The role of the spin tensor in spin hydrodynamics

Enrico Speranza

E.S. and N. Weickgenannt, arXiv:2007.00138 (2020)

N. Weickgenannt, E.S., X.-l. Sheng, Q. Wang, and D. H. Rischke, arXiv:2005.01506 (2020)

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

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Spin and Hydrodynamics in Relativistic Nuclear Collisions

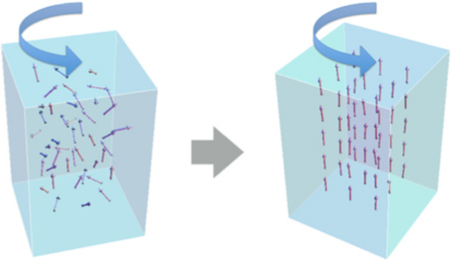
ECT*, October 12, 2020

Outline

- Spin physics in heavy-ion collisions
- How to include spin degrees of freedom
  - Thermodynamics
  - Kinetic theory
  - Hydrodynamics

Rotation and polarization

- Condensed matter: **Barnett effect**

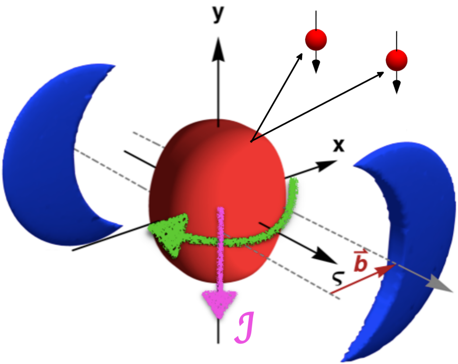


Picture by Mamoru Matsuo

Ferromagnet gets magnetized when it rotates

Polarization effects through rotation in heavy-ion collisions? Yes!

Noncentral heavy-ion collisions

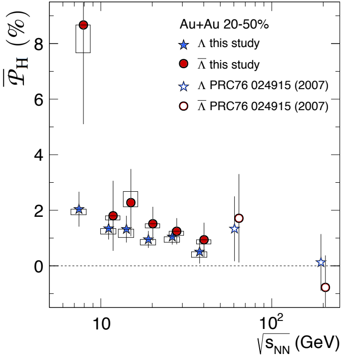


Picture from W. Florkowski, R. Ryblewski and A. Kumar, Prog. Part. Nucl. Phys. **108**, 103709 (2019)

Noncentral nuclear collisions $\Rightarrow$ Large global angular momentum
$\Rightarrow$ Vorticity of hot and dense matter $\Rightarrow$ particle polarization along vorticity

Experimental observation - Global Λ polarization

- Polarization along global angular momentum



L. Adamczyk et al. (STAR), Nature 548 62-65 (2017)

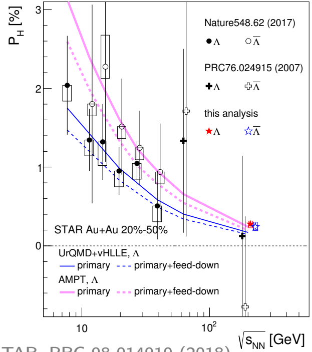
- Weak decay: $\Lambda \to p + \pi^-$ angular distr.: $dN/d\cos\theta = \frac{1}{2}(1 + \alpha|\vec{\mathcal{P}}_H|\cos\theta)$
- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega = (P_\Lambda + P_{\bar{\Lambda}})k_B T/\hbar \approx (9+1) \times 10^{21}\mathrm{s}^{-1}$$

Great Red Spot of Jupiter $10^{-4}\,\mathrm{s}^{-1}$,
Turbulent flow superfluid He-II $150\,\mathrm{s}^{-1}$, Superfluid nanodroplets $10^{7}\,\mathrm{s}^{-1}$

Experiments vs theory: Λ polarization

Global - along J

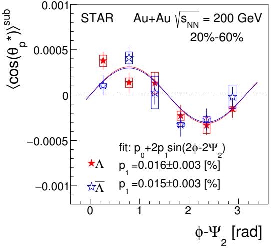


STAR, PRC 98 014910 (2018)

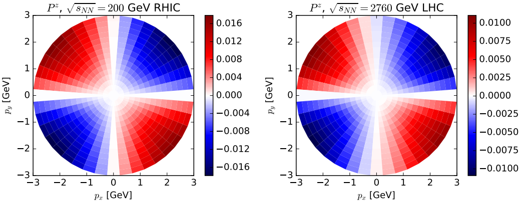
$$\Pi^{\mu}(x,p) \propto (1-n_F)\epsilon^{\mu\nu\rho\tau} p_{\nu} \varpi_{\rho\tau}$$

$$\varpi_{\rho\tau} = -\frac{1}{2}(\partial_{\rho}\beta_{\tau} - \partial_{\tau}\beta_{\rho})$$

Longitudinal - along beam axis



J. Adam *et al.* [STAR Collaboration], PRL **123**, 132301 (2019)



F. Becattini, I Karpenko, PRL **120**, 012302

- Theory assumes local equilibrium of spin degrees of freedom
- "Sign problem" between theory and experiments for longitudinal polarization!

Does spin play a dynamical role in hydro?

- Relativistic hydrodynamics is a good effective theory: $\partial_\mu T^{\mu\nu} = 0$

Goal: Relativistic hydrodynamics (classical) with spin (quantum) as dynamical variable

W. Florkowski, B. Friman, A. Jaiswal, and E. S., Phys. Rev. C **97**, no. 4, 041901 (2018)
W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. S., Phys. Rev. D **97**, no. 11, 116017 (2018)
W. Florkowski, F. Becattini, and E. S., Acta Phys. Polon. B **49**, 1409 (2018)
F. Becattini, W. Florkowski, E. S., Phys. Lett. B **789**, 419 (2019)
S. Bhadury, W. Forkowski, A. Jaiswal, A. Kumar, R. Ryblewski, 2002.03937, 2008.10976 (2020)
S. Shi, C. Gale, S. Jeong, 2008.08618 (2020)

Starting point: Quantum field theory

N. Weickgenannt, X.l. Sheng, E. S., Q. Wang, and D. H. Rischke, Phys. Rev. D **100**, no. 5, 056018 (2019)
N. Weickgenannt, E. S., X.l. Sheng, Q. Wang, and D. H. Rischke, 2005.01506 (2020)

- Alternative approaches: Lagrangian formulation, entropy current
  D. Montenegro, L. Tinti, G. Torrieri, PRD 96, 056012 (2017)
  D. Montenegro, G. Torrieri, 2004.10195 (2020)
  K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, PLB795, 100 (2019)
  K. Fukushima, S. Pu, 2010.01608 (2020)

Orbital-to-spin angular momentum conversion

Our results:

- How do we describe the orbital-to-spin angular momentum conversion in kinetic theory?

  Nonlocal particle scatterings (finite impact parameter)

- And in hydrodynamics?

  Antisymmetric part of energy-momentum tensor

Nonrelativistic kinetic theory

Kinetic Theory for a Dilute Gas of Particles with Spin

S. Hess and L. Waldmann

Institut für Theoretische Physik der Universität Erlangen-Nürnberg, Erlangen

(Z. Naturforschg. **21 a,** 1529—1546 [1966]; received 6 April 1966)

The kinetic theory of particles with spin previously developed for a Lorentzian gas is extended to the case of a pure gas. In part A the transport (Boltzmann) equation for the one particle distribution operator is stated and discussed (conservation laws, H-theorem). A magnetic field acting on the magnetic moment of the particles is incorporated throughout. In part B the pertaining linearized collision operator and certain bracket expressions linked with this operator are considered. Part C deals with the expansion of the distribution operator and of the linearized transport equation with respect to a complete set of composite irreducible tensors built from the components of particle velocity and spin. Thus, the distribution operator is replaced by a set of tensors depending only on time and space-coordinates. The physical meaning of these tensors (expansion coefficients) is invoked. They obey a set of coupled first-order differential equations (transport-relaxation equations). The reciprocity relations for the relaxation matrices are stated. Finally a detailed discussion of angular momentum conservation is given.

- "There is another effect which we cannot describe with a local collision operator: the orientation of the spin by a local or uniform rotation of the system (Barnett effect)"
- **Caveat**: Nonrelativistic Barnett effect $\Longrightarrow$ Magnetization
  Heavy-ion collisions $\Longrightarrow$ Spin polarization without magnetization
  (Spin particles + Spin antiparticles)

Canonical energy-momentum and spin tensors

Lagrangian $\Rightarrow$ Poincaré symmetry $\Rightarrow$ Noether's th. $\Rightarrow$ Conservation laws

- **Conservation of energy and momentum:**
  Canonical energy-momentum tensor $\hat{T}_C^{\mu\nu}(x)$

$$\partial_\mu \hat{T}_C^{\mu\nu}(x) = 0$$

- **Conservation of total angular momentum:**
  Canonical total angular momentum tensor ("orbital"+"spin")

$$\hat{J}_C^{\lambda,\mu\nu}(x) = x^\mu \hat{T}_C^{\lambda\nu}(x) - x^\nu \hat{T}_C^{\lambda\mu}(x) + \hat{S}_C^{\lambda,\mu\nu}(x)$$

$$\partial_\lambda \hat{J}_C^{\lambda,\mu\nu}(x) = 0 \Longrightarrow \partial_\lambda \hat{S}_C^{\lambda,\mu\nu}(x) = \hat{T}_C^{\nu\mu}(x) - \hat{T}_C^{\mu\nu}(x)$$

Pseudo-gauge transformations

Densities are not uniquely defined $\Longrightarrow$ Relocalization

F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976)

$$\hat{T}'^{\mu\nu}(x) = \hat{T}_C^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda\left[\hat{\Phi}^{\lambda,\,\mu\nu}(x) + \hat{\Phi}^{\mu,\,\nu\lambda}(x) + \hat{\Phi}^{\nu,\,\mu\lambda}(x)\right]$$

$$\hat{S}'^{\lambda,\,\mu\nu} = \hat{S}_C^{\lambda,\,\mu\nu}(x) - \hat{\Phi}^{\lambda,\,\mu\nu}(x) + \partial_\rho \hat{Z}^{\mu\nu,\lambda\rho}(x)$$

$\hat{\Phi}^{\lambda,\,\mu\nu} = -\hat{\Phi}^{\lambda,\,\nu\mu}$, $\hat{Z}^{\mu\nu,\lambda\rho} = -\hat{Z}^{\nu\mu,\lambda\rho} = -\hat{Z}^{\mu\nu,\rho\lambda}$

- Leave global charges invariant

$$\hat{P}^\mu = \int d^3\Sigma_\lambda\, \hat{T}_C^{\lambda\mu}(x) \qquad \hat{J}^{\mu\nu} = \int d^3\Sigma_\lambda\, \hat{J}_C^{\lambda,\,\mu\nu}(x)$$

- However, different global spin

$$\hat{S}_C^{\mu\nu} = \int d\Sigma_\lambda \hat{S}_C^{\lambda,\mu\nu} \neq \hat{S}'^{\mu\nu}$$

- Conservation laws $\partial_\mu \hat{T}'^{\mu\nu} = 0$, $\partial_\lambda \hat{S}'^{\lambda,\mu\nu} = \hat{T}'^{,\nu\mu}$

- Belinfante's case ($\hat{\Phi}^{\lambda,\,\mu\nu}(x) = \hat{S}_C^{\lambda,\,\mu\nu}(x)$ and $\hat{Z}^{\lambda\rho,\mu\nu} = 0$)

$$\hat{T}_B^{\mu\nu}(x) = \hat{T}_C^{\mu\nu}(x) + \tfrac{1}{2}\partial_\lambda\left[\hat{S}_C^{\lambda,\,\mu\nu}(x) + \hat{S}_C^{\mu,\,\nu\lambda}(x) + \hat{S}_C^{\nu,\,\mu\lambda}(x)\right]$$

$$\hat{S}_B^{\lambda,\,\mu\nu}(x) = 0$$

Canonical - Free Dirac theory

$$\hat{T}_C^{\mu\nu}(x) = \frac{i\hbar}{2}\bar{\psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \psi(x) - g^{\mu\nu}\mathcal{L}(x)$$

$$\hat{S}_C^{\lambda,\,\mu\nu}(x) = \frac{\hbar}{4}\bar{\psi}(x)(\gamma^\lambda\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\lambda)\psi(x) = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}\gamma_\alpha\gamma_5\psi$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- Global spin

$$\hat{S}_C^{\mu\nu} \equiv \int_\Sigma d\Sigma_\lambda\, \hat{S}_C^{\lambda,\mu\nu}$$

  does not transform as a tensor since $\partial_\lambda \hat{S}_C^{\lambda,\mu\nu} \neq 0$
- In a general frame: $\hat{S}^{0i} = 0$ and $\hat{S}^{ij} = \epsilon^{ijk}\hat{S}_C^k$

$$\hat{S}_C^k = \int d^3x\, \psi^\dagger \frac{\hbar}{2}\mathfrak{S}^k\psi, \quad \mathfrak{S}^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

We require covariant description of spin for free fields

Hilgevoord-Wouthuysen currents

J. Hilgevoord, S. Wouthuysen, NP 40 (1963) 1

- Apply Noether's theorem to Klein-Gordon Lagrangian for free spinors

$$\mathcal{L}_{KG} = \frac{1}{2m}(\hbar^2 \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi}\psi)$$

and use Dirac equation as subsidiary condition

$$\hat{T}^{\mu\nu}_{HW} = \frac{\hbar^2}{2m}\left(\partial^\mu \bar{\psi} \partial^\nu \psi + \partial^\nu \bar{\psi} \partial^\mu \psi\right) - g^{\mu\nu} \mathcal{L}_{KG}$$

$$\hat{S}^{\lambda,\mu\nu}_{HW} = \frac{i\hbar^2}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi$$

- $\partial_\lambda \hat{S}^{\lambda,\mu\nu}_{HW} = 0 \Longrightarrow$ Covariant global spin $\hat{S}^{\mu\nu}_{HW} \equiv \int_\Sigma d\Sigma_\lambda \, \hat{S}^{\lambda,\mu\nu}_{HW}$
- Compatible with Frenkel theory $\hat{P}_\mu \hat{S}^{\mu\nu}_{HW} = 0$
- Pseudo-gauge transformation with

$$\hat{\Phi}^{\lambda,\mu\nu} = \hat{M}^{[\mu\nu]\lambda} - g^{\lambda[\mu} \hat{M}_\rho{}^{\nu]\rho}$$

$$\hat{Z}^{\mu\nu,\lambda\rho} = -\frac{\hbar}{8m} \bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi$$

where $\hat{M}^{\lambda\mu\nu} \equiv \frac{i\hbar^2}{4m} \bar{\psi} \sigma^{\mu\nu} \overleftrightarrow{\partial}^\lambda \psi$

Other choices

- de Groot-van Leeuwen-van Weert currents
  Relativistic Kinetic Theory. Principles and Applications (North-Holland, 1980)

$$\hat{T}^{\mu\nu}_{GLW} = -\frac{\hbar^2}{4m}\bar{\psi}\overleftrightarrow{\partial}^{\mu}\overleftrightarrow{\partial}^{\nu}\psi$$
$$\hat{S}^{\lambda,\mu\nu}_{GLW} = \frac{i\hbar^2}{4m}\left(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^{\lambda}\psi - \partial_\rho\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma^5\psi\right)$$

  Pseudo-gauge transformation with

$$\Phi^{\lambda,\mu\nu} = \frac{i\hbar^2}{4m}\bar{\psi}(\sigma^{\lambda\mu}\overleftrightarrow{\partial}^{\nu} - \sigma^{\lambda\nu}\overleftrightarrow{\partial}^{\mu})\psi \qquad Z^{\mu\nu,\lambda\rho} = 0.$$

- Klein-Gordon Lagrangian with second-order derivatives

$$\hat{T}^{\mu\nu}_{KG} = \hat{T}^{\mu\nu}_{GLW} \qquad \hat{S}^{\lambda,\mu\nu}_{KG} = \hat{S}^{\lambda,\mu\nu}_{HW}$$

  with

$$\Phi^{\lambda,\mu\nu} = \frac{i\hbar^2}{4m}\bar{\psi}(\sigma^{\lambda\mu}\overleftrightarrow{\partial}^{\nu} - \sigma^{\lambda\nu}\overleftrightarrow{\partial}^{\mu})\psi$$
$$Z^{\mu\nu,\lambda\rho} = \frac{i\hbar^2}{4m}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma^5\psi$$

- Both choices lead to covariant global spin

Polarization observable in heavy-ion collisions

- Pauli-Lubanski vector for particle with momentum p^μ
  F. Becattini, 2004.04050; E.S., N. Weickgenannt, 2007.00138; L. Tinti, W. Florkowski, 2007.04029

$$\hat{\Pi}^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \hat{j}_{\alpha\beta}(p)$$

Total angular momentum $\hat{J}^{\mu\nu} = \int d^4p\, \hat{j}^{\mu\nu}(p)$

Operator $\hat{\Pi}(p)$ is pseudo-gauge invariant by definition

- We want to calculate thermal average $\langle \hat{\Pi}(p) \rangle = \mathrm{tr}(\hat{\rho}\, \hat{\Pi}(p))$

What are the physical implications of pseudo-gauges in statistical physics?

Statistical operator - Canonical

Zubarev, 1979, Ch, Van Weert 1982 , F. Becattini, L. Bucciantini, E. Grossi and L. Tinti, Eur. Phys. J. C **75**, no. 5, 191 (2015), F. Becattini, M. Buzzegoli, E. Grossi, Particles 2 (2019) 2

- Maximization of entropy

$$S = -\mathrm{tr}(\hat{\rho}_C \log \hat{\rho}_C)$$

- Constraints on energy and momentum

$$n_\mu \mathrm{tr}\left[\hat{\rho}_C\, \hat{T}^{\mu\nu}_C(x)\right] = n_\mu T^{\mu\nu}_C(x)$$

n^μ - vector orthogonal to hypersurface Σ

Spin tensor $\Longrightarrow$ Constraint on total angular momentum

$$n_\mu \mathrm{tr}\left(\hat{\rho}_C\, \hat{J}^{\mu,\lambda\nu}_C\right) = n_\mu \mathrm{tr}\left[\hat{\rho}_C\left(x^\lambda\, \hat{T}^{\mu\nu}_C - x^\nu\, \hat{T}^{\mu\lambda}_C + S^{\mu,\lambda\nu}_C\right)\right] = n_\mu J^{\mu,\lambda\nu}_C$$

- Density operator

$$\hat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu\left(\hat{T}^{\mu\nu}_C(x) b_\nu(x) - \frac{1}{2}\hat{J}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x)\right)\right]$$
$$= \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu\left(\hat{T}_C(x)^{\mu\nu}\beta_\nu(x) - \frac{1}{2}\hat{S}^{\mu,\lambda\nu}_C(x)\Omega_{\lambda\nu}(x)\right)\right]$$

$\Omega_{\lambda\nu}$ - Spin potential: Lagrange multiplier for conservation of total angular momentum

Global equilibrium - Canonical

- Asymmetric EM tensor $\hat{T}_C^{\mu\nu} = \hat{T}_S^{\mu\nu} + \hat{T}_A^{\mu\nu}$ with $\hat{T}_S^{\mu\nu} = \hat{T}_S^{\nu\mu}$, $\hat{T}_A^{\mu\nu} = -\hat{T}_A^{\nu\mu}$
- Density operator must be stationary

$$\frac{1}{2}\hat{T}_S^{\mu\nu}(\partial_\mu\beta_\nu+\partial_\nu\beta_\mu)+\frac{1}{2}\hat{T}_A^{\mu\nu}(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu)-\frac{1}{2}(\partial_\mu\hat{S}_C^{\mu,\lambda\nu})\Omega_{\lambda\nu}-\frac{1}{2}\hat{S}_C^{\mu,\lambda\nu}(\partial_\mu\Omega_{\lambda\nu}) = 0$$

- Global equilibrium conditions:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0$$
$$\beta_\nu = b_\nu + \Omega_{\nu\lambda}x^\lambda$$
$$\Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu) = \mathrm{const}$$

We used $\partial_\mu\hat{S}_C^{\mu,\lambda\nu} = -2\hat{T}_A^{\lambda\nu}$

F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012)

Statistical operator and pseudo-gauges

F. Becattini, W. Florkowski, E.S. PLB **789**, 419 (2019)

- Start with **Canonical**

$$\hat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}_C^{\mu\nu}\beta_\nu - \frac{1}{2}\hat{S}_C^{\mu,\lambda\nu}\Omega_{\lambda\nu}\right)\right]$$

- Use general transformation $(\hat{T}_C^{\mu\nu}, \hat{S}_C^{\lambda,\mu\nu}) \to (\hat{T}'^{\mu\nu}, \hat{S}'^{\lambda,\mu\nu})$:
- **Canonical** density operator becomes

$$\hat{\rho}_C = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}'^{\mu\nu}\beta_\nu - \hat{S}'^{\mu,\lambda\nu}\Omega_{\lambda\nu}\right.\right.$$
$$\left.\left. + \frac{1}{2}\xi_{\lambda\nu}\left(\hat{\Phi}^{\lambda,\mu\nu} + \hat{\Phi}^{\nu,\mu\lambda}\right) - \frac{1}{2}(\Omega_{\lambda\nu} - \varpi_{\lambda\nu})\hat{\Phi}^{\mu,\lambda\nu} - \hat{Z}^{\lambda\nu,\mu\rho}\partial_\rho\Omega_{\lambda\nu}\right)\right]$$

with $\varpi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda - \partial_\lambda\beta_\nu), \qquad \xi_{\lambda\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\lambda + \partial_\lambda\beta_\nu)$

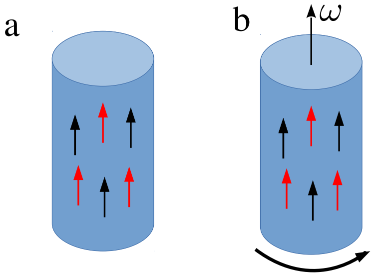
- When is $\hat{\rho}_C = \hat{\rho}'$?

$$\hat{\rho}' = \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}'^{\mu\nu}\beta_\nu - \frac{1}{2}\hat{S}'^{\mu,\lambda\nu}\Omega_{\lambda\nu}\right)\right]$$

1. β_μ is the same in both cases
2. $\xi_{\lambda\nu} = 0$ or $\hat{S}_C^{\lambda,\mu\nu} + \hat{S}_C^{\nu,\mu\lambda} = 0$
3. $\Omega_{\lambda\nu}$ coincides with thermal vorticity, $\Omega_{\lambda\nu} = \varpi_{\lambda\nu} =$ constant

Equivalence in global equilibrium!

Polarized neutral system - Physical meaning of $\Omega_{\mu\nu}$



- a) Fluid at rest with constant temperature with particles and antiparticles polarized in the same direction $\beta^\mu = (1/T)(1, \mathbf{0}) \Rightarrow \varpi = \frac{1}{2}(\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) = 0$; b) Polarized system with rotation
- Belinfante's pseudo-gauge does not imply that polarization vanishes, but rather it is locked to thermal vorticity
- Spin tensor and spin potential to describe hydro evolution, but only if spin density relaxes "slowly" compared to the microscopic interaction scale
- In general, away from equilibrium $\Omega_{\lambda\nu} \neq \varpi_{\lambda\nu}$
- It is crucial to calculate spin relaxation times see e.g. J. Kapusta, E. Rrapaj, S. Rudaz PRC 101, 024907; S. Li, H. U. Yee PRD 100, 056022; D.L. Yang, K. Hattori, Y. Hidaka, JHEP 07 (2020) 070

Spin hydrodynamics

- Hydrodynamic densities

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \qquad S^{\lambda,\mu\nu} = \langle \hat{S}^{\lambda,\mu\nu} \rangle,$$

- 10 equations of motion 4 usual hydro + 6 due to total angular momentum conservation

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- 10 unknowns: 4 + 6 additional independent fields (spin potential)

$$\beta^\mu \qquad \Omega^{\mu\nu}$$

Plus dissipative quantities

Wigner function

$$W(x,p) = \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \langle : \bar{\Psi}(x+\frac{y}{2})\Psi(x-\frac{y}{2}) : \rangle$$

- Dirac equation $\Longrightarrow$ Equation of motion for Wigner function
  H.-Th. Elze, M. Gyulassy, and D. Vasak, Ann. Phys. 173 (1987) 462
  de Groot, van Leeuwen, van Weert, Relativistic Kinetic Theory. Principles and Applications

$$\left[\gamma \cdot \left(p + i\frac{\hbar}{2}\partial\right) - m\right] W(x,p) = \hbar\mathcal{C} = \hbar \int \frac{d^4 y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} p\cdot y} \left\langle : \mathcal{J}(x-\frac{y}{2})\bar{\psi}(x+\frac{y}{2}) : \right\rangle$$

$\mathcal{J} = -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, $\mathcal{L}_I$ = interaction Lagrangian

- $\Longrightarrow$ Boltzmann equation and on-shell modification

$$p\cdot\partial W(x,p) = C\ , \qquad \left(p^2 - m^2 - \frac{\hbar^2}{4}\partial^2\right) W(x,p) = \hbar\delta M$$

- Idea: Find approximate solution by expanding in powers of $\hbar$
  and truncate at first order

$$W = W^{(0)} + \hbar W^{(1)} + \mathcal{O}(\hbar^2)$$

Calculating the Wigner function

- Clifford decomposition

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)$$

- Determine $\mathcal{V}^\mu$ and $\mathcal{A}^\mu$ from equations of motion
- Assumption: polarization effects at least $\mathcal{O}(\hbar)$

$$\mathcal{V}^\mu = \frac{1}{m}p^\mu\bar{\mathcal{F}} + \mathcal{O}(\hbar^2), \qquad \bar{\mathcal{F}} \equiv \mathcal{F} - \frac{\hbar}{m^2}p^\mu \mathrm{ReTr}(\gamma_\mu\mathcal{C})$$

- Transport equations:

$$p\cdot\partial\bar{\mathcal{F}} = m\,C_F, \qquad p\cdot\partial\mathcal{A}^\mu = m\,C_A^\mu$$

with $C_F = 2\mathrm{Im}\,\mathrm{Tr}(\mathcal{C})$, $C_A^\mu \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu \mathrm{Im}\,\mathrm{Tr}(\sigma_{\alpha\beta}\mathcal{C})$

Spin in phase space

- It is convenient to introduce new phase-space variable $\mathfrak{s}^\mu$
  J. Zamanian, M. Marklund, and G. Brodin, NJP 12, 043019 (2010)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)
  N. Weickgenannt, E.S., X.-l. Sheng, Q. Wang, and D. H. Rischke, arXiv:2005.01506 (2020)

$$\mathfrak{f}(x,p,\mathfrak{s}) \equiv \frac{1}{2}\left[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)\right] .$$

  Such that

$$\bar{\mathcal{F}} = \int dS(p)\, \mathfrak{f}(x,p,\mathfrak{s}) , \quad \mathcal{A}^\mu = \int dS(p)\, \mathfrak{s}^\mu \mathfrak{f}(x,p,\mathfrak{s})$$

  with $dS(p) \equiv \frac{\sqrt{p^2}}{\sqrt{3}\pi} d^4\mathfrak{s}\, \delta(\mathfrak{s}^2+3)\delta(p\cdot\mathfrak{s})$.
- All dynamics in the scalar Boltzmann equation

$$p\cdot\partial\, \mathfrak{f}(x,p,\mathfrak{s}) = m\, \mathfrak{C}[\mathfrak{f}] ,$$

  with $\mathfrak{C}[\mathfrak{f}] \equiv \frac{1}{2}(C_F - \mathfrak{s}\cdot C_A)$
- We parametrize: $\mathfrak{f}(x,p,\mathfrak{s}) = m\delta(p^2 - m^2 - \hbar\delta m^2) f(x,p,\mathfrak{s})$
- $\mathfrak{C}[\mathfrak{f}]$ is calculated including **nonlocal** collisions
  (See talks by Nora Weickgenannt and Xin-li Sheng)

Condition for $\mathfrak{C}[\mathfrak{f}] = 0 \Longrightarrow$ **Global equilibrium**

Related works: Yang, Hattori, Hidaka, JHEP (2020), Wang, Guo, Zhuang, 2009.10930 (2020)

Spin hydrodynamics with canonical currents

- Canonical currents

$$T_C^{\mu\nu} = \int d^4p\, p^\nu \mathcal{V}^\mu \qquad S_C^{\lambda,\mu\nu} = -\frac{\hbar}{2}\epsilon^{\lambda\mu\nu\rho}\int d^4p\, \mathcal{A}_\rho$$

- From kinetic theory

$$T_C^{\mu\nu} = \int dP\, dS\, p^\mu p^\nu f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)$$

$$S_C^{\lambda,\mu\nu} = \hbar\frac{m^2}{2}\int dP\, dS\, \frac{1}{p^2}\left(p^\lambda \Sigma_{\mathfrak{s}}^{\mu\nu} + p^\mu \Sigma_{\mathfrak{s}}^{\nu\lambda} + p^\nu \Sigma_{\mathfrak{s}}^{\lambda\mu}\right) f(x,p,\mathfrak{s})$$

$\Sigma_{\mathfrak{s}}^{\mu\nu} \equiv -(1/m)\epsilon^{\mu\nu\alpha\beta}p_\alpha \mathfrak{s}_\beta$ and $dP = d^4p\,\delta(p^2 - m^2 - \hbar\delta m^2)$

- Equations of motion

$$\partial_\mu T_C^{\mu\nu} = \int dP dS(p)\, p^\nu\, \mathfrak{C}[f] = 0$$

Relation to collisional invariant p^μ

$$\partial_\lambda S_C^{\lambda,\mu\nu} = \int dP dS(p)\, \frac{\hbar}{2}\left(\Sigma_{\mathfrak{s}}^{\mu\nu}\,\mathfrak{C}[f] + p^{[\mu}\Sigma_{\mathfrak{s}}^{\nu]\lambda}\partial_\lambda f(x,p,\mathfrak{s})\right) = T_C^{[\nu\mu]}$$

Relation to collisional invariant for the spin tensor not apparent

Global equilibrium $\Longrightarrow \partial_\lambda S_C^{\lambda,\mu\nu} \neq 0$
$\Longrightarrow$ Orbital-to-spin conversion should stop at global equilibrium

HW currents and interactions

- Modification of pseudo-gauge transformations transformations for interacting fields ($\rho = -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$, $\mathcal{L}_I$ = interacting Dirac Lagrangian)

$$\Phi^{\lambda,\mu\nu}_{HW} = M^{[\mu\nu]\lambda} - g^{\lambda[\mu} M_\rho{}^{\nu]\rho} + Q^{\lambda\mu\nu}$$
$$Z^{\mu\nu\lambda\rho}_{HW} = -\frac{\hbar^2}{8m}\langle\bar{\psi}(\sigma^{\mu\nu}\sigma^{\lambda\rho} + \sigma^{\lambda\rho}\sigma^{\mu\nu})\psi\rangle$$

with

$$M^{\lambda\mu\nu} \equiv \frac{i\hbar^2}{4m}\langle\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi\rangle \qquad Q^{\lambda\mu\nu} \equiv -\frac{\hbar^2}{4m}\langle\bar{\mathcal{J}}\gamma^\lambda\sigma^{\mu\nu}\psi\rangle - \frac{\hbar^2}{4m}\langle\bar{\psi}\sigma^{\mu\nu}\gamma^\lambda\mathcal{J}\rangle$$

- HW Currents (exact at any order in $\hbar$)

$$T^{\mu\nu}_{HW} = \frac{1}{m}\int d^4p\left[p^\nu\left(p^\mu\mathcal{F} - \hbar D^\mu_{\mathcal{V}}\right) + \frac{\hbar^2}{4}\partial^\nu\partial^\mu\mathcal{F} - \frac{\hbar^2}{4}g^{\mu\nu}\partial^2\mathcal{F} + \frac{\hbar^2}{4}\epsilon^{\lambda\mu\nu\alpha}\partial_\lambda D_{\mathcal{A}\alpha}\right]$$
$$S^{\lambda,\mu\nu}_{HW} = \frac{\hbar}{2m}\int d^4p\, p^\lambda\mathcal{S}^{\mu\nu}$$
$$D^\mu_{\mathcal{V}} = \mathrm{Re\,Tr}\left(\gamma^\mu\mathcal{C}\right),\ D^\mu_{\mathcal{A}} = \mathrm{Re\,Tr}\left(\gamma^\mu\gamma_5\mathcal{C}\right)$$

$T^{[\nu\mu]}_{HW} = \partial_\lambda S^{\lambda,\mu\nu}_{HW} \neq 0$ only in presence of interaction

Spin hydrodynamics with HW currents

- From kinetic theory

$$T^{\mu\nu}_{\rm HW} = \int dP\, dS(p)\, p^\mu p^\nu f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)$$

$$S^{\lambda,\mu\nu}_{\rm HW} = \hbar \int dP\, dS(p) p^\lambda \left(\frac{1}{2}\Sigma^{\mu\nu}_{\mathfrak{s}} - \frac{\hbar}{4m^2} p^{[\mu}\partial^{\nu]}\right) f(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2)$$

- Equations of motion

$$\partial_\mu T^{\mu\nu}_{\rm HW} = \int d\Gamma\, p^\nu\, \mathfrak{C}[\mathfrak{f}] = 0$$

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm HW} = \int d\Gamma\, \frac{\hbar}{2} \Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[\mathfrak{f}] = T^{[\nu\mu]}_{\rm HW}$$

- Energy-momentum conserved in a collision

Spin not conserved in nonlocal collisions $\Longrightarrow$ $T^{[\nu\mu]}_{\rm HW} \neq 0$
$\Longrightarrow$ Conversion between spin and orbital angular momentum

- $T^{[\nu\mu]}_{\rm HW} = 0$: (i) for local collisions (spin is collisional invariant)
  (ii) in **global equilibrium** ($\mathfrak{C}[\mathfrak{f}] = 0$)
- Nonlocal collisions away from global equilibrium $\Longrightarrow$ Dissipative dynamics

Unpolarized fluid gets polarized through rotation

Nonrelativistic limit

- $p^\mu \to m(1, \boldsymbol{v})$, $\Sigma_{\mathfrak{s}}^{\mu\nu} \to \epsilon^{ijk}\mathfrak{s}^k$

$$T^{[ji]}_{\mathrm{HW}} = m\epsilon^{ijk}\partial^0 \left\langle \frac{\hbar}{2}\mathfrak{s}^k \right\rangle + m\epsilon^{ijk}\partial^l \left\langle v^l \frac{\hbar}{2}\mathfrak{s}^k \right\rangle$$

with $\langle ... \rangle \equiv (m^2/2\pi\sqrt{3}) \int d^3v\, d^3\mathfrak{s}\, \delta(\mathfrak{s}^2 - 3)\,(...)f$

- Agreement with phenomenological result of nonrelativistic kinetic theory.

  S. Hess and L. Waldmann, Zeitschrift für Naturforschung A 26, 1057 (1971)

- Comparison with micropolar fluids - Viscous fluids made of particle with internal angular momentum with mass density ϱ and velocity u^i

  G. Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\varrho \left(\partial^0 + u^j \partial^j\right) \ell^i = \partial^j C^{ji} + \epsilon^{ijk} T^{jk}$$

$\Longrightarrow$ Internal angular momentum $\varrho\,\ell^i = m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle$,

$\Longrightarrow$ Couple stress tensor $C^{ji} = -\left\langle \frac{\hbar}{2}\mathfrak{s}^i p^j \right\rangle + m\left\langle \frac{\hbar}{2}\mathfrak{s}^i \right\rangle u^j$.

- Change of internal angular momentum due to C^{ji} and $\epsilon^{ijk} T^{jk}$
- Many applications in condensed matter e.g. spintronics and chiral active fluids

  R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okayasu, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nature Physics 12, 52 (2016)

  D. Banerjee, A. Souslov, A. G. Abanov, and V. Vitelli, Nature communications 8, 1 (2017)

Discussion on polarization observable

- Pauli-Lubanski vector for particle with momentum p^μ
  F. Becattini, 2004.04050; E.S., N. Weickgenannt, 2007.00138; L. Tinti, W. Florkowski, 2007.04029

$$\hat{\Pi}^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \hat{j}_{\alpha\beta}(p)$$
$$= -\frac{\hbar}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu \int d\Sigma^\lambda \, p_\lambda \hat{\mathcal{S}}_{\alpha\beta}(x,p) = \frac{\hbar}{2m}\int d\Sigma_\lambda \, p^\lambda \hat{\mathcal{A}}^\mu(x,p)$$

Total angular momentum $\hat{J}^{\mu\nu} = \int d^4p \, \hat{j}^{\mu\nu}(p)$

Operator $\hat{\Pi}(p)$ is pseudo-gauge invariant by definition

However, thermal average $\langle\hat{\Pi}(p)\rangle$ is pseudo-gauge dependent!

- Equivalence restored in **global equilibrium**
- Polarization in heavy-ion collisions $\Longrightarrow$ Preferred pseudo-gauge?
- What is the effect of spin hydrodynamic evolution on $\langle\hat{\Pi}(p)\rangle$?

More about pseudo-gauges

E.S., N. Weickgenannt, 2007.00138 (2020)

- Ambiguity in the definition of relativistic center of inertia q^μ
  $\Longrightarrow$ Redefinition of global spin $\Longrightarrow$ Pseudo-gauge transformations

$$J^{\mu\nu} = q^{[\mu} P^{\nu]} + S^{\mu\nu}$$

  - Massless case: q^μ cannot be a covariant vector $\Longrightarrow$ Side-jumps

    Chen, Son, Stephanov PRL (2015); Stone, Dwivedi, Zhou PRL (2015)

  - Massive case: covariant definition of q^μ possible (at least without nonlocal collisions) $\Longrightarrow$ $S^{\mu\nu} = S^{\mu\nu}_{HW}$

- Spin tensor in conventional general relativity and Einstein-Cartan theory: **Belinfante** VS **Canonical**

Conclusions

Summary

- Thermal expectation value of operators depends in general on pseudo-gauge
  - Equivalence in **global** equilibrium
- Spin hydrodynamics with Hilgevoord-Wouthuysen pseudo-gauge
  - **Antisymmetric** part of energy-momentum tensor
    $\Longrightarrow$ Orbital-to-spin angular momentum conversion
    $\Longrightarrow$ **Vanishes** with **local collisions** or in **global equilibrium**

Outlook

- Derive second-order dissipative spin hydrodynamics
- Study nonequilibrium effects for polarization vector
- Possible explanation for "sign problem"?