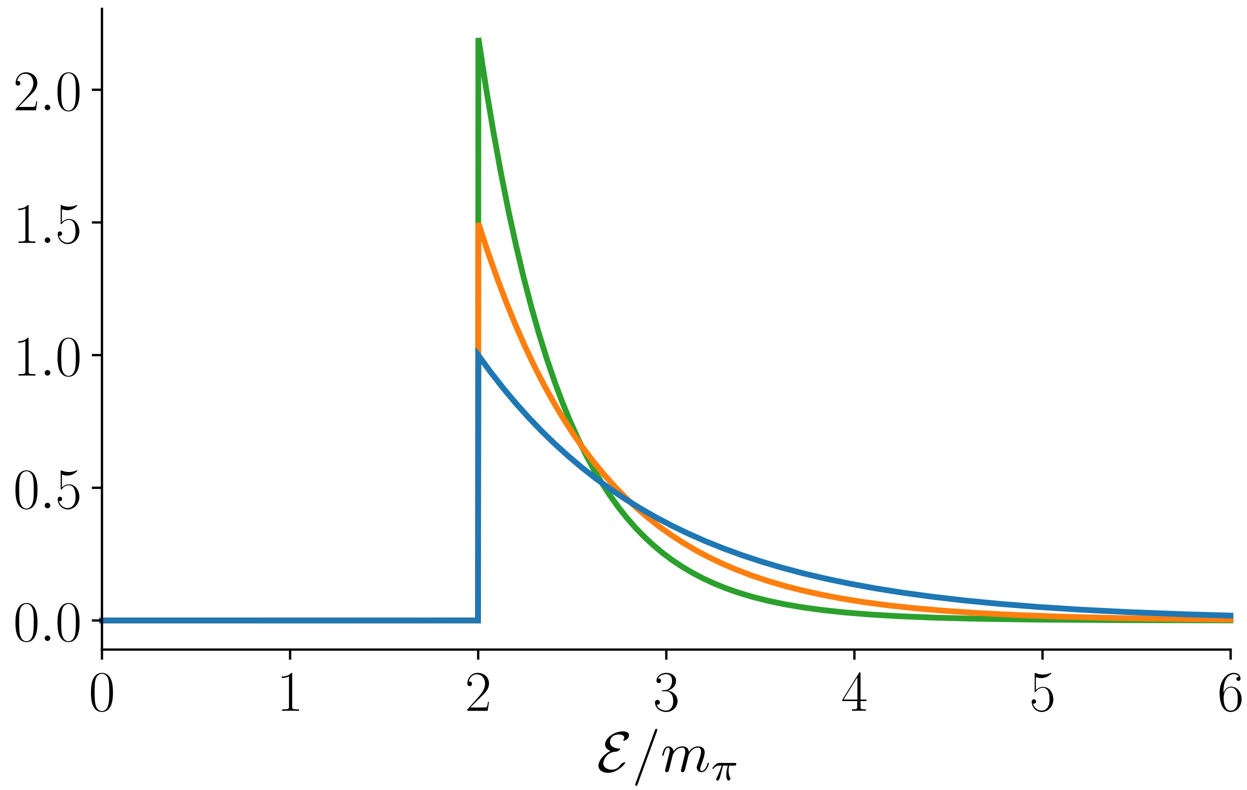




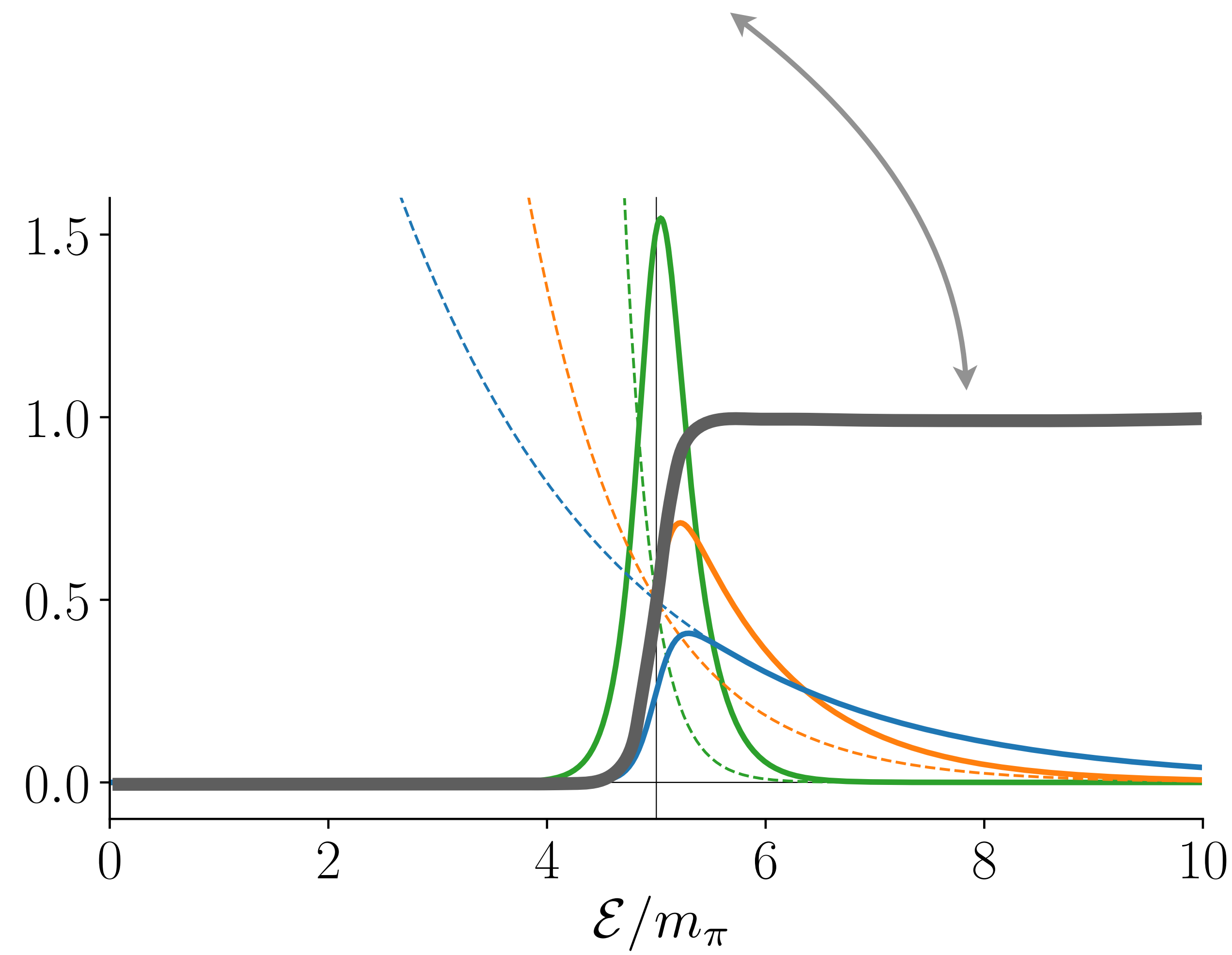
Bruno, MTH (2020)

Shift the peak!

$$\widehat{\delta}(\mathcal{E},\tau) = \theta(\mathcal{E} - 2m_\pi)\exp[-\mathcal{E}\tau]$$



$$\widehat{\delta}^{\Theta}(\mathcal{E},\tau) = \Theta(\mathcal{E} - 2m_\pi, \Delta)\exp[-\mathcal{E}\tau]$$



Bruno, MTH (2020)

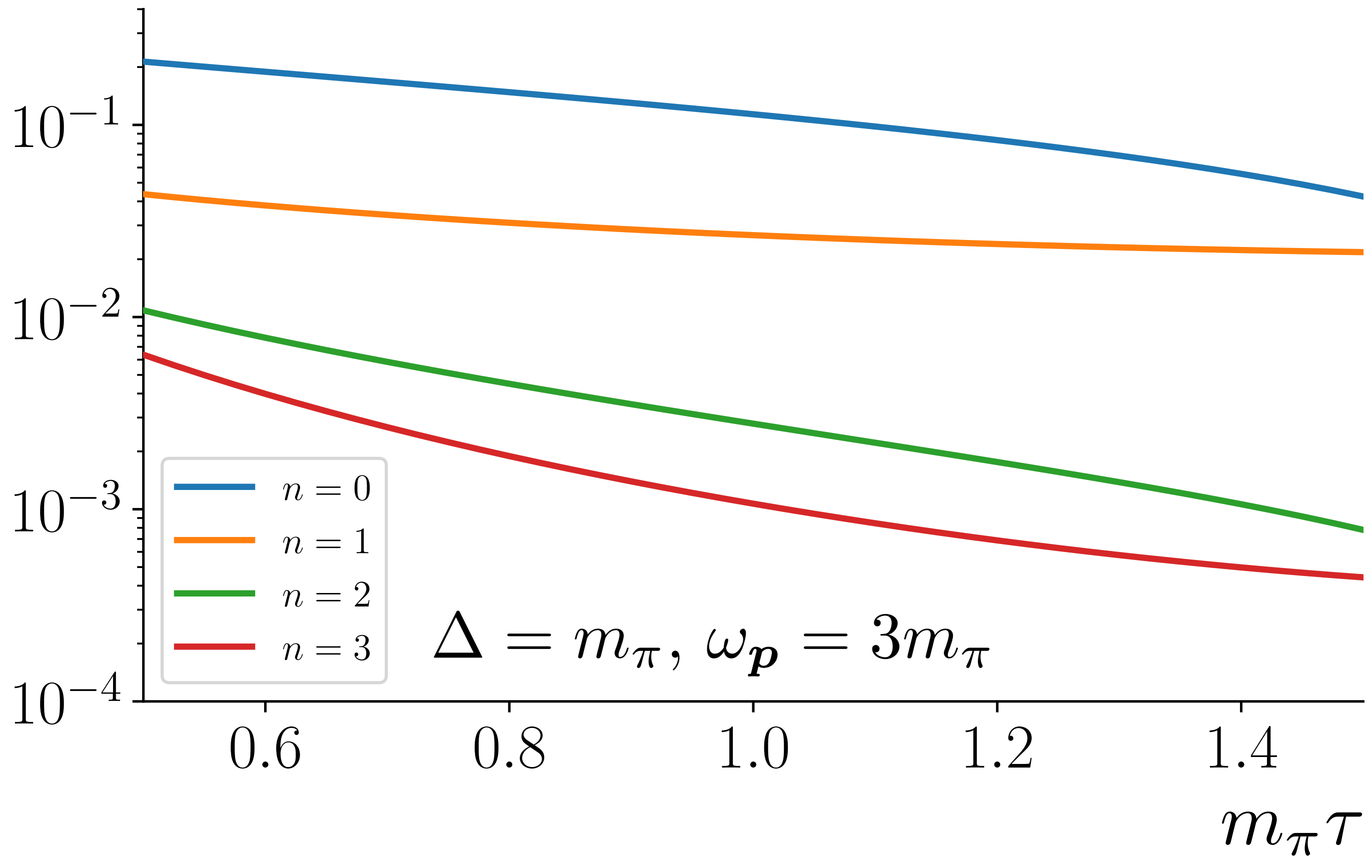
Reaching above threshold

- So the Maiani and Testa correlator becomes

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0)|0\rangle$$

- Now separating the fields gives something useful above threshold

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \frac{\sqrt{Z_\pi}}{2\omega_{\boldsymbol{p}}} e^{-\omega_{\boldsymbol{p}}\tau} \left[\Theta(0,\Delta)\, \mathrm{Re} f(4\omega_{\boldsymbol{p}}^2) - 2\mathcal{J}^{(0)}(\tau,\Delta)\, \mathrm{Im} f(4\omega_{\boldsymbol{p}}^2) + \cdots\right]$$



Hierarchy of $\mathcal{J}^{(n)}$ provides a useful fit function

Constructing the Θ correlator

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(\tau)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{p}}, \Delta)\, J(0)|0\rangle$$

- Two main methods:
  - Backus-Gilbert and HLT method
  - GEVP

$$G^{\Theta}_{[\boldsymbol{p}]}(\tau) = \sum_n \Theta(E_n - 2\omega_{\boldsymbol{p}}, \Delta) e^{-(E_n - \omega_{\boldsymbol{p}})\tau} \langle \pi_{-\boldsymbol{p}} | \pi_{\boldsymbol{p}}(0)|n\rangle \langle n|J(0)|0\rangle$$

combine finite-volume energies and matrix elements

- Volume effects?… Suppressed as $e^{-\Delta L}$, but more investigation is needed

More about volume effects…

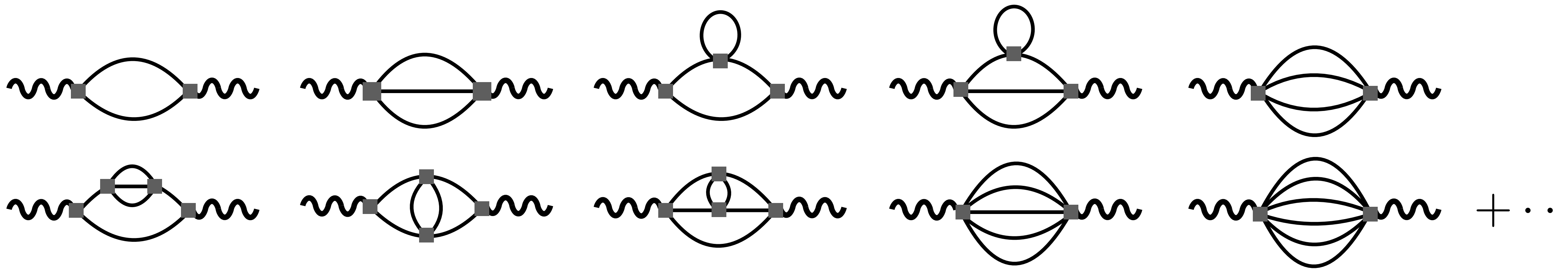
Two-point function in a box

$$C_L(t) = \int_0^L \mathrm{d}\boldsymbol{x}\, \langle j_1^a(x) j_1^a(0) \rangle_L = \int_0^\infty \mathrm{d}\omega\, e^{-\omega t}\, \rho_L(\omega)$$

$$\rho^{\times}_{\epsilon,L}(E) = \int_0^\infty \mathrm{d}\omega\, \delta_\epsilon^x(E-\omega)\, \rho_L(\omega)$$

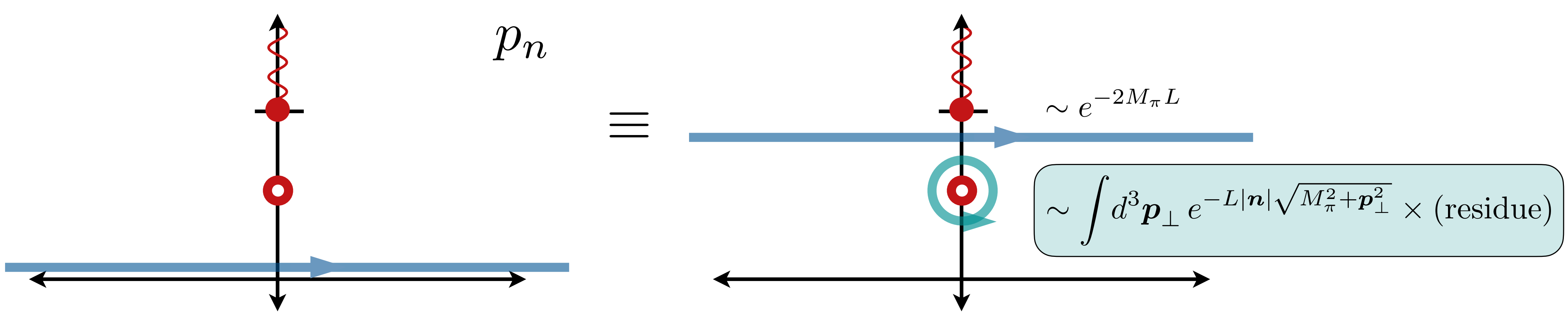
- ❑ $G(\tau)$ has exponentially suppressed volume effects
- ❑ Playing with 1+1 O(3) model
- ❑ Instructive cancellation of $1/L$ effects in $G(\tau)$ and $\widehat{\rho}_\epsilon(E)$
- ❑ $\widehat{\rho}_\epsilon(E)$ has exponentially suppressed volume effects

$C_L(t)$ has exponentially suppressed volume effects



- Loop momenta are summed… rewrite using ***Poisson summation***

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} \longrightarrow \sum_{\boldsymbol{n}} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{iL\boldsymbol{n}\cdot\boldsymbol{k}}$$



Lüscher (1986)   Hansen, Patella (2019,2020)

1+1 O(3) Model

- Integrable model… scattering phase, spectral function known exactly

• M. Karowski, P. Weisz (1978) •
• A. B. Zamolodchikov, A. B. Zamolodchikov (1978) • J. Balog, M. Niedermaier, (1997) •

- Can predict two-particle part of smeared finite-volume spectral function

$$\rho^{(2)}_{\epsilon,L}(E) = \sum_n c_n^{(2)}(L)\delta_\epsilon(E - E_n^{(2)}(L))$$

$$2\delta(E) + L\sqrt{E^2/4 - m^2} = 2\pi n$$

Lüscher

$$c_n^{(2)}(L) = \pi\left(\frac{\partial[\delta(E)+\phi(E,L)]}{\partial E}\right)^{-1}\rho^{(2)}(E)\Big|_{E=E_n(L)}$$

$$\phi(E,L) = kL/2 = \sqrt{E^2/4 - m^2}L/2$$

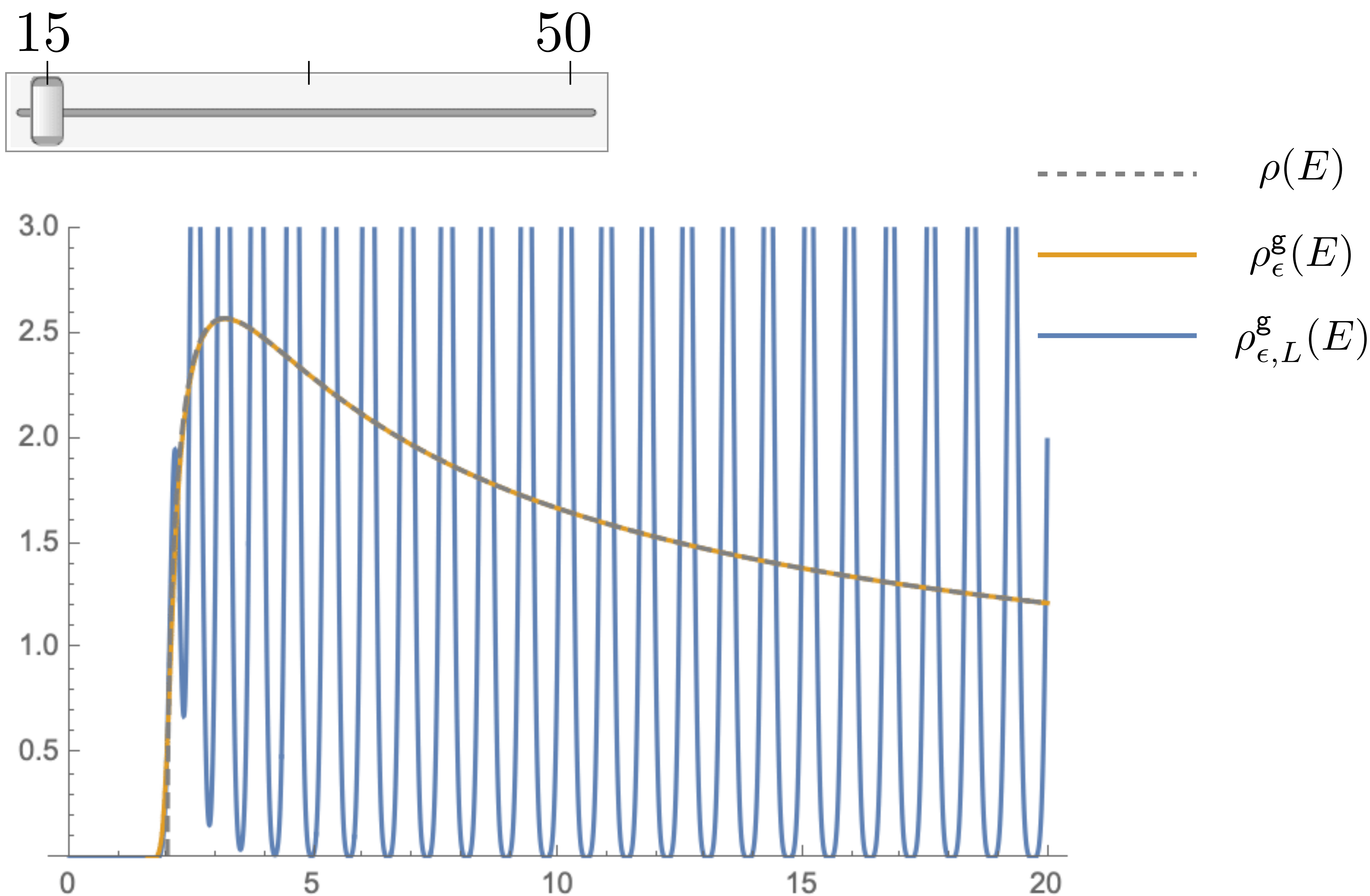
Lellouch and Lüscher

two-particle part is only unambiguous for states below $4m$

Playing with L

- Gaussian resolution function… $\epsilon/m = 0.1$

$mL = 15$



Playing with L

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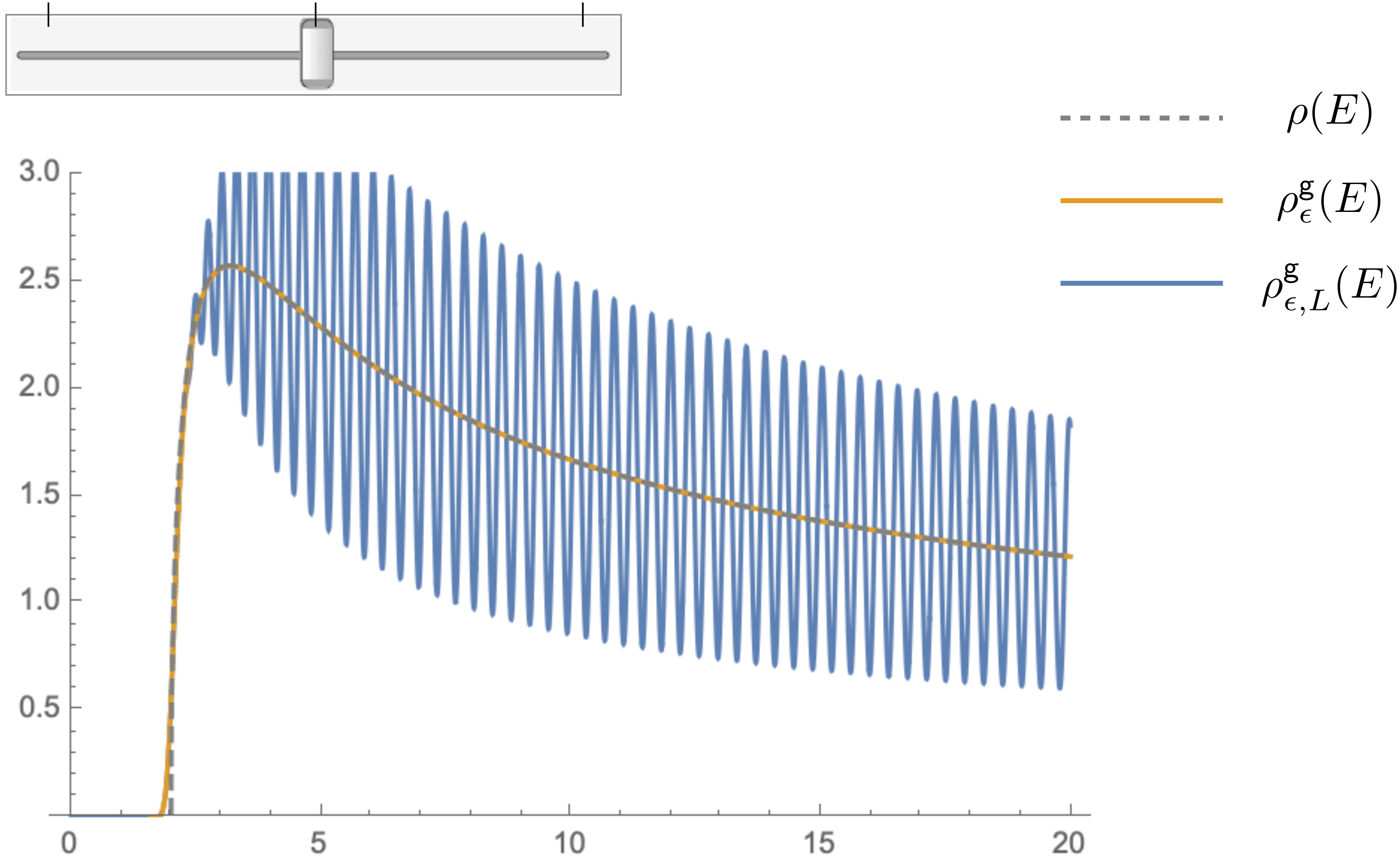
$mL = 15$ 50



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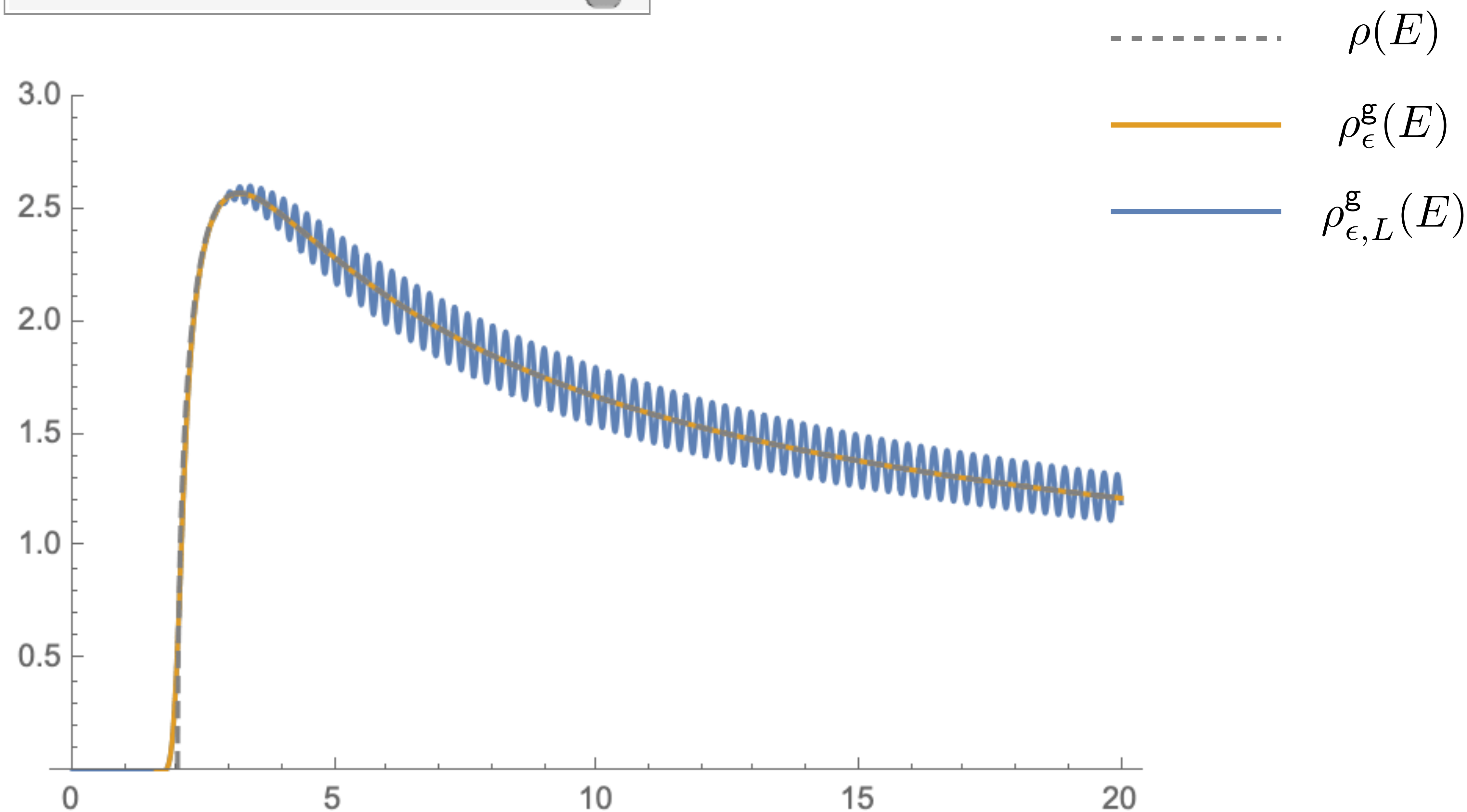
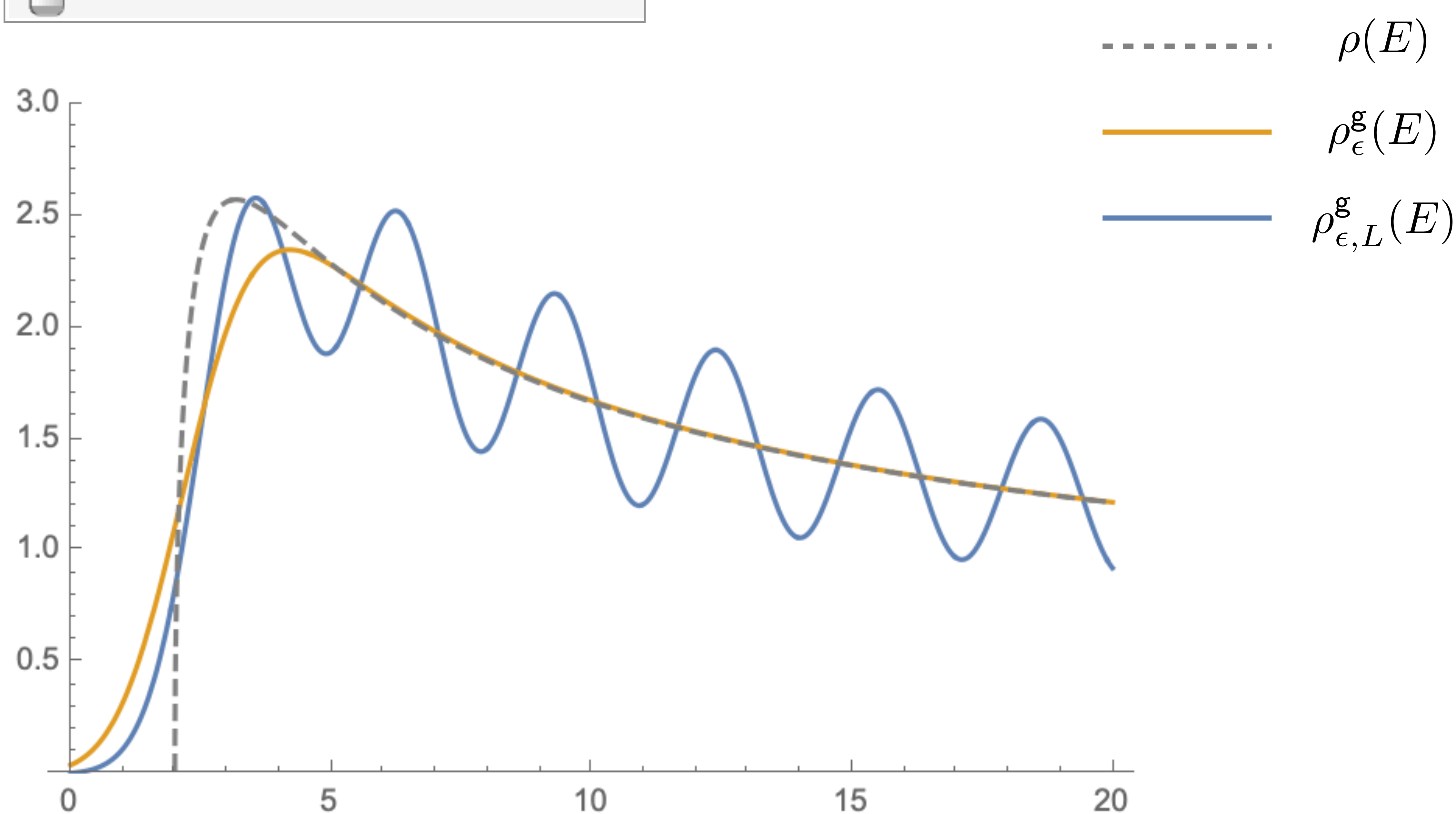


Playing with L

- Gaussian resolution function... $\epsilon/m = 1.0$

$mL = 4.0$ 6.0   8.0



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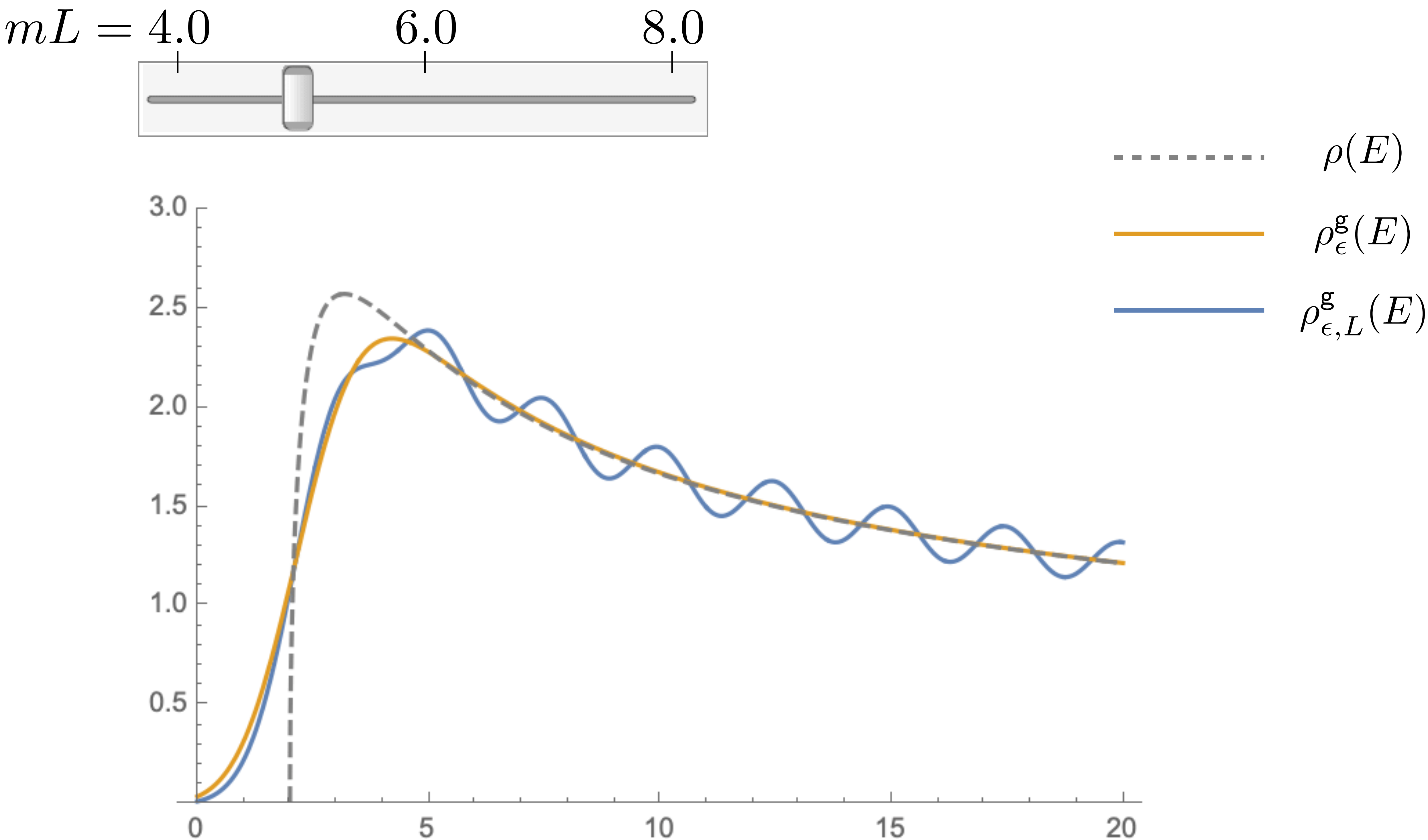

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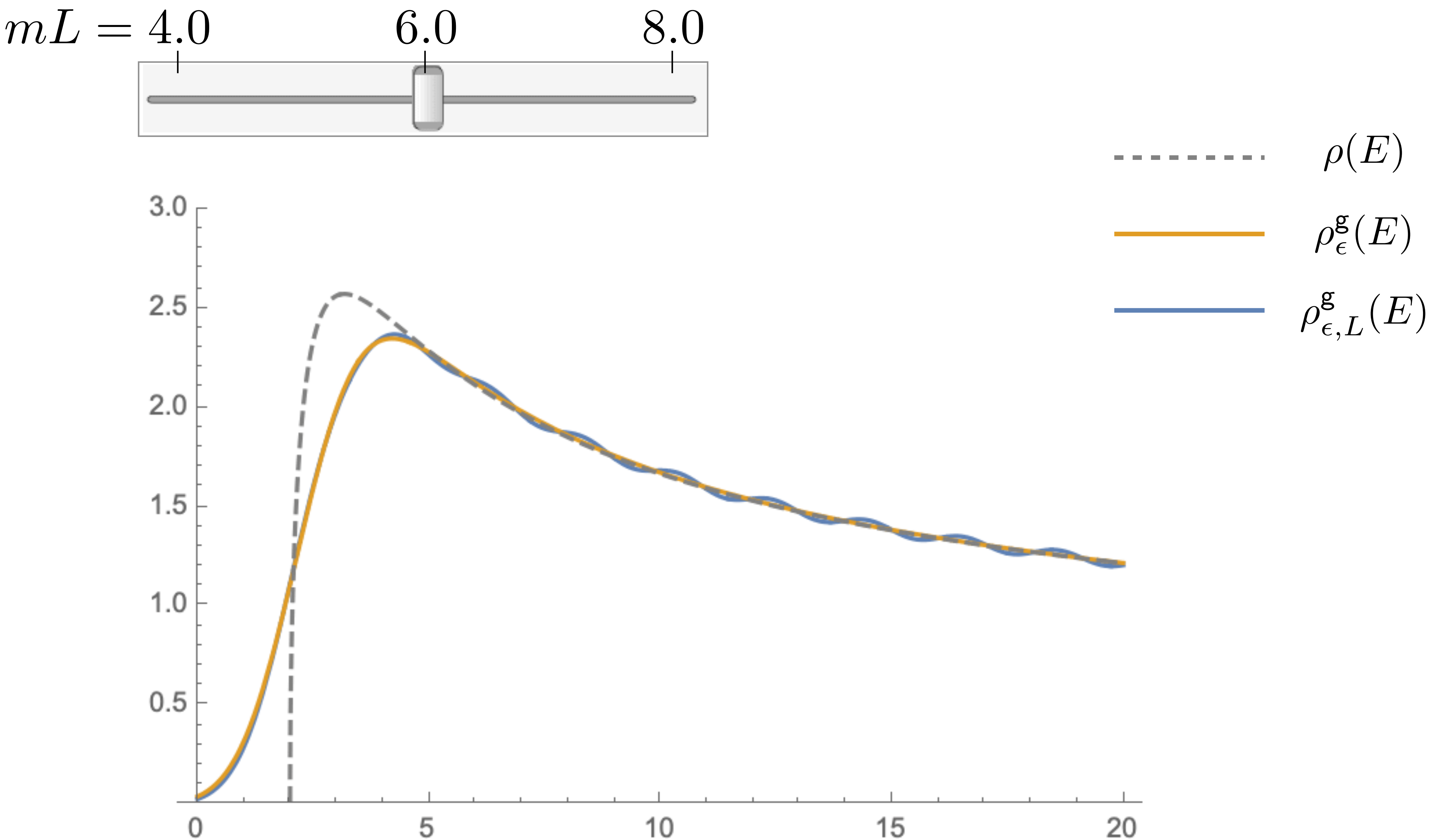


Two-point function in a box

$$C_L(t) = \sum_n c_n(L) e^{-E_n(L)t} \qquad\qquad C_\infty(t) = \int \mathrm{d}E\, \rho(E)\, e^{-Et}$$

$$c_n(L) = \pi \left(\frac{\partial[\delta(E) + \phi(E,L)]}{\partial E} \right)^{-1} \rho(E) \bigg|_{E=E_n(L)} \qquad E_n(L) = E_n^{(0)}(L) - \frac{8\sqrt{E^2/4 - m^2}}{EL} \delta(E) \bigg|_{E=E_n^{(0)}(L)} + \mathcal{O}(1/L^2)$$

$$\phi(E,L) = kL/2 = \sqrt{E^2/4 - m^2}\, L/2$$

- ❑ Infinite-volume limit behaves as expected

$$\lim_{L\to\infty} C_L(t) = \lim_{L\to\infty} \frac{1}{L} \sum_n \frac{8\pi k}{E} e^{-Et} \rho(E) = \int dk \frac{dE}{dk} e^{-Et} \rho(E) = C_\infty(t)$$

- ❑ $1/L$ correction cancels between energies and matrix elements

$$\lim_{L\to\infty} L\big[C_L(t) - C_\infty(t)\big] = -\int \frac{\mathrm{d}k}{2\pi} \left[4\pi\, e^{-Et}\, \rho(E) \frac{\partial \delta(E)}{\partial E} + \frac{8k\delta(E)}{E} \frac{\partial}{\partial E} \left(\frac{8\pi k}{E} e^{-Et} \rho(E) \right) \right]$$

$$= -8 \int_{2m}^{\infty} \mathrm{d}E\, \frac{\partial}{\partial E} \left[e^{-Et} \rho(E) \frac{k}{E} \delta(E) \right]$$

Two-point function in a box

$$\rho_{\epsilon,L}(E) = \sum_n c_n(L)\delta_\epsilon(E - E_n(L)) \qquad \rho_{\epsilon,\infty}(E) = \int \mathrm{d}\omega\, \rho(\omega)\, \delta_\epsilon(E-\omega)$$

$$c_n(L) = \pi\left(\frac{\partial[\delta(E)+\phi(E,L)]}{\partial E}\right)^{-1}\rho(E)\bigg|_{E=E_n(L)} \qquad E_n(L) = E_n^{(0)}(L) - \frac{8\sqrt{E^2/4-m^2}}{EL}\delta(E)\bigg|_{E=E_n^{(0)}(L)} + \mathcal{O}(1/L^2)$$

$$\phi(E,L) = kL/2 = \sqrt{E^2/4-m^2}\,L/2$$

- Infinite-volume limit behaves as expected

$$\lim_{L\to\infty}\rho_{\epsilon,L}(E) = \lim_{L\to\infty}\frac{1}{L}\sum_n \frac{8\pi k}{\omega}\delta_\epsilon(E-\omega)\rho(\omega) = \int dk\frac{d\omega}{dk}\delta_\epsilon(E-\omega)\rho(\omega) = \rho_\epsilon(E)$$

- $1/L$ correction cancels between energies and matrix elements

$$\lim_{L\to\infty} L\big[\rho_{\epsilon,L}(E) - \rho_{\epsilon,\infty}(E)\big] = -\int\frac{\mathrm{d}k}{2\pi}\left[4\pi\,\delta_\epsilon(E-\omega)\,\rho(\omega)\frac{\partial\delta(\omega)}{\partial\omega} + \frac{8k\delta(\omega)}{\omega}\frac{\partial}{\partial\omega}\left(\frac{8\pi k}{\omega}\delta_\epsilon(E-\omega)\rho(\omega)\right)\right]$$

$$= -8\int_{2m}^{\infty}\mathrm{d}\omega\,\frac{\partial}{\partial\omega}\left[\delta_\epsilon(E-\omega)\rho(\omega)\frac{k}{\omega}\delta(\omega)\right]$$

Two-point function in a box

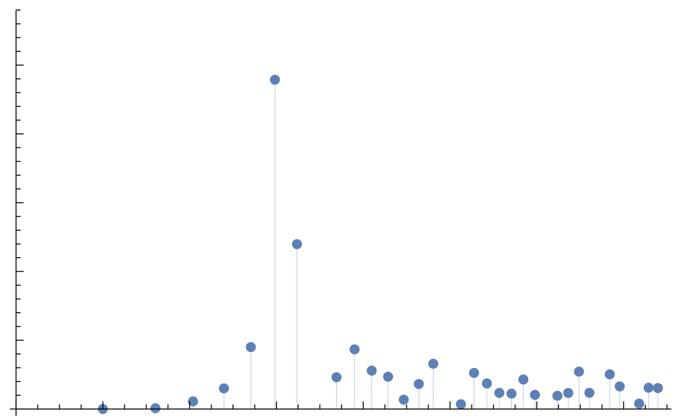
$$C_L(t) = \int_0^L \mathrm{d}\boldsymbol{x}\, \langle j_1^a(x) j_1^a(0) \rangle_L = \int_0^\infty \mathrm{d}\omega\, e^{-\omega t}\, \rho_L(\omega)$$

$$\rho^{\times}_{\epsilon,L}(E) = \int_0^\infty \mathrm{d}\omega\, \delta^x_\epsilon(E-\omega)\, \rho_L(\omega)$$

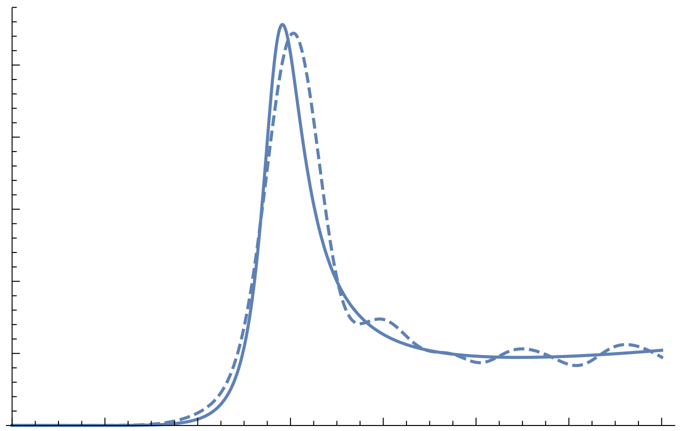
- ❑ $G(\tau)$ has exponentially suppressed volume effects
- ❑ Playing with 1+1 O(3) model
- ❑ Instructive cancellation of $1/L$ effects in $G(\tau)$ and $\widehat{\rho}_\epsilon(E)$
- ❑ $\widehat{\rho}_\epsilon(E)$ has exponentially suppressed volume effects

Conclusions

- Cannot solve the inverse problem, we can get $\widehat{\rho}_{L,\Delta}(\bar{\omega}) \equiv \int_0^\infty d\omega\, \widehat{\delta}_\Delta(\bar{\omega},\omega)\, \rho_L(\omega)$
- Smearing is needed anyway to ***suppress volume effects***



$$1/L \ll \Delta \ll \mu_{\text{physical}}$$



- Generalized Backus-Gilbert takes $\widehat{\delta}_\Delta^{\text{target}}(\bar{\omega},\omega)$ as input

This has unlocked a ***playground*** of calculations that we ***just beginning*** to explore

Thanks!