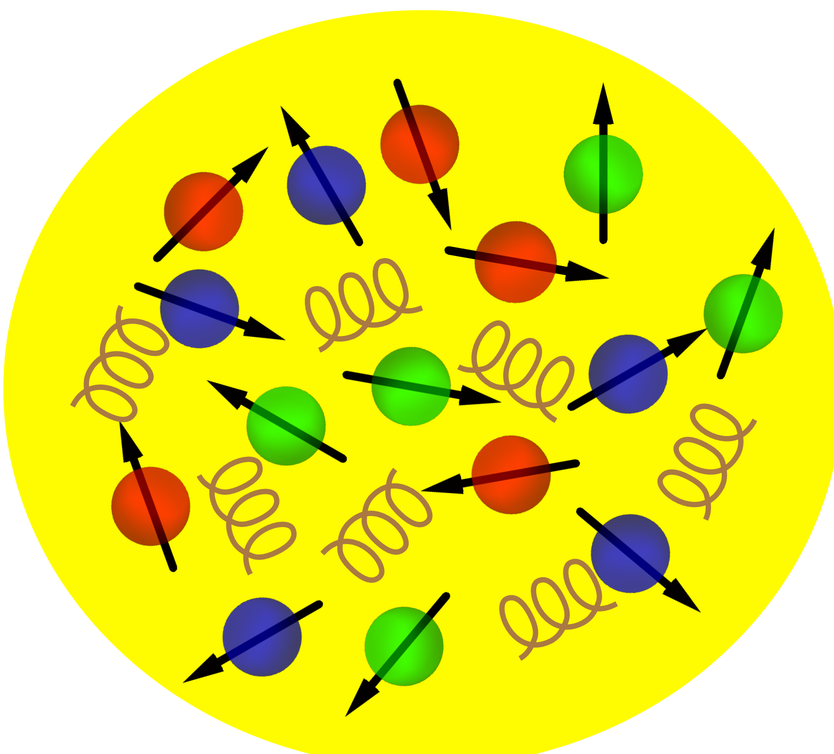
# Entropy-current analysis of spin hydrodynamics



$\overset{?}{\simeq}$ Hydro + Spin

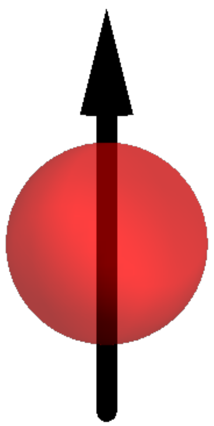
Masaru Hongo (Univ. of Illinois at Chicago)

2020/10/12, Spin and hydrodynamics, ECT* Online workshop

[Ref. Hattori-MH-Huang-Matsuo-Taya: PLB795,100 (2019)]

# Spin in Hydro?

- <u>Spin as a quantum number</u>

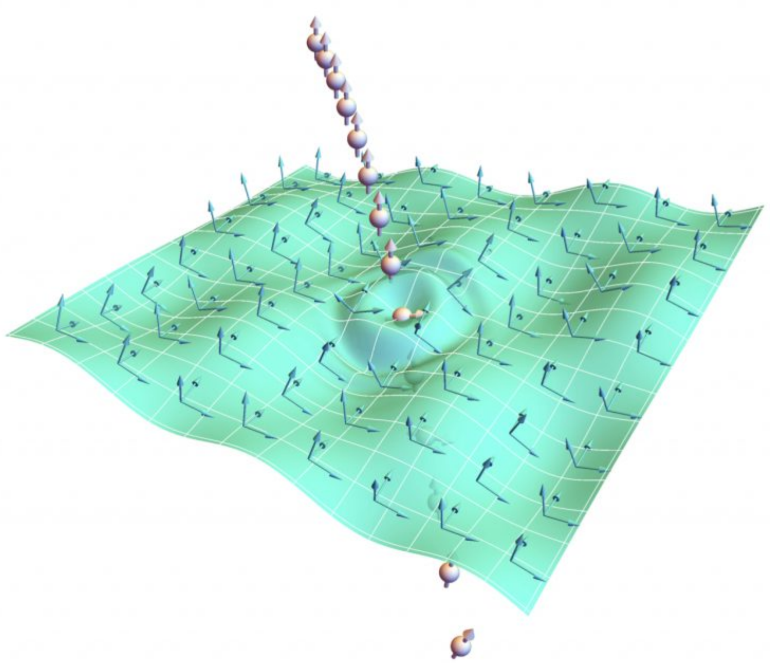


Spin ≠ good quantum # in ~~non~~relativistic theory

→ Transport phenomena of spin ?

- <u>Where and Why ?</u>

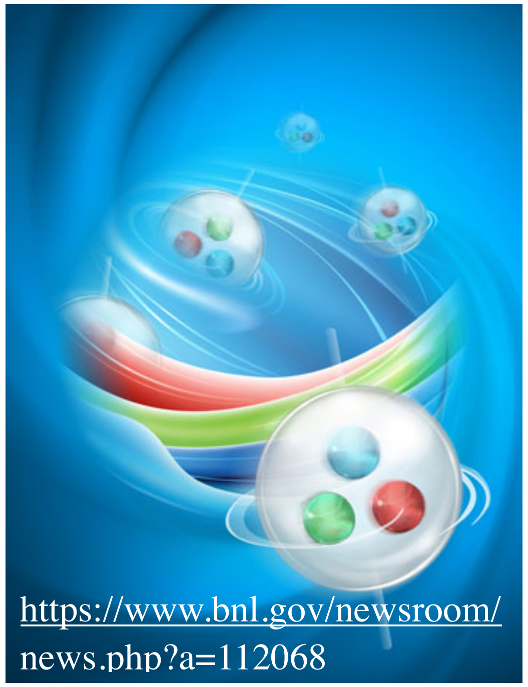
Spintronics



http://mmatsuo.com/

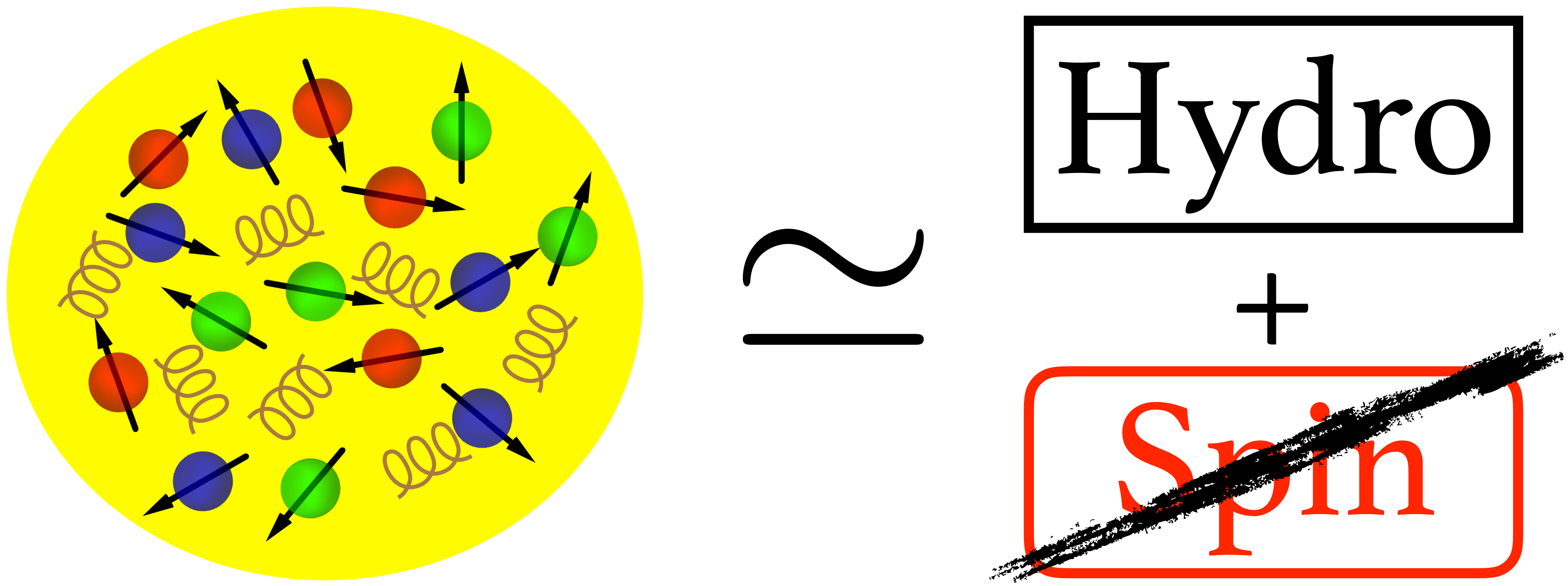
**Possibility of QGP spintronics!?**

Heavy-ion collision



# One-page Summary

**Phenomenological derivation of spin-hydro**




**Three main messages:**

(1) **Coupled dynamics** of hydro & spin is available

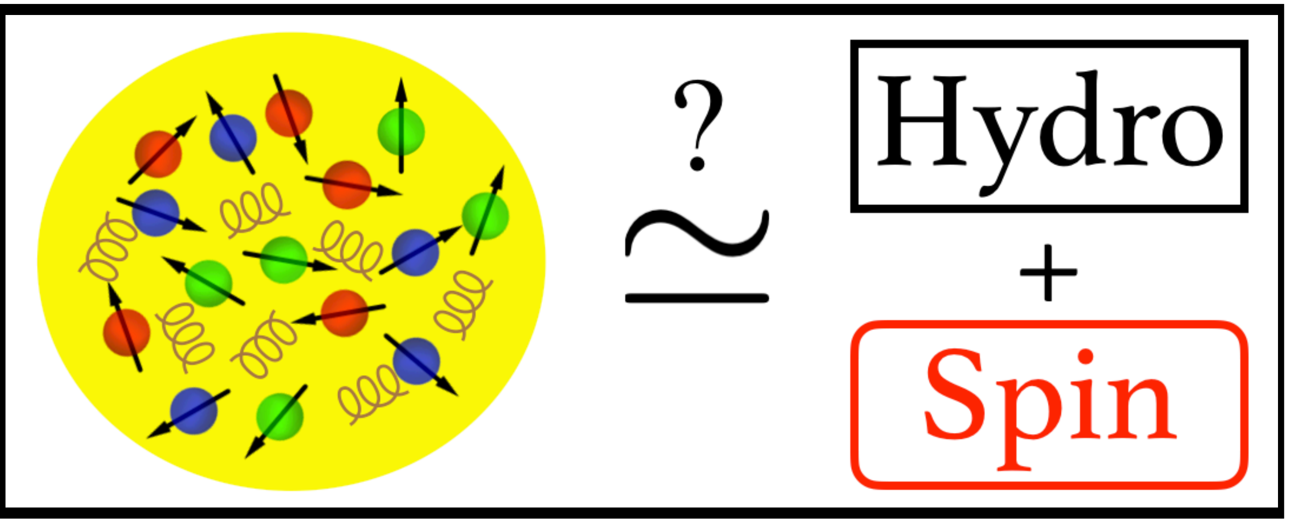
(2) Spin density shows (gapped) **relaxational dynamics**

(3) How to make spin hydro as **well-defined Hydro+**

# Outline

## Motivation:

Hydrodynamics of
a relativistic spinful fluid?




## Approach:

Phenomenological
entropy-current analysis

## Result:

(1) Coupled dynamics of hydro & spin

(2) Diffusive nature of spin:

# Phenomenological derivation of hydrodynamic equation

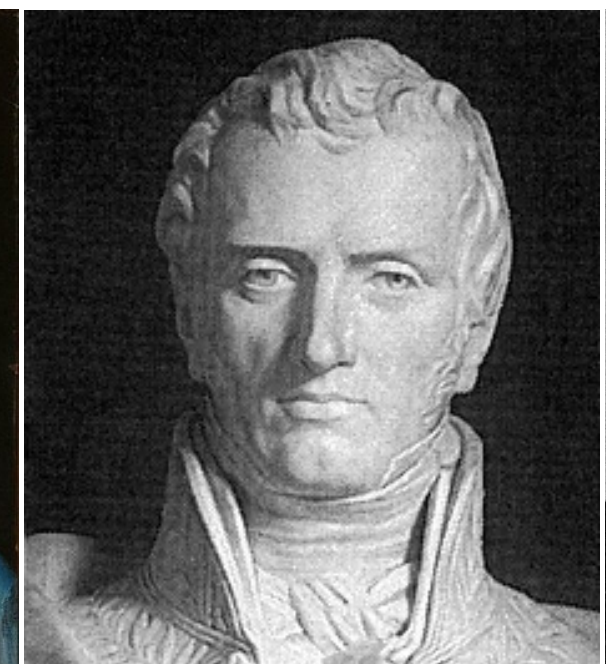
# What is **hydrodynamics?**

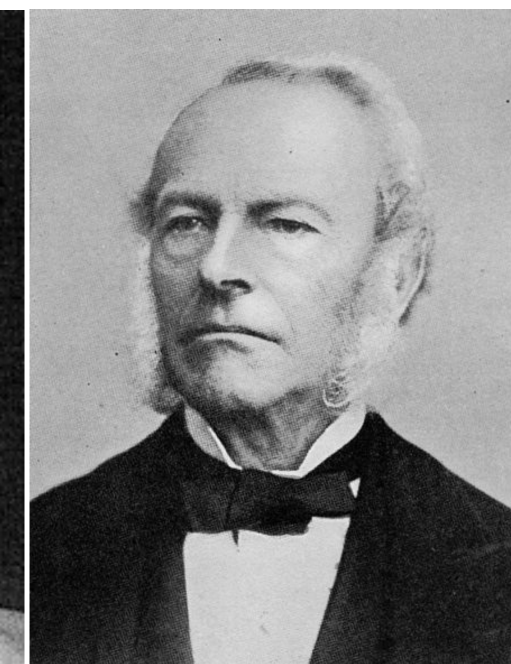
## The oldest but **state-of-the-art phenomenological** field theory


B. Pascal (1623-1662)


D. Bernoulli (1700-1782)


L. Euler (1707-1783)


C-L. Navier (1785-1836)


G. Stokes (1819-1903)

**Pascal's** law

Hydro*dynamics*

**Euler** equations
(Perfect fluid)

**Navier-Stokes** equations
(Viscous fluid)

1600 → 1700 → 1800 → 1900

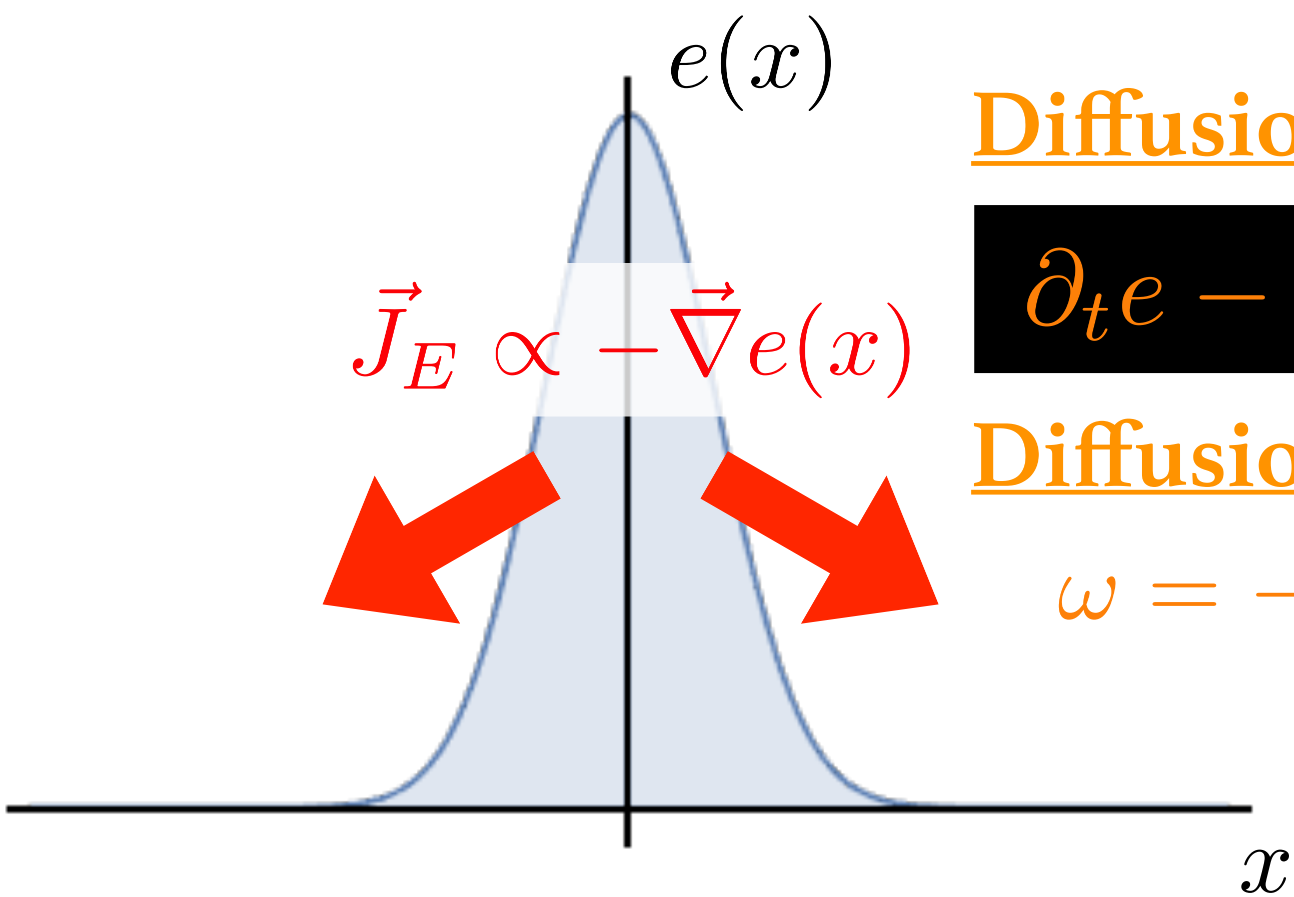
# Prototype: Energy diffusion

**◆Bulding blocks of hydrodynamic equation**

(1) **Conservation law:** $\partial_t e + \vec{\nabla} \cdot \vec{J}_E = 0$

(2) **Constitutive relation:** $\vec{J}_E = -D_E \vec{\nabla} e$

(3) **Physical properties:** Value of diffusion constant $D_E$

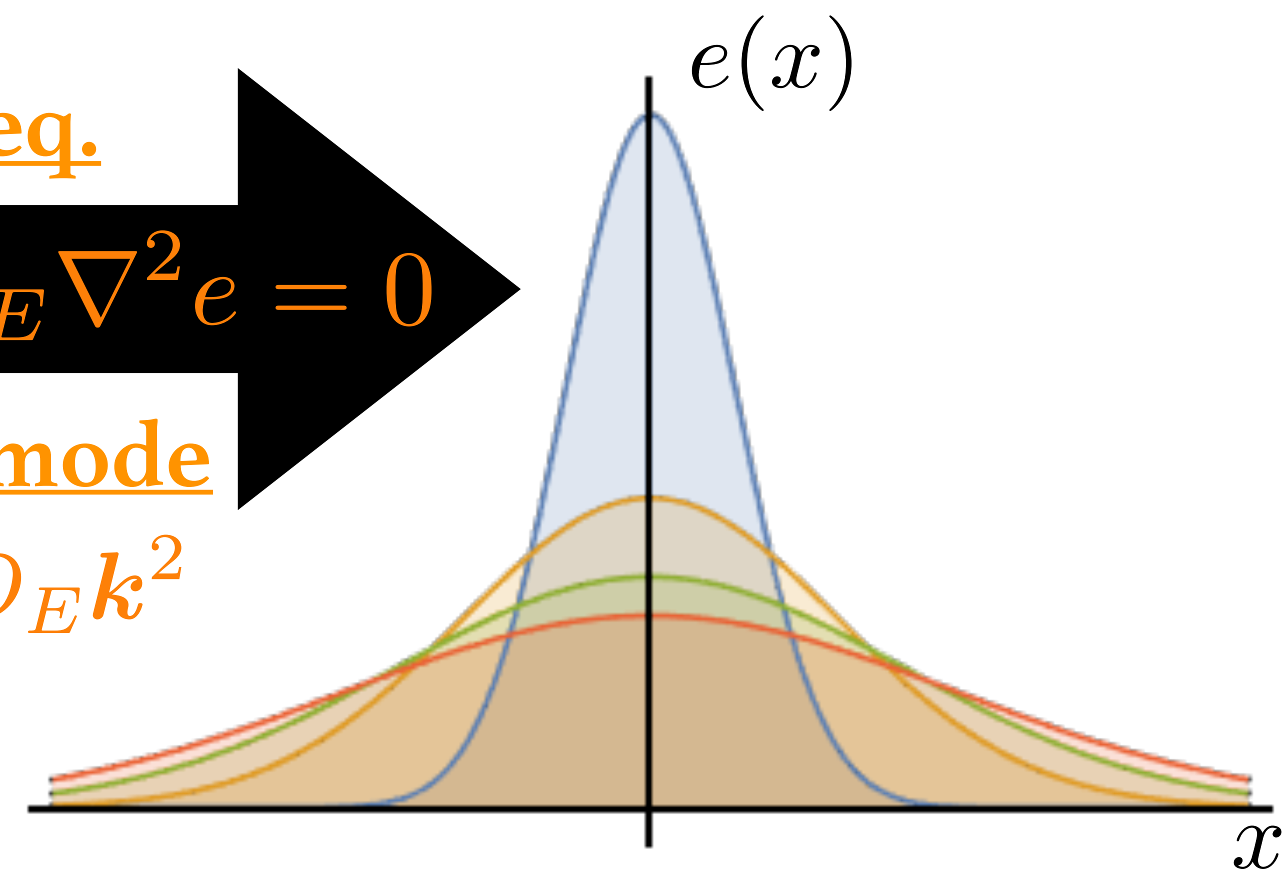



**Diffusion eq.**

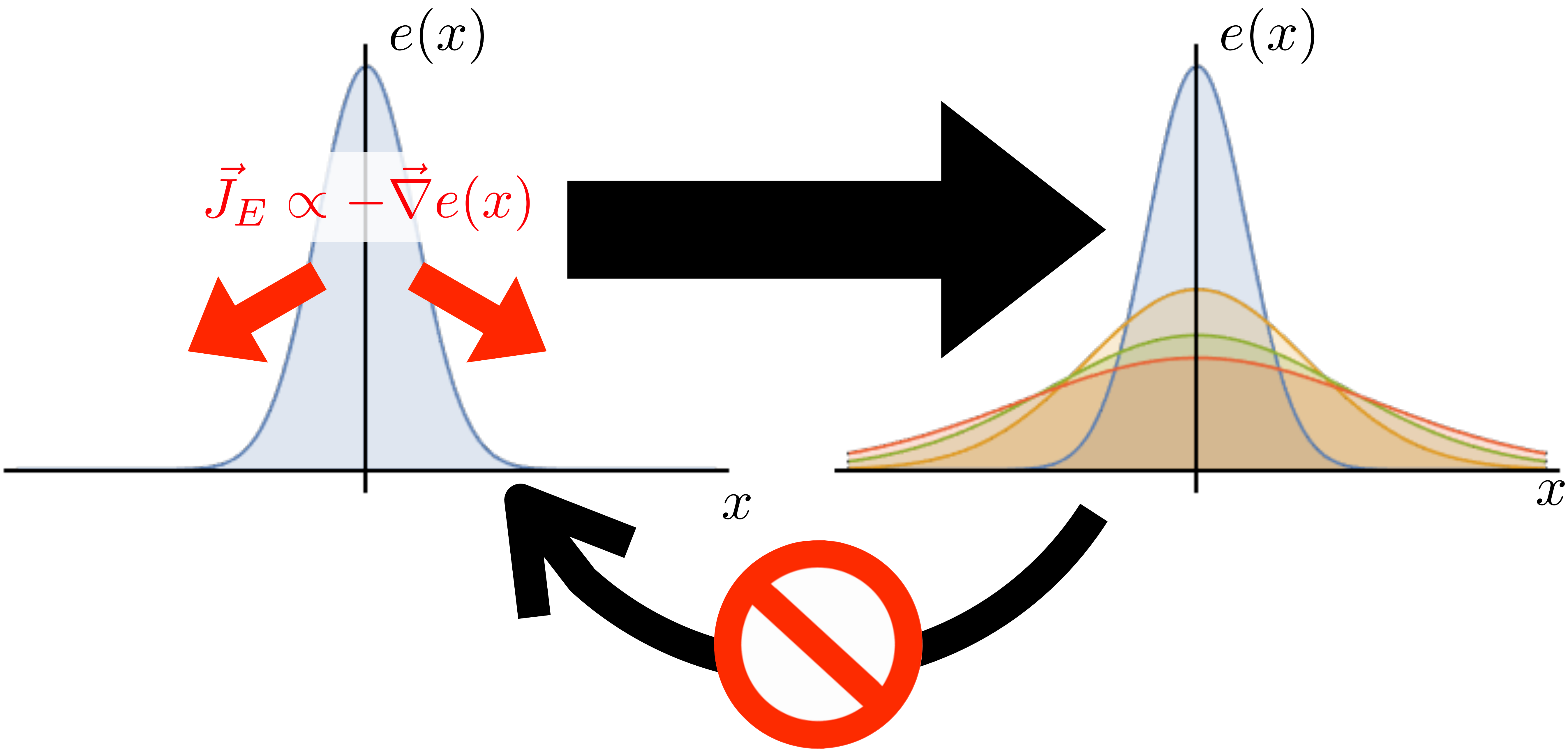
$$\partial_t e - D_E \nabla^2 e = 0$$

**Diffusion mode**

$$\omega = -\mathrm{i} D_E \boldsymbol{k}^2$$



# Irreversiblity of diffusion




**No-go for time-reversal process!**

**Thermodynamic concepts, especially, 2nd law, should appear!!**

# Closer look at derivation

**Global thermodynamics**

**1st law:** $dE = TdS + pdV$

**2nd law:** $dS \geq 0$

**Local thermodynamics**

**1st law:** $\beta de = ds, \ d = \partial_t$

**2nd law:** $^{\exists} s^\mu$ s.t. $\partial_\mu s^\mu \geq 0$

**What is $\vec{J}_E$ in conservation law: $\partial_t e + \vec{\nabla} \cdot \vec{J}_E = 0$ ?**

$$0 = \beta \partial_t e + \beta \vec{\nabla} \cdot \vec{J}_E = \partial_t s + \vec{\nabla} \cdot (\beta \vec{J}_E) - \vec{J}_E \cdot \vec{\nabla} \beta$$

$$\Leftrightarrow \partial_t s + \vec{\nabla} \cdot (\beta \vec{J}_E) = \vec{J}_E \cdot \vec{\nabla} \beta$$

$$(\beta \vec{J}_E) = \vec{s} \qquad \vec{J}_E = \kappa_E \vec{\nabla} \beta \ (\kappa_E \geq 0)$$

$$\Leftrightarrow \text{For} \begin{cases} s^\mu \equiv (s, \beta \vec{J}_E), & \partial_\mu s^\mu = \kappa_E (\vec{\nabla} \beta)^2 \geq 0 \\ \vec{J}_E = \kappa_E \vec{\nabla} \beta, & \end{cases}$$

**2nd law!**

$\vec{\nabla} \beta = -\chi_E^{-1} \vec{\nabla} e$ ⇒ $\vec{J}_E = -D_E \vec{\nabla} e, \ D_E = \dfrac{\kappa_E}{\chi_E}$

**Constitutive relation!**

# Flowchart

**Step 1. Determine dynamical d.o.m (& its equation of motion)**

Energy density: $e$ EoM: $\partial_t e + \vec{\nabla} \cdot \vec{J}_E = 0$

**Step 2. Introduce entropy & conjugate variable**

Entropy density: $s(e)$ ⇒ $ds = \beta de$ ⇒ Temperature: $\beta \equiv \dfrac{\partial s}{\partial e}$

**Step 3. Write down all possible terms with finite derivatives**

Current: $\vec{J}_E = 0 + \kappa_E \vec{\nabla}\beta + O(\vec{\nabla}^2) = \kappa_E \vec{\nabla}\dfrac{\partial s}{\partial e} + O(\vec{\nabla}^2)$

**Step 4. Restrict terms to be compatible with local 2nd law**

$\exists\, s^\mu$ such that $\partial_t s + \vec{\nabla} \cdot \vec{s} \geq 0$ ⇒ $\kappa_E \geq 0$ with $\vec{s} = \beta \vec{J}_E$

# Application: Hydrodynamics

◆**Bulding blocks of hydrodynamic equation**

**(1) Energy-momentum conservation laws:** $\partial_\mu \Theta^{\mu\nu} = 0$

**(2) Constitutive relations:** $\Theta^{\mu\nu} = \Theta^{\mu\nu}(\Theta^{0\nu})$

**(3) Physical properties:** EoS, Values of transport coeff.

**Complicated but the same analysis perfectly works as follows:**

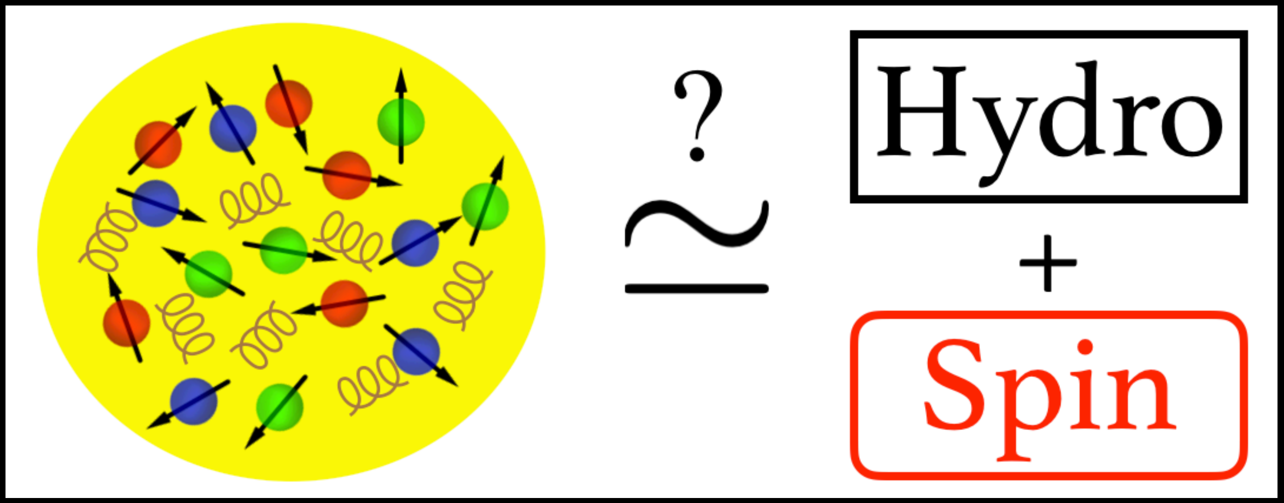
- **Step 1:** Dynamical d.o.f.: ${\Theta^0}_\mu$ with EoM: $\partial_\mu {\Theta^\mu}_\nu = 0$
- **Step 2:** Entropy: $s({\Theta^0}_\mu) \Rightarrow$ Conjugate variable: $\beta u^\mu \equiv \dfrac{\partial s}{\partial {\Theta^0}_\mu}$
- **Step 3:** EM tensor: $\Theta^{\mu\nu} = e u^\mu u^\nu + p \Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1)}$
- **Step 4:** $e + p = Ts\,,\quad \Theta^{\mu\nu}_{(1)} = -2\eta \partial_\perp^{\langle\mu} u^{\nu\rangle} - \zeta(\partial_\alpha u^\alpha)\Delta^{\mu\nu}$ etc.

➡ **Relativistic Euler/Navier-Stokes equation!**

# Outline

## Motivation:

Hydrodynamics of
a relativistic spinful fluid?




## Approach:

Phenomenological
entropy-current analysis

1st law:

2nd law:

## Result:

(1) Coupled dynamics of hydro & spin

(2) Diffusive nature of spin:

# Spin in a relativistic fluid

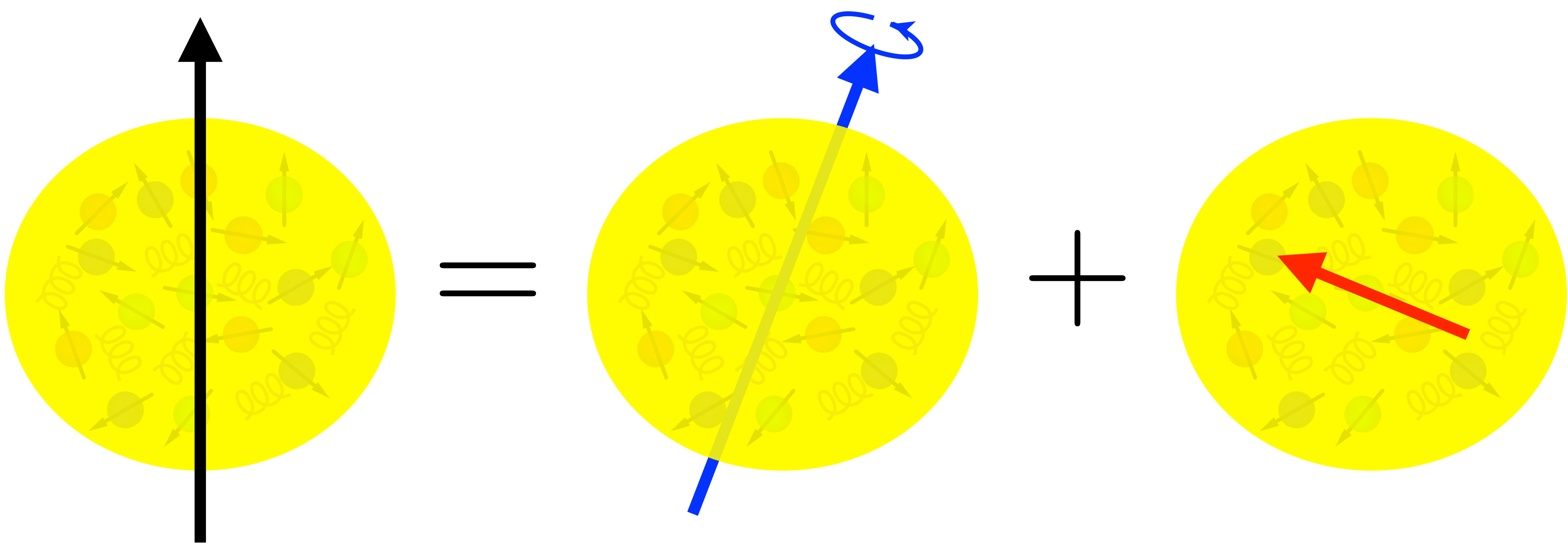
# Angular momentum conservation

◆ **All we need for angular momentum:**

**Conservation law:** $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu J^{\mu\nu\rho} = 0$

**Decomposition:** $J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}$

Total AM　　Orbital AM　　Spin AM



# Dynamics of spin density

**◆ All we need for angular momentum:**

**Conservation law:** $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu J^{\mu\nu\rho} = 0$

**Decomposition:** $J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}$

**Total AM** **Orbital AM** **Spin AM**

**- Spinless case:**

$\Sigma^{\mu\nu\rho} = 0$ $\xrightarrow{\partial_\mu J^{\mu\nu\rho} = 0}$ $\Theta^{\mu\nu} = \Theta^{\nu\mu}$ **: EM tensor is symmetric!**

# Dynamics of spin density

◆ **All we need for angular momentum:**

**Conservation law:** $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu J^{\mu\nu\rho} = 0$

**Decomposition:** $J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}$

Total AM    Orbital AM    Spin AM

**- Spinless case:**

$\Sigma^{\mu\nu\rho} = 0$ $\xrightarrow{\partial_\mu J^{\mu\nu\rho} = 0}$ $\Theta^{\mu\nu} = \Theta^{\nu\mu}$ : EM tensor is symmetric!

**- Spinful case:**

$\Sigma^{\mu\nu\rho} \neq 0$ $\xrightarrow{\partial_\mu J^{\mu\nu\rho} = 0}$ $\partial_\mu \Sigma^{\mu\nu\rho} = -(\Theta^{\nu\rho} - \Theta^{\rho\nu}) \equiv -2\Theta^{\nu\rho}_{(a)}$

$\overset{!}{=}$ Equation of motion for spin density: $\Sigma^{0\nu\rho}$

# Entropy-current analysis

## ◆ Setup

**Dynamical d.o.f.:**
- Canonical energy-momentum density: $\Theta^{0\nu}$
- Spin-angular momentum density: $\Sigma^{0\nu\rho}$

**Equation of motion:** $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu \Sigma^{\mu\nu\rho} = -2\Theta^{\nu\rho}_{(a)}$

**Expansion (assumption):**

$$\begin{cases} \Theta^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1)} \\ \Sigma^{\mu\nu\rho} = u^\mu S^{\nu\rho} + \Sigma^{\mu\nu\rho}_{(1)} \end{cases}$$

## ◆ Extension of local thermodynamics

**1st law:** $\beta(de \boxed{- \omega_{\mu\nu} dS^{\mu\nu}}) = ds, \ e + p \boxed{- \omega_{\mu\nu} S^{\mu\nu}} = Ts, \ d = u^\mu \partial_\mu$

($-\omega_{\mu\nu} dS^{\mu\nu} \sim \mu dn$, $-\omega_{\mu\nu} S^{\mu\nu} \sim \mu n$)

( $\omega_{\mu\nu}$ : Conjugate variable to characterize spin density $S^{\mu\nu}$
**Power-counting scheme:** $\omega_{\mu\nu} \sim S^{\mu\nu} = O(\partial^1)$ )

**2nd law:** $^\exists s^\mu$ s.t. $\partial_\mu s^\mu \geq 0$

# Flowchart

**Step 1. Determine dynamical d.o.m (& its equation of motion)**

d.o.f. : $\{\Theta^{0\nu}, \Sigma^{0\nu\rho}\}$   EoM: $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu \Sigma^{\mu\nu\rho} = -2\Theta^{\nu\rho}_{(a)}$

**Step 2. Introduce entropy & conjugate variable**

Entropy: $s(\Theta^{0\nu}, \Sigma^{0\nu\rho})$ ⇨ $\beta_\nu \equiv \dfrac{\partial s}{\partial \Theta^{0\nu}}, \quad \omega_{\nu\rho} \equiv \dfrac{\partial s}{\partial \Sigma^{0\nu\rho}}$

**Step 3. Write down all possible terms with finite derivatives**

$$\Theta^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1s)} + \Theta^{\mu\nu}_{(1a)}, \quad \Sigma^{\mu\nu\rho} = u^\mu S^{\nu\rho} + \Sigma^{\mu\nu\rho}_{(1)}$$

**Step 4. Restrict terms to be compatible with local 2nd law**

$\exists\, s^\mu$ s. t. $\partial_\mu s^\mu \geq 0$ ⇨ $e + p - \omega_{\mu\nu} S^{\mu\nu} = Ts, \quad \Theta^{\mu\nu}_{(1)} = \cdots$

# Constitutive relations

**Entropy-production rate up to $O(\partial^2)$:**

$$\partial_\mu \boxed{\left(su^\mu + s^\mu_{(1)}\right)} = \boxed{-\Theta^{\mu\nu}_{(1s)}}\,\partial_\mu\beta_\nu \boxed{-\Theta^{\mu\nu}_{(1a)}}\left(\partial_\mu\beta_\nu - 2\beta\omega_{\mu\nu}\right) \Leftrightarrow \partial_\mu s^\mu \geq 0$$

$\equiv s^\mu$ $\qquad \propto \partial^{(\mu}\beta^{\nu)}$ $\qquad \propto \left(\partial^{[\mu}\beta^{\nu]} - 2\beta\omega^{\mu\nu}\right)$

◆ **Constitutive relation** $\left(\Theta^{\mu\nu}_{(1s)} = \Theta^{\nu\mu}_{(1s)},\ \Theta^{\mu\nu}_{(1a)} = -\Theta^{\nu\mu}_{(1a)}\right)$

- **EM tensor:** $\Theta^{\mu\nu} = eu^\mu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1s)} + \Theta^{\mu\nu}_{(1a)}$
- **Spin-AM tenror:** $\Sigma^{\mu\nu\rho} = u^\mu S^{\nu\rho} + \Sigma^{\mu\nu\rho}_{(1)}$

with

$$\begin{cases} \Theta^{\mu\nu}_{(1s)} = -2\eta\partial^{\langle\mu}_\perp u^{\nu\rangle} - \zeta(\partial_\alpha u^\alpha)\Delta^{\mu\nu} \\ \Delta^\mu_\rho u_\sigma \Theta^{\rho\sigma}_{(1a)} = \lambda\left(\dfrac{\partial^\mu_\perp p}{e+p} - 4\omega^{\mu\nu}u_\nu\right) \\ \Delta^\mu_\rho\Delta^\nu_\sigma\Theta^{\rho\sigma}_{(1a)} = -2\gamma\left(\partial^{[\mu}_\perp u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\sigma\omega^{\rho\sigma}\right) \\ \Sigma^{\mu\nu\rho} = O(\partial^2) \end{cases}$$

: **Shear** & **Bulk** **viscosity**

: ***Boost heat conductivity***

: ***Rotational viscosity***

# Relativistic spin-hydro

## ◆Bulding blocks of hydrodynamic equation

**(1) (Non-)Conservation laws:** $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu \Sigma^{\mu\nu\rho} = -2\Theta^{\nu\rho}_{(a)}$

Leading!

**(2)+(3): Constitutive relation + Physical properties:**

- **EM tensor:** $\Theta^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1s)} + \Theta^{\mu\nu}_{(1a)}$
- **Spin-AM tenror:** $\Sigma^{\mu\nu\rho} = u^\mu S^{\nu\rho} + \Sigma^{\mu\nu\rho}_{(1)}$

with

$$\begin{cases} \Theta^{\mu\nu}_{(1s)} = -2\eta\partial_\perp^{\langle\mu}u^{\nu\rangle} - \zeta(\partial_\alpha u^\alpha)\Delta^{\mu\nu} \\ \Delta^\mu_\rho u_\sigma \Theta^{\rho\sigma}_{(1a)} = \lambda\left(\frac{\partial_\perp^\mu p}{e+p} - 4\omega^{\mu\nu}u_\nu\right) \\ \Delta^\mu_\rho \Delta^\nu_\sigma \Theta^{\rho\sigma}_{(1a)} = -2\gamma\left(\partial_\perp^{[\mu}u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\sigma\omega^{\rho\sigma}\right) \\ \Sigma^{\mu\nu\rho} = O(\partial^2) \end{cases}$$

: **Shear** & **Bulk** viscosity

: *Boost heat conductivity*

: *Rotational viscosity*

➡ **We can describe coupled dynamics of hydro & spin!**

# Linear-mode analysis
on spin-hydro

# Linearized spin-hydro

Perturbation on the top of **global static** thermal equilibrium:

$$
\begin{cases}
e(x) = e_0 + \delta e(x), \\
p(x) = p_0 + \delta p(x), \\
v^i(x) = 0 + \delta v^i(x), \\
S^{\mu\nu}(x) = 0 + \delta S^{\mu\nu}(x), \\
\omega^{\mu\nu}(x) = 0 + \delta\omega^{\mu\nu}(x),
\end{cases}
$$

Pickup $O(\delta)$-terms only

◆ **Linearized spin-hydrodynamic equations:**

$$\partial_0 \delta e + \partial_i \delta\pi^i - 2(c_s^2 \lambda' \partial_i \partial^i \delta e + D_b \partial^i \delta S^{0i}) = 0$$

$$(\partial_0 \delta\pi^i + c_s^2 \partial^i \delta e) - \gamma_\parallel \partial^i \partial_j \delta\pi^j - (\gamma_\perp + \gamma')(\delta^i_j \boldsymbol{\nabla}^2 - \partial^i \partial_j)\delta\pi^j + D_s \partial_j \delta S^{ji} = 0$$

$$\partial_0 \delta S^{ij} + 2\{D_s \delta S^{ij} - \gamma'(\partial^i \delta\pi^j - \partial^j \delta\pi^i)\} = 0$$

$$\partial_0 \delta S^{0i} + 2(c_s^2 \lambda' \partial^i \delta e + D_b \delta S^{0i}) = 0$$

with
$$
\begin{cases}
c_s^2 \equiv \dfrac{\partial p}{\partial e}, \quad \chi_s \equiv \dfrac{\partial S^{ij}}{\partial \omega^{ij}}, \quad \chi_b \equiv \dfrac{\partial S^{i0}}{\partial \omega^{i0}}, \quad D_s \equiv \dfrac{4\gamma}{\chi_s}, \quad D_b \equiv \dfrac{4\lambda}{\chi_b}, \\
\gamma' \equiv \dfrac{\gamma}{e_0 + p_0}, \quad \lambda' \equiv \dfrac{2\lambda}{e_0 + p_0}, \quad \gamma_\parallel \equiv \dfrac{1}{e_0 + p_0}\left(\zeta + \dfrac{4}{3}\eta\right), \quad \gamma_\perp \equiv \dfrac{\eta}{e_0 + p_0}.
\end{cases}
$$

# Linear-mode analysis

**Linearized eom can be solved by the use of Fourier tr.!**

$$\delta\mathcal{O}(x) = e^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\delta\tilde{\mathcal{O}}(k) \quad \Rightarrow \quad \textbf{EoM:}\ M(\omega,\boldsymbol{k})\delta\tilde{\mathcal{O}}(k) = 0$$

$$(M(\omega,\boldsymbol{k}) : \ 10\times 10\,\text{matrix})$$

---

**Characteristic equation:** $\det M(\omega,\boldsymbol{k}) = 0$

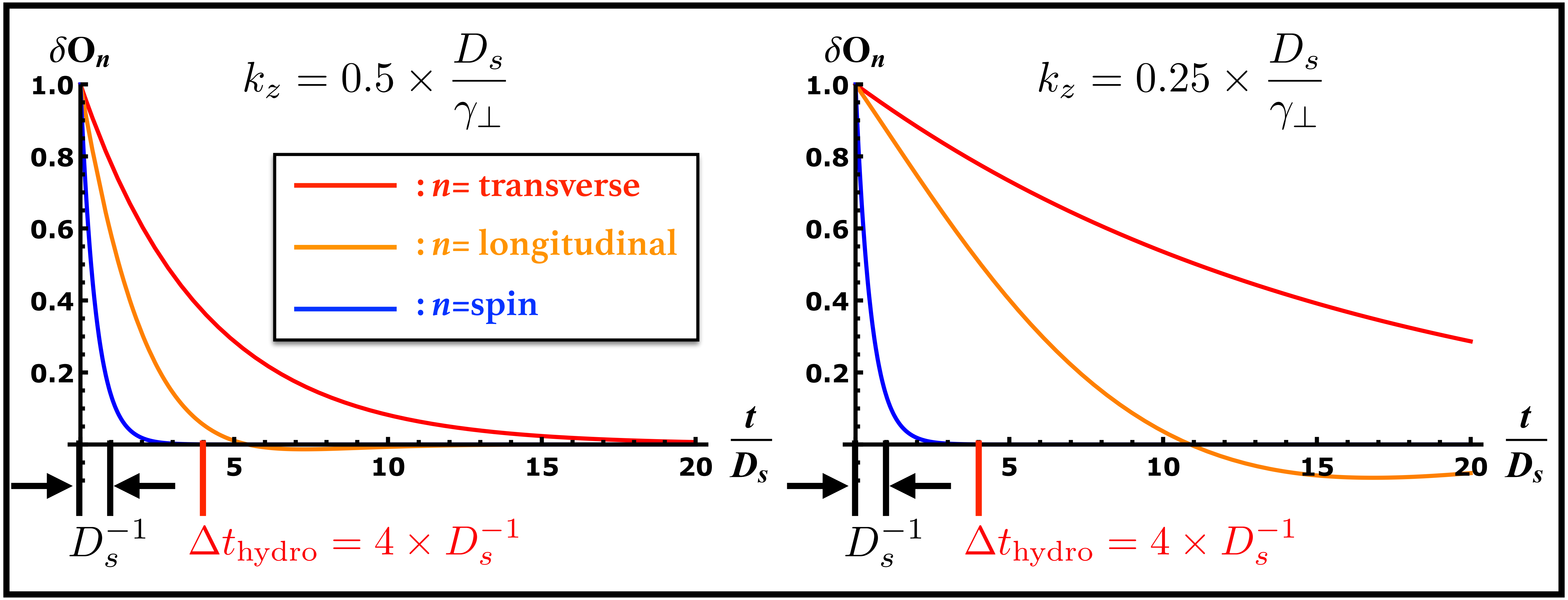
◆ **Solutions:**

$$\begin{cases} \omega = -2\mathrm{i}D_s + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -2\mathrm{i}D_b + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -\mathrm{i}\gamma_\perp k_z^2 + O(k_z^4) : (\times 2 \text{ gapless transverse modes}) \\ \omega = \pm c_s k_z - \mathrm{i}\dfrac{\gamma_\parallel}{2}k_z^2 + \mathcal{O}(k_z^3) : (\times 2 \text{ gapless longitudinal modes}) \end{cases}$$

➡ **4 gapless (hydro) modes + 6 gapped (non-hydro) modes**

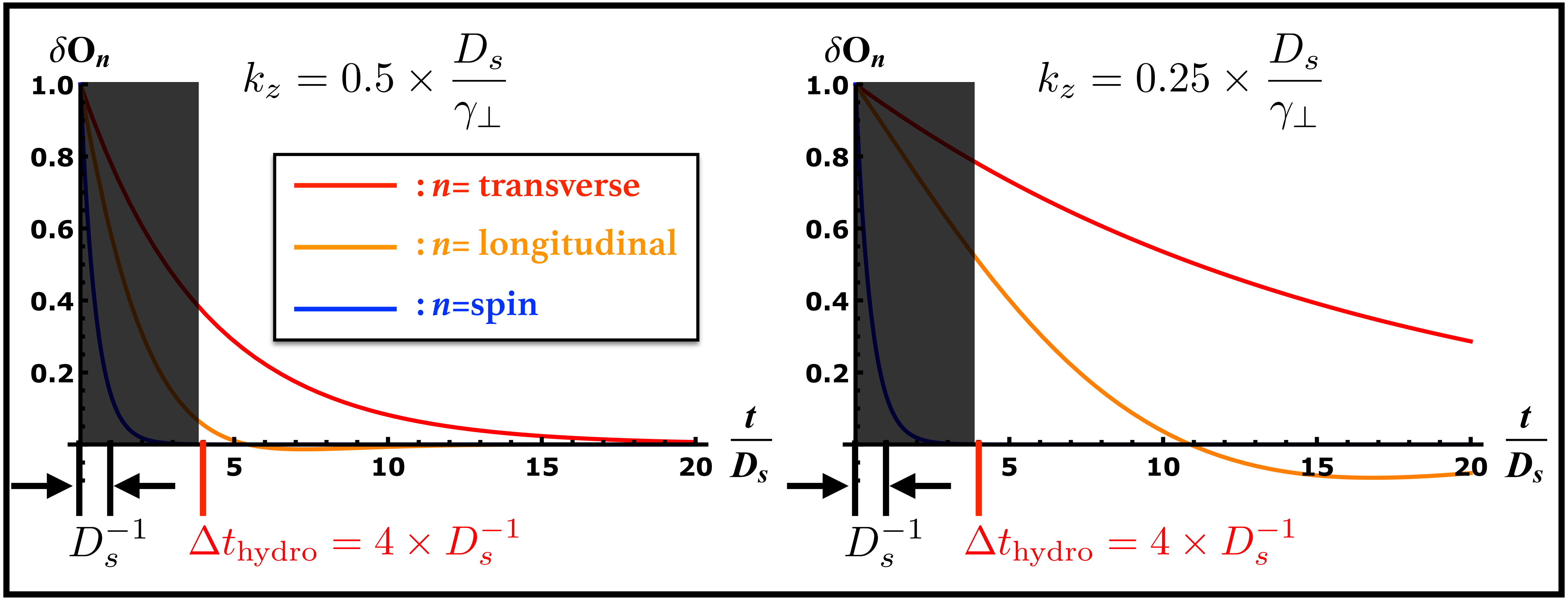
# Fate of spin polarization

$$\delta\mathcal{O}_n(x) \propto e^{-\mathrm{i}\omega t} \begin{cases} \omega = -2\mathrm{i}D_s + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -2\mathrm{i}D_b + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -\mathrm{i}\gamma_\perp k_z^2 + O(k_z^4) : (\times 2 \text{ gapless transverse modes}) \\ \omega = \pm c_s k_z - \mathrm{i}\frac{\gamma_\parallel}{2}k_z^2 + \mathcal{O}(k_z^3) : (\times 2 \text{ gapless longitudinal modes}) \end{cases}$$



# Fate of spin polarization

$$\delta\mathcal{O}_n(x) \propto e^{-\mathrm{i}\omega t} \begin{cases} \omega = -2\mathrm{i}D_s + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -2\mathrm{i}D_b + O(k_z^2) : (\times 3 \text{ gapped modes}) \\ \omega = -\mathrm{i}\gamma_\perp k_z^2 + O(k_z^4) : (\times 2 \text{ gapless transverse modes}) \\ \omega = \pm c_s k_z - \mathrm{i}\frac{\gamma_\parallel}{2} k_z^2 + \mathcal{O}(k_z^3) : (\times 2 \text{ gapless longitudinal modes}) \end{cases}$$




**Spin will disappear after characteristic time** $\simeq D_s^{-1}$

# Implication for QGP

## ◆ What spin hydro will do for heavy-ion collisions?

**input**

$$\begin{pmatrix} \Theta^{0\nu}(\tau_0, \boldsymbol{x}) \\ \Sigma^{0\nu\rho}(\tau_0, \boldsymbol{x}) \end{pmatrix} \rightarrow \textbf{Spin hydro} \rightarrow \begin{pmatrix} \Theta^{0\nu}(\tau_{\rm fo}, \boldsymbol{x}) \\ \Sigma^{0\nu\rho}(\tau_{\rm fo}, \boldsymbol{x}) \end{pmatrix}$$

**Spin hydro**

(EOS, Kinetic coefficient)

**output**

Cooper-Frye formula enables us to compute particle spectrum!

$\Theta^{0\nu}$ is **conserved** $\Rightarrow$ Multiplicity knows initial amount of energy!

$\Sigma^{0\nu\rho}$ is **not** conserved $\Rightarrow$ Information on initial amount of spin **could be lost** due to rotational viscosity!

## ◆ Question

What is lifetime of spin density? Is it large/small compared to e.g. $\tau_{\rm fo}$??

$\rightarrow$ need to evaluate rotational viscosity (or spin damping rate) of QGP!!

# Is spin hydrodynamics really **well-defined?**

# Flowchart

**Step 1. Determine dynamical d.o.m (& its equation of motion)**

$$\text{d.o.f.}: \{\Theta^{0\nu}, \Sigma^{0\nu\rho}\} \qquad \text{EoM: } \partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu \Sigma^{\mu\nu\rho} = -2\Theta^{\nu\rho}_{(a)}$$

**Step 2. Introduce entropy & conjugate variable**

$$\text{Entropy: } s(\Theta^{0\nu}, \Sigma^{0\nu\rho}) \Rightarrow \beta_\nu \equiv \frac{\partial s}{\partial \Theta^{0\nu}}, \quad \omega_{\nu\rho} \equiv \frac{\partial s}{\partial \Sigma^{0\nu\rho}}$$

**Step 3. Write down all possible terms with finite derivatives**

$$\Theta^{\mu\nu} = e u^\mu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1s)} + \Theta^{\mu\nu}_{(1a)}, \quad \Sigma^{\mu\nu\rho} = u^\mu S^{\nu\rho} + \Sigma^{\mu\nu\rho}_{(1)}$$

**Step 4. Restrict terms to be compatible with local 2nd law**

$$\exists\, s^\mu \ \text{s.t.}\ \partial_\mu s^\mu \geq 0 \Rightarrow e + p - \omega_{\mu\nu} S^{\mu\nu} = Ts, \quad \Theta^{\mu\nu}_{(1)} = \cdots$$

# Flowchart

## Step 1. Determine dynamical d.o.m (& its equation of motion)

d.o.f. : $\{\Theta^{0\nu}, \phi\}$ EoM: $\partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu \phi = f_\mu$

## Step 2. Introduce entropy & conjugate variable

Entropy: $s(\Theta^{0\nu}, \phi) \Rightarrow \beta_\nu \equiv \frac{\partial s}{\partial \Theta^{0\nu}}, \quad \pi \equiv \frac{\partial s}{\partial \phi}$

## Step 3. Write down all possible terms with finite derivatives

$$\Theta^{\mu\nu} = e u^\nu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1)}, \ f_\mu = q u_\mu + f^{(1)}_\mu$$

## Step 4. Restrict terms to be compatible with local 2nd law

$\exists\, s^\mu$ s. t. $\partial_\mu s^\mu \geq 0 \Rightarrow q = -\gamma\pi = -\gamma \frac{\partial s}{\partial \phi}$ This gives EoM in **Hydro+**!!

# Spin hydro as Hydro+

[See Stephanov-Yin, PRD, **98**, 036006 (2018) , ...]

**Hydro+ is a general framework describing both**

**◆ Hydrodynamic (gapless) mode**

- Conserved charge densities: Normal hydrodynamics
- Nambu-Goldstone mode: Superfluid hydrodynamics

**◆ Non-hydrodynamic (gapped) mode**

- Critical fluctuation around $T \sim T_c$: Original Hydro+
- $SU(2)_A$ charge density in QCD: Chiral hydrodynamics

} **well-defined**

- Spin density: Spin hydrodynamics
- Stress tensor: Muller-Israel-Stewart theory
- $U(1)_A$ charge density in QCD: Chiral hydrodynamics

} **ill-defined**

➡ **There are well-defined and (possibly) ill-defined Hydro+!**

# Caution from old paper

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## Unified Hydrodynamic Theory for Crystals, Liquid Crystals, and Normal Fluids*

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Liquid crystal can have spin density!

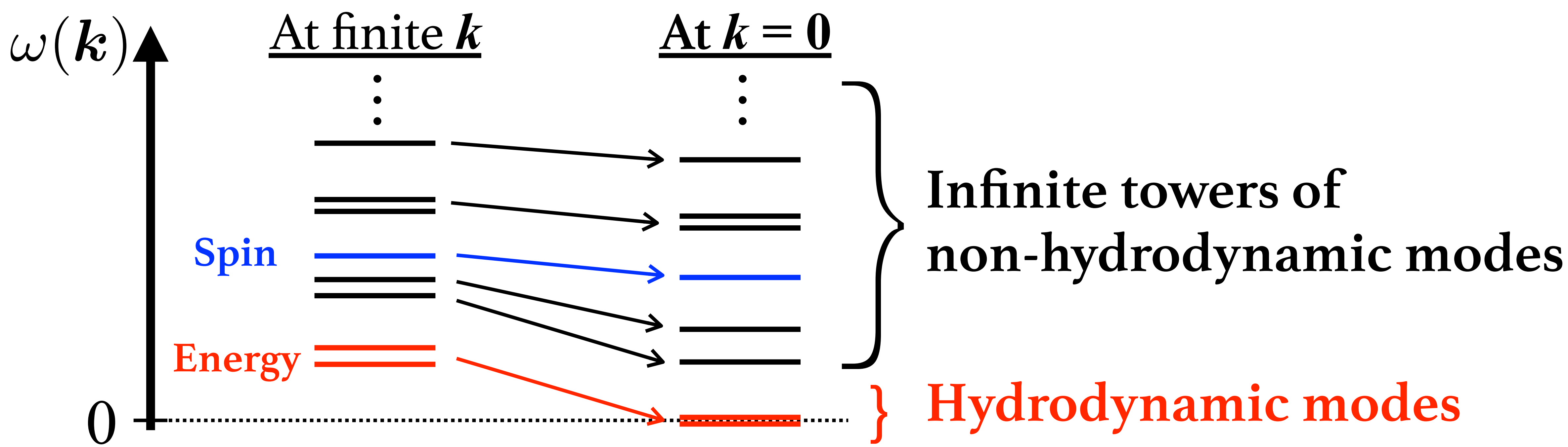
A unified hydrodynamic theory is presented that is appropriate for crystals; smectic, cholesteric, and nematic liquid crystals; glasses; and normal fluids. In the theory, the increased spatial degeneracy as the system progresses from crystalline and mesomorphic phases to the isotropic fluid phase is marked by successive reductions in the number of first-order elastic constants and in the number of transport coefficients. Distinction between local lattice dilations and local mass changes, and recognition of processes like vacancy diffusion that this difference makes possible, are crucial for understanding the connection between theories in different phases. Formulas are derived that give the number of hydrodynamic modes and the frequencies, lifetimes, and intensities of these modes in all of the above systems. In the nematic and cholesteric phases, the results agree with some found previously. In more complex systems, they are new. An attempt is made to explain the differences between the present hydrodynamic theory and other phenomenological proposals.

# Caution from old paper

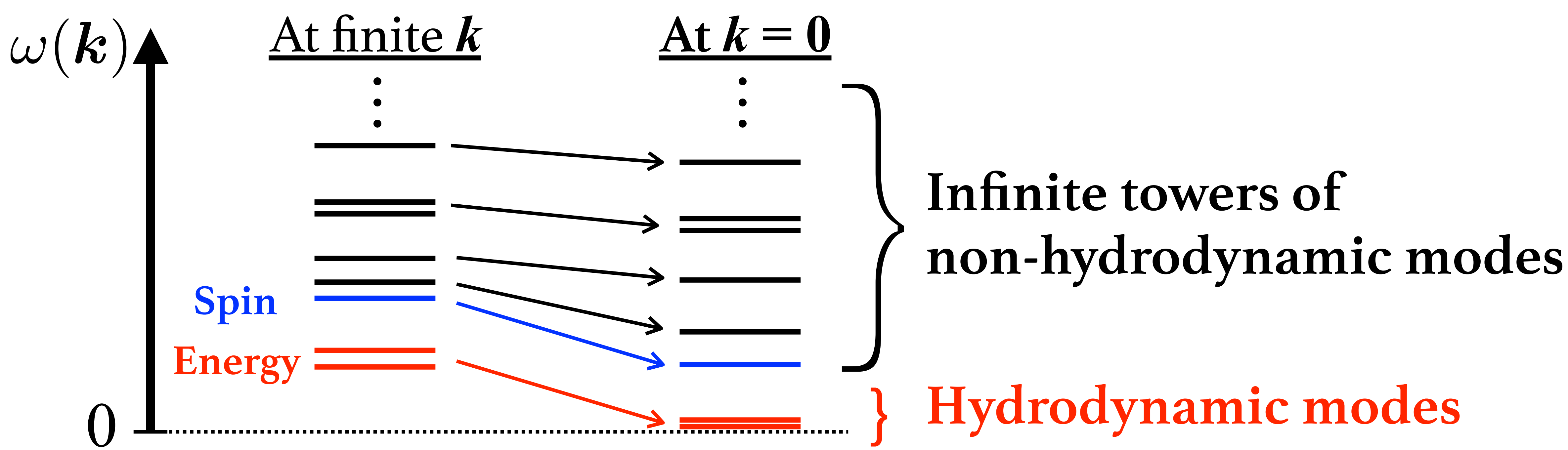
[17]In the hydrodynamic regime for nematics, the "extension" of H. W. Hwang, Phys. Rev. Letters 26, 1525 (1971), is equivalent to FLMPS. Outside of the hydrodynamic regime, the terms he keeps in addition are *ad hoc* and incomplete and there is no reason to think experiments would necessarily give the line shapes they predict *even if* the experiments could be performed. They are just the "irrelevant transport coefficients" which should be discarded as discussed in Ref. 11. Some readers may object to our use of the word irrelevant, since under certain circumstances nonhydrodynamic modes are slow and measurable, e.g., near phase transitions. We agree but point out in response that the same arguments apply in such cases to other variables that have been omitted (e.g., to the magnitude of the order parameter as well as its direction).

# Spin hydro is ill-defined

◆ Scenario 1 (Bad: Spin hydro = Hydro++++?)

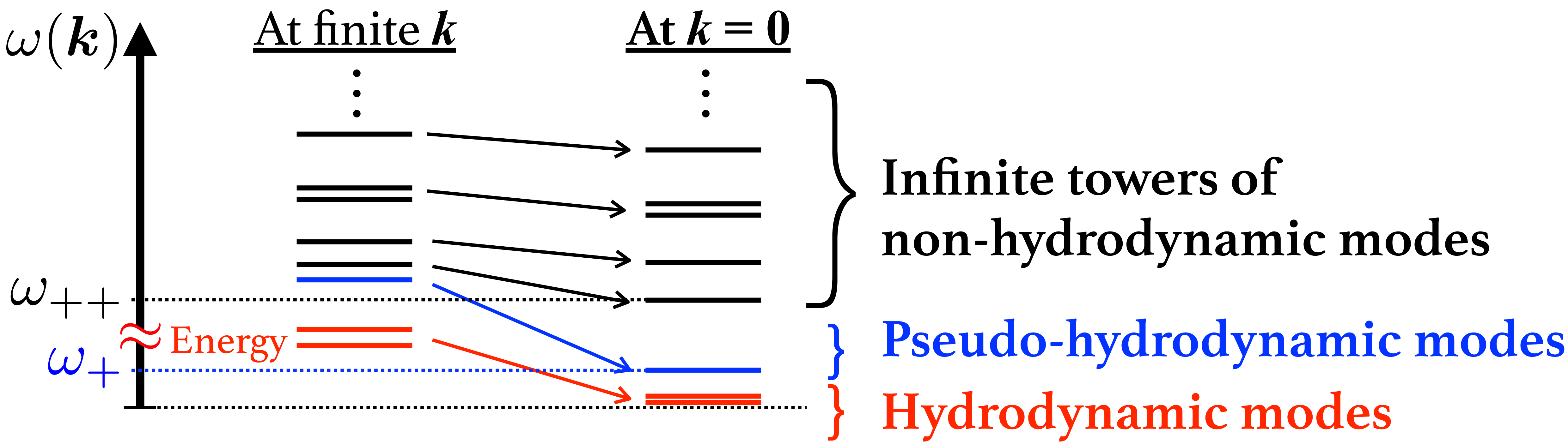



◆ Scenario 2 (Better but still not good: Spin hydro = Hydro+?)



# Well-defined HYDRO+

**◆ When Hydro+ is well-defined**




**If** $\omega_{+} \ll \omega_{++}$ **is satisfied, Hydro+ becomes well-defined!!**

This generally happens when

**emergent symmetry appears by tuning parameters ($T$, $m$, ...)!**

( - Critical fluid: Scale symmetry emerges at $T = T_c$
- SU(2)$_A$ chiral fluid: SU(2)$_A$ symmetry emerges at $m_q = 0$ )

# HQ-spin hydro is well-defined

[38] If for some reason the coupling between "spin" and orbital angular momentums vanishes, or can be neglected, a separate conservation for "spin" angular momentum will follow from the microscopic Hamiltonian. This is actually the case for a number of models employed to describe magnetic problems.

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When we consider **heavy quark limit:** $m_Q \to \infty$,

**emergent heavy quark symmetry** appears!

◆ Heavy quark spin hydrodynamics

**Heavy quark spin damping rate is suppressed by $1/m_Q$,**

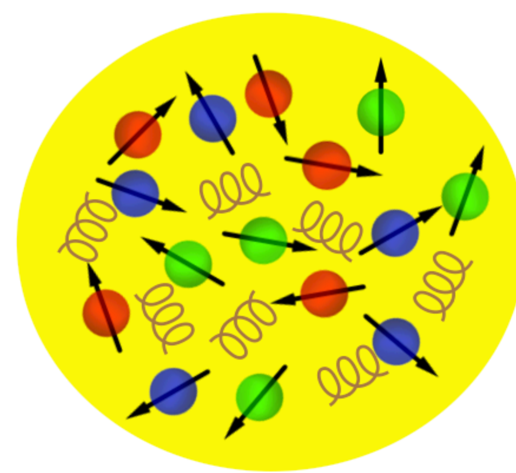
**so that HQ-spin hydro is well-defined Hydro+!**

(But I do not know whether there is enough # of heavy quarks...)

# Summary

## Motivation:

Hydrodynamics of
a relativistic spinful fluid?



$\overset{?}{\simeq}$ Hydro + Spin

## Approach:

Phenomenological
entropy-current analysis

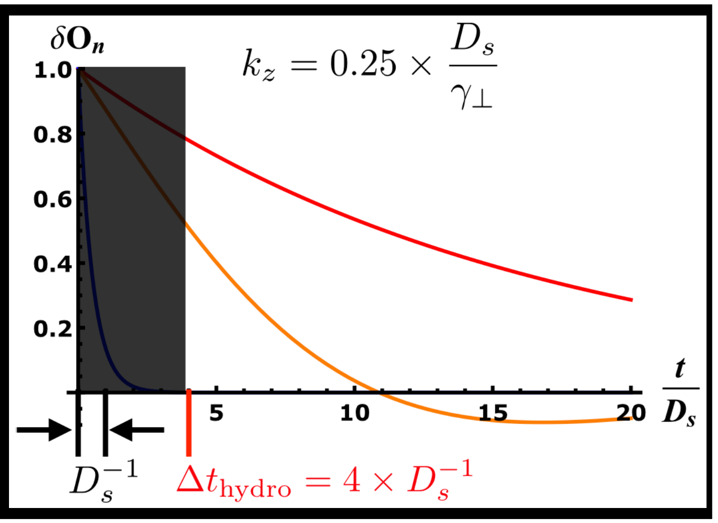
1st law: $\beta(de-\omega_{\mu\nu}dS\mu\nu) = ds$

2nd law: $\exists s^\mu$ s.t. $\partial_\mu s^\mu \geq 0$

## Result:

(1) Coupled dynamics of hydro & spin

(2) Diffusive nature of spin: $\omega = -2\mathrm{i}D$



# Outlook

## ◆ Microscopic derivation & Green-Kubo formula

Derivation of (HQ) spin hydro based on field/kinetic theory

Calculation of (HQ) spin damping coefficients

## ◆ Extension to more general situation

Spin-hydro under strong vorticity

Spin-hydro with dynamical/background electromagnetic field

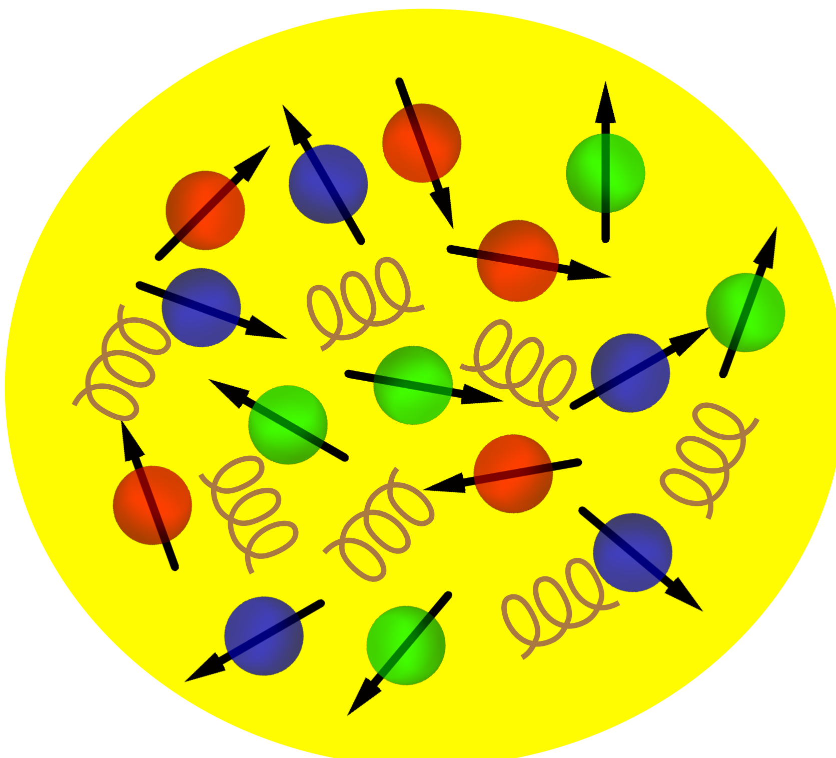
## ◆ Application to QGP/cond-mat spintronics

Possibility of QGP spintronics in heavy-ion collision?

Application of spin-hydro to e.g. clean graphene?

# One-page Summary

**Phenomenological derivation of spin-hydro**



$\simeq$ Hydro + ~~Spin~~

**Three main messages:**

(1) **Coupled dynamics** of hydro & spin is available

(2) Spin density shows (gapped) **relaxational dynamics**

(3) How to make spin hydro as **well-defined Hydro+**