Proton rms-radius from electron scattering

Ingo Sick

Proton form factors studied for long time

since the time of Bob Hofstadter
measured up to the largest q's
with σ's as accurate as 0.5%

RMS-radius R

result of fits to data
not much of an issue
changed in 2010

- **R measured via Lambshift in muonic hydrogen**

  energy difference gave R in $0.84fm$ region
    much more precise than value from (e,e)
    disagrees with $R \sim 0.88fm$ from (e,e)

- **R from hyper-precise transition energies in electronic hydrogen**

  determine R despite very small effect
  situation at present not clear
    Munich experiment close to muonic radius
    Paris experiment close to value from (e,e)

Has generated intense interest

- embarrassing to not know R accurately
  - affects definition of fundamental Rydberg constant
- many re-analyses of data
  - often sloppy (see Phys. Rev. C 95 (2017) 012501)
- several new experiments
- partly with non-optimal approaches
  - 12GeV accelerator to measure $q = 0$ property?
  - with initial-state radiation?
- with muons instead of electrons (PSI experiment)
- many speculations about new physics

Will not enter this discussion

- my job: do the best one can to get R from existing data on (e,e)
- subject is tricky enough all by itself

Published results on R: disturbing

- large scatter of results
- values between 0.84 and 0.92fm
  - with error bars of typically 0.015fm
- the more serious ones near 0.88fm disagree with muonic hydrogen .8409$\pm$.0004fm

Main problem: interpretation of data

smaller problem: differences between data sets

Goal of talk

go to bottom of discrepancies
understand causes for differences
redo to-be-taken-seriously analyses to locate origin
will not discuss obviously flawed ones

What to expect

survey of methods used
identification of main problems occurring
determination of average of trustworthy results $\rightarrow \; R \pm \delta R$

Main insight

apparently "simple" task of determining R **surprisingly complex**
"naive" extrapolation to $q = 0$ **very model-dependent**

State right away

results do not fix discrepancy with μH

Generalities

Sachs form factors $G_e(q), G_m(q)$ from

$$\frac{d\sigma}{d\Omega} = \sigma_{Mott}\ f_{recoil}\left[(G_e^2 + \tau G_m^2)/(1+\tau) + 2\tau\ G_m^2\ tg^2(\theta/2)\right]$$

with momentum transfer

$$q^2 = 4\ E\ E'\ sin^2(\theta/2), \qquad \tau = q^2/4m^2$$

E and E' incident and scattered electron energies, and θ scattering angle

Separation of G_e and G_m

data at same q but variable θ

difficult for G_e at large q, difficult for G_m at low q

helped by polarization transfer which yields G_e/G_m

Two-photon exchange corrections

Coulomb distortion (second soft photon)

important at low q, $\quad \Delta R \sim 0.01 fm$

second hard photon, important at large q

fixes problems with G_e from Rosenbluth $\leftrightarrow$ polarization transfer

Optimal approach to get G's

parameterize G_e and G_m, fit to *world* data, L/T during fit

Charge radius and density

Radius R defined via

$$R^2 \equiv \int_0^\infty \rho(r)\ r^4\ 4\pi\ dr$$

with (non-relativistically) $\rho(r)$ from

$$G_e(q) = \frac{4\pi}{q} \int_0^\infty \rho(r)\ sin(qr)\ r\ dr$$

or, inverted

$$\rho(r) = \frac{1}{2\pi^2 r} \int_0^\infty G_e(q)\ sin(qr)\ q\ dq$$

Not practical as

1. Maximum $q \neq \infty$

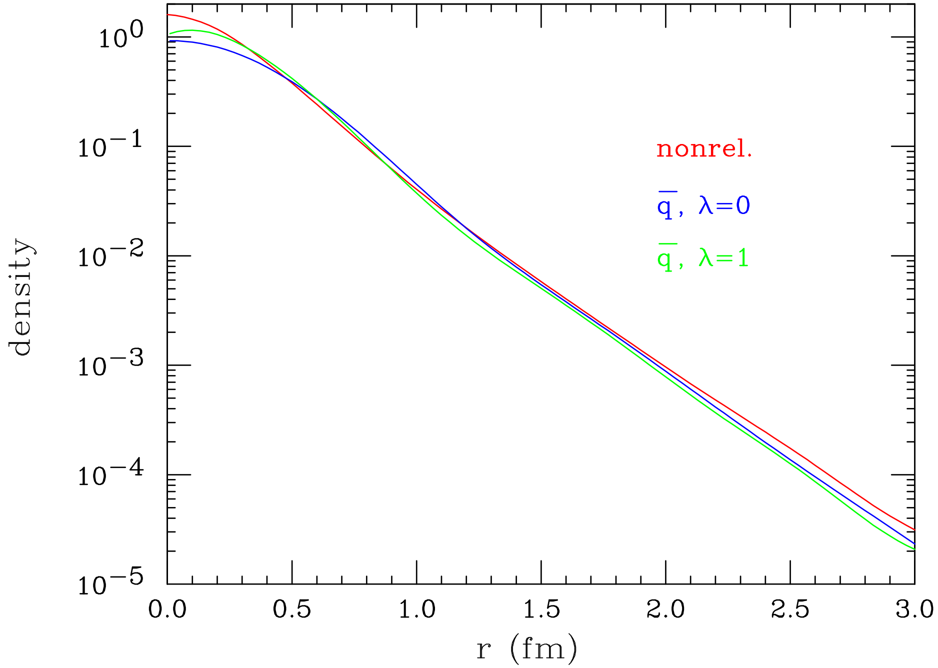
    fit model-$\rho(r)$ or model-$G(q)$ to data

2. Relativistic corrections

- Relevant coordinate system = Breit frame, not nucleon rest frame
    Lorentz contraction
    corrected using $\tilde{q}^2 = q^2(1 + q^2/4m^2)$
- Boost operator in some theories interaction dependent
    additional multiplicative correction $(1 + q^2/4m^2)^\lambda$
    $\lambda = 0$ or 1 for G_e

Effect of relativistic corrections

explore using [3/5]Pade fit to G_e, $q_{max} = 10 fm^{-1}$
determine $\rho(r)$ with/without relativistic corrections and $\lambda = 0, 1$



Result

important change at $r \sim 0$, fixes problems with cusp
minor effect at large r, hardly affects shape of $\rho(r > 1 fm)$

Despite relativistic corrections $\rho(r)$ at large r remains well-defined
this $\rho(r)$ strongly affects R! (see below)

Standard idea to get R

from slope of G_e at $q = 0$, without ever considering $\rho(r)$

$$R^2 = -6 \, \frac{dG_e(q^2)}{dq^2}\Big|_{q=0}$$

This is the origin of many problems of R-determinations

- $q = 0$ slope cannot be measured
- model dependence of $q = 0$ extrapolation from $q's$ that are measurable *and* sensitive to R

Peculiarities and difficulties for proton

1. Importance of $\rho(r)$ at large r

charge at radius r_0 generates Fourier component $sin(qr_0)/(qr_0)$
for large r_0 generates curvature of $G(q)$ at low $q_0 \sim \pi/(2r_o)$
this curvature affects extrapolation to $q = 0$

Density very different from standard Woods-Saxon as $G(q) \sim$ dipole

$$G_D(q) = 1/(1 + q^2 R_D^2/12)^2$$

Hence density close to exponential

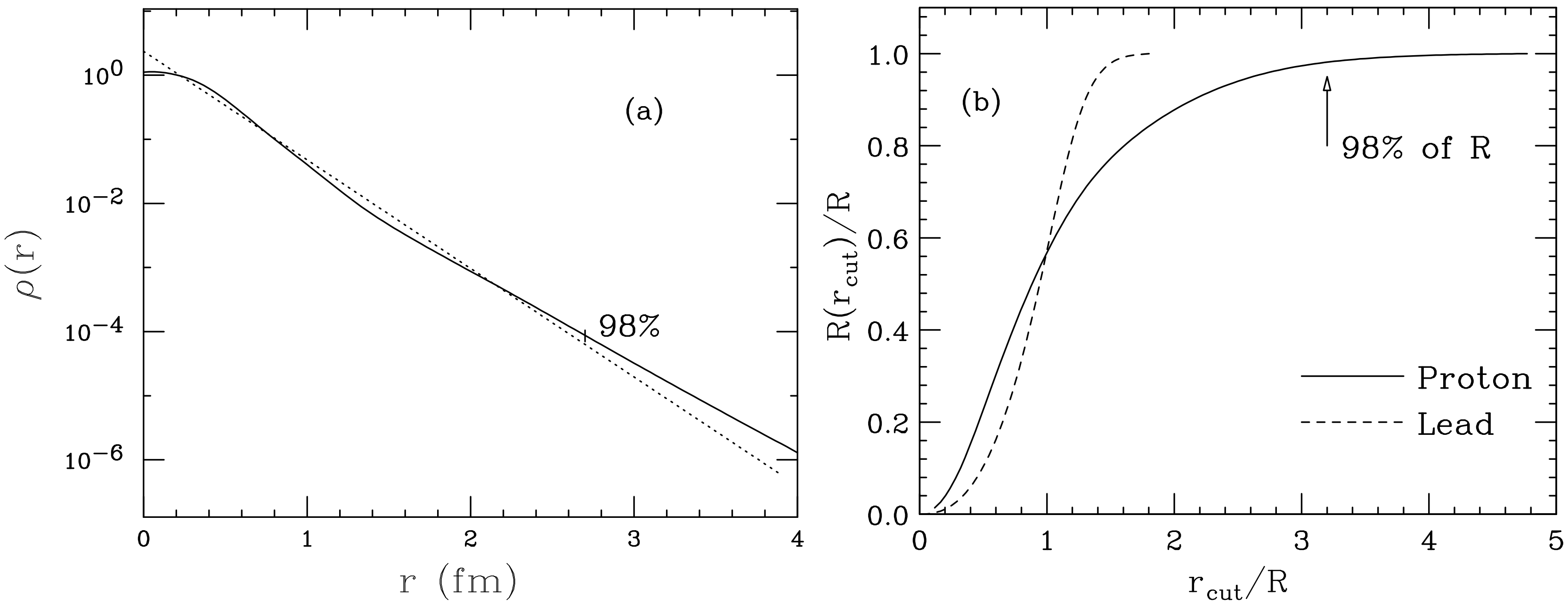
$$\rho_D(r) \propto e^{-\sqrt{12} \; r/R_D}$$

exhibits long tail towards large r which contributes a lot to R

Study partial integral determining R

$$R(r_{cut}) = \left[\int_0^{r_{cut}} \rho(r)\ r^4\ dr\ \Big/ \int_0^{\infty} \rho(r)\ r^4\ dr \right]^{1/2}$$

with $R = R(r_{cut} = \infty)$

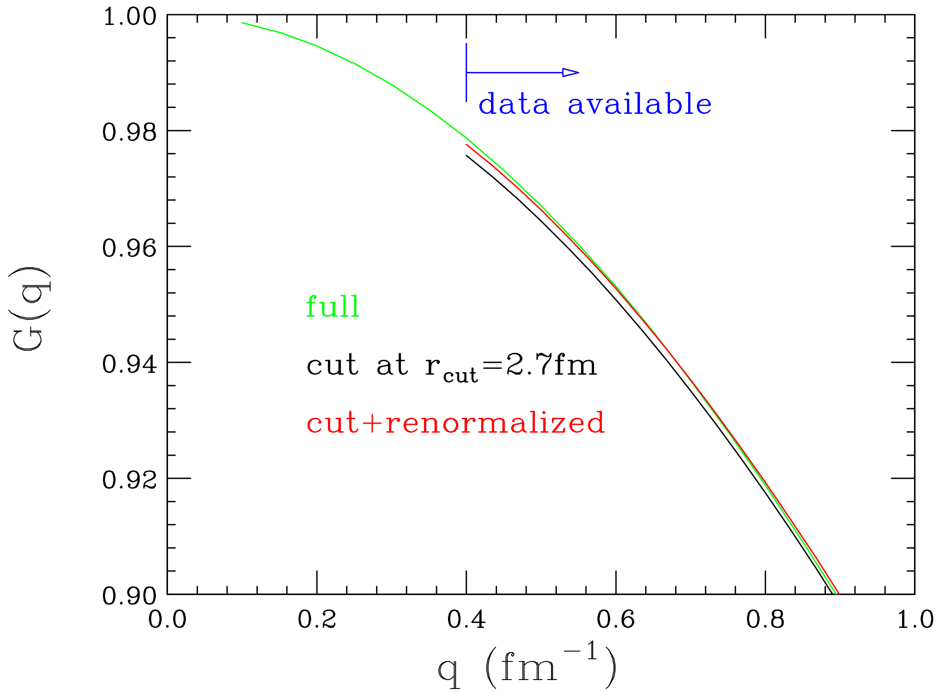


dotted: dipole, solid: realistic

To get 98% of R must integrate out to $2.7 fm$

there $\rho(r)$ has dropped to $\sim 10^{-4}$ of central value!

Effect of $\rho(r > 2.7fm)$ upon $G(q)$



green=dipole black=cut at $2.7fm$ red=cut+renormalized

Difference green-red <0.12%, not measurable!

same argument applicable to $r_{cut} = 2.4fm \rightarrow 4\%$ error of R

$\rightarrow$ cannot be fixed by curve-fitting of data of realistic precision

$\rightarrow$ need to constrain $\rho(r)$ at large r

2. Smallness of contribution of R to $G(q)$

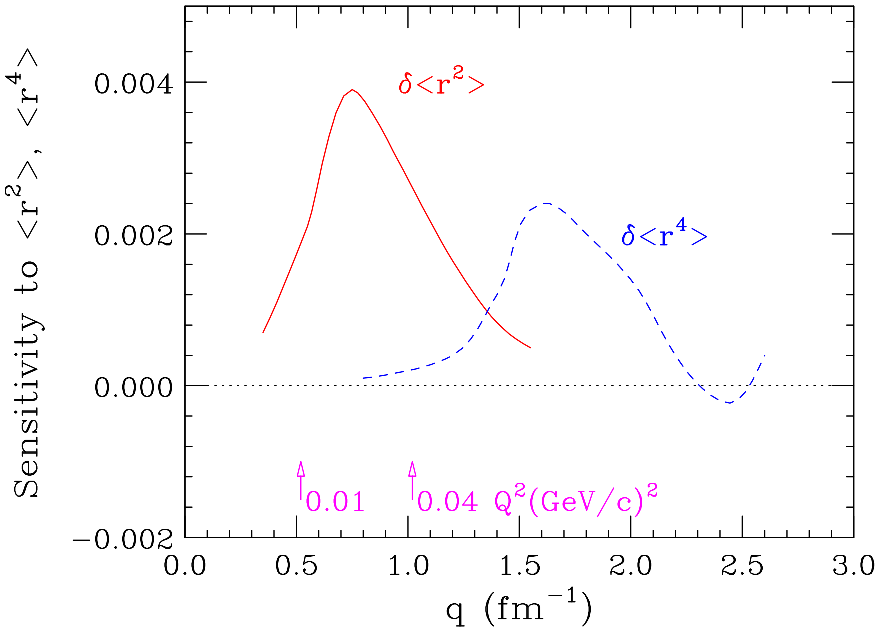
where is it maximal? how large?

Notch-test

change G(q) in narrow region around q_0

refit data, get ΔR

plot ΔR as function of q_0



Data sensitive at $0.5 < q < 1.2$ fm^{-1} $\quad$ ($0.01 < Q^2 < 0.04 GeV^2/c^2$)

at $0.8 fm^{-1}$ effect of $R \sim q^2 R^2/6 \sim 0.08$

To measure R to 1% must measure $G(q)$ to ± 0.0016, i.e. 0.17%!

Consequence

Fits aiming at $\delta R \sim$1% must reproduce data to <0.17%

requires best χ^2 possible

requires look at difference data-fit with <0.1% resolution

visual fits (often standard) not good enough

fits achieving small χ^2 by rescaling error bars neither: need small $G_{exp} - G_{fit}$

3. Parameterizations restricted to q-space are problematic

standard in analysis of data

$\rho(r)$ systematically ignored

can generate uncontrolled effects

Example

Fit of Bernauer data up to $2fm^{-1}$ using Pade

$$G(q) = \left[1 + a_1 q^2\right] / \left[1 + b_1\ q^2 + b_2\ q^4 + b_3\ q^6\right]$$

Fit has none of diseases often encountered

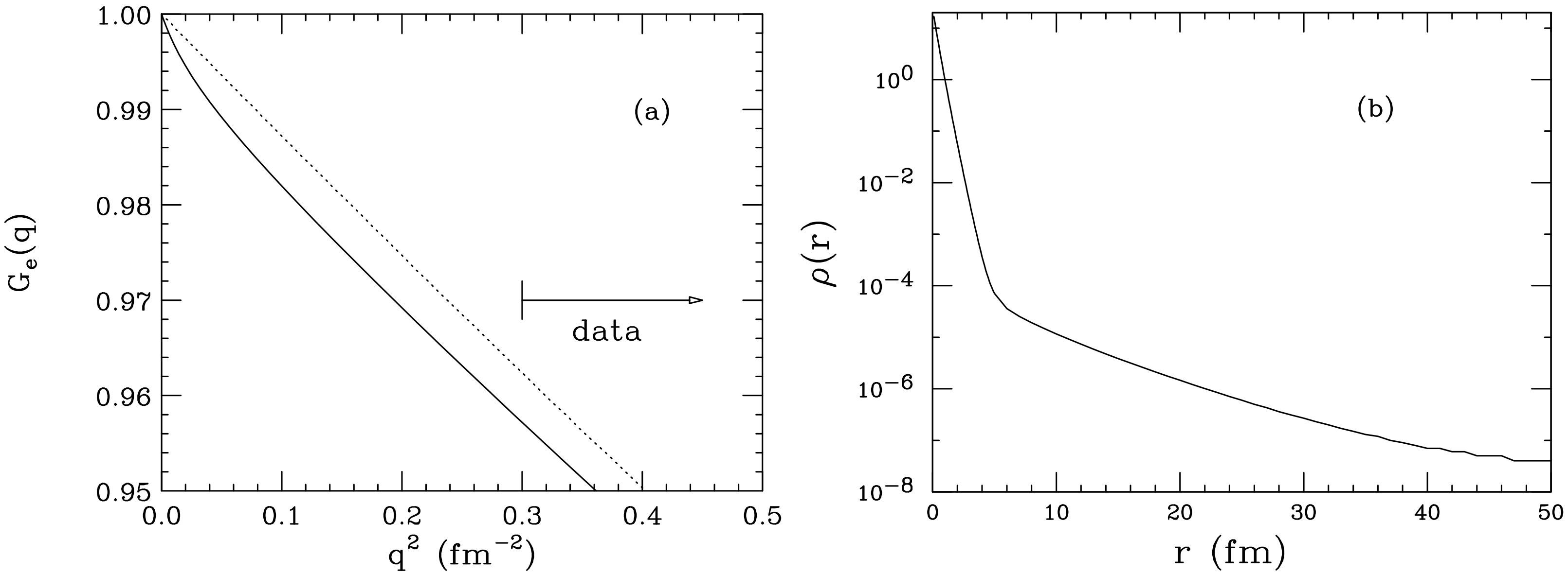
no poles, no unphysical behavior for $q \to \infty$

achieves χ^2 as low as spline-fit

analytical form as acceptable as standard parameterizations of $G(q)$

Yields $R = 1.48fm$!

Reason for large *R*: curvature of $G(q)$ at *very* low q



Note: above $0.2fm^{-2}$ Pade and standard fit parallel

Pade and standard fit have same χ^2 as data floating

Problem immediately seen when looking at $\rho(r)$

outrageously long tail

Generic problem with q-space parameterizations

most do not have Fourier transform, therefore no $\rho(r)$

cannot certify absence of anomalies of above type

Second example

some parameterization have poles at $q > q_{max}$

power series, inverse polynomials, some Pade

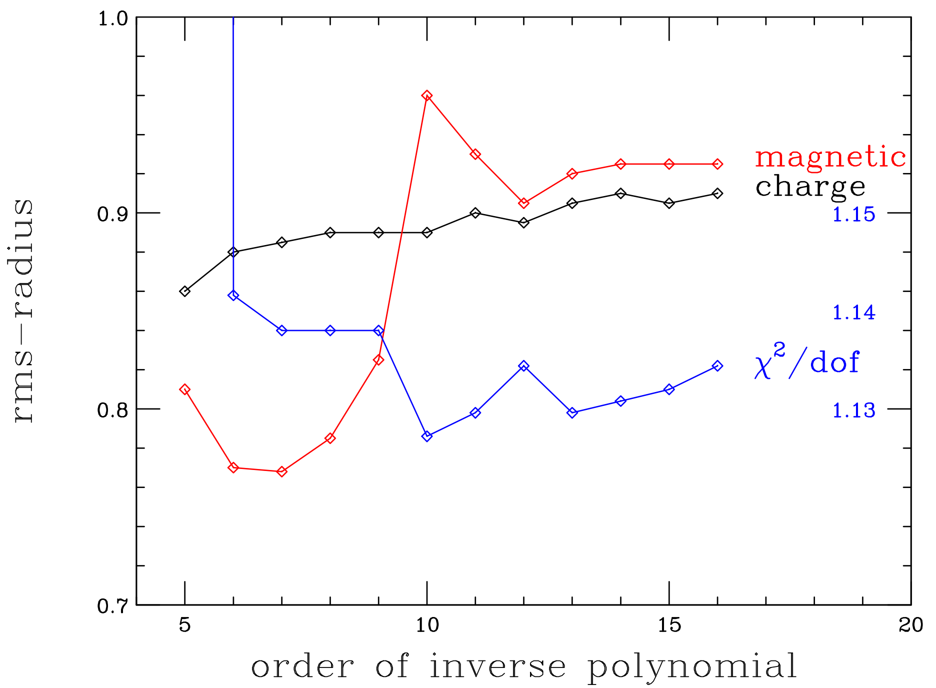
can cause problems

poles $\rightarrow$ oscillations in $\rho(r)$ out to extremely large r

consequences

large-r contributions can have adverse effects on R

cannot judge if large-r behavior of $\rho(r)$ sensible



example: IP fit Bernauer

jump at N=10 due to close pole

note: choice of N $\rightarrow$ arbitrariness of R

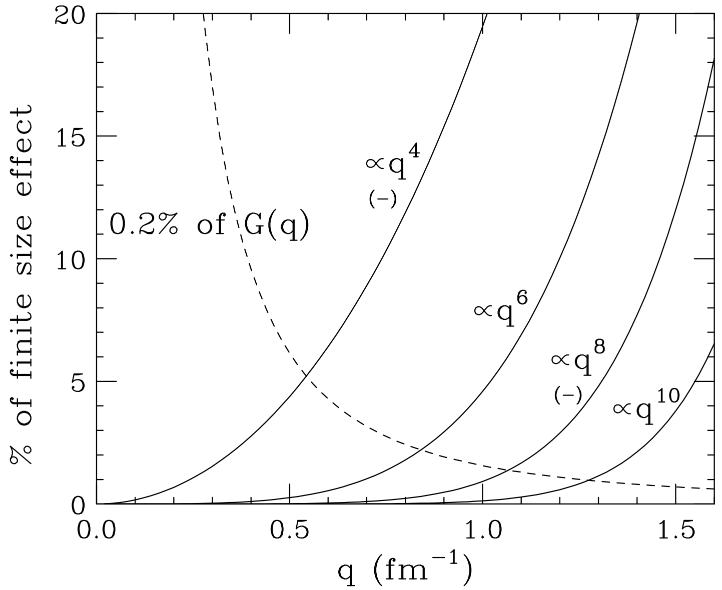
4. R from low-q data impractical

$$G(q) = 1 - q^2R^2/6 + ...$$

Can get R from low-q-data without worrying about higher moments?

Exponential shape of $\rho(r) \rightarrow$ *large* $\langle r^n \rangle$

illustrated by contributions of $\langle r^n \rangle$ to finite size effect $1 - G(q)$



at $q = 0.8fm^{-1}$ effect $\langle r^4 \rangle$ 15%

To reduce contribution of $n \geq 2$ to <1% in R need $q_{max} = 0.34fm^{-1}$

there $q^2R^2/6$ is extremely small: 0.015

but experimental uncertainties are (at best) fraction of 0.01

A 1% measurement of R would need δG of 0.03%! Not realistic!

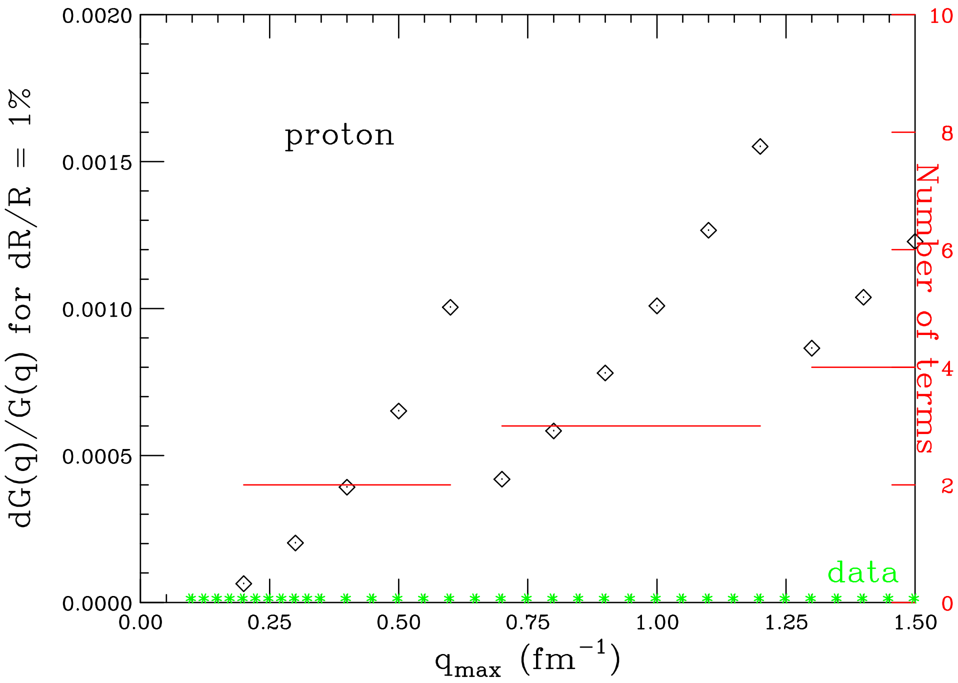
Accurate $q = 0$ slope *without* dealing with higher moments hopeless

Numerical example

invent $G(q) \pm \delta G(q)$ data (at green q's in plot)
fit with suitable function, *e.g.* N'th order polynomial in q^2

adjust N such that a_{N+1} has significance $< 5\sigma$
adjust values $\delta G(q)$ such that δR becomes 1%

plot $\delta G(q)$ as function of q_{max}



jumps of δG due to jump in N

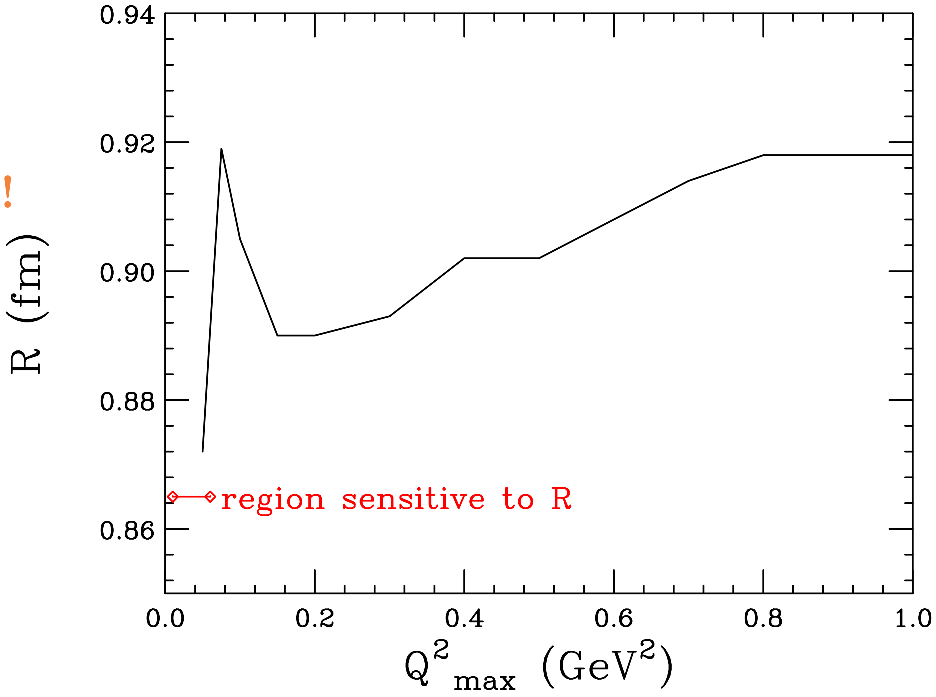
required δG extremely small
not achievable in practice

true: slope determined by R
but: slope not accurately measurable
must get R from $\rho(r)$

5. A counter-intuitive observation

in many analyses R depends on q_{max}
fits with *large* q_{max}, $5fm^{-1} - 10fm^{-1}$, tend to give $R > 0.88fm$
show as example q_{max}-dependence of Lee *et al.*

How can these large q's affect R?
supposedly R 'measured' at $q = 0$!



This behavior calls for explanation!

Understanding: effect of large-r tail of $\rho(r)$

remember: R sensitive to *large* r due to r^4 weight

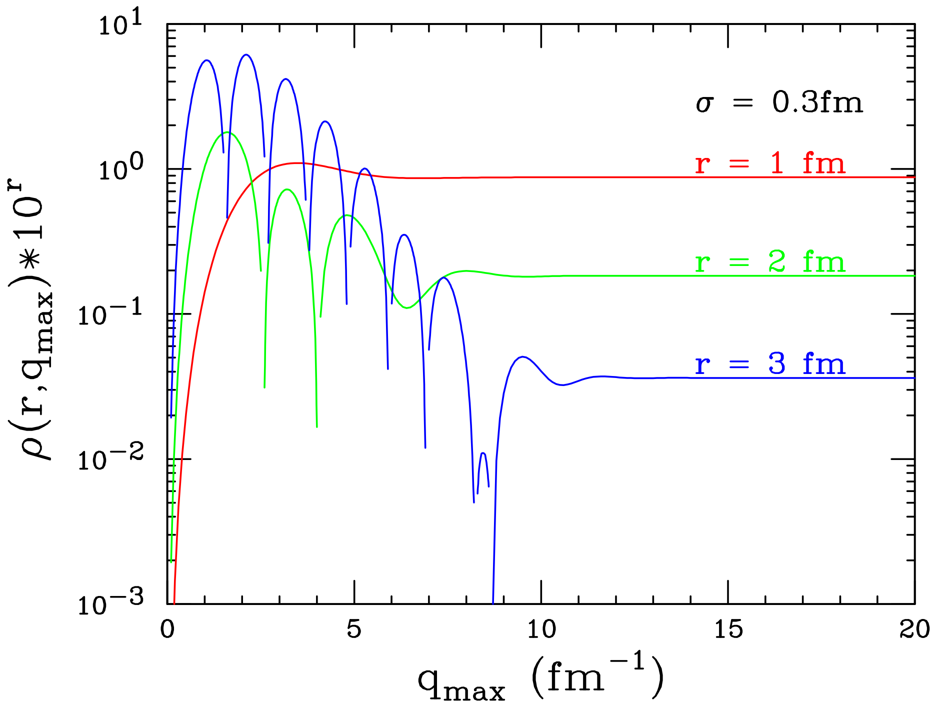
large r affect low q and curvature of $G(q)$ below q_{min}

Data up to large q fix *shape* of $\rho(r)$ *including* large-r tail

this reduces arbitrariness of shape of fitted $G(q)$ at low q

this leads to more reliable extrapolation from q_{min} to $q = 0$

For demonstration study $\rho(r, q_{max}) = ... \frac{1}{r} \int_0^{q_{max}} G(q)\ sin(qr)\ q\ dq$



converged $\rho(r)$ for larger r

needs higher q_{max}

Important observation: to fix $\rho(r)$ at the larger r must include $G(q)$ at the higher q's

Fits with maximal q_{max} yield the most reliable extrapolation to $q = 0$

Importance of high q data for $dG/dq^2(q=0)$ is not a contradiction

low-q data important to fix $G(q)$ in region where data sensitive to R

high-q data fix *shape* of $\rho(r)$ *i.e.* *shape* of $G(q)$ needed for extrapolation to $q=0$

Or more simply said:

rms-radius depends strongly on density at large r: $R^2 = ... \int \rho(r) r^4 dr$

to fix (implicitly) this density need $G(q)$ up to large q

$G(q)$ at high q small, fixes small Fourier components important for $\rho(r)$ at large r

In following discuss fits that:

include world data up to large $q \sim 10 fm^{-1}$
parameterizations that do correspond to densities
do allow to check for sensible ρ at large r

Above considerations special case of more general point:

6. Lack of an important *physics* constraint in $G(q)$-fits

Do parameterizations of $G(q)$ guarantee $\rho(r > 3fm) = 0$?

does $G(q)$ contain components of $sin(qr)/qr$ with $r > 3fm$?
would be undesirable as we believe charge of p to be confined to $r < 3fm$

can be checked if $G(q)$ has Fourier Transform
is difficult if $G(q)$ has *no* FT ($\pm$ standard in literature)

Lenz, 1969, investigated model-independent information from (e,e)
is contained in first moment function

$$T(Q) = \int_0^Q r(Q')\, dQ'$$

integral over $\rho(r)$ up to fractional charge Q

Convenient distribution for ρ and $T(Q)$:

$$\rho(r) = \sum_i a_i\ \delta(r_i) \rightarrow T(Q_i) = \sum_{l=1}^{i} a_l\ r_l$$

Can approximate moment function to any desired accuracy

same $T(Q) \longrightarrow$ same cross sections

Test model-$G(q)$ via fit with $\sum a_i \ \delta(r_i)$

for $G(q)$ from $\rho(r)$ with $\rho(r > 3fm)$=0 expect $a_i = 0$ for $r_i > 3fm$

for $G(q)$ *not* from $\rho(r)$ could get $a_i \neq 0$ for $r_i > 3fm$

would indicate that $G(q)$ contains unphysical Fourier components

Practicalities

fit model-$G(q)$ with $\sum a_i \ \delta(r_i)$, 10 terms for $r_i < 7fm$, uniformly distributed
sum contributions to linear moment from $\delta(r_i > 3fm)$

Results

Find no contribution for:

MD, [3/5]Pade, Laguerre, Borisyuk, [1/3]Pade Kelly

Find contributions of several % for:

Lee, Horbatsch+Hessels, Inv.Polynomial, Polynomial Higibotham/Griffioen

$G(q)$ *without* FT often contain *unphysical* contributions implying charge at $r > 3fm$!
..... which affect curvature of $G(q)$ below q_{min}

$\longrightarrow$ Use $G(q)$'s that do correspond to ρ's with $\rho(r > 3fm)$=0

Parameterizations and fits

1. [3/5] Pade, Arrington *et al*

Often "best" approximation of curve by rational function of given order

$$G(q) = \left(1 + \sum_{i=1}^{I} a_i \; q^{2i}\right) \Bigg/ \left(1 + \sum_{j=1}^{J} b_j \; q^{2j}\right)$$

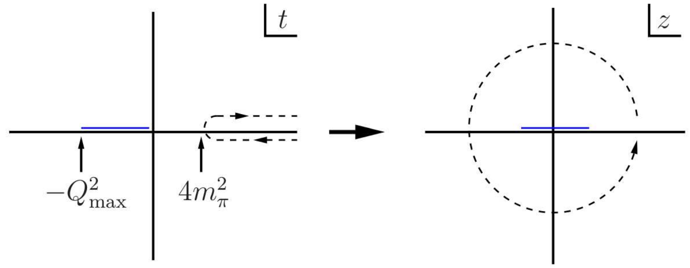
use $b_j > 0$ and $J \geq I+2$
avoids poles and divergences

Fit of *world* data up to $10 fm^{-1}$, excellent χ^2
without Bernauer data (disagree)
includes two-photon corrections
finds 'well behaved' large-r density (shown above)

yields $R = 0.878 fm$.

2. Conformal mapping, Lee *et al*

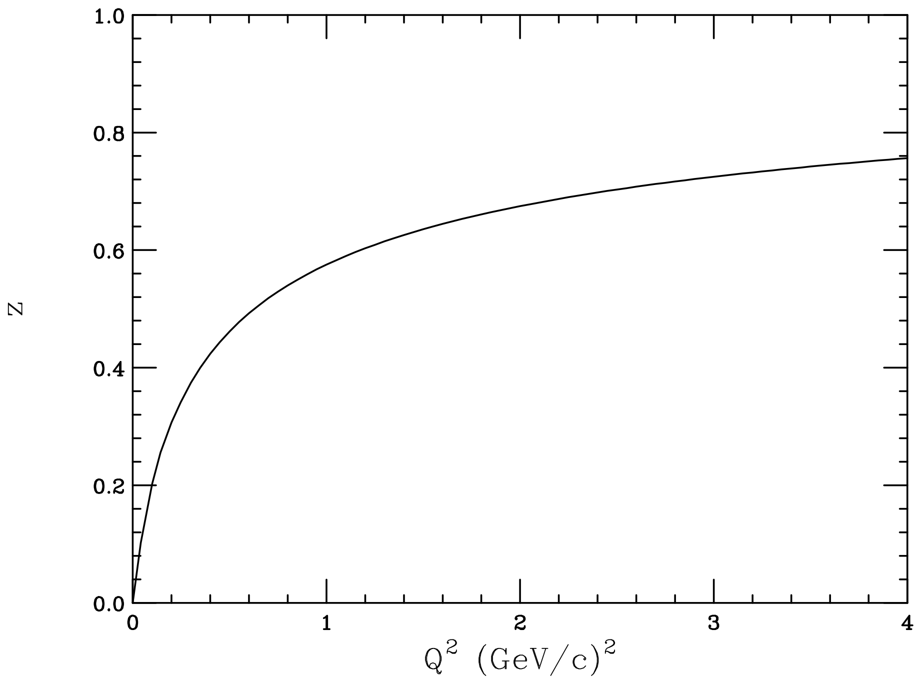
flexible expansions of $G(q)$ in terms of q not optimal
expansion parameter can become >1
variable-transformation could help

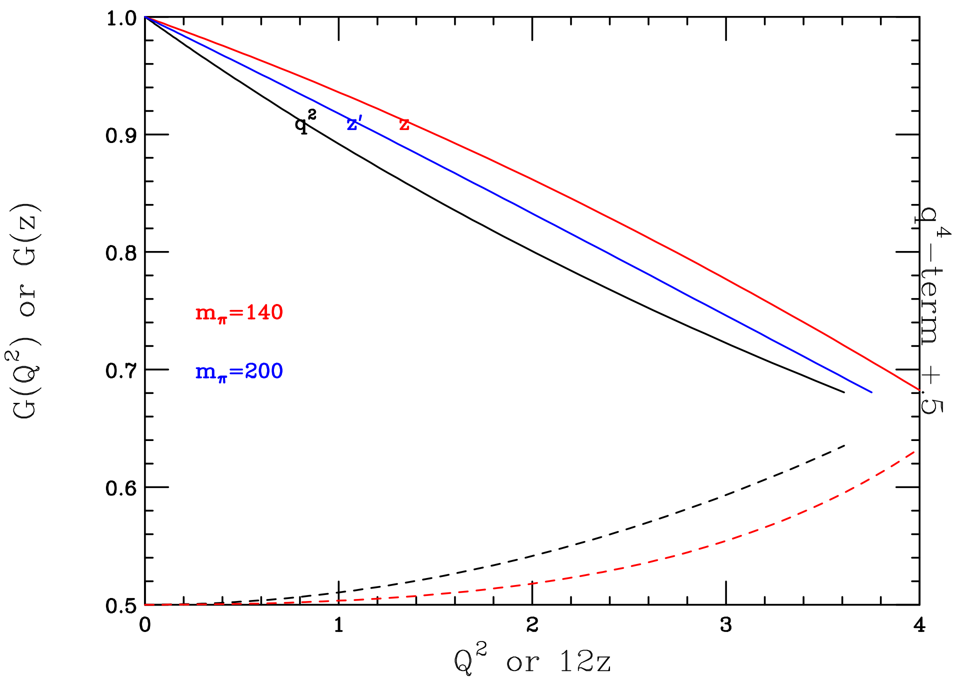


Standard choice $z = \frac{\sqrt{t_c - t} - \sqrt{t_c - t_0}}{\sqrt{t_c - t} + \sqrt{t_c + t_0}}$, $\quad t = -q^2$

most often with $t_0 = 0$ and $t_c = 4m_\pi^2$
yields expansion parameter $z < 1$, see figure

Claimed to 'linearize' extrapolation: only to small extent

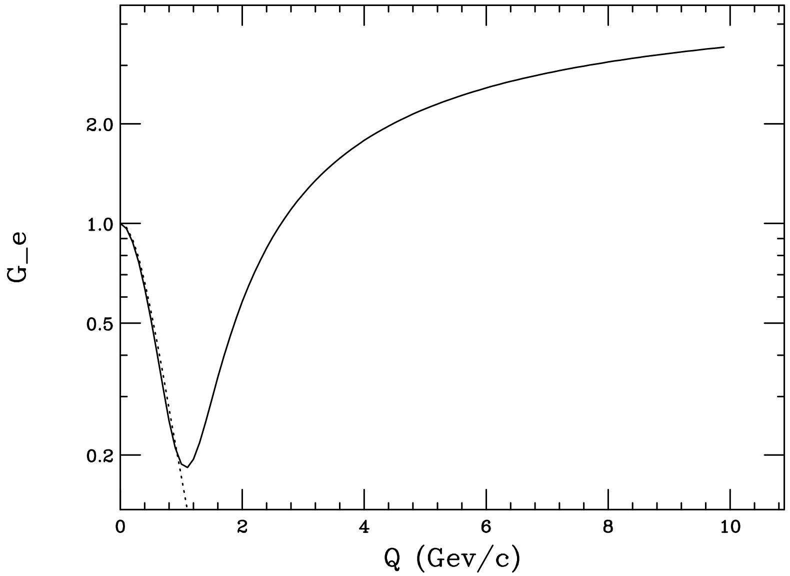




From careful analysis of world data Lee *et al* find $R = 0.916 \pm 0.024 fm$

what is reason for large R?

Even in terms of z, use of power-series is highly unwise!



Standard disease of power-expansions

must generate *small* $G(q)$ at large q via cancellations

large contribution at q_{max} of low-order terms

for $q_{max} = 5 fm^{-1}$ term yielding R contributes 70% of FSE!

Better choice: parameterization a la Borisyuk

$G(q)$ = polynomial in z *times* dipole in q

then q^{-4} fall-off guaranteed, polynomial remains of order 1

polynomial only parameterizes deviation from dominant dipole

Analysis following closely Lee *etal*, but with better parameterization:

find R systematically $0.03fm$ lower
find large-r density close to MD, VDM densities (see below)
$\rightarrow$ large R of Lee not due to use of z, but due to unphysical $G(z(q))$

3. Polynomial in ξ times dipole, Borisyuk

$$\xi = q^2/(1+q^2/\xi_0), \qquad \xi_0 = 0.71 GeV^2/c^2$$

ξ very similar to z, maximum value 0.7

$$G(q) = (1-\xi/\xi_0)^2 \sum a_k \xi^k$$

then q^{-4} fall-off guaranteed, polynomial remains of order 1

Repeating analysis of Borisyuk with
world data up to $10fm^{-1}$
2-photon exchange corrections

yields $0.880 \pm 0.011 fm$

large-r density close to MD, VDM density (see below)

4. R from Bayesian inference, Graczyk+Juszcak

form factors come from large class of models

sigmoid functions times dipole

combination determined from neural network with 1 hidden layer

fixed via maximum a posteriori weights

likelihood via χ^2 plus Occam factor

Applied to

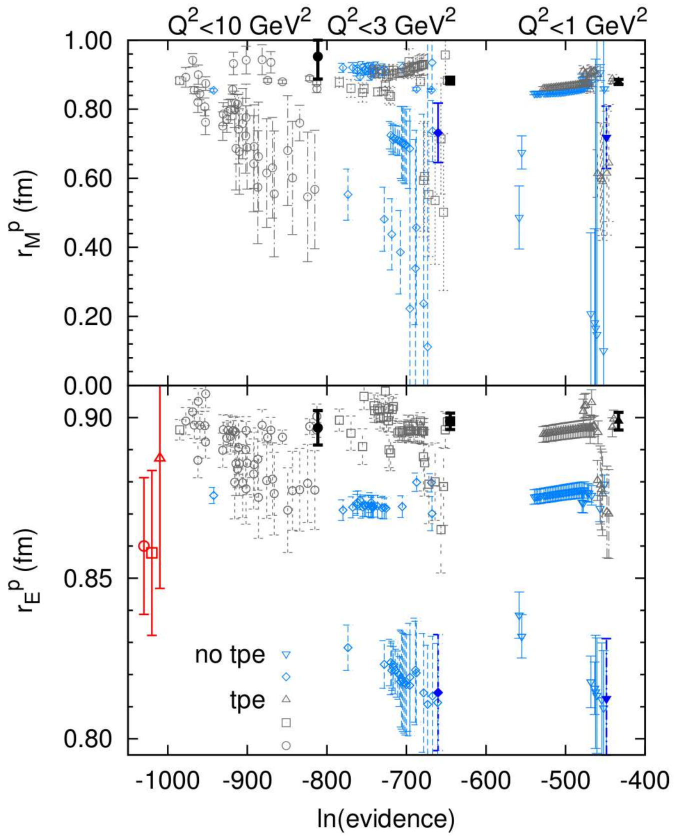
world data up to $10 fm^{-1}$

including 2-photon corrections

find $R = 0.899 \pm 0.003 fm$

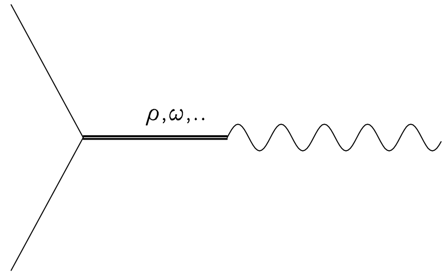
well-behaved density out to $2.7 fm$

0.003 seems low given systematic errors



5. *R* from Vector Dominance Model fits

Basic assumption of VDM



leads a priori to form factor $G(q) = \sum_i a_i/(1+q^2\gamma_i)$, $\rho(r) = \sum a_i e^{-\gamma_i r}/r$

γ_i^{-1} = masses squared of vector mesons

The promise: VDM could fix problem with large-r behavior

tail $\sim e^{-\gamma r}/r$ is given by *physics*

Complication

'pole' closest to physical region (responsible for low q) is *not* a pole

it is a cut starting at $4m_\pi^2$

accounts for interaction with pion tail of N (triangle diagram)

Strength distribution in cut: difficult to come by

determined by Hoehler *et al.* in 1976 using dispersion relations

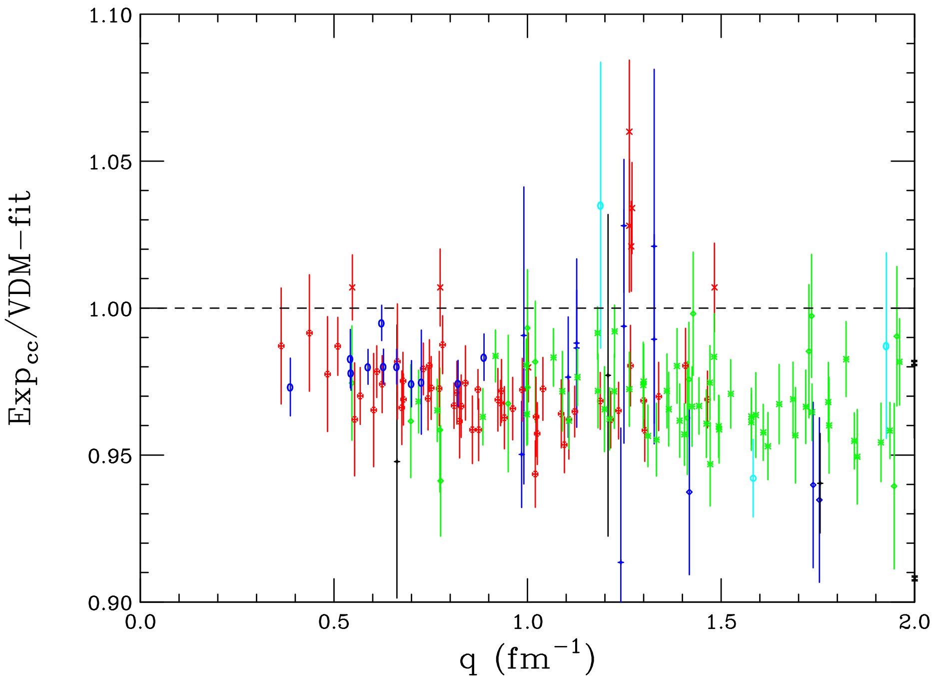
only partly updated since

Result of VDM-fits: $R \sim 0.84 fm$

Generic problem of VDM analyses

χ^2 is too large

systematic differences to data at low q: fit Mergel *et al.*



difference to true R

can be trivially read off figure

±same for all VDM fits

since Hoehler's time

Reason for too large χ^2

VDM spectral function is too strong a constraint
has not enough flexibility to allow good fit of (e,e)

Demonstration of constraining role of spectral function

VDM analysis of Adamuscin *et al.*
want to fit (e,e) data over largest q^2-range ($< 0, > 0$)

Difficulty: TPE in (e,e) cross sections (G_e from L/T and PT disagree)
their 'solution': omit σ, fit only polarization transfer data
result: $R = 0.848 \pm 0.007 fm$
amazingly small error bar!

BUT...

PT data measure only *ratio* G_e/G_m
contain no information whatsoever on charge form factor

Conclusion

spectral function all by itself fixes R to amazing precision
adding cross section data only leads to bad χ^2!
explains the problem of VDM analyses since Hoehler's time
find always poor χ^2 and $0.84 fm$ + systematic deviations from data

6. VDM-motivated parameterization: MD

VDM adherents claim that analytic structure of $G(q)$ important
indeed: helps to control large-r tail

use VDM-type form factor:

sum of monopoles times dipole = M·D-parameterization

$$G(q) = \sum_i a_i/(1 + q^2\gamma_i) \;\; 1/(1 + q^2\Gamma)^2$$

with free a_i, γ_i with VDM-constraint $\gamma_i^{-1} > 4m_\pi^2$
ensures physical fall-off of large-r density
$\Gamma < \gamma_i/5$ (such as not to affect shape of tail)
has been very successful in past: IJL, BZ, ...

Result of fit of *world* data up to $10 fm^{-1}$

variation of γ_i's (±uniform distribution) not even needed
fit of parameters a_i enough
χ^2 as low as other best fits (SOG, Laguerre) of same data

Find $R = 0.891 \pm 0.013 fm$

M·D parameterization optimal for (partial) control of large-r density

7. Laguerre function fits

often difficult to find parameters for multi-parameter functions
local minima of χ^2, many failed fits in literature
orthogonal basis should help

Laguerre functions

optimal since incorporates $e^{-\mu r}/r$ shape expected from pion tail

$$\rho(r) = \sum_{n=0}^{N} a_n \; e^{-x} \; L_n(x) = \sum_{n=0}^{N} a_n \sum_{m=0}^{n} c_{nm} \; x^m \; e^{-x}$$

with $x = r/\beta$, L_n=Laguerre polynomial

Similar to other multi-parameter expansions

sensitive to N
too many parameters $\rightarrow$ correlations between higher-order a_n's
avoided with penalty in χ^2

Fits to world data

$q_{max} = 10 fm^{-1}$, two-photon corrections
for 2·7+1 parameters, 604 data get χ^2=540

find $R = 0.879 \pm 0.02 fm$

$\rho(r)$ out to $2.6 fm$ agrees with VMD, SOG, MD

8. Sum-Of-Gaussians with tail constraint

SOG often employed for $A > 1$

width Γ limits fine structure of ρ

decouples densities at different r

Best used together with tail constraint

at $r < 1fm$ quark/gluon structure of p complicated

at large r ρ dominated by Fock component $n + \pi^+$

example: cloudy bag model for $r > 0.8fm$

Shape of density at large r :

asymptotic w.f. of pion $W_{-\eta,3/2}(2\kappa r)/r$

can be used as constraint on *shape*

used extensively for $A \geq 2$

Corrections+sophistications

CM, π-size, $\pi + \Delta$ components, ...

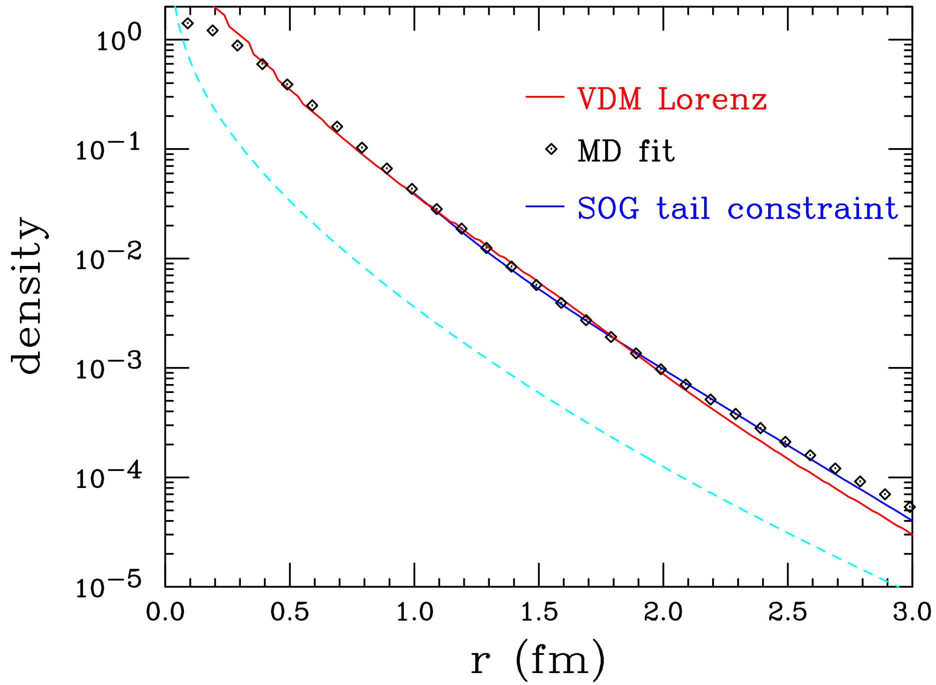
investigated, minor numerical importance

Physics $\equiv$ 2π triangle diagram in VDM

Fit of world data up to $10fm^{-1}$, 2γ corrections,

constraint for $r > 1.2fm$

find $R = 0.886 \pm 0.008fm$



Summary

Pointed out in this talk;

difficulties of standard $q \to 0$ extrapolation
curvature of parameterized function model-dependent
lacks constraint that $\rho(r)$ confined to $r \leq 3fm$

Emphasized

curvature at low q related to $\rho(r)$ at large r
there have knowledge: density dominated by least-bound Fock state
for reliable R: $\rho(r)$ must be close to this behavior

Three consequences

- Use physical $G(q)$ which does correspond to density

- Fit data up to largest q_{max}
  data fix shape $\rho(r)$ *including* large-r tail

- Verification that shape ρ at large r is *physical*
  fix shape using *physics* constraint

Unweighted average of fits respecting above insights:

[3/5]Pade, Dip.·Poly(z), Dip.·Poly(ξ), Bayesian, MD, Laguerre, SOG
radii .878, .886, .880, .899, .891, .879, .886fm

$R = 0.887 \pm 0.012 fm$, disagrees with $.8409 \pm .0004 fm$ from μH

Reason for difference: many speculations!

address only a popular one: $e \leftrightarrow \mu$

discuss by looking at other nuclei

Deuteron

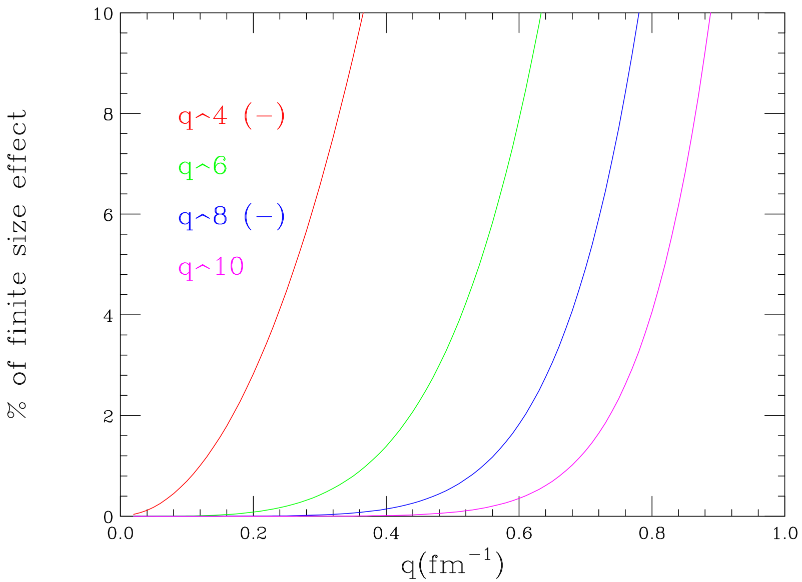
Problematic in past: large scatter of results

main problem same as for proton: large-r tail

actually much worse than for proton

last 2% of R come from r >7 fm!

corresponding contribution in $G(q)$ not measurable



demonstration case for R *not* from $q = 0$ slope

q_{min} of data $\sim 0.4 fm^{-1}$

precise R would be of high interest as theory predicts very accurate radius from V_{NN}

Added complication: 3 form factors, need T_{20} data

separation of C0 enhances uncertainty

Determination of R: use same approach as for proton

- use *world* data
  - up to largest q_{max} possible
- use tail constraint
  - shape of large-r density well known from theory
  - Hankel function determined by binding energy
- include 2-photon corrections (Coulomb distortion)
  - previously always neglected
  - significantly change R

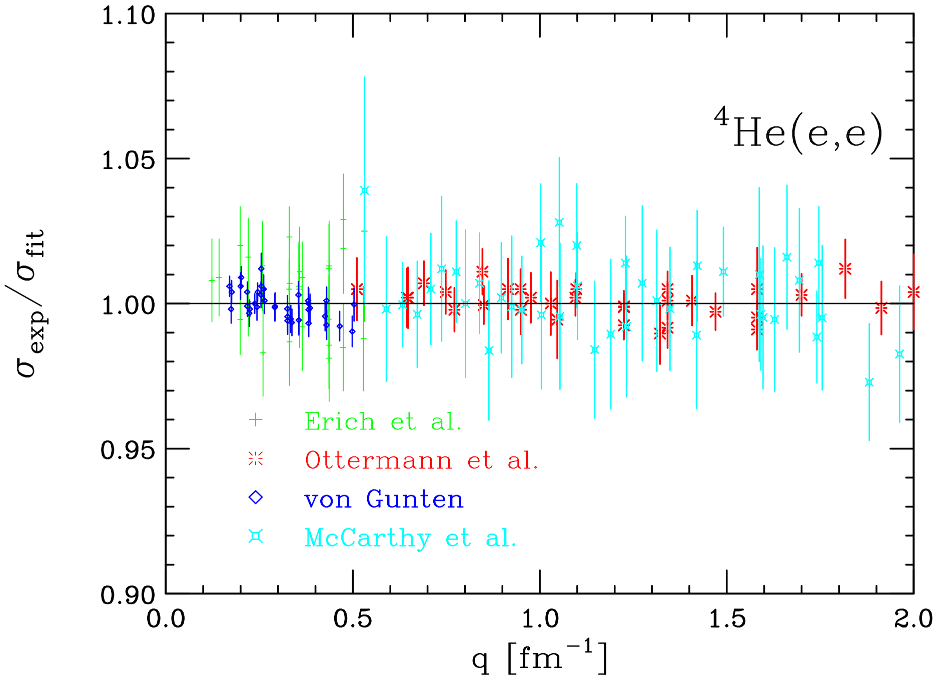
(e, e)	$2.130 \pm 0.010\ fm$ (Sick 1998)
μH	$2.1289 \pm 0.0012 fm$ (prelim.)
a_{n-p}	$2.131 fm$

Find perfect agreement with μX data from Pohl et al.

agreement within 0.5% significant given 4% discrepancy for proton

Helium 4

Great: ^{4}He data most precise of all light nuclei



Simple-most case: only *one* form factor

no error-enhancing L/T-separation needed

Perform exactly same type analysis as for proton

parameterize ρ in r-space using SOG
employ data up to largest q_{max}
use tail constraint

Most helpful: FDR analysis of *world* data on p -^{4}He scattering

determines residuum of closest singularity
corresponding to exchange scattering at 0°
yields *absolute* normalization of tail to ±10%
can use in fitting (e,e) data

Consequence: get most precise rms-radius of all nuclei

R=1.681±0.004 fm (Sick 2008)

R=1.6783±0.0005 fm μ^4H, Antognini et al. (prelim.)

Highly significant: agreement (e,e)-μX within 0.25%

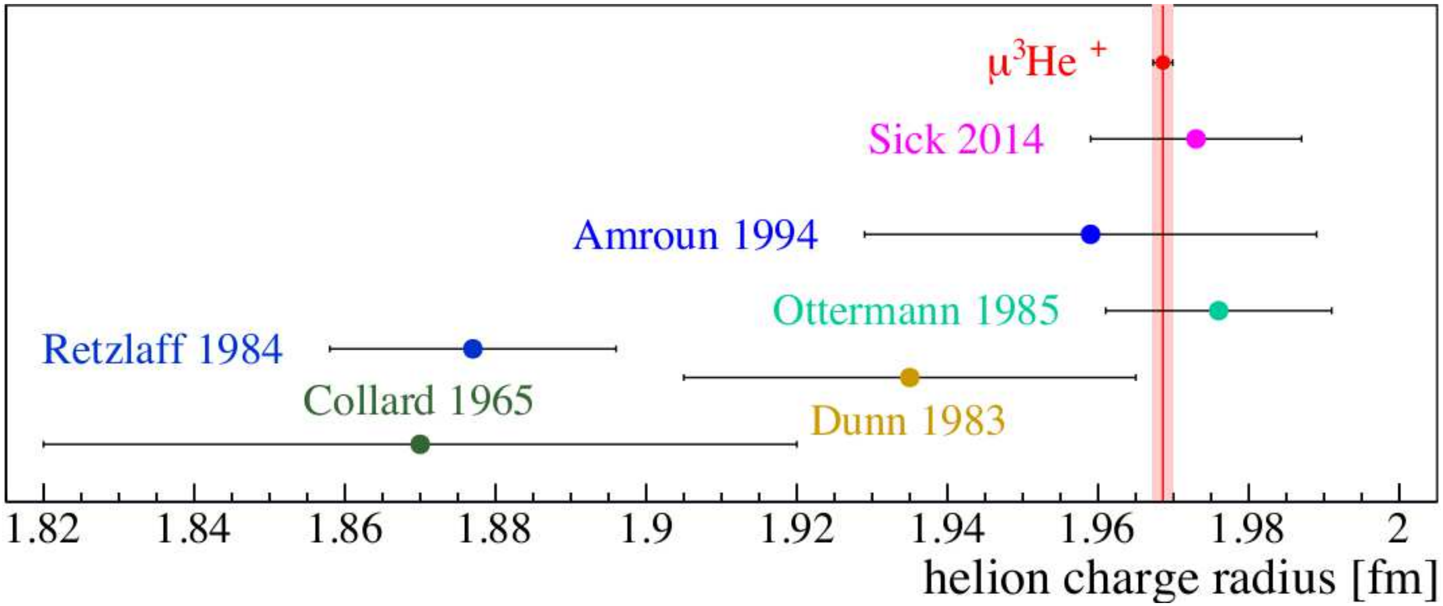
as compared to 4% for proton

Conclude: problem is not e $\leftrightarrow$ μ, problem is with *proton*

Similar agreement for ^3He

though less precise

recent data from Antognini etal



Generic problem for proton:

extremely large higher moments due to $\rho(r) \sim$ exponential

	$\langle r^4 \rangle / \langle r^2 \rangle^2$	$\langle r^6 \rangle / \langle r^2 \rangle^3$	
naive estimate	$\sim$1.		
exponential density	2.49	8.82	
experimental value	4.32	64.2	fit of $q \leq 5 fm^{-1}$ data Bernauer 2010
Horbatsch,Hessels,Pineda	1.25	14.5	

large-r contribution even worse than for exponential density

Consequence: at $q \sim 0.8 fm^{-1}$ of maximal sensitivity to $\langle r^2 \rangle$:

contribution of $\langle r^4 \rangle \sim 15\%$ of finite-size effect, even $\langle r^6 \rangle$ contributes 4%

even at q^2=0.6, where finite size effect only 0.077, $\langle r^4 \rangle$ contributes 10%

$\rightarrow$ serious interference of higher moments

Wrong $\langle r^4 \rangle$ or wrong $\langle r^6 \rangle \rightarrow$ wrong R

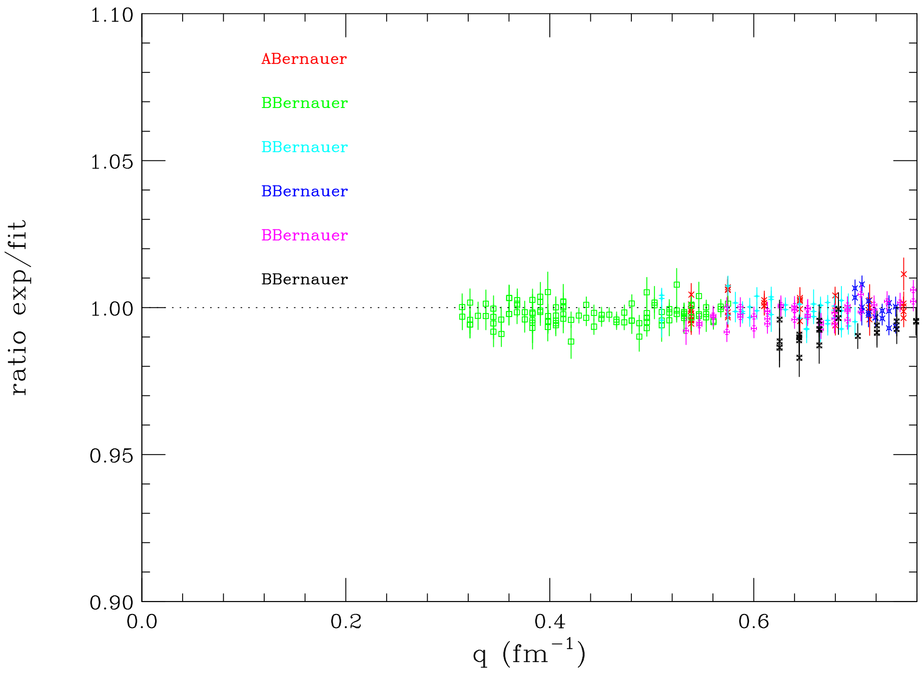
= short version why some determinations of R, discussed below, are wrong

Example: fit of Horbatsch, Hessels, Pineda

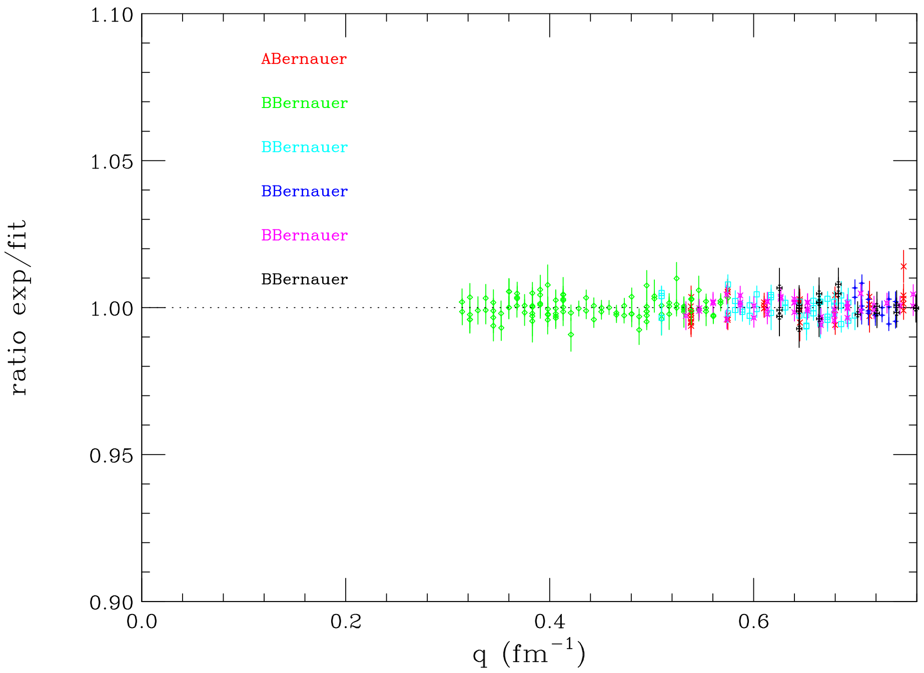
polynomial in q^2
higher moments from CPT
R fit to Bernauer data, low q

$\langle r^4 \rangle$ extremely low
disagrees with fit to (e,e) data

fit HHP



fit CF

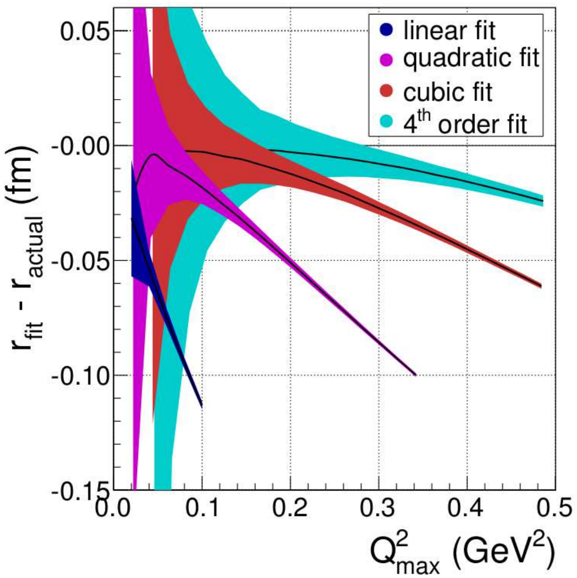


difference in χ^2: factor 1.5

$G(q)$ as polynomial in q^2: $1 - q^2\langle r^2\rangle/6 + q^4\langle r^4\rangle/120 - ...$??

used by many authors in past
shown already in 2003 that *not* suitable, Phys. Lett. B 576 (2003) 62
in 2014 quantitatively studied by Kraus *et al.*

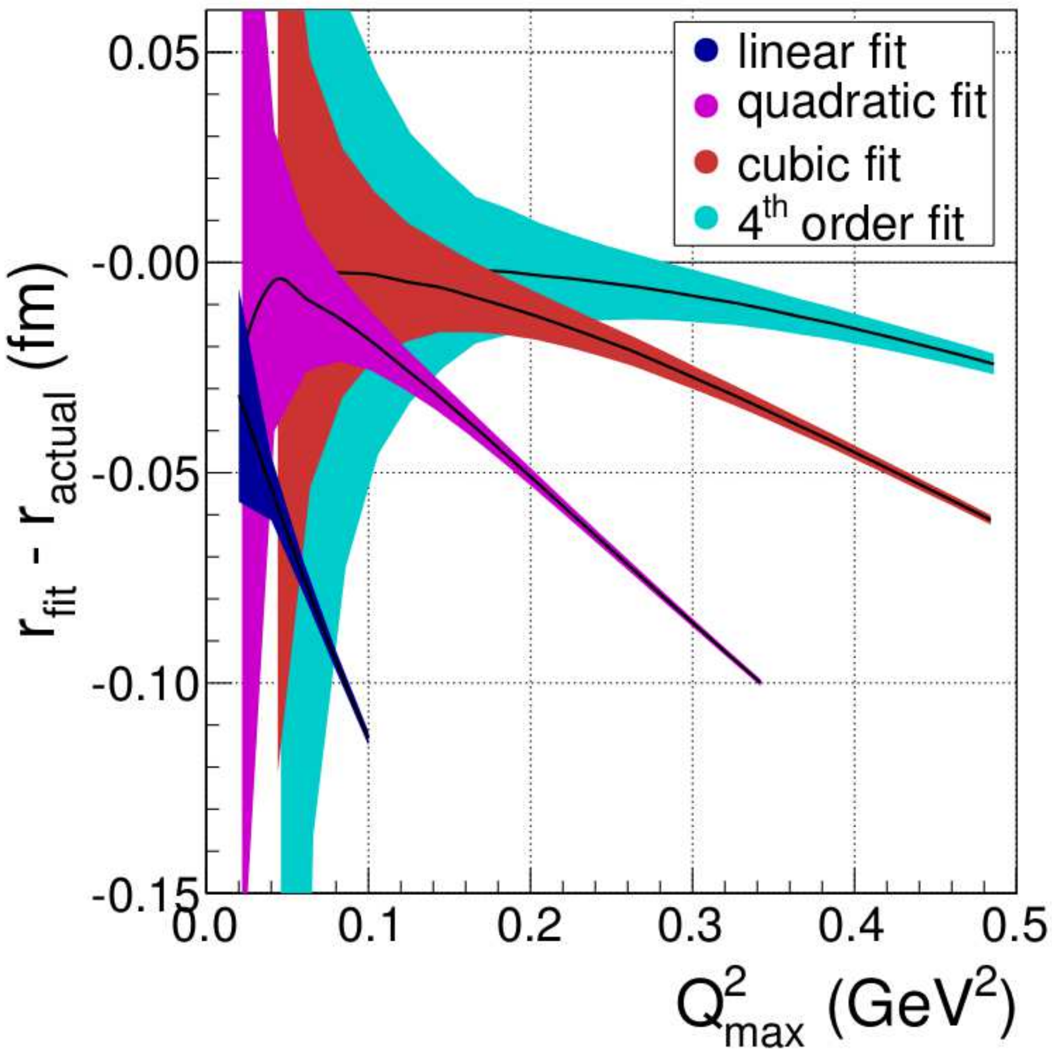
parameterized $G(q) \rightarrow$ pseudo data $\pm 0.4\%$ $\rightarrow$ power series fit $\rightarrow R_{fit}$
always gives low R_{fit}, and R_{fit} depends strongly on q_{max}



e.g. for $Q^2 = 0.03$ (typical) and linear fit
defect according to figure $= 0.04 fm$
as large as discrepancy (e,e)...μX

must be dumb to use low-order polynomial
but some authors do it anyway

..



linear fit
quadratic fit
cubic fit
4th order fit
0.05
-0.00
-0.05
-0.10
-0.15
0.0
0.1
0.2
0.3
0.4
0.5
r_fit - r_actual (fm)
Q²_max (GeV²)

Understanding of low R's

discuss for "very low-q" linear fit in q^2

corresponds to $\langle r^4 \rangle = 0$

how can produce $\langle r^2 \rangle \neq 0$ with $\langle r^4 \rangle = 0$?

What would a physicist think?

would try to think how corresponding $\rho(r)$ would look like:
positive inside, negative at very large r
then negative tail can compensate positive part in $\langle r^4 \rangle$ to yield $\langle r^4 \rangle = 0$
given r^6 weight in $\langle r^4 \rangle$-integral

But: negative tail also impacts $\langle r^2 \rangle$
will yield too small value for R

remember: $\langle r^4 \rangle$ contributes ∼15% of finite-size effect
at q of maximal sensitivity to R

= physical explanation of above results of Kraus *et al.*

= general argument why truncated polynomial (also higher order) generates problems

Illustration of problems with low-order polynomial fits

recent fit of Higinbotham *et al.*
Mainz80+Saskatoon data
$q^2_{max} = 0.8 fm^{-2}$ for R with smallest δR
$G(q) = a_0(1 + a_1 q^2)$

Find $R = 0.844 \pm 0.009 fm$

conclude that is compatible with μX result

Wrong, as a trivial back-of-the-envelope estimate shows! For q^2=0.8

$$\left.\begin{array}{ll} q^2R^2/6 & = 0.094 \\ q^4\langle r^4\rangle/120 & = 0.0138 \end{array}\right\}$$ q^4 contribution is 14.7% of q^2 contribution

$\rightarrow q^2$ contribution wrong by $\sim$14.7%
$\rightarrow R^2$ is wrong by $\sim$14.7%

... and this sort of analysis is claimed to provide insight on radius-puzzle!

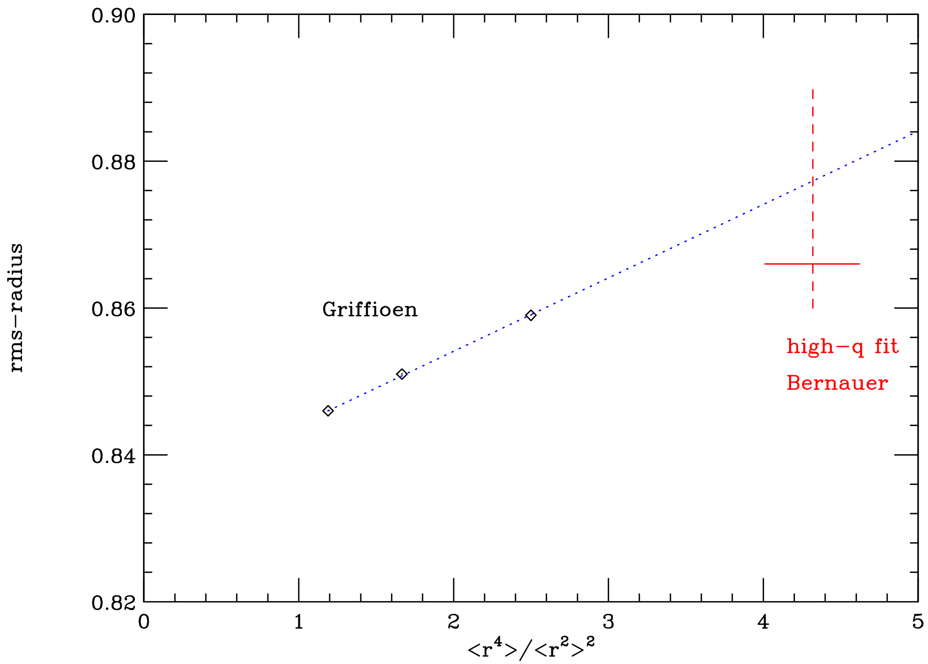
Another illustration: recent fit by Griffioen, Carlson, Maddox

use $G(q) = 1 - q^2\langle r^2\rangle/6 + q^4\langle r^4\rangle/120$
$Q^2 \leq 0.02 GeV^2$, "Bernauer" data, $R = 0.850 \pm 0.019 fm$ compatible with μX?

find contribution q^4-term 0.0018
experimental $\langle r^4\rangle$ yields 0.011 } $\Delta \sim$ 15% error in $\langle r^2\rangle$. R=0.850 simply wrong!

Another demonstration of effect of $\langle r^4 \rangle$

$\langle r^2 \rangle$ *vs* $\langle r^4 \rangle$ for different densities
$\delta(r - c)$, exponential, gaussian
Griffioen *et al.*



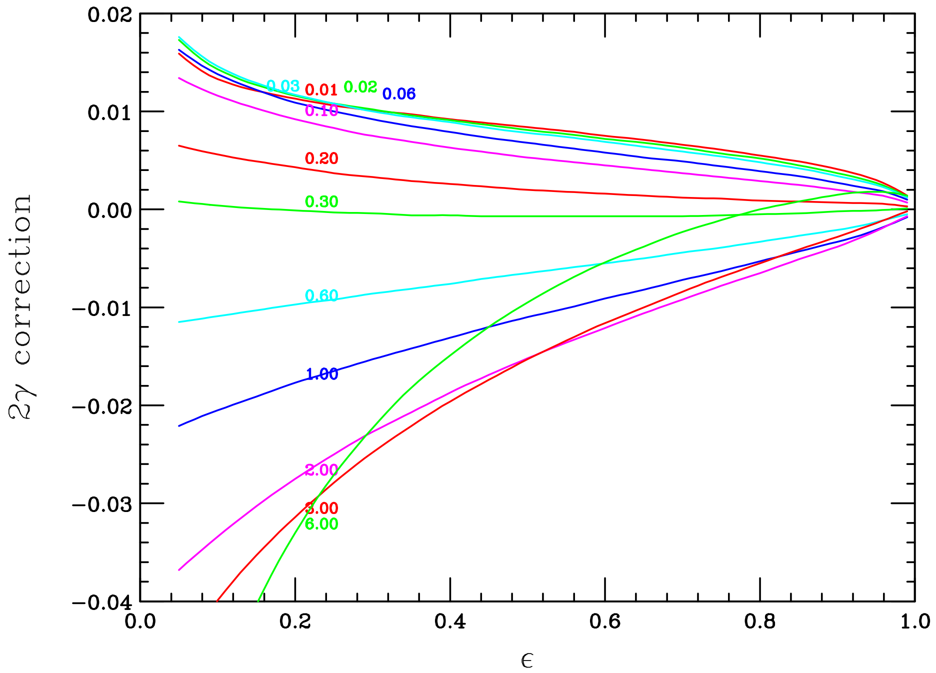
R linear function of $\langle r^4 \rangle / \langle r^2 \rangle^2$

extrapolate to true $\langle r^4 \rangle$ (Bernauer high-q fit)

get $R \sim 0.88 fm$

Two-photon effects

PWIA relation $\sigma \leftrightarrow G(q)$ complicated by 2-γ exchange TPE
at low q mainly Coulomb distortion, well under control



At q of maximum sensitivity to R $1 - G(q) \sim 0.2$, so TPE$\sim$0.01 do matter!

Inappropriate TPE:

no corrections, corrections for point nucleus (McKinley-Feshbach)
phenomenological corrections (not determined at low q)

For valid result on R must use valid TPE

Good χ^2

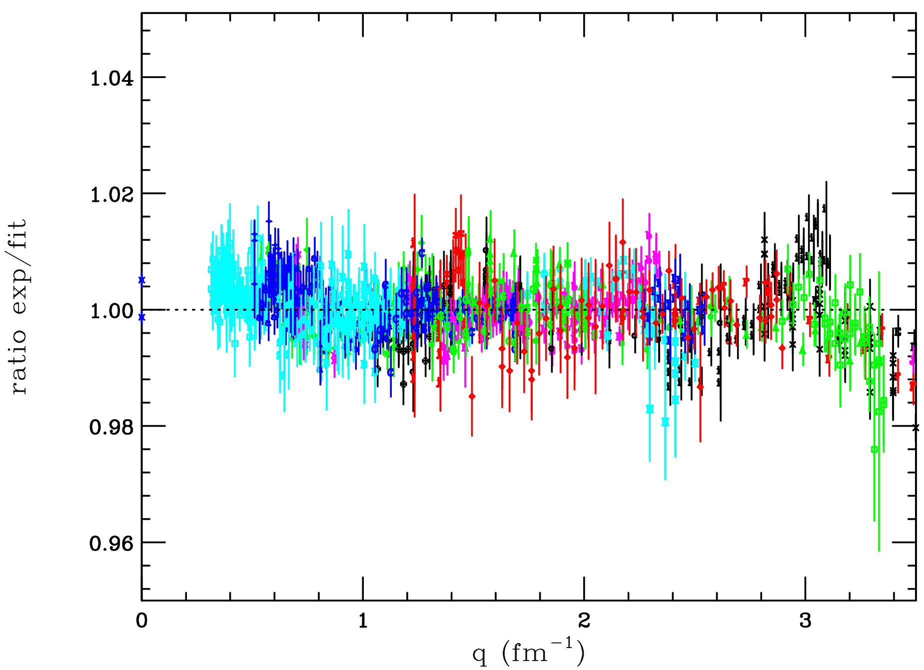
Remember: one standard deviation σ corresponds to $\Delta\chi^2 = 1$

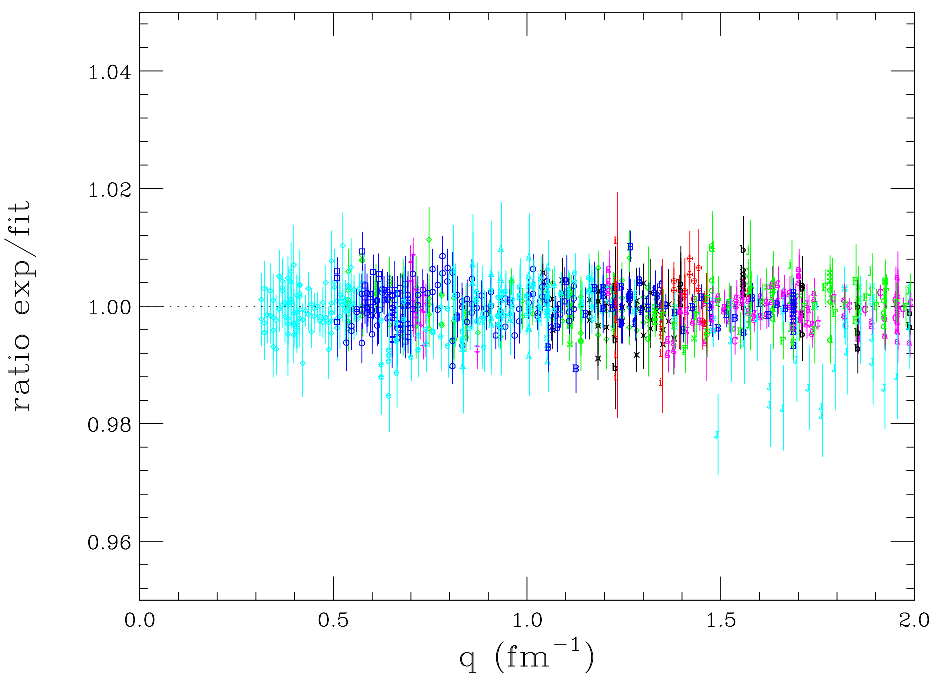
many analyses take cavalier-attitude about χ^2
accept χ^2 of (say) 1.5 *per degree of freedom*
while good fits give 1.1

With (typically) 500 data points

difference 1.1 $\leftrightarrow$ 1.5 corresponds to $\Delta\chi^2$ of 200 corresponding to 14σ!
which is not acceptable when discussing a 5σ difference (e,e)$\leftrightarrow$ μH

compare fit Lorenz fit Bernauer





Important distinction: absolute value of χ^2 is not the main issue

depends on optimism of experimentalist assigning $\delta\sigma$
depends on eventual rescaling of $\delta\sigma$

Really relevant: comparison of fits to *same* $\sigma \pm \delta\sigma$

if fit "A" gives significantly larger χ^2 than fit "B"
then fit "A" has *systematic* differences to data
then fit "A" must be discarded

Many published fits have $\chi^2 \gg \chi^2_{min}$, hence are irrelevant

Illustration

recent fit of Higinbotham *et al.*
use dipole form factor
fit Mainz80+Saskatoon+Stanford+JLab data

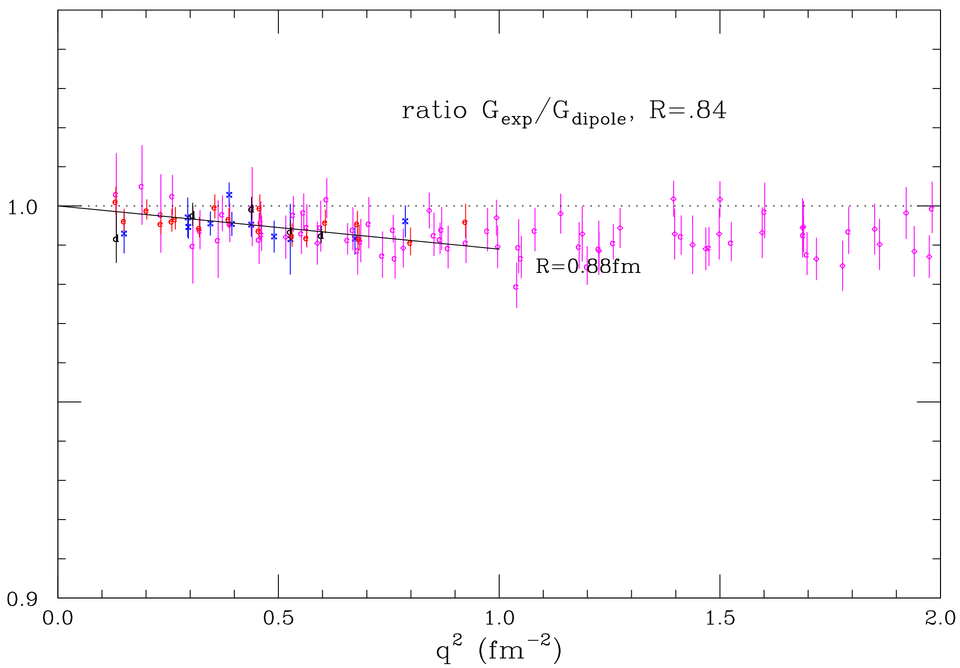
fit has reduced χ^2 of 1.25
find $R = 0.849 \pm 0.006 fm$

conclude that R is compatible with μX

But χ^2 is much too large

take one of my fits of ***world*** **data (603 data points)**
find reduced χ^2 of 0.96, *not* 1.25

Consequence: systematic deviation of Higigbotham dipole fit from cross section data

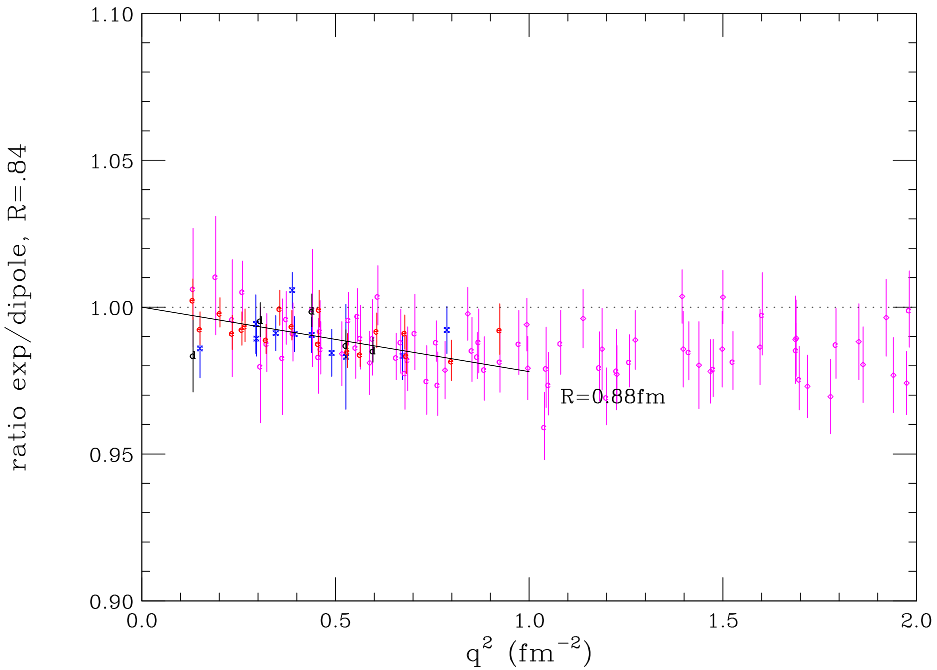


solid line shows change of low-q slope to $R = 0.88 fm$
fits data!

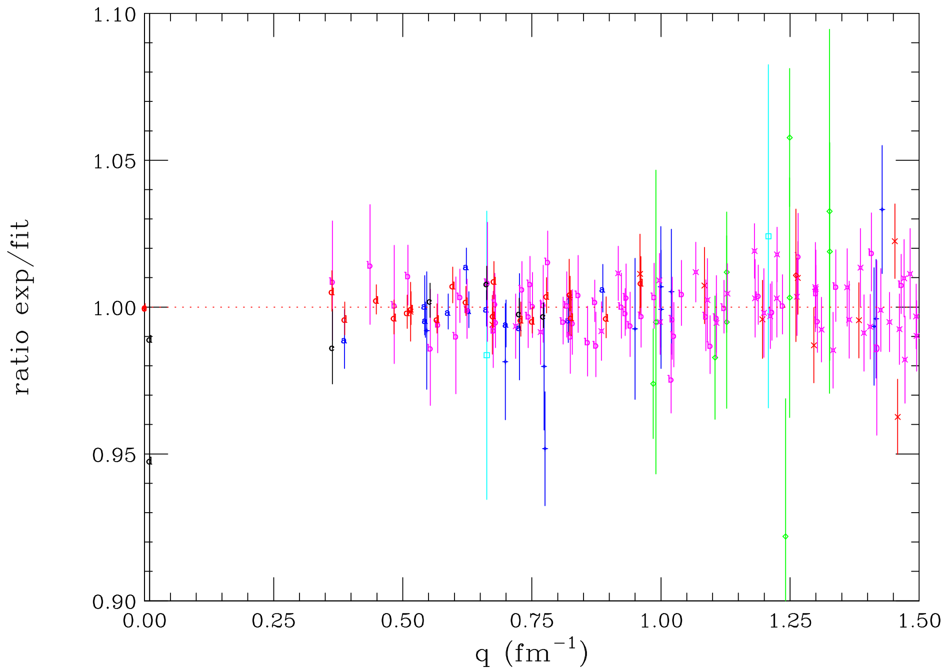
the dipole "fit" (dotted line) is simply wrong

A direct comparison of cross section ratios of "fit" and fit

Higginbotham dipole



MD



Another illustration: CF fit of Griffioen *et al.*

fit Bernauer data
$q_{max} = 5 fm^{-1}$

get reduced χ^2 of 1.61 (+pole ...)
find $R = 0.8389 \pm 0.0004 fm$

conclude that agrees with μX

But Bernauer data can be fit with reduced χ^2 of 1.14
shown years ago

For 1400 data points difference 1.61 .. 1.14 is $\Delta\chi^2 = 660$!

Who in his right mind would call that a "fit"?

Model-dependence due to choice of fit-function

some authors use 1- parameter fits at low q

power of q^2, linear in z, single dipole, ...

then $\langle r^4\rangle/\langle r^2\rangle$ is fixed by fit-function, not data

$\langle r^4\rangle \neq$ true value known from fit of data over whole q-range

then $\langle r^2\rangle$ must compensate for wrong $\langle r^4\rangle \rightarrow$ wrong value of R

also gives larger χ^2

Illustration: low-q fits of Horbatsch+Hessels

fit Bernauer data, $q_{max} \sim 1.6 fm^{-1}$

use 1-parameter dipole resp. 1-parameter linear function in z

find $R = 0.842(2)$ resp. $0.888(1) fm$

fit-function fixes $\langle r^4\rangle$ to $1.244 fm^4$ resp. $2.15 fm^4$. True value $2.58 fm^4$ (fit to all data)

explains both low and discrepant radii

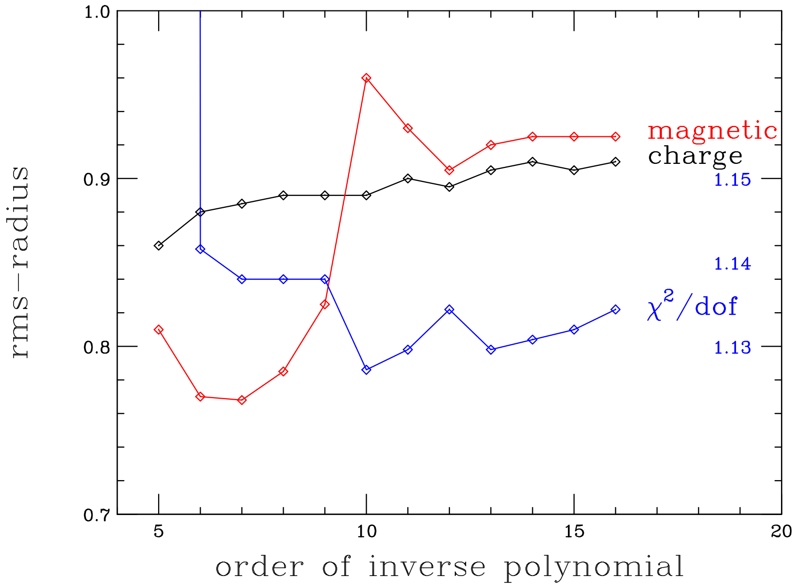
is reason why $\chi^2/dof = 1.11$ instead of 1.03

for 761 points $\Delta\chi^2 = 61!$ ($\sim 8\sigma's$)

'Fits' 8σ's from minimum are irrelevant when discussing 5σ difference (e,e)$\leftrightarrow$ μX!

Inverse Polynomial, Bernauer

$$G(q^2) = 1/(1 + a_1 q^2 + a_2 q^4 + ...)$$



Curious behavior:

between order N=7 and N=10 R^M jumps from $0.76 fm$ to $0.96 fm$

χ^2 best for N=10

would nominally be *the* best fit!

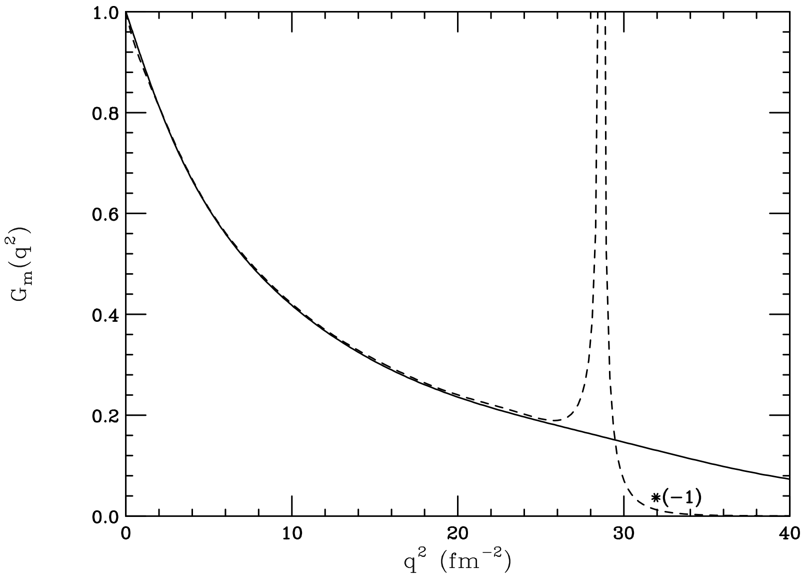
Bernauer *et al.* chose order N=7 (χ^2 ±stabilized)

Question remains:

what is responsible for jump?

how can the q^{20}-term affect the *rms*-radius?

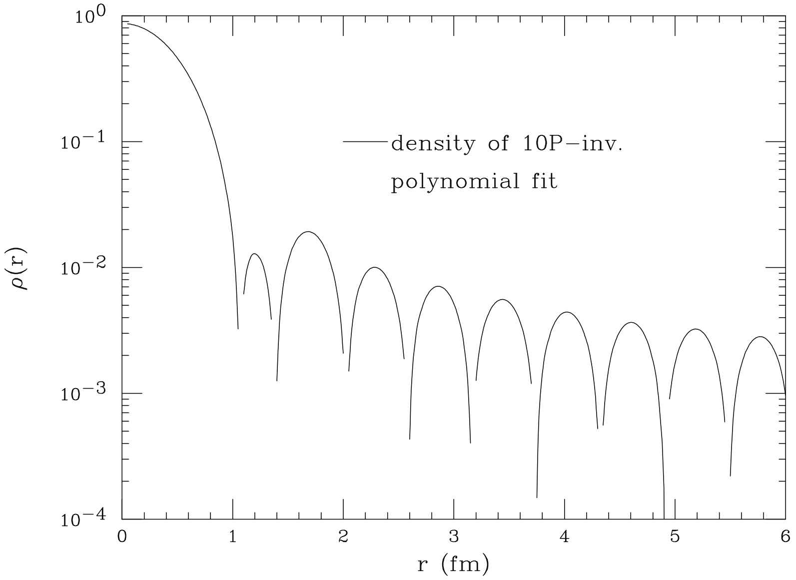
Understanding

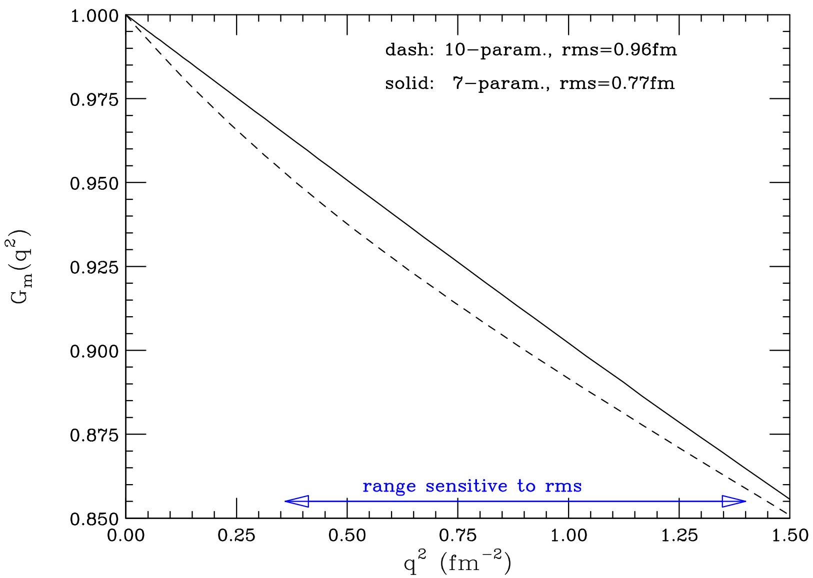


G_M for N=10 has pole at $q > q_{max}$

In $\rho(r)_m$ this leads to oscillatory tail extending to *very* large r, see next page

Density from $G(q)$ with pole





Tail affects $G_m(q)$ at *very low q* <u>below</u> q_{min} of data

N=10 fit pathological

N=7 better? Has pole too, at larger q

Cannot believe either radius

Failed fits with too large χ^2

continued fraction fits by Lorenz *et al.*

$$G(q) = \cfrac{1}{1 + \cfrac{q^2 b_1}{1 + \cfrac{q^2 b_2}{1 + \cdots}}} \quad (1$$

many fits of Bernauer data with variable q_{max}

for *e.g.* 5 terms and $q_{max} = 3.5 fm^{-1}$ find charge-rms-radius $0.84 fm$

disagrees with "accepted" result of $0.88 fm$

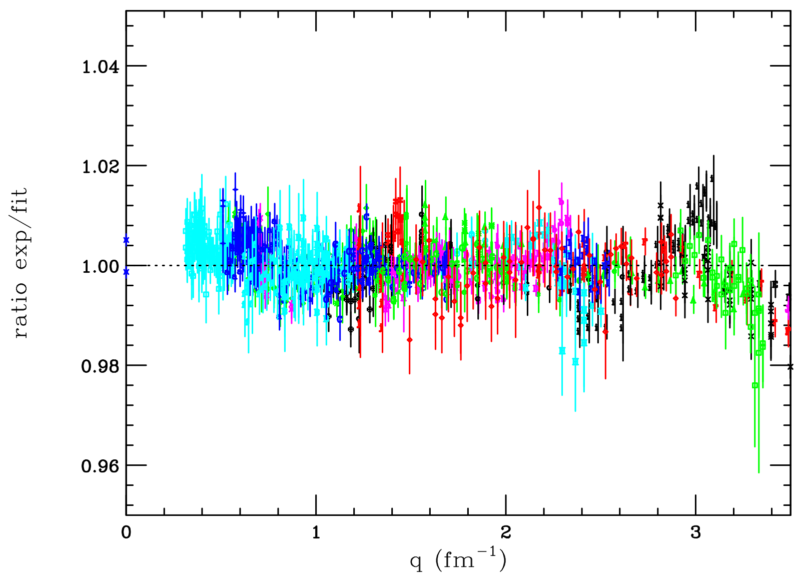
One reason

$\chi^2 \sim 1.4/dof$ much to big

$\rightarrow$ systematic deviations at low q

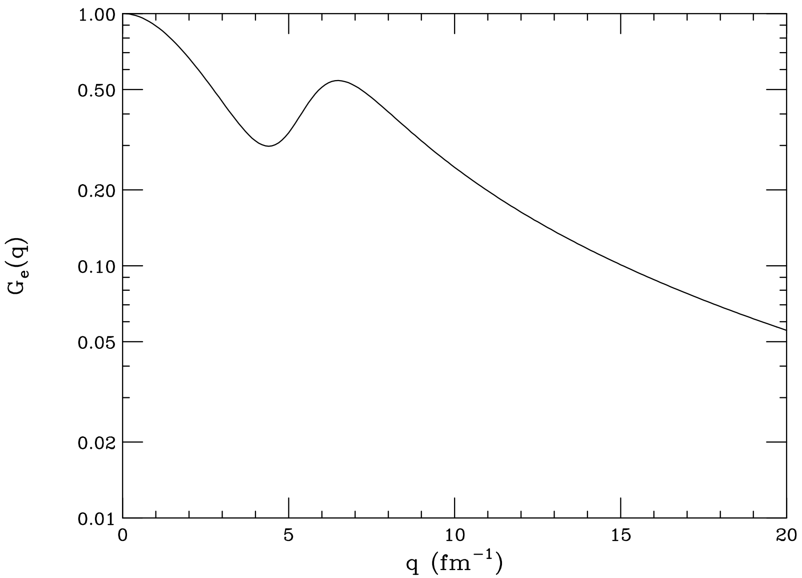
Spline fit gives $1.06/dof$

from such a fit cannot draw conclusions



Main problem of Lorenz *et al.*

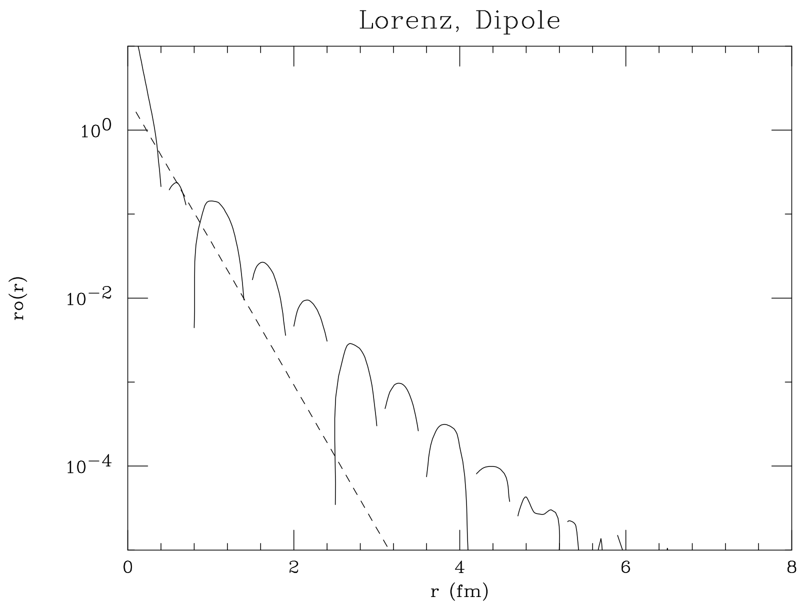
Unphysical behavior of G at $q > q_{max} = 3.5 fm^{-1}$



large $G(q)$ at large q

falls *very* slowly
$\rightarrow$ structure of $\rho(r)$ at very large r

large contribution to rms-radius
main effect on $G(q)$ at $q < q_{min}$



Physical model for large r

least-bound Fock state: p = n + π^+

dominates $\rho(r)$ completely at large-enough r ($> 0.8fm$ in cloudy bag model)

To maximally fix $\rho(r)$ from $G(q)$-data

want to use data up to largest q's measured

must account for relativistic corrections

$\rho(r)_{exp}$ from (e,e) *vs* relativistic corrections

non-relativistic: $\rho(r)$ = Fourier-transform of $G_e(q)$

Relativistic corrections:

1. Determine $\rho(r)$ in Breit-frame, accounts for Lorentz contraction

   use as momentum transfer $\kappa^2 = q^2/(1+\tau), \quad \tau = q^2/4M^2$

2. For composite systems boost operator depends on structure

   various theoretical results (Licht, Mitra, Ji, Holzwarth,...), all of form
   $G_e(q) \rightarrow G_e(q)(1+\tau)^\lambda$, λ=1 or 2

   numerical test: λ=1 or 2 makes little difference for ρ at large r
   (but fixes unphysical behavior at $r \sim 0$)

Calculation of density at very large r

a priori: asymptotic form = Whittaker function $W_{-\eta,3/2}(2\kappa r)/r$
with physical masses m_N, m_π, l=1
with separation energy = m_π, include CM-correction

makes sense only at *large* n-π relative distance: $R^p = 0.89 fm$, R^π=$0.66 fm$
only at large r overlap of n and π small

potential difficulty

need to fold W^2/r^2 with charge distribution of n, π
could get into trouble with $r = 0$ divergence of $W(2\kappa r)/r$

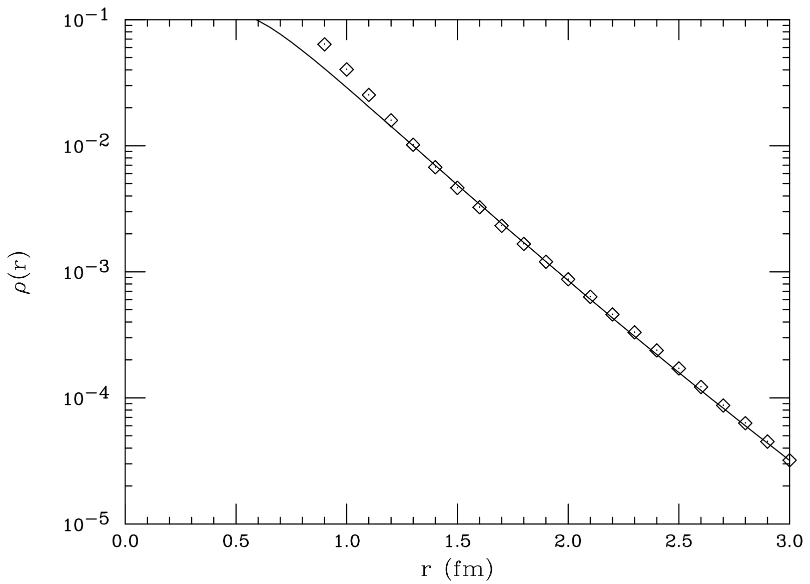
In practice

calculate w.f. in square well potential, $V(r > r_0) = 0$ (courtesy D.Trautmann)
radius $r_0 = 0.8 fm$ (not important), depth adjusted to separation energy

for $r > r_0$ shape of $\psi^2 \equiv$ shape of Whittaker function
can easily fold

Result

excellent agreement with shape of $\boldsymbol{\rho_{exp}(r)}$ ◇
(= fit *world* data with [3][5] Pade)
(normalization fitted to $\boldsymbol{\rho_{exp}}$)



"Refinements" of model (not needed, nice consistency check)

allow also for $\Delta + \pi$ contribution

coefficients of various terms from Dziembowski,...,Speth

'Pionic contribution to nucleon EM properties in light-front approach'

include all states: $\pi^+ n$, $\pi^- p$, $\pi^- \Delta^{++}$, $\pi^+ \Delta^0$, $\pi^- \Delta^+$, $\pi^+ \Delta^-$

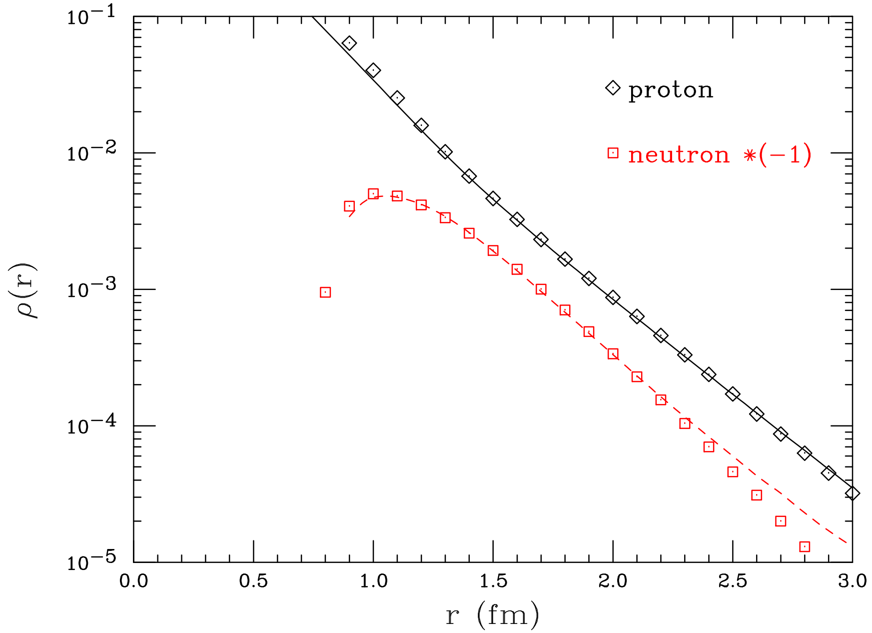
effect on p-tail: small, tail even a bit closer to ρ_{exp} at small r

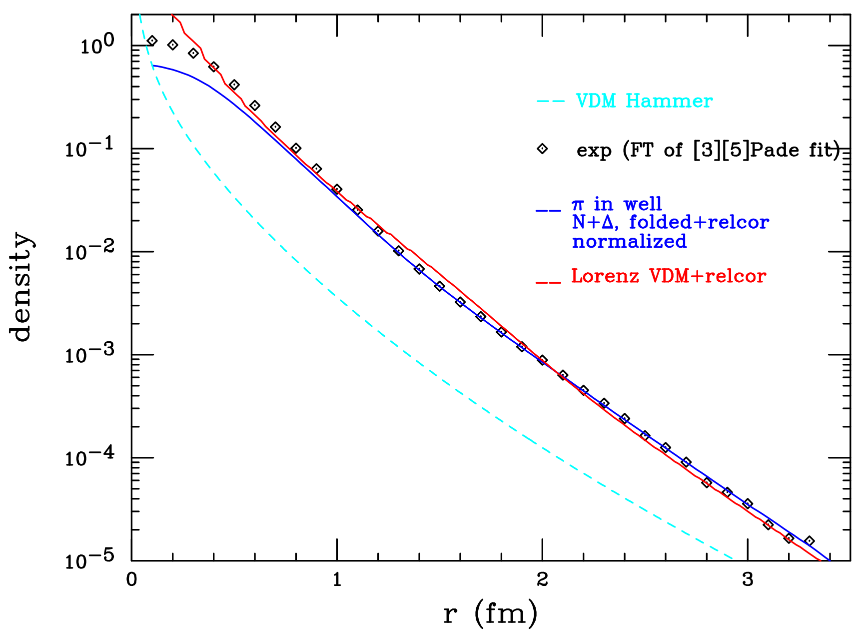
effect on n-tail: larger, gets close to ρ_{exp} with exactly *same* parameters

nice consistency check

will ignore n since components $\neq \pi^- p$ too important

◇ $\rho_p(r)$, ◇ $-\rho_n(r)$, ——— shape tail, compares nicely to (new) VDM





Details of SOG fit

Data used

- *world* (e,e) data up to 12 fm^{-1}
  - both cross sections and polarization data, 605 data points
- for some fits add Bernauer σ with 0.4% quadr. added to $\delta\sigma$
  - accounts for problems with background subtraction, target offsets
- two-photon exchange corrections
  - needed to make G_{ep} from σ and P agree
  - includes both soft+hard photons, Melnitchouk+Tjon
- (relative) tail density for $r > 1.3 fm$

Parameterization for G_e and G_m

- use r-space SOG parameterization to implement constraint
- equivalent results with Laguerre

Results

- average over various combinations of data sets
- floating or not of normalization

$R^{ch} = .886 \pm 0.008\ fm$ $\qquad$ $R^{m} = .858 \pm .024\ fm$

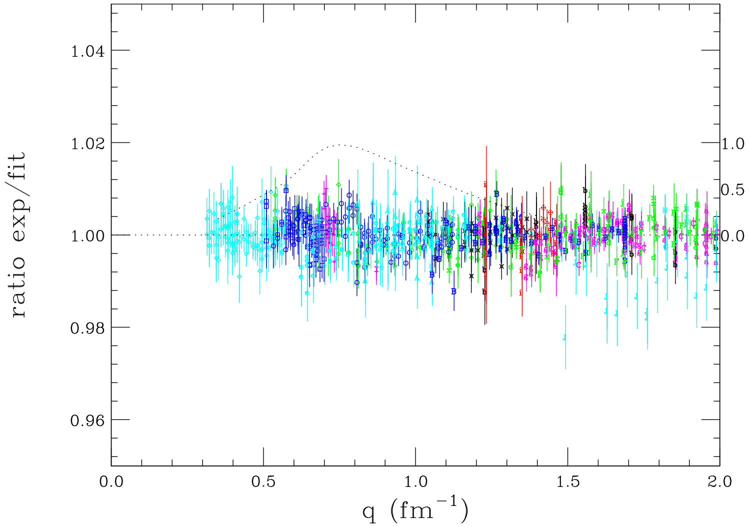
Conclusion: disagreement with μ-H confirmed

Question: to which degree could fit (e,e) with $R_{\mu X}$?

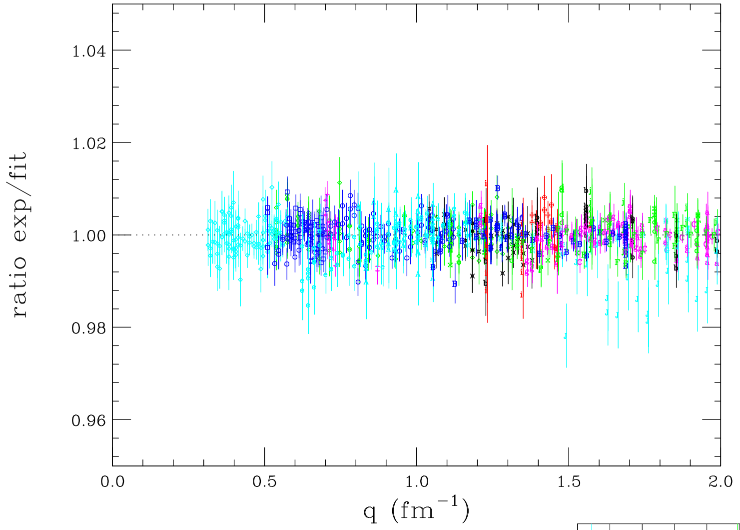
redo analysis with various combinations of data, normalization

1. Bernauer data alone: $\Delta\chi^2 \sim 5\%$, not visible in $\sigma_{exp}/\sigma_{fit}$

$R{=}0.84fm$



$R{=}0.88fm$



2. *world*(floated) + Bernauer: $\Delta\chi^2 \sim 8\%$

3. *world*(floated) + tail: $\Delta\chi^2 \sim 10\%$

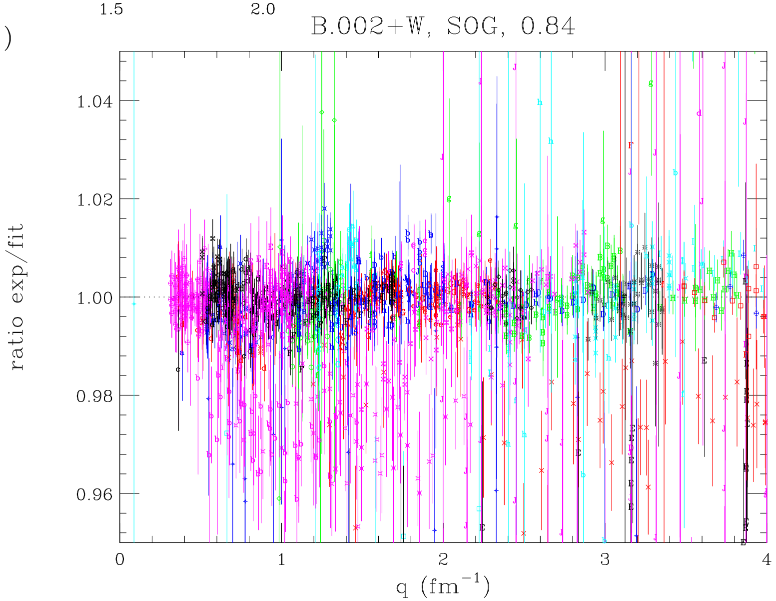
4. *world* + tail: $\Delta\chi^2 \sim 24\%$

5. *world* + Bernauer + tail: $\Delta\chi^2 \sim 24\%$ $\longrightarrow$

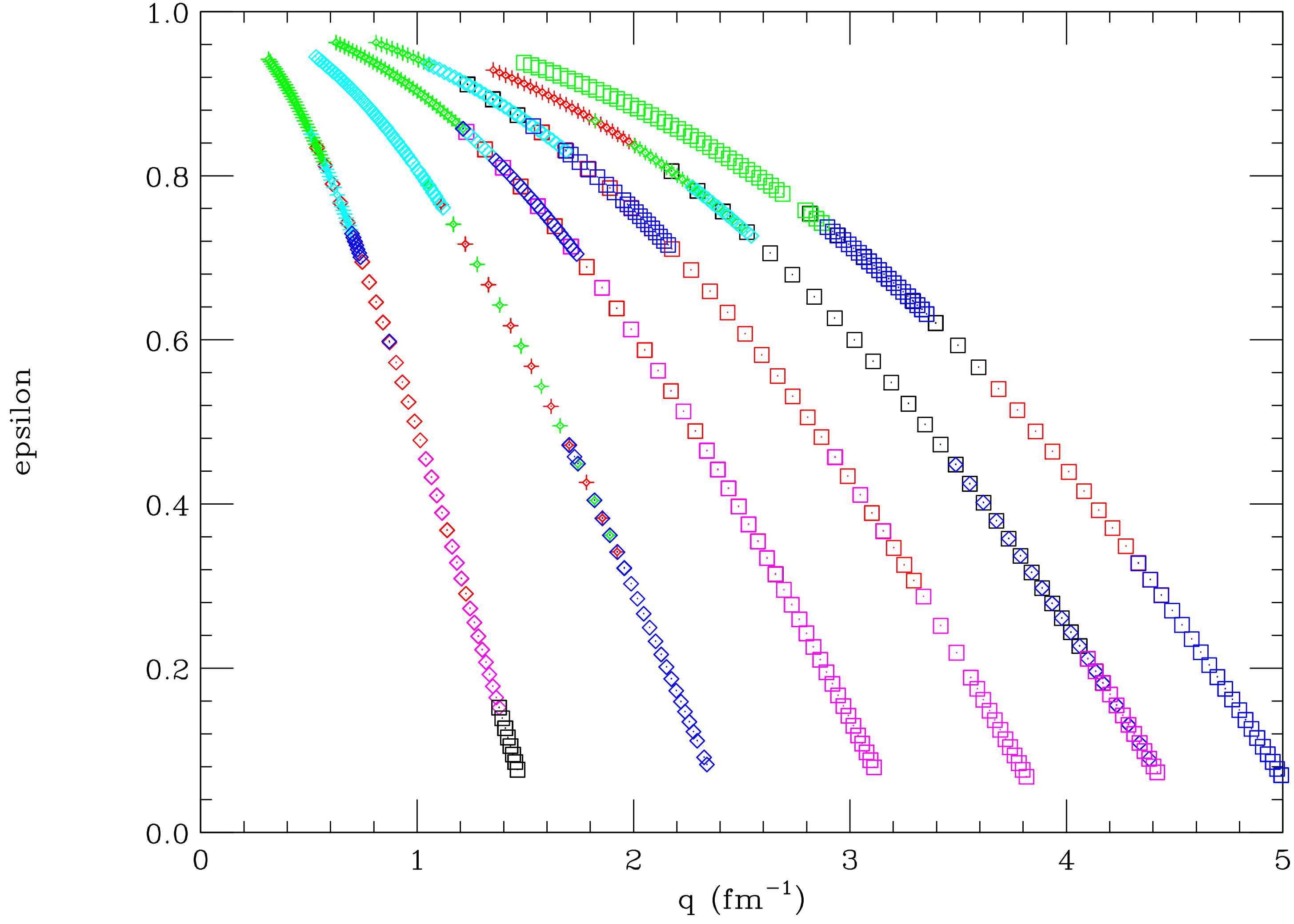
visible disagreement data/fit

***world* 2-3% below fit**

$\rightarrow$ *absolute* data + tail information most helpful



Kinematics of Bernauer data



Bernauer data, background subtraction

$R^{ch} = 0.879 \pm 0.007\ fm, \quad R^m = 0.777 \pm 0.02\ fm$
at first sight nice confirmation of previous R^{ch}

Problematic: disagreement with *world* value $R^m = 0.855 \pm 0.035 fm$

Note: R^m-discrepancy only 0.3% of σ at q of maximal sensitivity to rms-radius

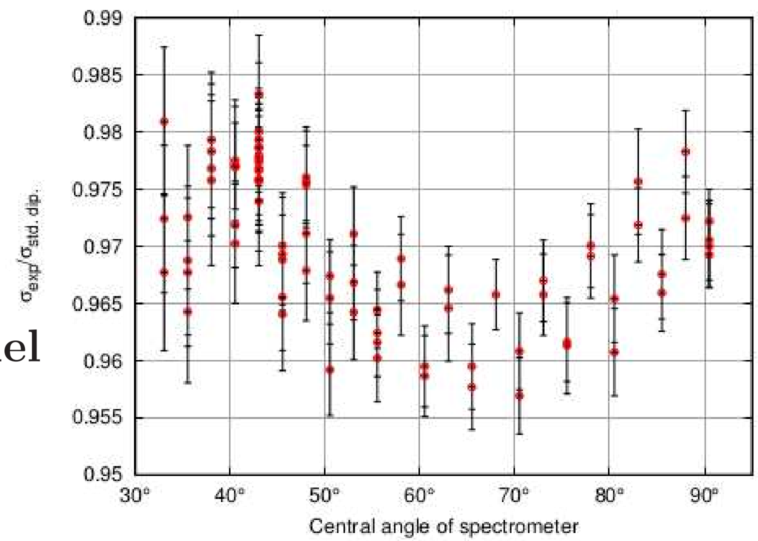
At this level target-window subtraction no good

background 4 ... 10% , not measured, but simulated!

Window subtraction via model wrong
data affected by detector inefficiency
window model *not* affected

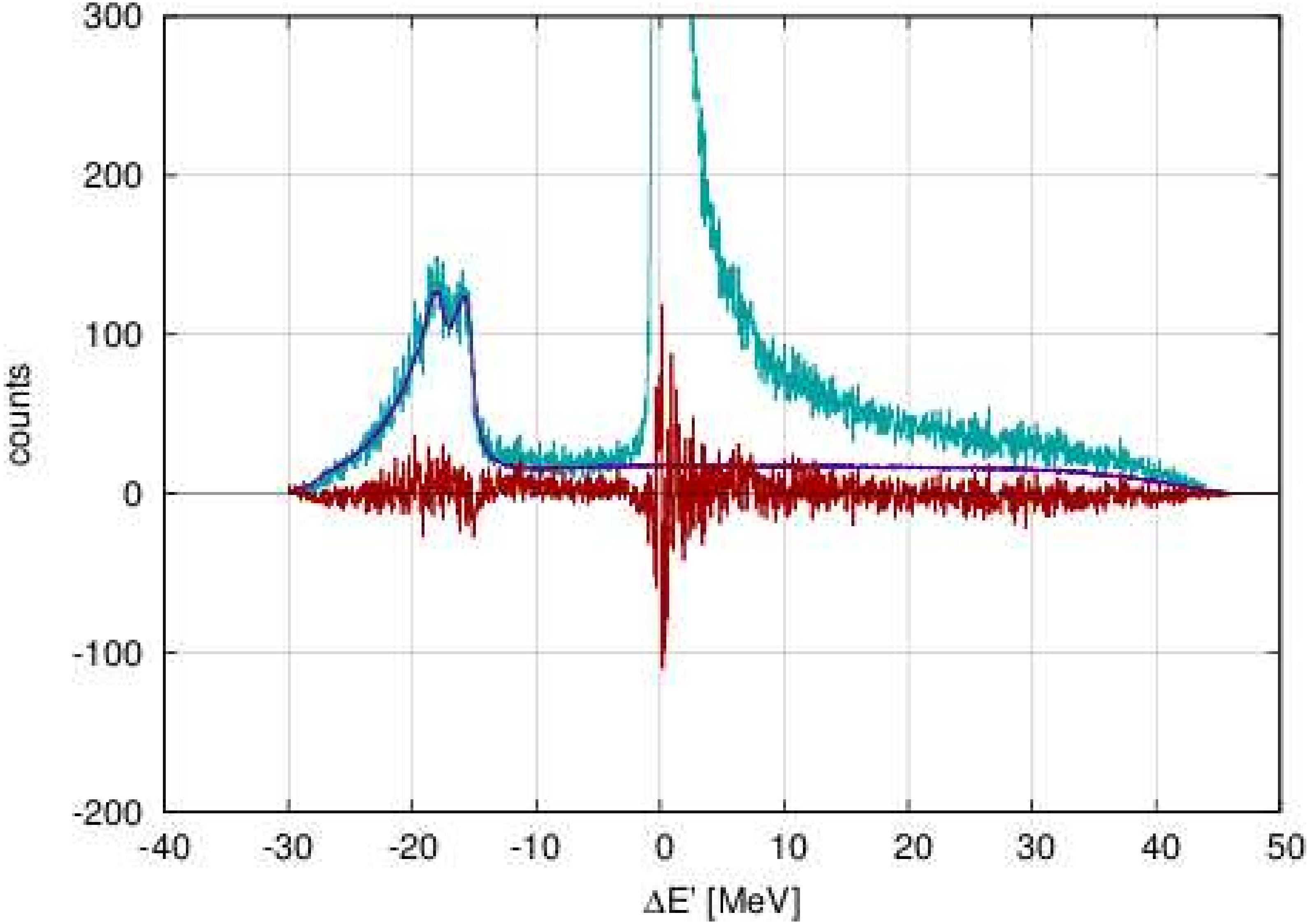
Window model much too primitive
radiative tail+q.e.-scattering in Fermi-Gas model
no inelastic scattering, F-G model no good

Fit of data poor
see next page



Spectrum shown in thesis

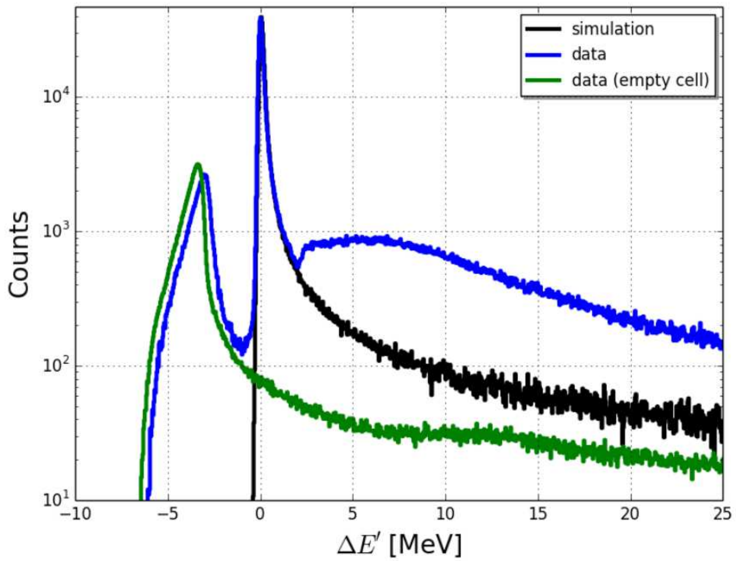
shows misfit amounting to 1.2% in cross section! Large compared to 0.3%!



Must be fixed before can believe results

Measured window contribution

MAMI d(e,e)-experiment



shows structure of background relevant in $1 fm^{-1}$ region

Problem of Bernauer data also shown by fits

Inverse polynomial fit of Bernauer

χ^2 as low as other best-fits

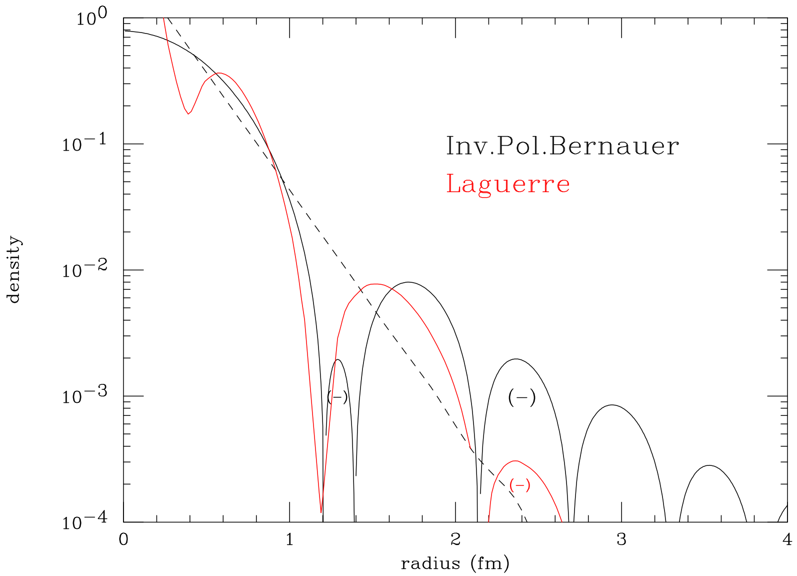
yields strongly oscillating density, same with Laguerre

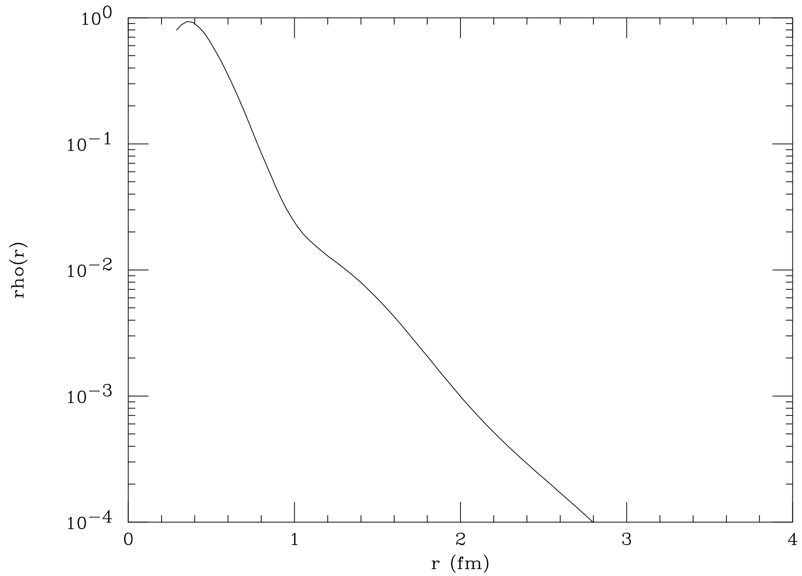
Laguerre fits with tail constraint

trying to reduce unphysical oscillations

χ^2 higher by factor $\sim$1.5

but density still shows remainders of oscillations

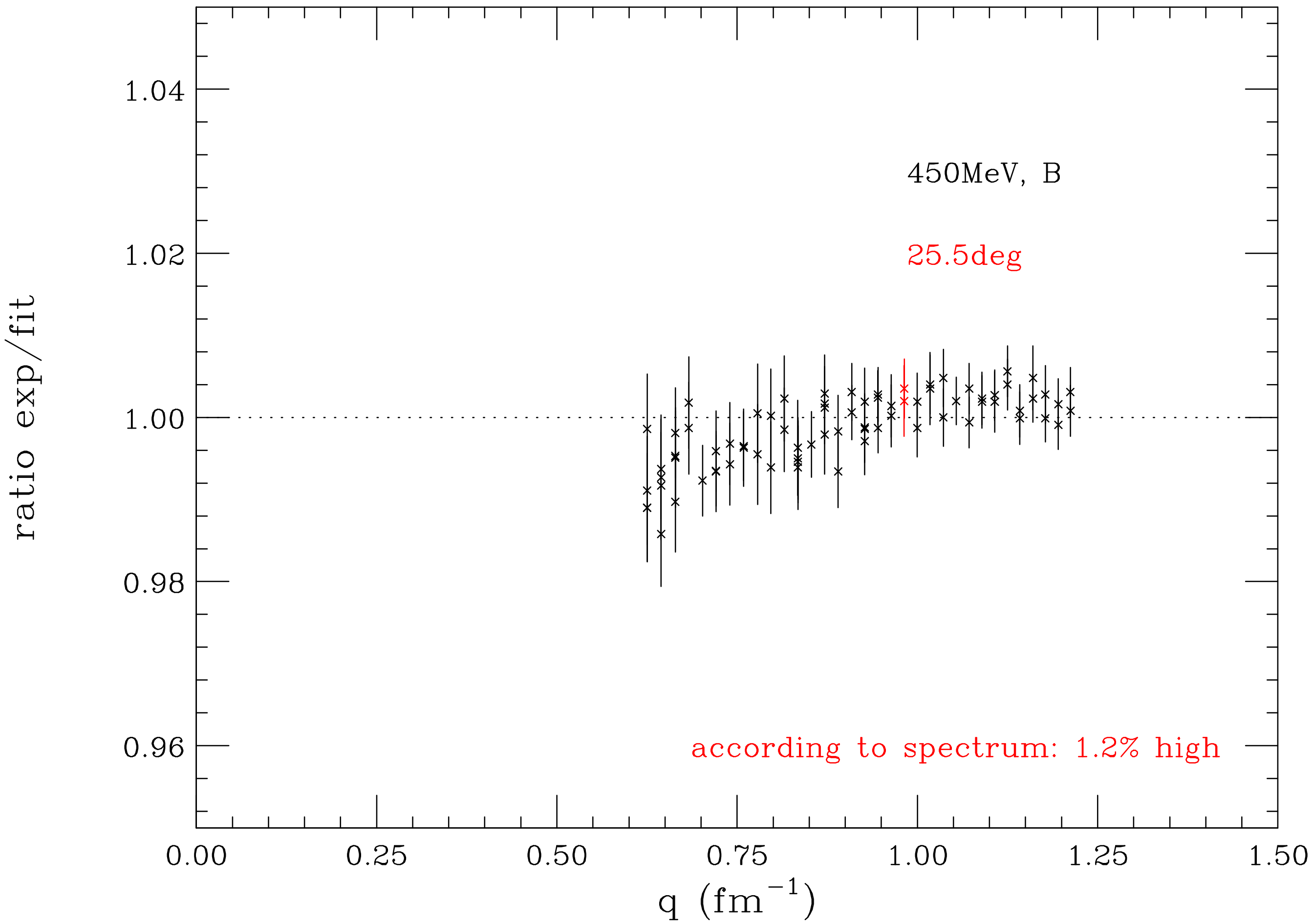




differences Bernauer-world of few % lead to unreasonable ρ

also true for other parameterizations MD, SOG, ..

1.2% problem is systematic

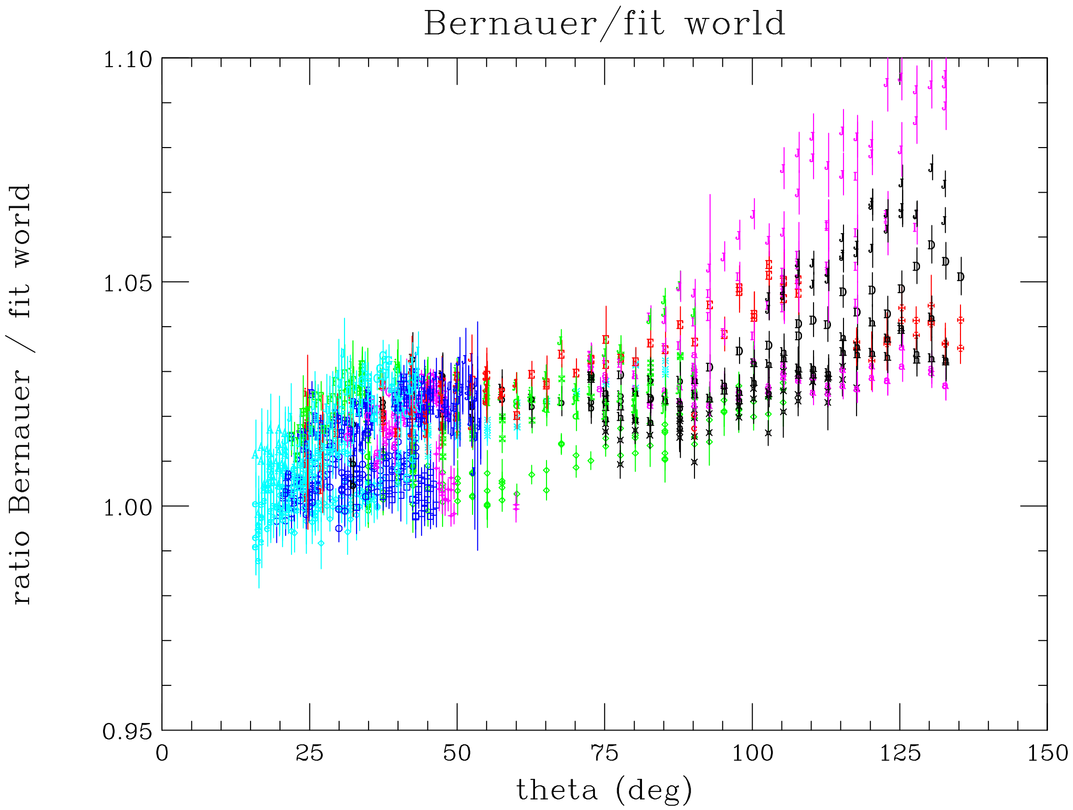


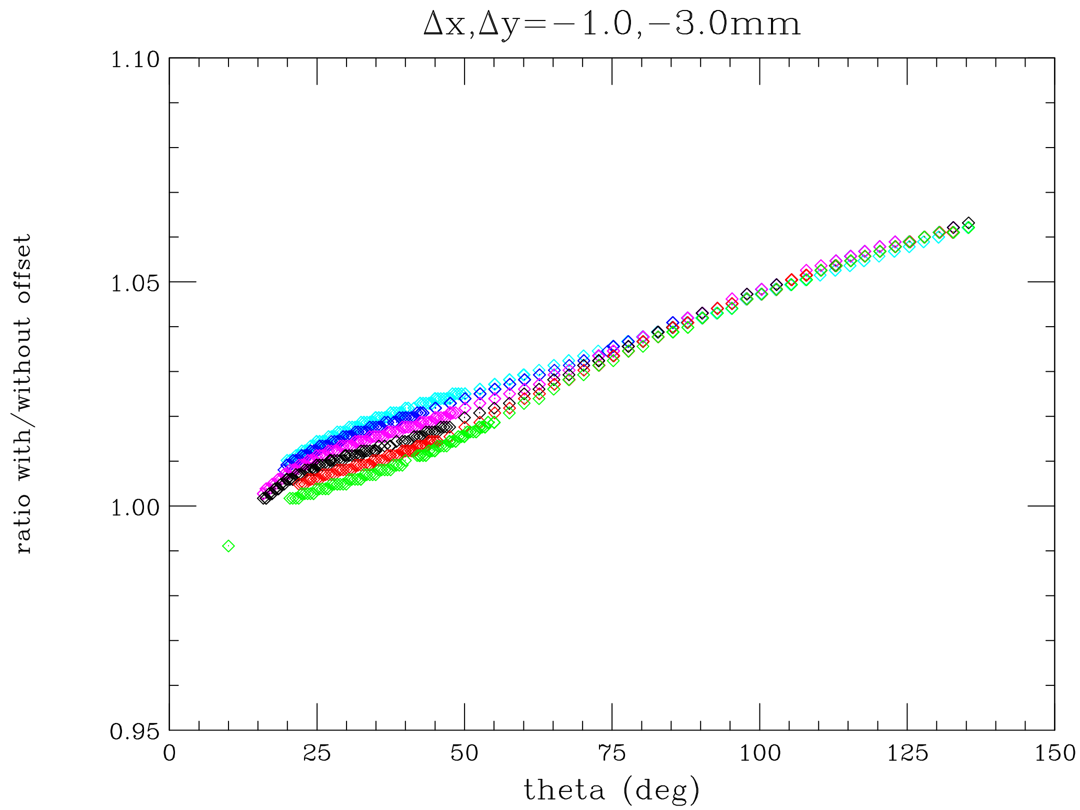
→ concerns entire *region* of q

≡ region of maximal sensitivity to *rms*-radius

Disagreement Bernauer ↔ world data: very poor χ^2 of common fit

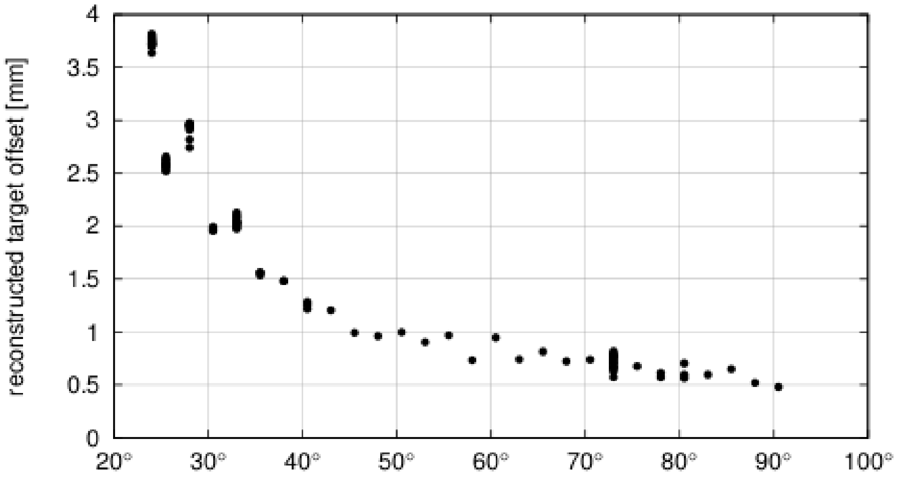
disagreement studied as function of different variables
systematic* difference seen in ratio as function of *angle
unlikely to be problem of world data (∼20 independent sets)





Most likely reason: x/y-offset of target from center of rotation

Off-set indeed seen by Bernauer et al.



according to thesis: corrected for in data analysis

BUT: there is a possible *different* reason

magnetic asymmetry of spectrometer relative to mid-plane
or misaligned detector system
would also lead to incorrect reconstruction of target position

$\rightarrow$ different correction to σ: basically none

One A1 spectrometer now known to have asymmetry
(partial short in coil)

not enough information available to make true correction