A spectral reconstruction test using the two-dimensional O(3)-model

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Spectral Properties From Euclidean Path Integrals
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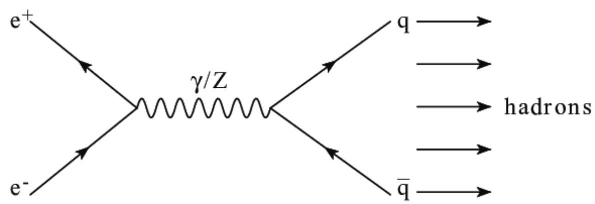
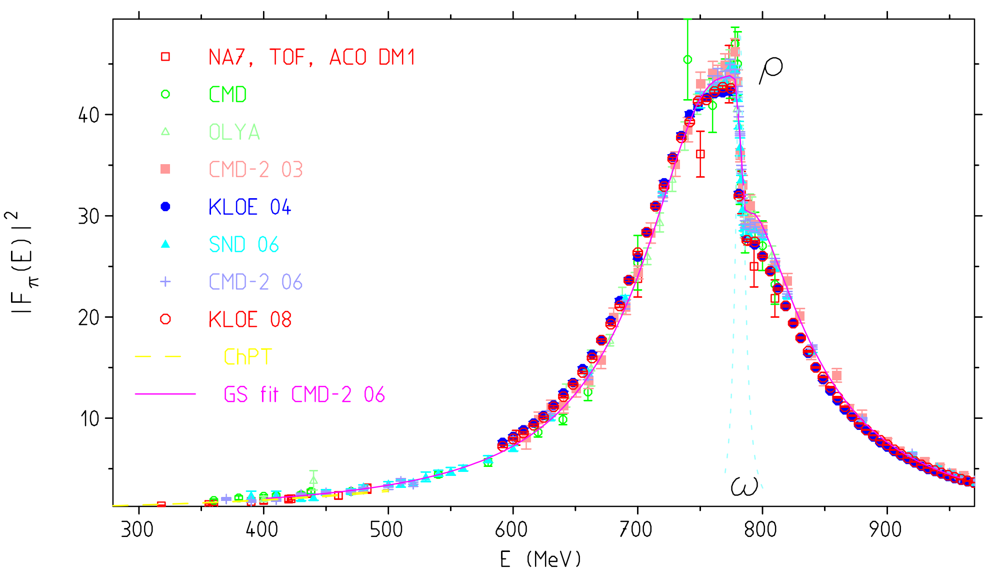
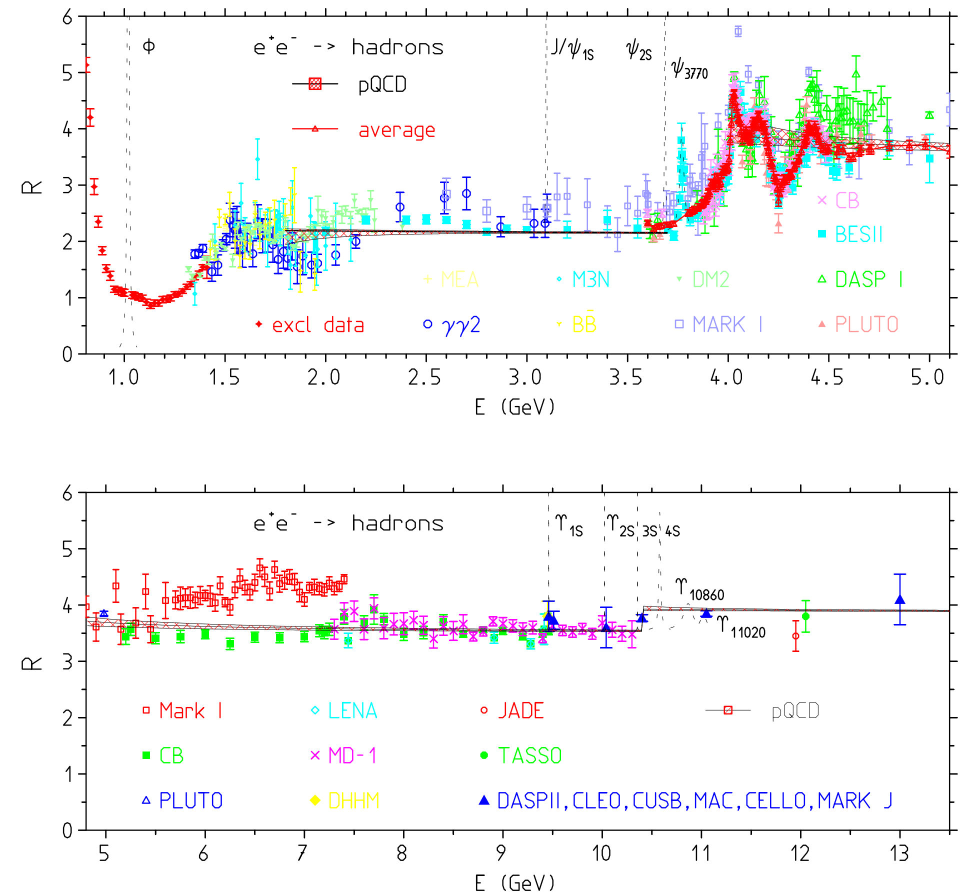
A "simple" lattice QCD spectral reconstruction



$$R(s) = \frac{R(e^+e^- \to \text{hadrons})}{4\pi\alpha_{\text{em}}(s)/(3s)}, \qquad \rho(s) = \frac{R(s)}{12\pi^2}$$





Jegerlehner, Nyffeler `09

Higher energies => lower resolution.

Useful ‘smeared’ spectral functions

$$G(\tau) = \int \mathrm{d}^3\mathbf{x}\, \langle\Omega|\hat{j}_i^{\mathrm{em}}(\mathbf{x},\tau)\, \hat{j}_i^{\mathrm{em}\dagger}(0)|\Omega\rangle = \int_0^\infty \mathrm{d}\omega\, \omega^2\, \rho(\omega^2)\, \mathrm{e}^{-\omega\tau}$$

Adler function:

$$D(Q^2) = Q^2 \int_0^\infty \mathrm{d}s \frac{\rho(s)}{(s+Q^2)^2}$$

Hadronic contribution to $(g-2)_\mu$:

$$a_\mu^{\mathrm{HVP}} = \int_0^\infty \mathrm{d}Q^2\, D(Q^2)\, g(Q^2, m_\mu)$$

Spectral Reconstruction

Backus, Gilbert ‘68, ‘70
F. Pijpers, M. Thompson ‘92
M. R. Hansen, A. Lupo, N. Tantalo, PRD99 (2019)

Linear ansatz:

$$\hat{\rho}_\epsilon(E) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, C(t), \qquad \hat{\delta}_\epsilon(E,\omega) = \sum_{t=a}^{t_{\max}} q_t(\epsilon, E)\, \mathrm{e}^{-\omega t}$$

Two criteria when choosing $\{q_t(\epsilon, E)\}$

- Accuracy: $$A[q] = \int_{E_0}^{\infty} \mathrm{d}\omega \left\{ \delta_\epsilon(E-\omega) - \hat{\delta}_\epsilon(E,\omega) \right\}^2$$
- Precision: $$B[q] = \mathrm{Var}\left\{\rho_\epsilon(E)\right\}$$

Optimal coeffs minimize:

$$W_\lambda[q] = (1-\lambda) A[q]/A[0] + \lambda B[q]$$

Controlled Test

JB, M. W. Hansen, M. T. Hansen, (M. R. Hansen), A. Patella, N. Tantalo, *in prep.*

2d O(3)-model: …, M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990),...

$$S[\sigma] = -\beta \sum_{x,\mu} \sigma(x) \cdot \sigma(x+\hat{\mu}), \qquad \sigma(x) \in \mathbb{R}^3, |\sigma(x)| = 1$$

Conserved (global) current:

$$j_\mu^a = \beta \epsilon^{abc} \sigma^b(x) \hat{\partial}_\mu \sigma^c(x)$$

Massive single-particle states. Target process: inclusive rate for $j \to X$

$$\rho(E) = \sum_\alpha \delta(\boldsymbol{P}_\alpha) \delta(E - E_\alpha) \, |{}_{\text{out}}\langle \alpha | \hat{j}(0) | 0 \rangle|^2$$

Controlled Test

Spectral function from current-current correlator:

$$C(\tau) = \int \mathrm{d}\boldsymbol{x}\, \langle j_1^a(\tau, \boldsymbol{x}) j_1^a(0) \rangle = \int_0^\infty \mathrm{d}\omega\, \mathrm{e}^{-\omega\tau} \rho(\omega)$$

All (even) particle-number sectors contribute

$$\rho(E) = \sum_{n=1}^{\infty} \rho^{(2n)}(E), \qquad \rho^{(2n)}(E) = 0 \,\mathrm{for}\, E < 2nm$$

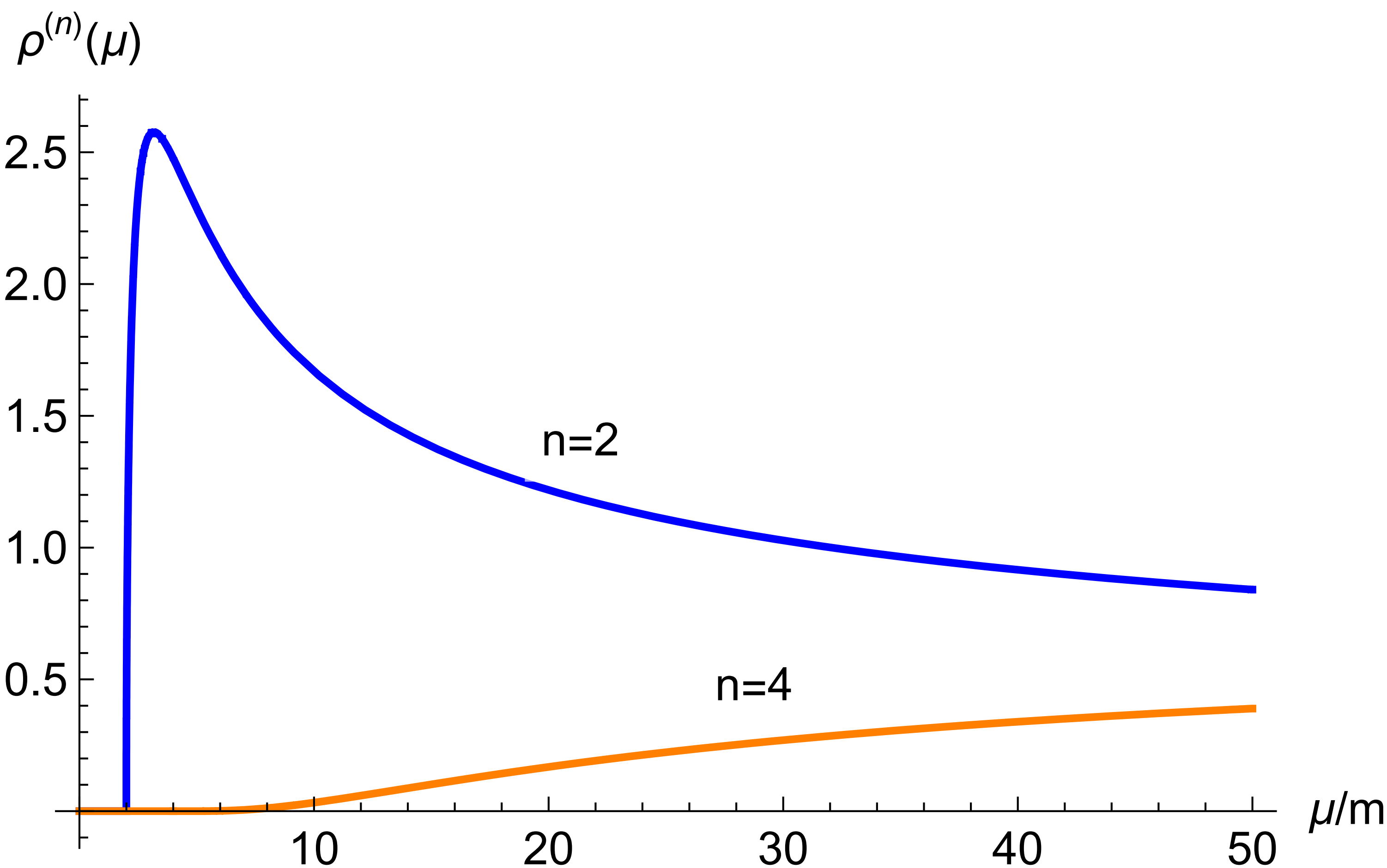
Elastic region E<4m can be computed using finite-volume formalism

(e.g. timelike pion form factor in QCD)

Controlled Test

Integrable model => spectral function known exactly:

M. Karowski, P. Weisz, Nucl. Phys. B139 (1978)
A. B. Zamolodchikov, A. B. Zamolodchikov, Nucl. Phys. B133 (1978)
J. Balog, M. Niedermaier, Nucl. Phys. B500 (1997)



Two-particle contribution dominant, four-particle ~2% near E = 10m

Controlled Test

ID	$(L/a) \times (T/a)$	am	mL	mT
A1	640×320	0.0453605(96)	29	15
A2	1280×640	0.0259551(49)	33	17
A3	1920×960	0.0176980(47)	34	17
A4	2880×1440	0.0112925(45)	33	16
B1	160×320	0.0623644(66)	10	20
B2	320×320		20	
B3	480×320		30	
B4	640×320		40	
B5	800×320		48	
B6	960×320		58	
C1	320×160	0.0623644(66)	20	10
C2	320×240			15
C3	320×320			20
C4	320×480			30

Simulations with Wolff two-cluster algorithm:

- **A1-A4**: continuum limit
- **B1-B6**: finite-L effects
- **C1-C4**: finite-T effects

Controlled Test

Five smearing kernels $\delta_\epsilon(E-\omega)$:

$$\delta_\epsilon^{\mathsf{g}}(x) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\mathsf{c0}}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

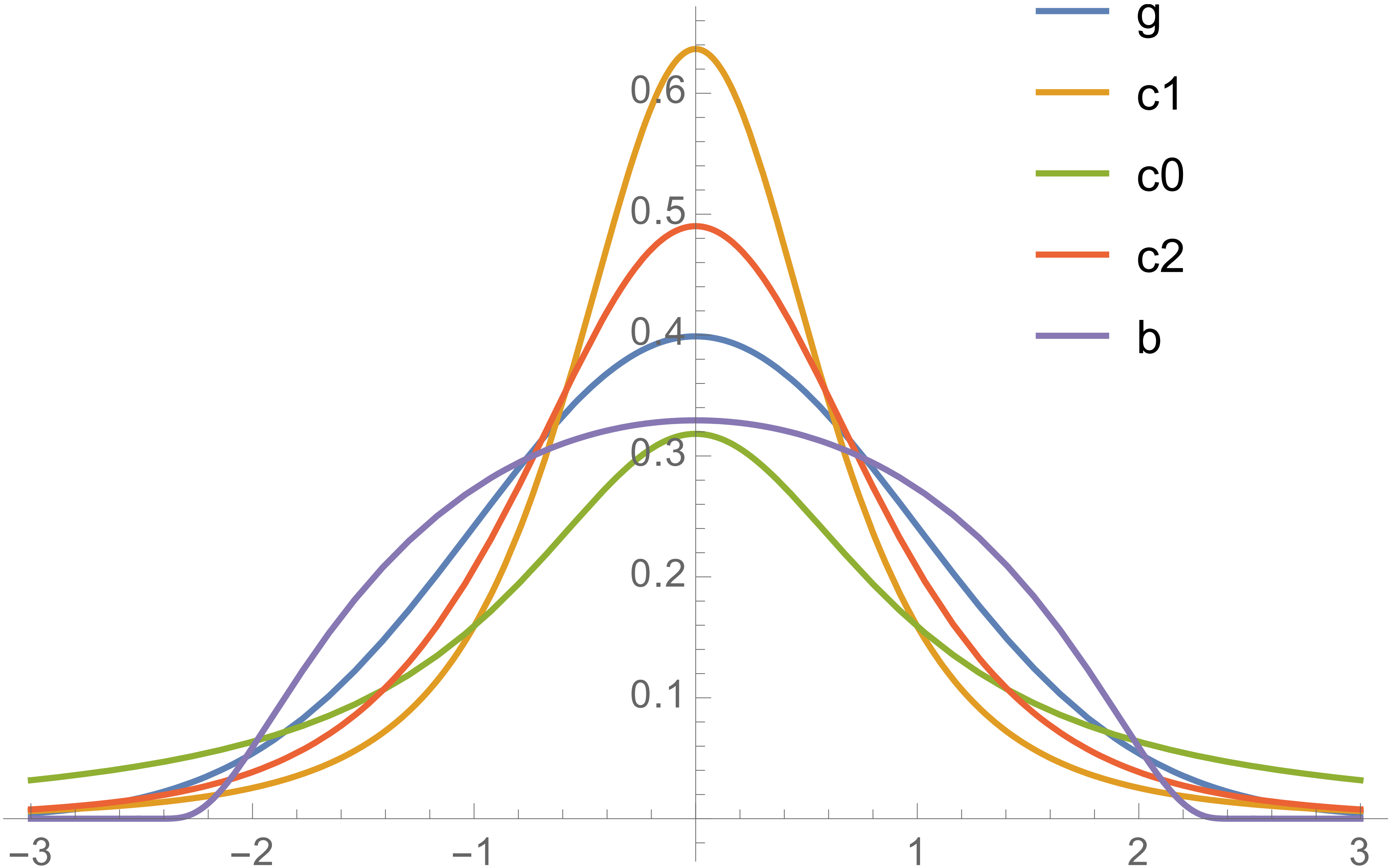
$$\delta_\epsilon^{\mathsf{c1}}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\mathsf{c2}}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3},$$

$$\delta_\epsilon^{\mathsf{b}}(x) = \frac{1}{N\epsilon}b(x/\epsilon), \quad b(y) = \begin{cases} \exp\left(-\frac{1}{1-y^2}\right), & -1<y<1, \\ 0, & \text{otherwise}, \end{cases}$$

Width of ‘b’ and ‘c2’ adjusted to coincide with Gaussian second moment.

Controlled Test

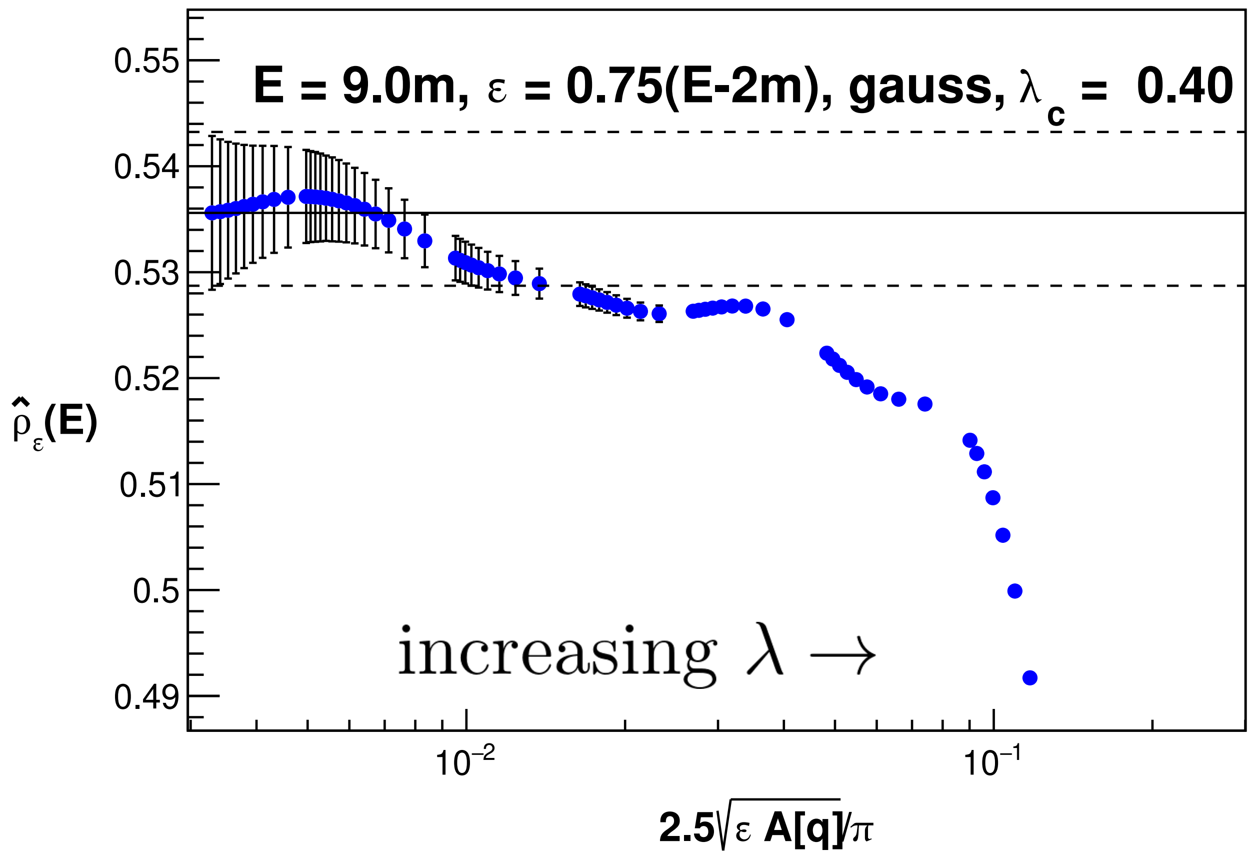
Five smearing kernels $\delta_\epsilon(E-\omega)$:



Spectral Reconstruction

$$W_\lambda[q] = (1-\lambda)A[q]/A[0] + \lambda B[q]$$

Tradeoff parameter (λ) balances systematic (A) and statistical (B) error

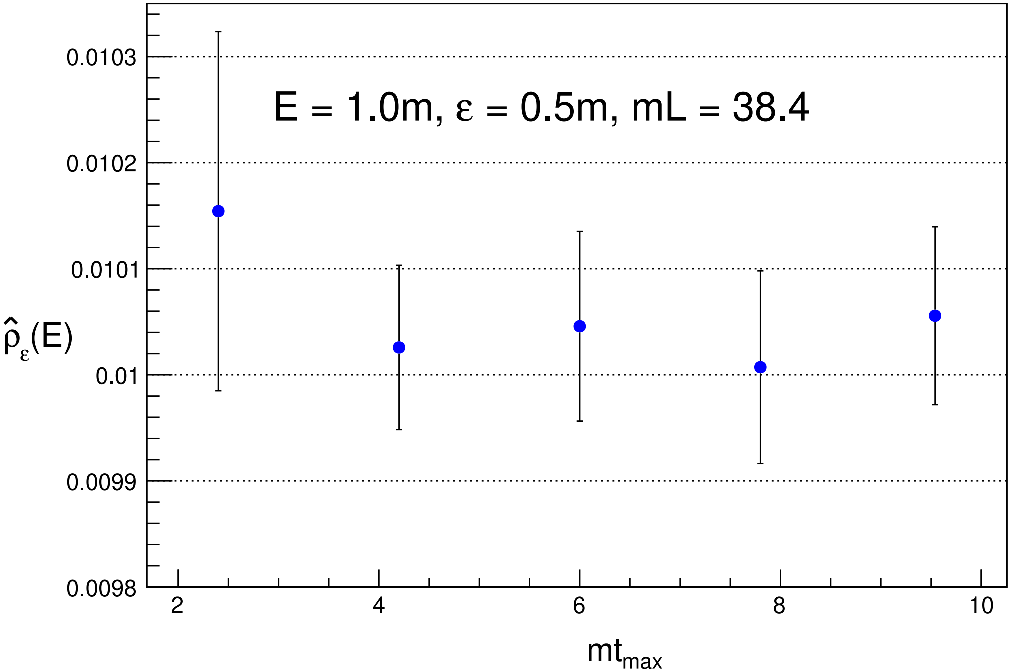


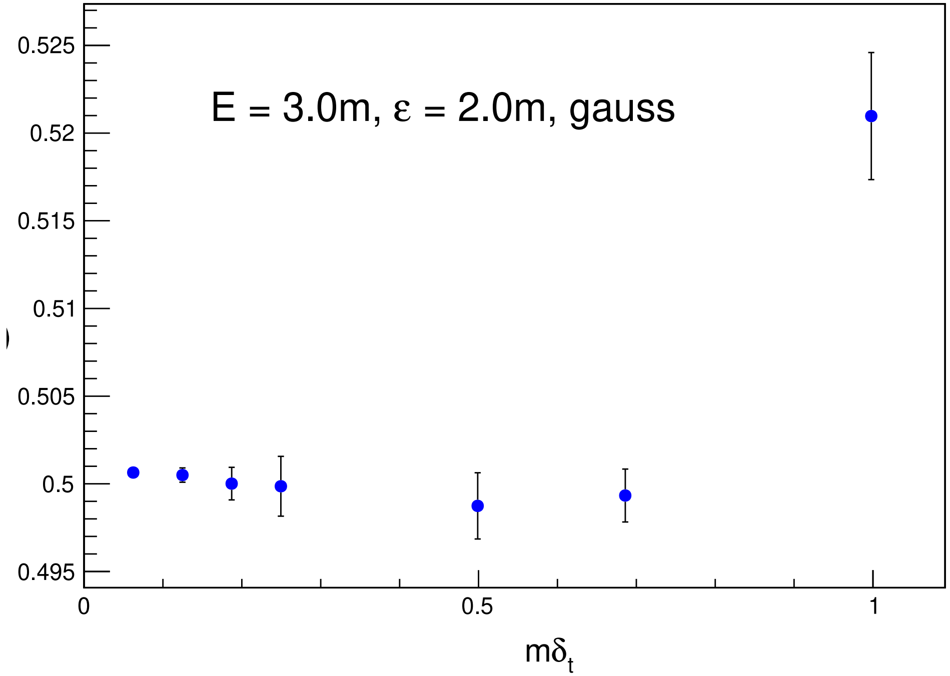
'Plateau' indicates statistics-limited regime. Choose point where:

$$q^* = \max_\lambda W_\lambda[q] \quad \Rightarrow \quad A[q^*]/A[0] = B[q^*]$$

Spectral Reconstruction

Variation of timeslices used in the reconstruction:



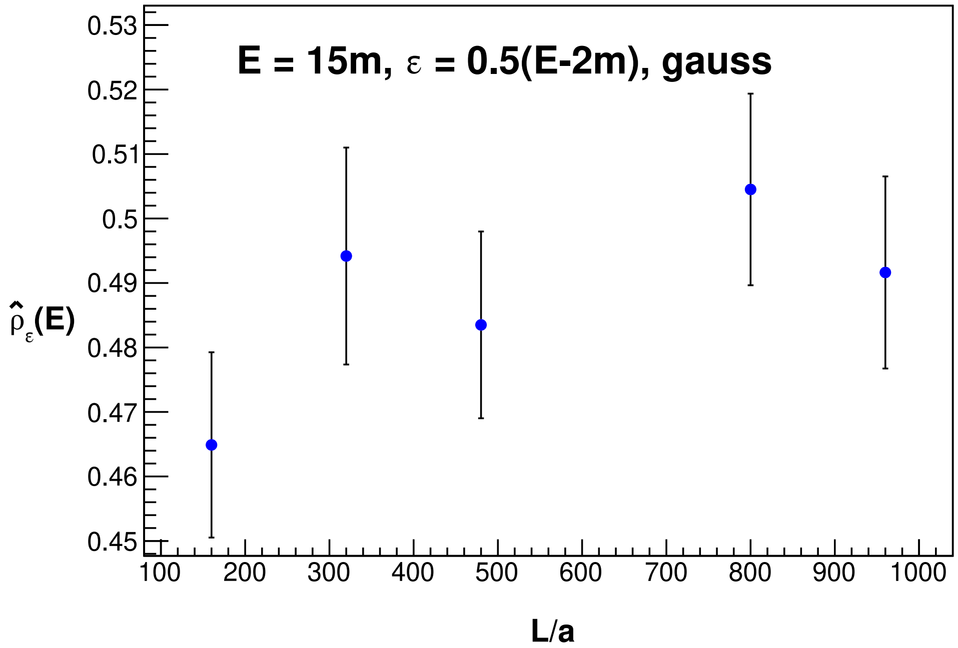


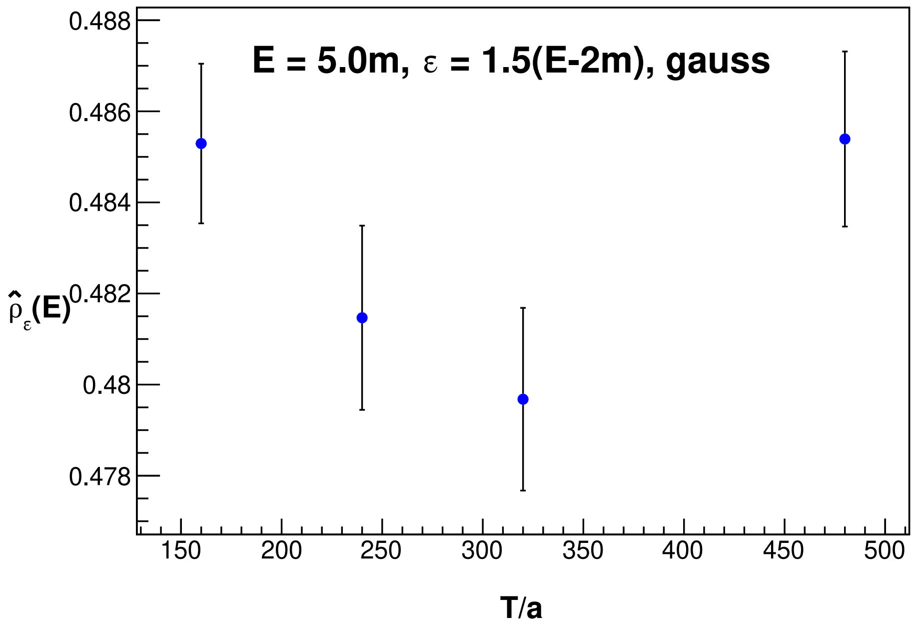
We take $t_{\max} \approx 160a$ and spacing $\delta_t = a$

Finite L and T effects

Data-driven estimate of sys. errors for each (E, ϵ) and kernel.

- Typically at or below few-percent level
- Can be large for large (E, eps) => investigation ongoing.



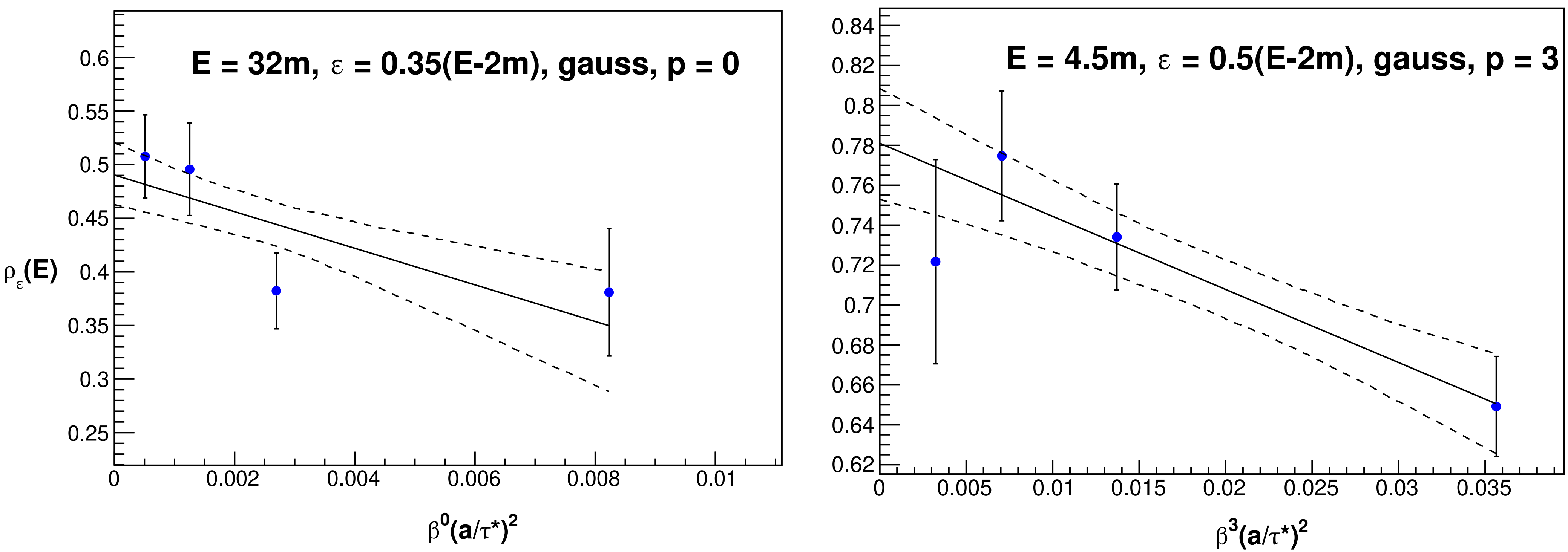


Theoretical investigations (cf. Max's talk tomorrow), higher statistics ongoing…

Continuum Limit

Long History! For spectral quantities: … , J. Balog, F. Niedermayer, P. Weisz, Nucl. Phys. B824 (2010)

$$\lim_{a\to 0} E(a) = E^{\mathrm{phys}} + A\beta^3 a^2 (1 + \frac{r}{\beta} + \frac{c}{\beta^2} + \dots)$$

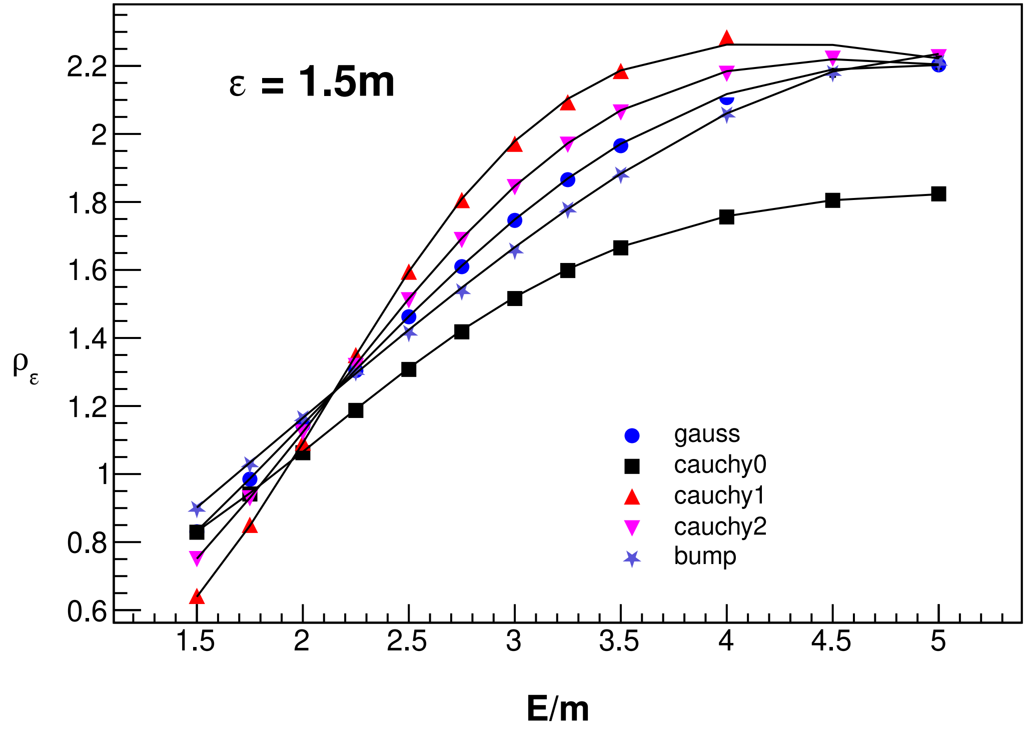


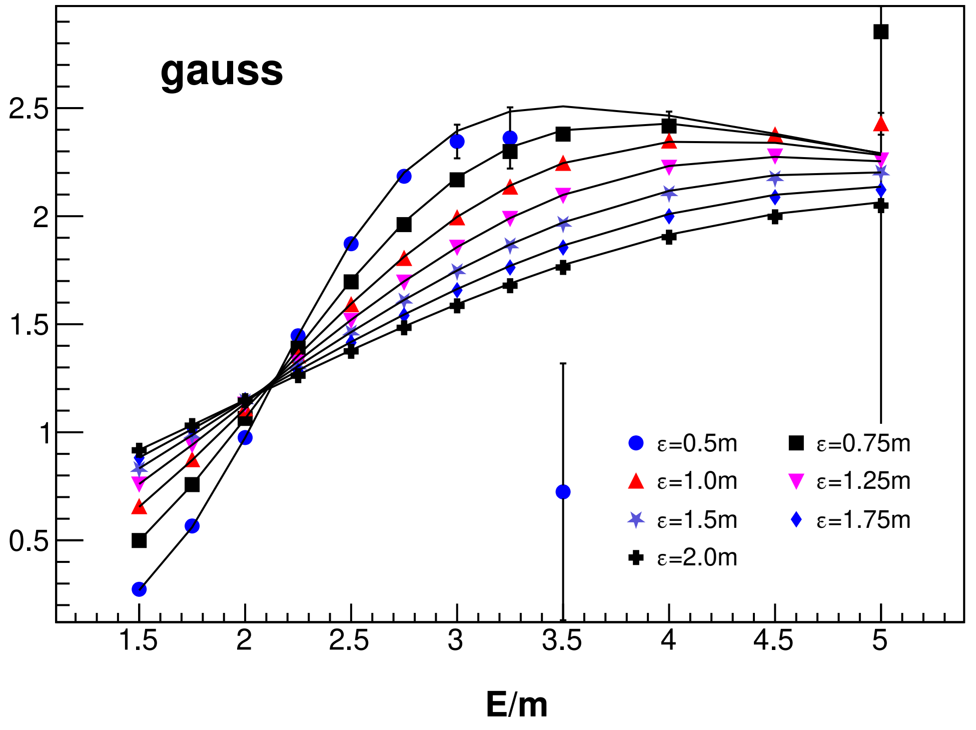
Not applicable here.

Sys. error estimate from three (arbitrary) fit forms: $\beta^p (a/m)^2, \quad p = 0,\, 3,\, 6$

Results: fixed smearing width

PRELIMINARY: systematic error estimates not finalized.





Solid lines: exact smeared spectral function, using N=2 and 4 particle contributions.

Results: extrapolation to zero width

All kernels have the same $\mathrm{O}(\epsilon^2)$ coefficient (up to a sign):

$$\rho_\epsilon^{\mathsf{x}}(E) = \rho(E) + \sum_{k=1}^{\infty} w_k^{(\mathsf{x})} a_k(E) \epsilon^k \,,$$

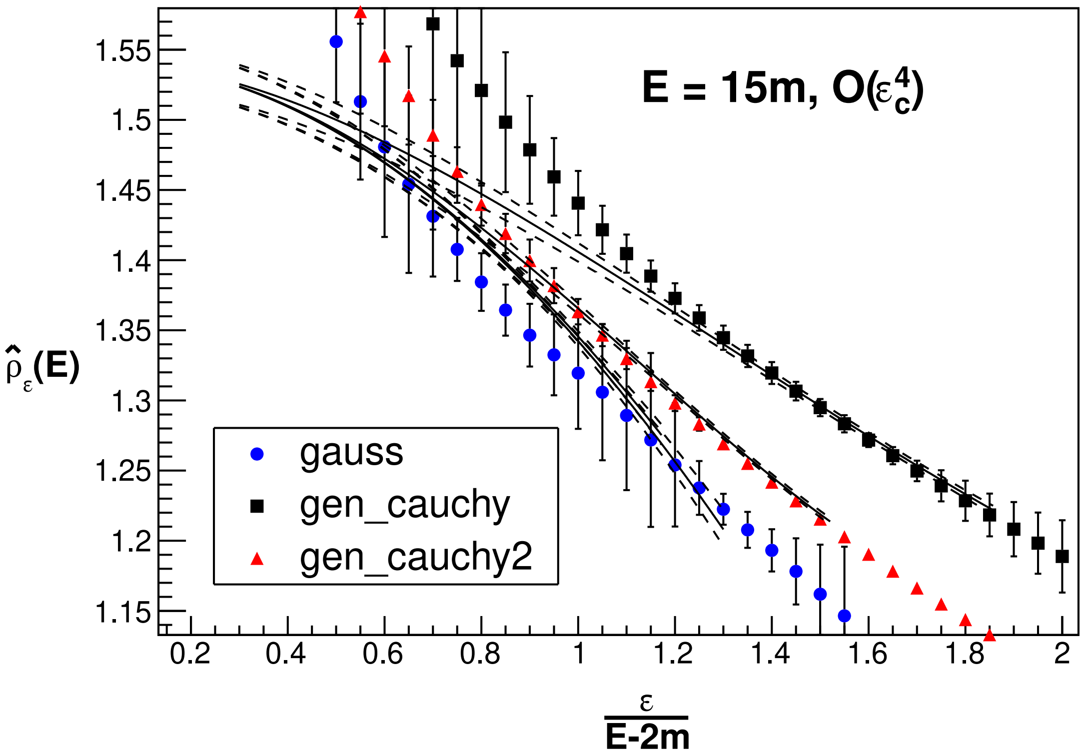
x	$w_1^{(\mathsf{x})}$	$w_2^{(\mathsf{x})}$	$w_3^{(\mathsf{x})}$	$w_4^{(\mathsf{x})}$
g	0	-1	0	1
c0	1	1	1	1
c1	0	-1	-2	-3
c2	0	-1	0	9
b	0	-1	0	2.119

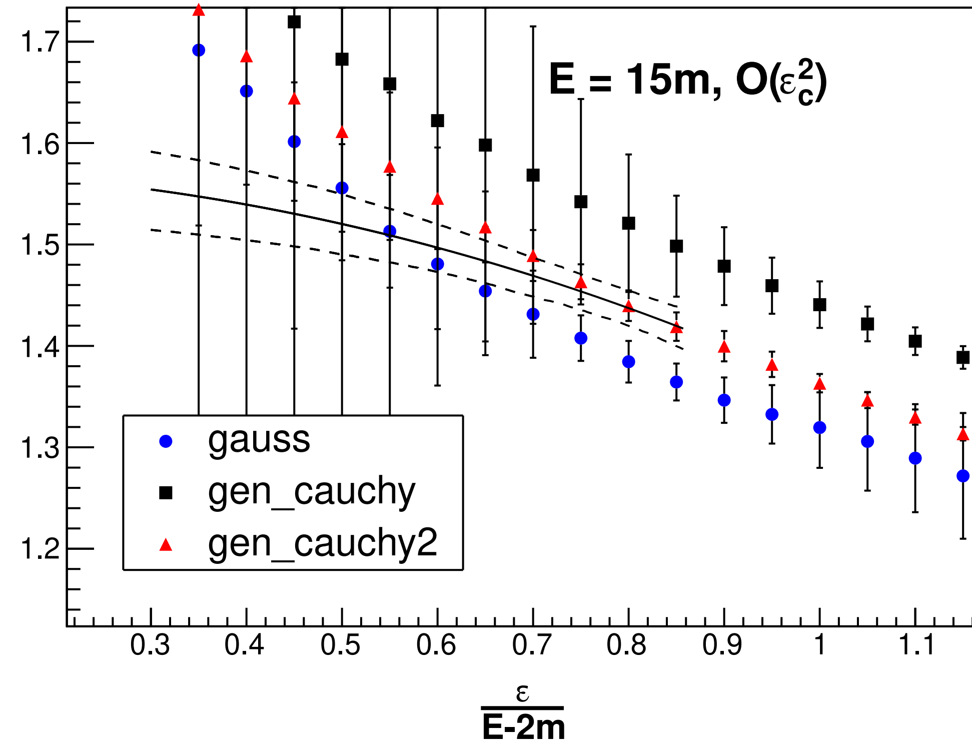
Expansion coefficients worked out to arbitrary orders. (Numerically for ‘bump’)

Results: extrapolation to zero width

Fits to determine $\{a_k(E)\}$:

systematics estimated from $\mathrm{O}(\epsilon^2),\ \mathrm{O}(\epsilon^3),\ \mathrm{O}(\epsilon^4)$

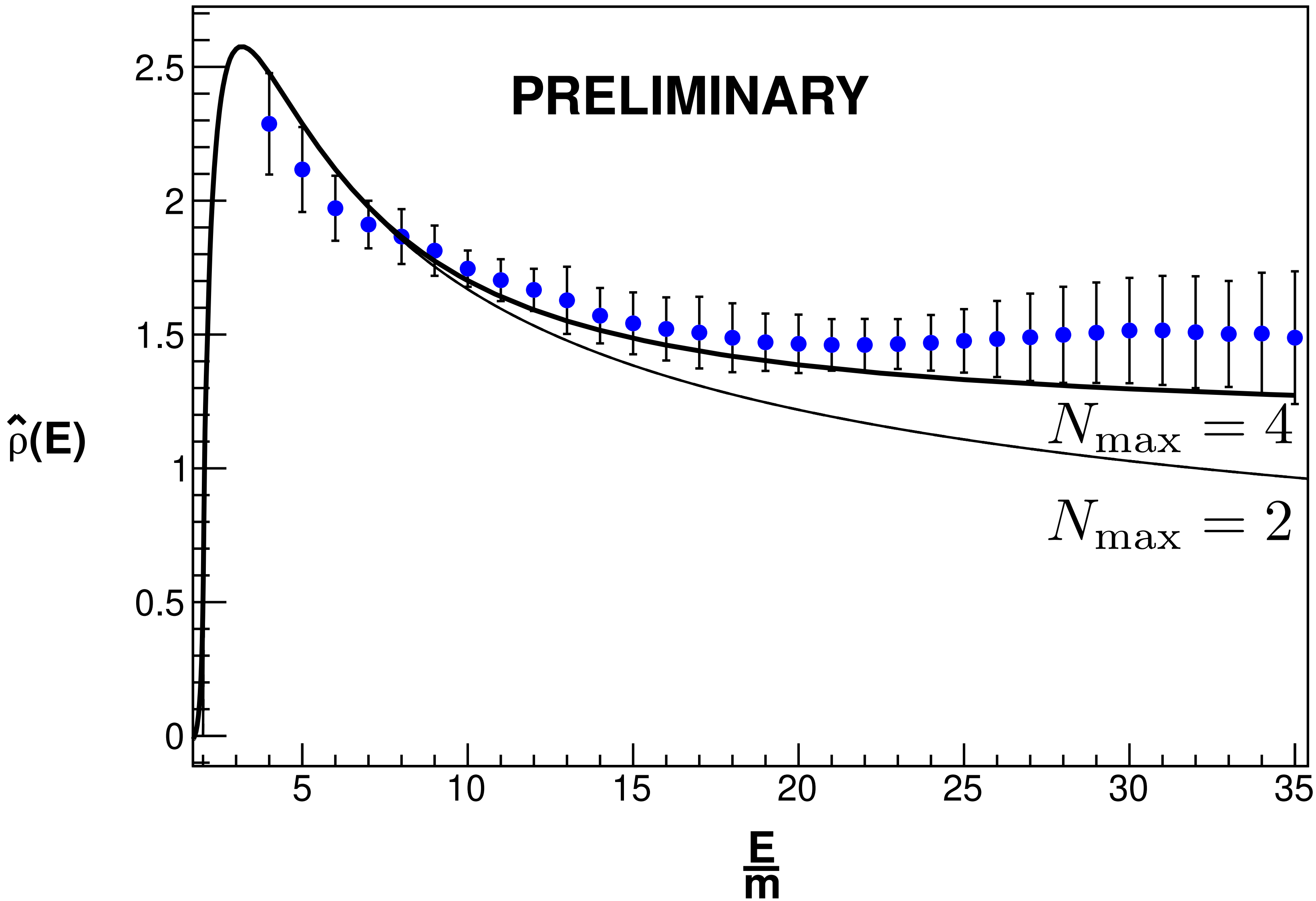




'Cauchy0' kernel not useful due to linear approach. 'Bump' noisy.

Results: extrapolation to zero width

Inelastic region only:



Extrapolation systematics not yet controlled below E=4m.

Conclusions

- Inelastic region accessible in the 2d O(3) model, using the eps→0 extrapolation.

- Our approach to the inverse problem is controlled and flexible, works with arbitrary smearing kernels and basis functions.

- Stay tuned for scattering amplitudes from spectral functions (cf. Max's talk tomorrow)

- Application to QCD:
  - ➔ Signal-to-noise and precision similar to vector-vector correlator
  - ➔ Masterfield paradigm ideal: see Marco Ce' and Patrick Fritzsch talks at Lattice21