Quantum relaxation operator of spin density matrix in Perturbative QCD

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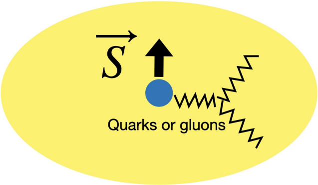
Reference: Phys.Rev. D100 (2019) no.5, 056022
Shiyong Li and HUY

Related Works

Kapusta-Rrapaj-Rudaz, Phys.Rev.C 101 (2020) 2, 024907
Yang-Hattori-Hidaka, JHEP 07 (2020) 070

How spins of quarks and gluons evolve in QGP in weakly coupled limit of QCD?



As a start, we consider a simple case of massive quark

N.B. $\Lambda = (uds)$ **spin mostly arises from the spin of the s-quark**

Why density matrix for spin $1/2$?

Two spin states are almost degenerate
$\Delta E \sim \boldsymbol{S}\cdot\omega + \boldsymbol{S}\cdot\boldsymbol{B} \sim \hbar$**.**
Quantum correlation time is classical

$$\tau_q \sim \frac{\hbar}{\Delta E} \sim \mathcal{O}(\hbar^0)$$

We need to keep 2×2 spin density matrix in the kinetic theory of time evolution in the classical domain $\Delta t \sim \mathcal{O}(\hbar^0)$.

quantum kinetic theory

We expect a Lindblad-type of kinetic equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H_0,\hat{\rho}] - L\hat{\rho}L^\dagger + \frac{1}{2}L^\dagger L\hat{\rho} + \frac{1}{2}\hat{\rho}L^\dagger L$$
$$= -\frac{i}{\hbar}[H_0,\hat{\rho}] - \Gamma\cdot\hat{\rho}$$

The first term contains free streaming advective flow and background EM field, and has been worked out in Refs: **Gao-Liang, Weickgenannt-Sheng-Speranza-Wang-Rischke, Hattori-Hidaka-Yang**

The $\Gamma\cdot\hat{\rho}$ is the relaxational "collision" operator, that we aim to construct in perturbative QCD framework to leading log in QCD coupling constant g

We consider the case of dilute (Boltzmann), massive quarks (strange, bottom), interacting with the background thermal QGP

1-particle quantum mechanics of a single quark moving in QGP (Improvement: Yang-Hattori-Hidaka)

Further simplification: Spatial homogeneity limit or "local collision" limit

$$\Gamma = \Gamma_0 + \mathcal{O}(\hbar\partial_x/T), \quad \omega \sim \partial_x u$$

(Improvement: Liu-Sun-Ko, Weickgenannt-Speranza-Sheng-Wang-Rischke)

See the talks by Yifeng Sun and Qun Wang

The equilibrium density matrix is expected to be Boltzmann $\hat{\rho}_{\rm eq} = ze^{-\beta E_p}\mathbf{1}_{2\times 2}$, where $E_p = \sqrt{p^2+m^2}$.

$$\Gamma_0 \cdot \hat{\rho}_{\rm eq} = 0$$

$\Gamma_0 \sim g^4 \log(1/g) T$ gives the relaxation of spin polarization to equilibrium

Spin density matrix

(Becattini-Chandra-Del Zanna-Grossi,

Florkowski-Friman-Jaiswal-Ryblewski-Speranza)

(See Leonardo Tinti's talk)

$$\hat{\rho} = \sum_{s,s'} \rho_{s,s'} |s >< s'| \tag{1}$$

In the s_z basis, $\hat{\rho}$ becomes a 2×2 matrix, and the (non-relativistic) spin expectation value is

$$< \vec{S} >= \frac{\hbar}{2} \mathrm{Tr}(\vec{\sigma}\hat{\rho}) \tag{2}$$

N.B. The relativistic "canonical" spin is

$$\langle \boldsymbol{S}^i \rangle_{\rm relativistic} = \frac{\hbar}{8} \epsilon^{ijk} \bar{\psi} \gamma^0 [\gamma^j, \gamma^k] \psi = K^{ij} \langle \boldsymbol{S}^j \rangle \tag{3}$$

where $K^{ij} = \frac{1}{4E_p} \mathrm{Tr}(\sqrt{p \cdot \sigma}\sigma^i \sqrt{p \cdot \sigma}\sigma^j + \sqrt{p \cdot \bar{\sigma}}\sigma^i \sqrt{p \cdot \bar{\sigma}}\sigma^j)$

(D.-L Yang) Other spin vectors can be computed as well

Position and momentum

Density matrix in the momentum basis

$$\hat{\rho} = \int_{\boldsymbol{p}_1} \int_{\boldsymbol{p}_2} \rho_{s,s'}(\boldsymbol{p}_1, \boldsymbol{p}_2) |\boldsymbol{p}_1, s\rangle \langle \boldsymbol{p}_2, s'| \tag{4}$$

Going to the position basis,

$$\hat{\rho} = \int_{\boldsymbol{p}_1} \int_{\boldsymbol{p}_2} \int_{\boldsymbol{x}_1} \int_{\boldsymbol{x}_2} e^{i(\boldsymbol{p}_1 \boldsymbol{x}_1 - \boldsymbol{p}_2 \boldsymbol{x}_2)} \rho_{s,s'}(\boldsymbol{p}_1, \boldsymbol{p}_2) |\boldsymbol{x}_1, s\rangle \langle \boldsymbol{x}_2, s'| \tag{5}$$

Write $\boldsymbol{p}_1 \boldsymbol{x}_1 - \boldsymbol{p}_2 \boldsymbol{x}_2 = \boldsymbol{p}_a \boldsymbol{x}_r + \boldsymbol{p}_r \boldsymbol{x}_a$ where

$$\boldsymbol{x}_r = \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)\,, \boldsymbol{p}_r = \frac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\,, \boldsymbol{x}_a = \boldsymbol{x}_1 - \boldsymbol{x}_2\,, \boldsymbol{p}_a = \boldsymbol{p}_1 - \boldsymbol{p}_2$$

i.e. $[x_r, p_a] = i\hbar$ and $[x_a, p_r] = i\hbar$, and (x_r, p_r) are simultaneously diagonalizable

We map $\langle p_2|$ to $|p_2\rangle^* \in \mathcal{H}^*$ in the conjugate space, i.e., the time-reversed (T) space (Thermo-Field Theory). The $\mathcal{H}$ and $\mathcal{H}^*$ are naturally described by the Schwinger-Keldysh contours, labeled by 1 and 2 respectively,

$$[x_1, p_1] = i\hbar, \quad [x_2, p_2] = -i\hbar$$

from which we have

$$[x_r, p_a] = i\hbar, \quad [x_a, p_r] = i\hbar, \quad [x_r, p_r] = 0$$

This means we can have the basis of simultaneous eigenstates of (x_r, p_r) in $\mathcal{H} \otimes \mathcal{H}^*$, and we introduce the Wigner function $\rho(\boldsymbol{x}_r, \boldsymbol{p}_r)$,

$$\hat{\rho} = \int_{\boldsymbol{x}_r} \int_{\boldsymbol{p}_r} \rho(\boldsymbol{x}_r, \boldsymbol{p}_r) |\boldsymbol{x}_r, \boldsymbol{p}_r\rangle$$

Since $[x_r, p_a] = i\hbar$, they are conjugate to each other

$$\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r) = \int_{\boldsymbol{p}_a} e^{i\boldsymbol{x}_r\cdot\boldsymbol{p}_a}\hat{\rho}(\boldsymbol{p}_1, \boldsymbol{p}_2)$$

We consider the cases where $\frac{\partial}{\partial x_r} \sim p_a \ll p_r \sim T$, that is, x_r-gradients can be neglected $\to$ homogeneous limit

This means that the density matrix in momentum space is approximately diagonal, $\boldsymbol{p}_a = 0$,

$$\hat{\rho} \;=\; \int_{\boldsymbol{p}} \rho_{s,s'}(\boldsymbol{p})|\boldsymbol{p}, s\rangle\langle\boldsymbol{p}, s'|$$

Relativistic massive fermion

Field quantization

$$\psi(\boldsymbol{x}) = \int_{\boldsymbol{p}} \frac{1}{\sqrt{2E_p}} \sum_s u(p,s) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p},s} + \text{anti quark}$$

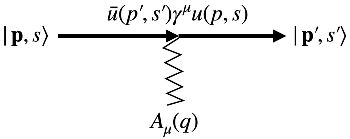
1-quark states: $|\boldsymbol{p}, s\rangle = a^\dagger_{\boldsymbol{p},s}|0\rangle$, $[a, a^\dagger] = 1$
It is most convenient to use the helicity basis,

$$u(\boldsymbol{p}, s) \sim \begin{pmatrix} \sqrt{E_p - sp}\,\xi_s(\boldsymbol{p}) \\ \sqrt{E_p + sp}\,\xi_s(\boldsymbol{p}) \end{pmatrix}, \quad (\boldsymbol{\sigma}\cdot\boldsymbol{p})\xi_s(\boldsymbol{p}) = s|\boldsymbol{p}|\xi_s(\boldsymbol{p})$$

For 1-quark states, this is the usual QM of spin $1/2$ particle, where $|\boldsymbol{p}, s\rangle \sim \xi_s(\boldsymbol{p})$. The only modification from relativity is that the interaction matrix elements are given by the overlap of spinors $\psi^*(x)$, i.e. $u(\boldsymbol{p}, s)$

The interaction vertices with the background gluon fields A_μ^a, through the relativistic spinors $u(\boldsymbol{p}, s)$

$$
\begin{aligned}
H_I(t) &= g \int d\boldsymbol{x}\, \bar{\psi}(\boldsymbol{x}) \gamma^\mu t^a \psi(\boldsymbol{x}) A_\mu^a(\boldsymbol{x}, t) \\
&\sim \int_{\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{q}} \bar{u}(\boldsymbol{p}', s') \gamma^\mu u(\boldsymbol{p}, s) A_\mu(\boldsymbol{q}) a^\dagger_{\boldsymbol{p}', s'} a_{\boldsymbol{p}, s} \delta_{\boldsymbol{p}' - \boldsymbol{p} - \boldsymbol{q}}
\end{aligned}
$$



The amplitudes $\bar{u}(\boldsymbol{p}', s') \gamma^\mu u(\boldsymbol{p}, s) A_\mu(\boldsymbol{q})$ give the matrix elements of H_I in the QM of 1-quark Hilbert space of spin 1/2.

Relation to the field theory Wigner function

$\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r)$ **is the Wigner transform of** $a_s(\boldsymbol{x}) = \int_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p},s}$

$$\hat{\rho}_{2\times 2}(\boldsymbol{x}_r, \boldsymbol{p}_r) = \int_{\boldsymbol{x}_a} e^{-i\boldsymbol{x}_a\cdot\boldsymbol{p}_r} \langle a_s(\boldsymbol{x}_r - \boldsymbol{x}_a/2) a^\dagger_{s'}(\boldsymbol{x}_r + \boldsymbol{x}_a/2)\rangle_{\hat{\rho}}$$

Recall $\psi(\boldsymbol{x}) = \int_{\boldsymbol{p}} \frac{1}{\sqrt{2E_p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} u(\boldsymbol{p}) a_{\boldsymbol{p}}$**, so** $\psi(\boldsymbol{x})$ **and** $a(\boldsymbol{x})$ **are non-locally related, and** $\hat{\rho}(\boldsymbol{x}_r, \boldsymbol{p}_r)$ **is not equal to the Wigner transform of** $\psi(\boldsymbol{x})$ **field. However, for the spatially homogeneous case, they are related by**

$$W_{\alpha\beta}(\boldsymbol{p}) = \int d\boldsymbol{x}_a \langle \psi_\alpha(x - x_a/2) \psi^\dagger_\beta(x + x_a/2)\rangle e^{i\boldsymbol{p}\cdot\boldsymbol{x}_a}$$

$$= \sum_{s,s'} \frac{1}{2E_p} u_\alpha(\boldsymbol{p}, s) u^\dagger_\beta(\boldsymbol{p}, s') \rho_{s,s'}(\boldsymbol{p})$$

The explicit 2×2 spin density matrix in momentum space in the helicity basis

$$\hat{\rho}_{2\times 2}(\boldsymbol{p}) = \sum_{s,s'} \xi_{\boldsymbol{p},s}\rho_{s,s'}(\boldsymbol{p})\xi^{\dagger}_{\boldsymbol{p},s'} \equiv \frac{1}{2}f(\boldsymbol{p}) + \boldsymbol{\sigma}\cdot\boldsymbol{S}(\boldsymbol{p})$$

and

$$\langle \boldsymbol{S}(\boldsymbol{p})\rangle = \frac{\hbar}{2}\mathrm{Tr}(\boldsymbol{\sigma}\hat{\rho}(\boldsymbol{p})) = \hbar\boldsymbol{S}(\boldsymbol{p})$$

This object is unambiguous under a phase redefinition of $\xi_{\boldsymbol{p},s}$, since $\hat{\rho}_{2\times 2}(\boldsymbol{p})$ is physical. Note that $u(\boldsymbol{p}, s)$ and $\xi_{\boldsymbol{p},s}$ share the same phase.

We are going to derive the evolution equation for this physical object

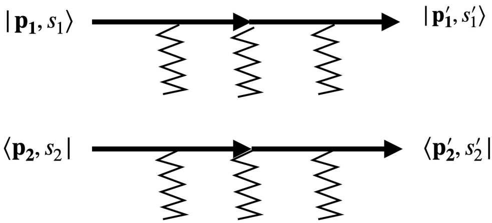
N.B. : The spin-traced object $f(\boldsymbol{p}) = \mathrm{Tr}(\hat{\rho}(\boldsymbol{p}))$ is the usual number distribution in momentum space

Time evolution of density matrix

$$\hat{\rho}(t+\Delta t) = U_1(\Delta t)\hat{\rho}(t)U_2^\dagger(\Delta t)$$

where $U_{1,2}$ are unitary evolution operators with QCD gluons in Schwinger-Keldysh contours 1 or 2, with the Hamiltonians

$$H_{1,2} = H_{\rm kinetic} + g\int d\boldsymbol{x}\bar{\psi}(\boldsymbol{x})\gamma^\mu\psi(\boldsymbol{x})A_\mu^{(1)/(2)}$$



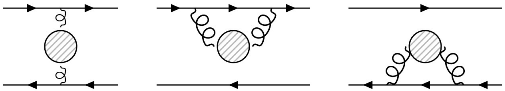
Time evolution of density matrix

$A_\mu^{(i)}$ **are the gluon fields on the Schwinger-Keldysh contour** $i = 1, 2$**. Note** $U_1 \neq U_2$**, and** $H_{1,2}$ **are time-dependent due to time-dependent gluon fields.**

We average over quantum/thermal fluctuating Schwinger-Keldysh gluon fields $A_\mu^{(i)}$**, given by equilibrium two-point functions of** $\langle A_\mu^{(i)}(\boldsymbol{p}) A_\nu^{(j)}(\boldsymbol{p})\rangle = G_{\mu\nu}^{(ij)}(\boldsymbol{p})$**, satisfying the thermal KMS relations** $G^{(12)}(q^0) = n_B(q^0)/(n_B(q^0)+1)G^{(21)}(q^0)$

N.B: The 1-point average is zero, $\langle A_\mu^{(i)}\rangle = 0$**, and we need to do the second order perturbation theory in the interaction** $H_I \propto A_\mu$

Do the second order perturbation theory in the interaction picture

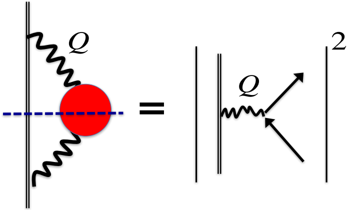


(a) (b) (c)

$$
\begin{aligned}
\hat{\rho}(\Delta t) &= U_0(\Delta t)\hat{\rho}(0)U_0^\dagger(\Delta t) \\
&+ \int_0^{\Delta t} dt_1 \int_0^{\Delta t} dt_2 U_0(\Delta t)\langle H_I^{\mathrm{int}(1)}(t_1)\hat{\rho}(0)H_I^{\mathrm{int}(2)}(t_2)\rangle_A U_0^\dagger(\Delta t) \\
&+ (-i)^2 U_0(\Delta t)\int_0^{\Delta t} dt_1 \int_0^{t_1} dt_1' \langle H_I^{\mathrm{int}(1)}(t_1)H_I^{\mathrm{int}(1)}(t_1')\rangle_A \hat{\rho}(0)U_0^\dagger(\Delta t) \\
&+ (+i)^2 U_0(\Delta t)\hat{\rho}(0)\int_0^{\Delta t} dt_2 \int_0^{t_2} dt_2' \langle H_I^{\mathrm{int}(2)}(t_2')H_I^{\mathrm{int}(2)}(t_2)\rangle_A U_0^\dagger(\Delta t)
\end{aligned}
$$

N.B. Compare this with the Lindblad form

The thermal average $\langle H_I^{(i)} H_I^{(j)} \rangle$ contains the gluon two-point functions $G_{\mu\nu}^{(ij)}(Q)$ that include the Hard Thermal Loop (HTL) self-energy. These contributions represent interactions with background hard thermal particles with t-channel gluon exchange.



$G^{12}(q^0) = n_B(q^0)\rho(q^0)$, **where** $\rho(q^0)$ **is the HTL spectral density**

$G^{(ij)}(t,t')$ have correlation time of $\tau_c \sim (gT)^{-1}$ because the leading log contribution comes from soft t-channel momentum exchange $gT \ll q \ll T$
When $\Delta t \gg \tau_c$ (but $\Delta t \ll 1/\Gamma \sim 1/(g^4 \log(1/g)T)$ to neglect multi-interactions within Δt), we have linear terms in Δt

$$\int_0^{\Delta t} dt \int_0^{\Delta t} dt'\, G^{(ij)}_{\mu\nu}(t-t')e^{i\omega(t-t')} \sim G^{(ij)}(\omega)\Delta t + \cdots$$

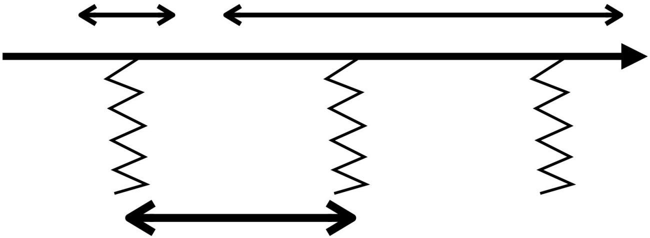
that gives the evolution equation first order in time.

N.B. In diagrammatic language, this corresponds to the ladder approximation, which is justified because of the scale separation
$$\tau_c \sim 1/gT \ll 1/\Gamma \sim 1/g^4 \log(1/g)T$$

Times scales of perturbative QCD in finite T plasma

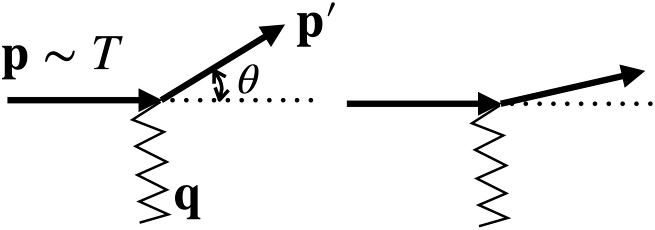
$$\tau_c = \frac{1}{gT}$$

$$\Delta t \sim \tau_q = \frac{\hbar}{\Delta E} \gg \tau_c$$



$$l_{mfp} = \frac{1}{g^4 \log(1/g) T} = \frac{1}{\Gamma}$$

Small angles versus large angles



$g < \theta \lesssim 1$

$(gT < \mathbf{q} \lesssim T)$

$g^2 < \theta < g$

$(g^2T < \mathbf{q} \lesssim gT)$

Small angle scatterings ($\Gamma \sim g^2 \log(1/g)T$):
Total damping rates, Color transports

Large angle scatterings ($\Gamma \sim g^4 \log(1/g)T$):
Conserved charge transports, Spin transports

Why small angle scatterings are suppressed for spin transports?
(From discussions with K. Hattori and D.-L. Yang in Kyoto, Dec. 2019)

Spin flipping interactions are magnetic moment interactions $\mu \cdot \boldsymbol{B} \sim \boldsymbol{\sigma} \cdot \boldsymbol{B}$

The Gordon identity

$$\bar{u}(p')\gamma^{\mu}u(p) = \frac{1}{2m}\left((p+p')^{\mu} + i\sigma^{\mu\nu}(p'-p)_{\nu}\right) \tag{6}$$

The second part flips spin, and vanishes when $q = p' - p \to 0$

Explicitly

$$\frac{d}{dt}\rho_{s,s'}(\boldsymbol{p},t) = g^2 C_2(F)(\Gamma_{\rm cross} + \Gamma_{\rm self\,energy})$$

$$\Gamma_{\rm cross} = \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\frac{1}{4E_pE_{p'}}\sum_{s'',s'''}[\bar{u}(\boldsymbol{p},s)\gamma^\mu u(\boldsymbol{p}',s'')]\rho_{s'',s'''}(\boldsymbol{p}')[\bar{u}(\boldsymbol{p}',s''')\gamma^\nu u(\boldsymbol{p},s')]G^{(1}_{\mu\nu}$$

$$\Gamma_{\rm self\,energy} = -\gamma\,\hat{\rho}(\boldsymbol{p})$$

where

$$\gamma = \frac{1}{2}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{4E_pE_{p'}}\sum_{s,s''}[\bar{u}(\boldsymbol{p},s)\gamma^\mu u(\boldsymbol{p}',s'')][\bar{u}(\boldsymbol{p}',s'')\gamma^\nu u(\boldsymbol{p},s)]G^{(21)}_{\mu\nu}(E_p - E_{p'},\boldsymbol{q})$$

The spin sum is challenging, but can be done with great effort

Expanding in soft momentum exchange $q \sim gT \ll p \sim T$, i.e. $\theta \sim q/p \sim g \ll 1$ ("momentum diffusion approximation")

we need to compute typically

$$J_n^{L/T} = \int_{q^0_{\rm min}}^{q^0_{\rm max}} \frac{dq^0}{(2\pi)} (q^0)^{2n-1} \rho_{L/T}(q^0, q)$$

$j_0^L = \frac{p}{E_p}$	$j_0^T = \frac{\eta_p}{2}$
$j_{1/2}^L = -\frac{p}{2E_p}$	$j_{1/2}^T = -\frac{pE_p}{4m^2}$
$j_1^L = \frac{p^3}{3E_p^3}$	$j_1^T = \frac{\eta_p}{2} - \frac{p}{2E_p}$
$j_{3/2}^L = -\frac{2p^3}{E_p^3}$	$j_{3/2}^T = -\frac{p^3}{4m^2 E_p}$
$j_2^L = \frac{p^5}{5E_p^5}$	$j_2^T = \frac{\eta_p}{2} - \frac{p}{2E_p} - \frac{p^3}{6E_p^3}$

$$J_n^{L/T} = \frac{m_D^2}{q^{(4-2n)}} j_n^{L/T} \quad (n = \text{integer}), J_n^{L/T} = \frac{m_D^2}{q^{(3-2n)}} \frac{m^2}{E_p^3} j_n^{L/T} \; (n = \text{half integer})$$

Result

Write $\hat{\rho}(\boldsymbol{p}) = \frac{1}{2} f(\boldsymbol{p}) \mathbf{1}_{2\times 2} + \boldsymbol{\sigma} \cdot \boldsymbol{S}(\boldsymbol{p})$

$$\frac{\partial f(\boldsymbol{p},t)}{\partial t} = C_2(F) \frac{m_D^2 g^2 \log(1/g)}{(4\pi)} \frac{1}{2pE_p} \Gamma_f$$

$$\frac{\partial \boldsymbol{S}(\boldsymbol{p},t)}{\partial t} = C_2(F) \frac{m_D^2 g^2 \log(1/g)}{(4\pi)} \frac{1}{2pE_p} \Gamma_S$$

where

$$m_D^2 = \frac{g^2 T^2}{6}(2N_c + N_F)$$

$$\frac{\Gamma_f}{2pE_p} = \nabla_{p^i}\Big(T(\frac{3}{4}-\frac{E_p^2}{4p^2}+\frac{\eta_p m^4}{4p^3E_p})\nabla_{p^i}f(\boldsymbol{p})+\frac{\boldsymbol{p}^i}{2p^2}(E_p-\frac{\eta_p m^2}{p})f(\boldsymbol{p})$$

$$+\ \boldsymbol{p}^i\frac{Tm^2}{4p^3E_p}(\eta_p+\frac{3E_p}{p}-\frac{3\eta_p E_p^2}{p^2})\boldsymbol{p}\cdot\nabla_p f(\boldsymbol{p})\Big)$$

$$\boldsymbol{\Gamma}_S^i = \left(2p+\frac{TE_p}{p}-\frac{\eta_p m^2 T}{p^2}\right)\boldsymbol{S}^i(p)+\left(pTE_p-\frac{m^2TE_p}{2p}+\frac{\eta_p m^4 T}{2p^2}\right)\nabla_p^2\boldsymbol{S}^i(\boldsymbol{p})$$

$$+\ \left(\frac{\eta_p m^2 T}{2p^2}\left(1-\frac{3E_p^2}{p^2}\right)+\frac{3m^2TE_p}{2p^3}\right)(\boldsymbol{p}\cdot\nabla_p)^2\boldsymbol{S}^i(\boldsymbol{p})$$

$$+\ \frac{1}{p^2}\left(pE_p^2-\frac{3m^2TE_p}{2p}+\eta_p m^2\left(-E_p-\frac{T}{2}+\frac{3TE_p^2}{2p^2}\right)\right)(\boldsymbol{p}\cdot\nabla_p)\boldsymbol{S}^i(\boldsymbol{p})$$

$$+\ 2T\left(\eta_p\left(\frac{1}{2}-\frac{E_p^2}{p^2}+\frac{mE_p}{2p^2}+\frac{E_p^3}{2p^2(E_p+m)}\right)+\frac{E_p}{p}-\frac{m}{2p}-\frac{m^2}{2p(E_p+m)}\right)\boldsymbol{p}^i$$

$$-\ 2T\left(\eta_p\left(\frac{1}{2}-\frac{E_p^2}{p^2}+\frac{mE_p}{2p^2}+\frac{E_p^3}{2p^2(E_p+m)}\right)+\frac{E_p}{p}-\frac{m}{2p}-\frac{m^2}{2p(E_p+m)}\right)\nabla$$

$$-\ \frac{T}{p^2}\left(\frac{E_p(E_p+2m)}{p(E_p+m)}+\frac{\eta_p mE_p}{E_p+m}\left(-\frac{3E_p}{p^2}+\frac{1}{E_p+m}\right)\right)\boldsymbol{p}^i(\boldsymbol{p}\cdot\boldsymbol{S}(\boldsymbol{p}))$$

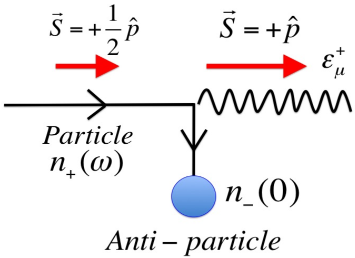
These results pass very non-trivial tests of
1) Detailed balance: $f(\boldsymbol{p}) = ze^{-E_p/T}$ **is equilibrium, that is,** $\Gamma_f = 0$ **for this**
2) Chirality in massless limit: When formally $m = 0$**, the density matrix factorizes as**

$$\hat{\rho}(\boldsymbol{p}) = f_+(\boldsymbol{p})\mathcal{P}_+(\boldsymbol{p}) + f_-(\boldsymbol{p})\mathcal{P}_-(\boldsymbol{p})$$

where $\mathcal{P}_\pm(\boldsymbol{p}) = \frac{1}{2}(\mathbf{1} \pm \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma})$ **are the chirality projection operators, and** $f_\pm(\boldsymbol{p})$ **satisfy the same equation in parity-even background. This means that it should admit the consistent Ansatz** $S(\boldsymbol{p}) = f_s(\boldsymbol{p})\hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}$**, and moreover** $f(\boldsymbol{p})$ **and** $f_s(\boldsymbol{p})$ **should satisfy the same evolution equation. Also,** $f_s(\boldsymbol{p}) = ze^{-|p|/T}$ **should be the equilibrium solution of** $\Gamma_S = 0$**.**
All these are true in the above result

Thank you very much !

We consider a hard scale quark mass $m \gg gT$. This justifies neglecting quark-gluon conversion process (Compton scattering) in our leading log computation, since the t-channel fermion exchange momentum becomes hard $q \gtrsim m \gg gT$, and leading log is absent



For light quarks, we need to consider both quark spins and gluon spins in leading log

It should be doable with fermion/boson statistics:

A nice future problem!