Spin, polarization and Wigner functions

Outline

- Introduction: a bridge between calculations and measurements
- Spin tensor as a polarization sensitive field
- The Wigner distribution as a generalization of the distribution function

CRC-TR 211
Strong-interaction matter under extreme conditions

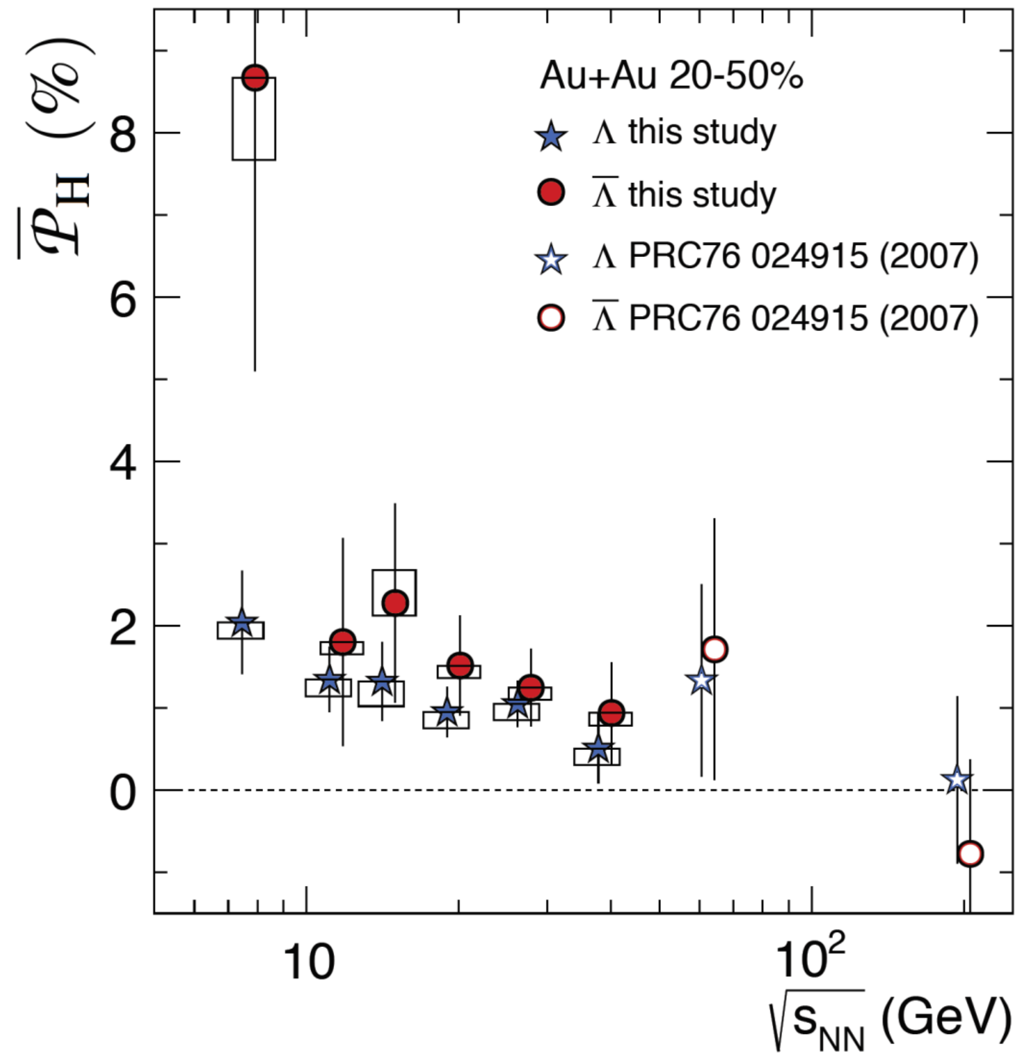
ECT*|SPIN AND HYDRODYNAMICS IN RELATIVISTIC NUCLEAR COLLISIONS *7 Oct. 2020*

Leonardo Tinti

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Motivations

From the STAR collaboration paper (2017) [arXiv:1701.06657]



Polarization measurements

- Qualitative relation between "rotation" and polarization (global equilibrium)
- Current understanding (polarization at local equilibrium)
- Possible improvements (spin degrees of freedom in hydrodynamics?)

Presentation based on
arXiv:2007.04029v2

Comparisons between theory and experiments

What we compute (e.g. hydrodynamics)

$$T^{\mu\nu}(x) = tr\left(\rho\, \hat{T}^{\mu\nu}(x)\right)$$

$$J_B^{\mu}(x) = tr\left(\rho\, \hat{J}_B^{\mu}(x)\right)$$

Tensor densities

What we measure

$$\frac{dN}{d^3p} \qquad \frac{d\bar{N}}{d^3p}$$

$$\langle \mathbf{\Pi} \rangle|_{\boldsymbol{p}} \qquad \langle \bar{\mathbf{\Pi}} \rangle|_{\boldsymbol{p}}$$

Spectra (momentum space)

It is important to translate from one picture to the other in the appropriate way

Relativistic kinetic theory and its limitations

The relativistic generalization of the Boltzmann equation

$$p \cdot \partial f(x,\boldsymbol{p}) = C[f,\bar{f}]$$
$$p \cdot \partial \bar{f}(x,\boldsymbol{p}) = \bar{C}[f,\bar{f}]$$

Well defined stress-energy tensor and barion current

$$T^{\mu\nu}(x) = \frac{g_S}{(2\pi)^3} \int \frac{d^3p}{E_{\boldsymbol{p}}}\, p^\mu p^\nu \left(f(x,\boldsymbol{p}) + \bar{f}(x,\boldsymbol{p})\right)$$

$$J_B^\mu(x) = \frac{g_S}{(2\pi)^3} \int \frac{d^3p}{E_{\boldsymbol{p}}}\, p^\mu \left(f(x,\boldsymbol{p}) - \bar{f}(x,\boldsymbol{p})\right)$$

Improper treatment of the spin through the degeneracy factor $g_S = 2S + 1$

Can we do better with quantum field theory?

Quantum field theory already embeds spin degrees of freedom

Polarization is included from the start in the conserved currents ($\widehat{T}^{\mu\nu}, J_B^{\mu}$)

$$\hat{J}^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$$

$$\Psi(x) = \sum_r \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left[U_r(\mathbf{p})a_r(\mathbf{p})e^{-ip\cdot x} + V_r(\mathbf{p})b_r^{\dagger}(\mathbf{p})e^{ip\cdot x}\right]$$

Can we do better with quantum field theory?

Unfortunately the space-time densities are rather complicated

$$J^\mu(x) = \langle : \hat{J}^\mu(x) : \rangle = \mathrm{tr}\left(\rho : \hat{J}^\mu(x) :\right) =$$

$$= \sum_{r,s} \int \frac{d^3p\, d^3p'}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{p}'}}} \Big[\langle a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') \rangle \bar{U}_r(\mathbf{p}) \gamma^\mu U_s(\mathbf{p}') e^{i(p-p')\cdot x}$$

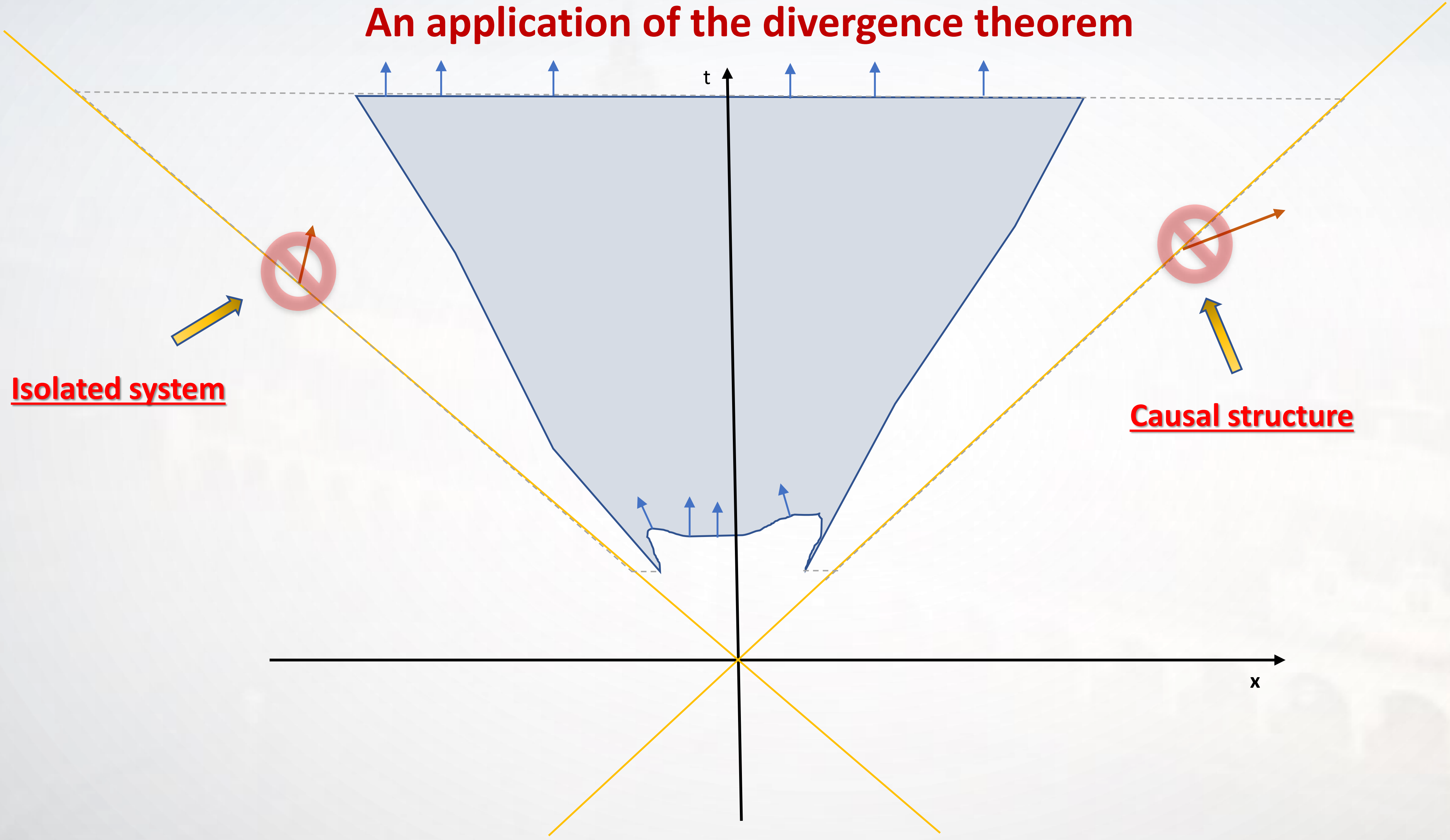
$$- \langle b_r^\dagger(\mathbf{p}) b_s(\mathbf{p}') \rangle \bar{V}_s(\mathbf{p}') \gamma^\mu V_r(\mathbf{p}) e^{i(p-p')\cdot x}$$

$$+ \langle a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \rangle \bar{U}_r(\mathbf{p}) \gamma^\mu V_s(\mathbf{p}') e^{i(p+p')\cdot x}$$

$$+ \langle b_r(\mathbf{p}) a_s(\mathbf{p}') \rangle \bar{V}_r(\mathbf{p}) \gamma^\mu U_s(\mathbf{p}') e^{-i(p+p')\cdot x} \Big]$$

Can we do better with quantum field theory?

An application of the divergence theorem



Can we do better with quantum field theory?

The conserved currents are better

$$\int d^3x \; J^0(x) = \int \frac{d^3p}{(2\pi)^3} \left[\sum_r \langle a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) \rangle - \sum_r \langle b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) \rangle \right]$$

$$\int d^3x \; T^{0\mu}(x) = \int \frac{d^3p}{(2\pi)^3} p^\mu \left[\sum_r \langle a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) \rangle + \sum_r \langle b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) \rangle \right]$$

Can we do better with quantum field theory?

$$\int d^3x\, J^0(x) = \int \frac{d^3p}{(2\pi)^3}\left[\sum_r \langle a_r^\dagger(\mathbf{p})a_r(\mathbf{p})\rangle - \sum_r \langle b_r^\dagger(\mathbf{p})b_r(\mathbf{p})\rangle\right]$$

$$\int d^3x\, T^{0\mu}(x) = \int \frac{d^3p}{(2\pi)^3}p^\mu\left[\sum_r \langle a_r^\dagger(\mathbf{p})a_r(\mathbf{p})\rangle + \sum_r \langle b_r^\dagger(\mathbf{p})b_r(\mathbf{p})\rangle\right]$$

We can recognize the invariant spectra:

$$N = \int \frac{d^3p}{(2\pi)^3}\sum_r \langle a_r^\dagger(\mathbf{p})a_r(\mathbf{p})\rangle$$

$$\bar{N} = \int \frac{d^3p}{(2\pi)^3}\sum_r \langle b_r^\dagger(\mathbf{p})b_r(\mathbf{p})\rangle$$

$$\begin{cases} E_{\boldsymbol{p}}\dfrac{dN}{d^3p} = \dfrac{1}{(2\pi)^3}\displaystyle\sum_r \langle a_r^\dagger(\boldsymbol{p})a_r(\boldsymbol{p})\rangle \\ E_{\boldsymbol{p}}\dfrac{d\bar{N}}{d^3p} = \dfrac{1}{(2\pi)^3}\displaystyle\sum_r \langle b_r^\dagger(\boldsymbol{p})b_r(\boldsymbol{p})\rangle \end{cases}$$

Can we do better with quantum field theory?

Indeed from the quantum wavefunction

$$\rho = \sum_i \mathsf{P}_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

Can we do better with quantum field theory?

Indeed from the quantum wavefunction

$$\rho = \sum_i \mathsf{P}_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

$$|\psi\rangle = \sum_{N,\bar{N}} \sum_{\underline{r},\underline{\bar{s}}} \int [d\underline{p}]^N [d\underline{\bar{q}}]^{\bar{N}} \alpha_{N,\bar{N}}(\underline{p},\underline{r};\underline{\bar{q}},\underline{\bar{s}}) \, |\underline{p},\underline{r};\underline{\bar{q}},\underline{\bar{s}}\rangle$$

Can we do better with quantum field theory?

Indeed from the quantum wavefunction

$$\rho = \sum_i \mathrm{P}_i \, |\psi_i\rangle \langle\psi_i|$$

$$\left(\int [d\underline{p}]^N [d\underline{\bar{q}}]^{\bar{N}} = \int \frac{d^3 p_1}{(2\pi)^3 2E_{\mathbf{p}_1}} \cdots \frac{d^3 p_N}{(2\pi)^3 2E_{\mathbf{p}_N}} \frac{d^3 \bar{q}_1}{(2\pi)^3 2E_{\bar{\mathbf{q}}_1}} \cdots \frac{d^3 \bar{q}_{\bar{N}}}{(2\pi)^3 2E_{\bar{\mathbf{q}}_{\bar{N}}}} \right)$$

$$|\psi\rangle = \sum_{N,\bar{N}} \sum_{\underline{r},\underline{\bar{s}}} \int [d\underline{p}]^N [d\underline{\bar{q}}]^{\bar{N}} \, \alpha_{N,\bar{N}}(\underline{p}, \underline{r}; \underline{\bar{q}}, \underline{\bar{s}}) \, |\underline{p}, \underline{r}; \underline{\bar{q}}, \underline{\bar{s}}\rangle$$

Can we do better with quantum field theory?

one has

$$\mathrm{tr}\left(|\psi\rangle\langle\psi|\, a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}')\right) = 0 +$$

$$+\frac{1}{\sqrt{2E_\mathbf{p} 2E_{\mathbf{p}'}}} \sum_{\bar{N},N>0} \sum_{j+1}^{N} \sum_{\underline{t}-t_j} \sum_{\underline{\bar{u}}} \int [d\underline{k}]^{(N-j)} [d\underline{\bar{q}}]^{\bar{N}} \times$$

$$\times\, \alpha^*_{N,\bar{N}}(\underline{k}-\mathbf{k}_j, \mathbf{p}, \underline{t}-t_j, r; \underline{\bar{q}}, \underline{\bar{u}})\, \alpha_{N,\bar{N}}(\underline{k}-\mathbf{k}_j, \mathbf{p}', \underline{t}-t_j, s; \underline{\bar{q}}, \underline{\bar{u}})$$

therefore

$$\sum_r \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}\left(|\psi\rangle\langle\psi| a_r^\dagger(\mathbf{p}) a_r(\mathbf{p})\right) = \sum_N N \sum_{\bar{N}} \|\alpha_{N,\bar{N}}\|^2$$

Can we do better with quantum field theory?

Still unconstrained by the polarization states

$$\int d^3x\, J^0(x) = \int \frac{d^3p}{(2\pi)^3} \left[\sum_r \langle a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) \rangle - \sum_r \langle b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) \rangle \right]$$

$$\int d^3x\, T^{0\mu}(x) = \int \frac{d^3p}{(2\pi)^3} p^\mu \left[\sum_r \langle a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) \rangle + \sum_r \langle b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) \rangle \right]$$

Spin sensitive object?

Conserved currents and charges

$$\mathcal{A}[\phi^a] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi, x)$$

gravitational tensor

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - g_{\mu\nu}\mathcal{L}$$

conserved charges and currents from the Noether Theorem…

$$\begin{cases} x^\mu \longrightarrow \xi^\mu = x^\mu + \epsilon\delta x^\mu, \\ \phi^a(x) \longrightarrow \alpha^a(\xi) = \phi^a(x) + \epsilon\delta\phi^a(x) + \epsilon\delta x^\mu \partial_\mu \phi^a(x), \end{cases}$$

$$Q^\mu = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\partial_\nu \phi^a - \mathcal{L}\delta^\mu_\nu\right]\delta x^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\left(\delta\phi^a(x) + \delta x^\nu \partial_\nu \phi^a(x)\right)$$

Conserved currents and charges

for instance space-time translations

$$x^\mu \to \xi^\mu = x^\mu + \epsilon\delta x^\mu,$$
$$\phi^a(x) \to \alpha^a(\xi) = \phi^a(x) + \epsilon\delta\phi^a(x) + \epsilon\delta x^\mu \partial_\mu \phi^a(x),$$

$$T_c^{\mu\nu}(x) = \frac{i}{2}\bar{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial^\nu}\Psi(x) - g^{\mu\nu}\mathcal{L} \equiv \frac{i}{2}\bar{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial^\nu}\Psi(x)$$

and the Lorentz group transformations

$$\begin{cases} \delta x^\mu = \omega^{\mu\nu} x_\nu \\ \delta\phi^a + \delta x^\mu \partial_\mu \phi^a = -\frac{1}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^a_b \phi^b \end{cases}$$

canonical angular momentum flux

$$M_c^{\lambda,\mu\nu} = x^\mu T_c^{\lambda\nu} - x^\nu T_c^{\lambda\mu} - i\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi^a)}(\Sigma^{\mu\nu})^a_b\,\phi^b$$

$$S_c^{\lambda,\mu\nu}(x) = \frac{i}{8}\bar{\Psi}(x)\left\{\gamma^\lambda,\left[\gamma^\mu,\gamma^\nu\right]\right\}\Psi(x)$$

spin tensor

Tensors transformations

Two ways to change the currents, without affecting the conserved charges

- **Changing the action (preserving the equations of motion)**
- **Direct change (pseudogauge transformation):**

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\mathcal{G}^{\lambda,\mu\nu} - \mathcal{G}^{\mu,\lambda\nu} - \mathcal{G}^{\nu,\lambda\mu}\right),$$

$$\mathcal{S}'^{\lambda,\mu\nu} = \mathcal{S}^{\lambda,\mu\nu} - \mathcal{G}^{\lambda,\mu\nu} - \partial_\alpha \Xi^{\alpha\lambda,\mu\nu}.$$

- E. Speranza, and N. Weickgenannt, arXiv:2007.00138

Tensors transformations

e.g. the Belinfante symmetrization

$$\partial_\mu T_B^{\{\mu\nu\}} = 0,$$

$$T_B^{[\mu\nu]} = 0 \Rightarrow \partial_\lambda S_c^{\lambda\mu\nu} = -\left(T_c^{\mu\nu} - T_c^{\nu\mu}\right),$$

$$P^\mu = \int d^3x\, T_B^{\{0\mu\}}, \qquad J^{\mu\nu} = \int d^3x\, \left(x^\mu T_B^{\{0\nu\}} - x^\nu T_B^{\{0\mu\}}\right).$$

Warning, not manifestly symmetric in the fields

$$T_B^{\mu\nu} = \frac{i}{2}\bar{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi(x) - \frac{i}{16}\partial_\lambda\left(\Psi(x)\left\{\gamma^\lambda, \left[\gamma^\mu, \gamma^\nu\right]\right\}\Psi(x)\right)$$

Angular momentum and spin

$$\hat{J}^{\mu\nu} = \int d^3x\, \Psi^\dagger(x)\left(\frac{i}{2}x^\mu \overleftrightarrow{\partial}^\nu - \frac{i}{2}x^\nu \overleftrightarrow{\partial}^\mu + \frac{i}{8}\gamma^0\left\{\gamma^0, \left[\gamma^\mu, \gamma^\nu\right]\right\}\right)\Psi(x)$$

The contribution from the spin tensor is surely sensitive to the spin states

$$\frac{1}{2}\varepsilon_{ijk}\int d^3x\, \mathcal{S}_c^{0jk} = \frac{1}{2}\sum_{r,s}\int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big[\langle a_r^\dagger(\mathbf{p})a_s(\mathbf{p})\rangle U_r^\dagger(\mathbf{p})\Sigma_i U_s(\mathbf{p})$$
$$- \langle b_r^\dagger(\mathbf{p})b_s(\mathbf{p})\rangle V_s^\dagger(\mathbf{p})\Sigma_i V_r(\mathbf{p})$$
$$+ \langle a_r^\dagger(\mathbf{p})b_s^\dagger(-\mathbf{p})\rangle U_r^\dagger(\mathbf{p})\Sigma_i V_s(-\mathbf{p})e^{2iE_{\mathbf{p}}t}$$
$$+ \langle b_s(-\mathbf{p})a_r(\mathbf{p})\rangle V_r^\dagger(-\mathbf{p})\Sigma_i U_s(\mathbf{p})e^{-2iE_{\mathbf{p}}t}\Big]$$

$$\Sigma_i = \frac{i}{4}\varepsilon_{ijk}[\gamma^j, \gamma^k] = \begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix}$$

Polarization in relativistic quantum systems

In ordinary quantum mechanics $\langle\psi|\boldsymbol{\sigma}|\psi\rangle$ relativistic extension following the classical case

$$j^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + s^{\mu\nu} \Rightarrow \frac{1}{2}\varepsilon_{ijk} j^{jk} = (\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{s})|^i$$

$$\Pi^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} j_{\nu\rho} p_\sigma = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} s_{\nu\rho} p_\sigma$$

making use of the corresponding quantum operators

$$\hat{\Pi}^\mu = -\frac{1}{2m}\varepsilon^{\mu\nu\rho\sigma} : \hat{J}_{\nu\rho} :: \hat{P}_\sigma :$$

one obtains the polarization of single particle states

Polarization in relativistic quantum systems

In particular

$$\langle\psi_1|\hat{\Pi}^\mu|\psi_1\rangle = -\frac{1}{4m}\varepsilon^{\mu ij\sigma}\varepsilon_{ijk}\sum_{r,r'}\int\frac{d^3p}{(2\pi)^3}\frac{\psi_1^*(p,r')\psi_1(p,r)}{2E_{\mathbf{p}}}\frac{p_\sigma}{2E_{\mathbf{p}}}U_{r'}^\dagger(\mathbf{p})\Sigma_k U_r(\mathbf{p})$$

In other words

$$\langle\psi_1|\hat{\mathbf{\Pi}}|\psi_1\rangle = \frac{1}{2}\sum_{r,s}\int\frac{d^3p}{(2\pi)^3}\frac{\psi_1^*(p,r)\psi_1(p,s)}{2E_{\mathbf{p}}}\left[\phi_r\boldsymbol{\sigma}\phi_s + \frac{\phi_r(\mathbf{p}\cdot\boldsymbol{\sigma})\phi_s}{m(E_{\mathbf{p}}+m)}\mathbf{p}\right]$$

using the familiar convention

$$U_r(\mathbf{p}) = \sqrt{E_{\mathbf{p}}+m}\begin{pmatrix}\phi_r\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}+m}\phi_r\end{pmatrix}$$

$$V_r(\mathbf{p}) = \sqrt{E_{\mathbf{p}}+m}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_{\mathbf{p}}+m}\chi_r\\ \chi_r\end{pmatrix}$$

$$\left(\ \mathbf{\Pi}_{\text{com.}} = \mathbf{\Pi} - \frac{\mathbf{\Pi}\cdot\mathbf{p}}{E_{\mathbf{p}}(E_{\mathbf{p}}+m)}\mathbf{p}\ \right)$$

Wigner functions

Wigner transform, originally developed for the wavefunction

$$|\psi(x)|^2 = \int dv\, \delta(v)\psi^*(x - v/2)\psi(x + v/2) = \int dv \int \frac{dp}{(2\pi)}\, e^{-ipv}\psi^*(x + v/2)\psi(x - v/2)$$

$$\begin{cases} W(x,p) = \int \frac{dv}{(2\pi)}\, e^{-ivp}\psi^*(x + v/2)\psi(x - v/2) \\ |\psi(x)|^2 = \int dp\; W(x,p), \qquad |\psi(p)|^2 = \int dx\; W(x,p) \end{cases}$$

it can be used for operators in QFT

$$\widehat{W}(x,k) = \int \frac{d^4v}{(2\pi)^4}\; e^{-iv\cdot k}\,\phi^\dagger(x + v/2)\,\phi(x - v/2)$$

not all properties are retained

$$W(x,k) = tr\big(\rho\!:\!\widehat{W}(x,k)\!:\big)$$

Wigner functions

- S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert, *Relativistic Kinetic Theory*

making use of the Klein-Gordon equation

$$\left[-\frac{1}{4}\hbar^2\Box W + \left(k^2 - m^2c^2\right)W\right] + i\hbar(p\cdot\partial)W = 0$$

and an $\hbar$ expansion

$\propto \delta(k^2 - m^2c^2)$

$$W = \delta\left(k^0 - E_{\boldsymbol{k}}\right)\frac{f(x,\boldsymbol{k})}{(2\pi)^3} + \delta\left(k^0 + E_{\boldsymbol{k}}\right)\frac{\bar{f}(x,\boldsymbol{k})}{(2\pi)^3}$$

summing up

$$\begin{cases} k\cdot\partial f(x,\boldsymbol{k}) = 0 \\ k\cdot\partial \bar{f}(x,\boldsymbol{k}) = 0 \end{cases}$$

very similar to free streaming particles

Fermion fields and polarization

$$\hat{W}_{AB}(x,k) = \int \frac{d^4v}{(2\pi)^4}\, e^{-ik\cdot v}\, \Psi^\dagger_B(x+v/2)\Psi_A(x-v/2)$$

any Dirac bilinear can be obtained

$$\mathrm{tr}\left(\rho : \bar{\Psi}(x)\gamma^{\nu_1}\cdots\gamma^{\nu_n}\,\frac{i}{2}\,\overleftrightarrow{\partial}^{\mu_1}\cdots\frac{i}{2}\,\overleftrightarrow{\partial}^{\mu_m}\,\Psi(x) :\right)$$
$$= \int d^4k\; k^{\mu_1}\cdots k^{\mu_m}\;\mathrm{tr}_4\left(W(x,k)\gamma^0\,\gamma^{\nu_1}\cdots\gamma^{\nu_n}\right)$$

Fermion fields and polarization

$$W(x,k) = \sum_{rs} \int \frac{d^4v}{(2\pi)^4} e^{-ik\cdot v} \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \Big[$$

$$\langle a_r^\dagger(\mathbf{p}) a_s(\mathbf{q})\rangle U_r^\dagger(\mathbf{p}) U_s(\mathbf{q}) e^{i(p-q)\cdot x} e^{i\left(\frac{p+q}{2}\right)\cdot v} +$$

$$- \langle b_r^\dagger(\mathbf{p}) b_s(\mathbf{q})\rangle V_s^\dagger(\mathbf{q}) V_r(\mathbf{p}) e^{i(p-q)\cdot x} e^{-i\left(\frac{p+q}{2}\right)\cdot v} +$$

$$+ \langle b_s(\mathbf{q}) a_r(\mathbf{p})\rangle V_s^\dagger(\mathbf{q}) U_r(\mathbf{p}) e^{-i(p+q)\cdot x} e^{i\left(\frac{p-q}{2}\right)\cdot v} +$$

$$+ \langle a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{q})\rangle U_r^\dagger(\mathbf{p}) V_s(\mathbf{q}) e^{i(p+q)\cdot x} e^{i\left(\frac{p-q}{2}\right)\cdot v} \Big],$$

Interesting fact, the current $k^\mu W(x,k)$ is conserved

$$\mathrm{tr}_4\left(\int d\Sigma_\mu\, k^\mu W(x,k)\right) = \delta(k^0 - E_{\mathbf{k}})\, E_{\mathbf{k}} \frac{dN}{d^3p}(\mathbf{k}) + \delta(k^0 + E_{\mathbf{k}})\, E_{\mathbf{k}} \frac{d\bar{N}}{d^3p}(-\mathbf{k})$$

Fermion fields and polarization

$$\frac{1}{2m}\mathrm{tr}_4\left[\left(\int d\Sigma_\mu\, k^\mu\, W(x,k)\right)\gamma^0\gamma^i\gamma_5\right] = \frac{1}{2}\sum_{r,s}\left\{\delta(k^0-E_{\mathbf{k}})\frac{\langle a_r^\dagger(\mathbf{k})a_s(\mathbf{k})\rangle}{(2\pi^3)}\left[\phi_r\sigma_i\phi_s+\frac{\phi_r(\mathbf{k}\cdot\boldsymbol{\sigma})\phi_s}{m(E_{\mathbf{k}}+m)}\,k_i\right]+\right.$$
$$\left.\delta(k^0+E_{\mathbf{k}})\frac{\langle b_r^\dagger(-\mathbf{k})b_s(-\mathbf{k})\rangle}{(2\pi^3)}\left[\chi_s\sigma_i\chi_r+\frac{\chi_s(\mathbf{k}\cdot\boldsymbol{\sigma})\chi_r}{m(E_{\mathbf{k}}+m)}\,k_i\right]\right\}$$

average polarization directly from the Wigner distribution

$$\frac{1}{2m}\mathrm{tr}_4\left[\left(\int d\Sigma_\lambda\, k^\lambda\, W(x,k)\right)\gamma^0\gamma^\mu\gamma_5\right] =$$
$$= \delta^4(k^0-E_{\mathbf{k}})\frac{dN}{d^3p}(\mathbf{k})\,\langle\Pi^\mu(\mathbf{k})\rangle - \delta^4(k^0+E_{\mathbf{k}})\frac{dN}{d^3p}(-\mathbf{k})\,\langle\bar{\Pi}^\mu(-\mathbf{k})\rangle$$

- **see also: F. Becattini arXiv:2004.04050**

Fermion fields and polarization

$$\frac{1}{2m}\int d^4k\, \mathrm{tr}_4\left[\left(\int d\Sigma_\lambda\, k^\lambda\, W(x,k)\right)\gamma^0\gamma^\mu\gamma_5\right] =$$

$$= \int d\Sigma_\lambda \langle : \frac{i}{4m}\bar{\Psi}\left(\overleftrightarrow{\partial}^\lambda\, \gamma^\mu\gamma_5\right)\Psi :\rangle$$

conserved polarization flux tensor

$$\frac{i}{4m}\bar{\Psi}(x)\left(\overleftrightarrow{\partial}^\lambda\, \gamma^\mu\gamma_5\right)\Psi(x)$$

Conclusions and outlook

- Relativistic kinetic theory must be generalized
- The Wigner distribution natively embeds polarization
- New dynamical degrees of freedom in extended hydro-transport?