From correlations to universality

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Open Quantum Systems:
From atomic nuclei to ultracold atoms and quantum optics

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Personal remarks

- Definition of correlations in the present context
- Appearance of universal behavior
- Definition of the universal window: Efimov physics
- Correlation between bound and scattering states in the low-energy regime
- Dynamics governed by a few parameters (control parameters)
- Continuous (or discrete) scale invariance

Interplay of two aspects

- Efimov physics put in evidence universal aspects of weakly bound systems
- Potential models are very detailed covering some times universal behavior
- Weakly bound systems are strongly correlated
- Are correlated systems and universal properties compatible?

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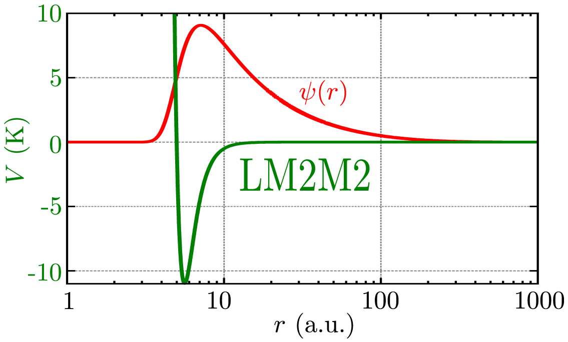
Correlated systems: the He-He system as example



$N = 2$: $\psi(r) \to 0$ if $r < r_c$

$N > 2$: $\psi(\ldots r_{ij} \ldots) \to 0$ if $r_{ij} < r_c$

The many body system is strogly correlated since $\psi \to 0$ when two particles are close independently of the position of the other particles

Ground state properties of helium drops

- He-He potentials has been used to calculate the ground state energies of drops with $3 \leq N \leq \infty$
- In V.R. Pandharipande et al., PRL 50, 1676 (1983) one of the first applications of the GFMC method appeared
- The motivations for that study were twofold:
  i) To compare theoretical results using potential models with experimental data
  ii) To analyze extrapolation formulas from calculations with fix number of atoms to the infinite system
- The E/N experimental value of liquid Helium (-7.14K) was well described. The calculations predicted -7.11K or -7.02K from an extrapolation using results in the range $20 \leq N \leq 112$
- In nuclear physics we have a different situation: experimental values exists for the clusters (nuclei) but not for the infinite system (nuclear matter)

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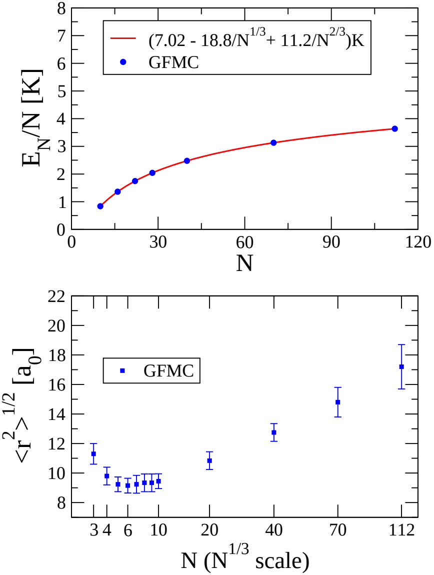
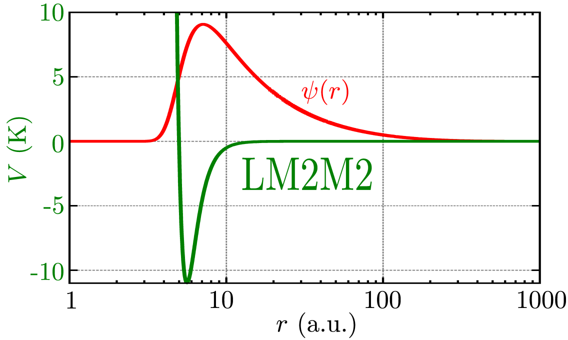


What about universal behavior?



The He dimer is a weakly bound system:

- Its binding energy is small $B(He - He) \approx 1$ mK
- Its energy length $a_B = 1/k_B = \sqrt{\hbar^2/mB} \approx 190 a.u. >> r_0$
- The energy length a_B, the scattering length a and the effective range r_e are correlated (ERE): $1/a_B = 1/a + r_e/2a_B^2$
- The particles are most of the time outside the interaction range r_0

Universality in the two-body system

The scaling (or zero-range) limit

- in the scaling limit the range of the interaction is zero

$$\phi_d = C_d e^{-k_d r}$$

$$\phi_0 = C_0 (r - a)$$

$$\phi_k = C_k (\sin kr - \tan\delta \cos kr)$$

- which is the relation between k_d, a and $\tan\delta$?

orthogonality of the states

$$\int_0^\infty \phi_d \phi_0 = 0 \Rightarrow k_d^{-1} = a_B = a \Rightarrow B = \hbar^2/ma^2$$

$$\int_0^\infty \phi_d \phi_k = 0 \Rightarrow k \cot\delta = -1/a_B \Rightarrow k \cot\delta = -1/a$$

In the scaling limit $a - a_B = r_B = 0$ (where r_B is the correlation length)

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In the scaling limit $a - a_B = r_B = 0$ (where r_B is the correlation length)

Increasing the correlation length

- In real systems r_B could be small but not zero
- we make the following model:

$$\phi_d = 0 \; (r < r_B) \text{ and } \phi_d = C_d e^{-k_d r} \qquad (r \geq r_B)$$
$$\phi_0 = 0 \; (r < r_B) \text{ and } \phi_0 = C_0 (r - a) \qquad (r \geq r_B)$$
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orthogonality of the states outside the cutoff r_B

$$\int_{r_B}^{\infty} \phi_d \phi_0 = 0 \rightarrow a - a_B = r_B$$

$$\int_{r_B}^{\infty} \phi_d \phi_k = 0 \rightarrow k \cot\delta = -1/a + (r_B a_B / a) k^2$$

Therefore $a - a_B = r_B$ and $r_e = 2 r_B a_B / a$

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Universal behavior in few-body systems

Examples

- The helium dimer (as given by the TTY potential):
  $E_d = 1.309$ mk
  $a = 188.78$ a.u., $a_B = 181.355$ a.u.
  $r_e = 13.845$ a.u., $r_B = 6.94$ a.u.
  $E(a, r_{eff}) = 1.311$ mk
- The deuteron:
  $E_d = 2.225$ MeV
  $a^1 = 5.419 \pm 0.007$ fm, $a^1_B = 4.317$ fm
  $r^1_e = 1.753 \pm 0.008$ fm, $r^1_B = 1.1$ fm
  $E(a, r_{eff}) = 2.223$ fm

From zero- to finite-range: a gaussian potential model

$$V(r) = V_0 e^{-(r/r_0)^2}$$

with V_0 fixed to describe a or E_d and the role of r_0 to be discussed.

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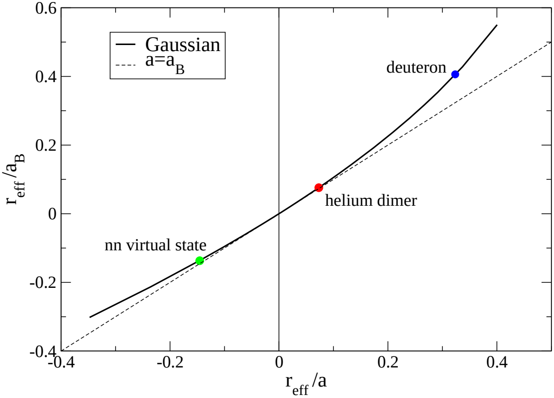
with V_0 fixed to describe a or E_d and the role of r_0 to be discussed.

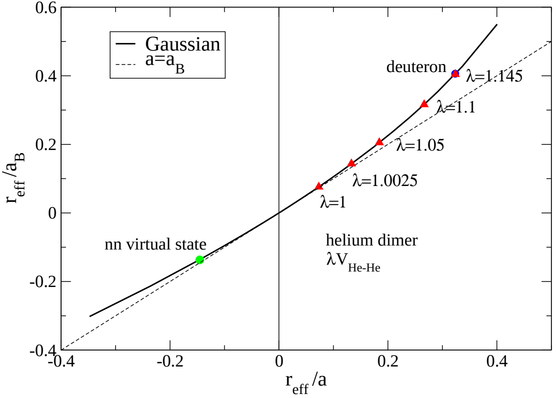
Universal behavior in few-body systems

- When a shallow state exists, a Gaussian potential gives a reasonable description of the low energy regime, bound and scattering states.



Travelling across the gaussian trajectory

- Varying the strength of the potential we can move allong the gaussian trajectory



Continuous Scale Invariance

- For $\lambda \approx 1.145$ the helium dimer and the deuteron overlap:

λV_{He}	B(mK)	$a(a_0)$	$r_e(a_0)$	$r_B(a_0)$
1.000	1.303	189.42	13.845	7.166
1.025	5.027	99.935	13.290	7.146
1.050	11.137	69.448	12.792	7.108
1.100	30.358	44.792	11.937	7.034
1.145	55.408	34.919	11.299	6.970

Studying the CSI with the asymptotic normalization constant

$$\psi(r \to \infty) \longrightarrow C_0 e^{-r/a_B}$$

If CSI is verified the ANC should be the same (in units of $\sqrt{a^{-1}}$)

system	$C_0\sqrt{a}$
$1.145 V_{He}$	2.03
deuteron	2.04
gaussian	2.04

The three-boson system

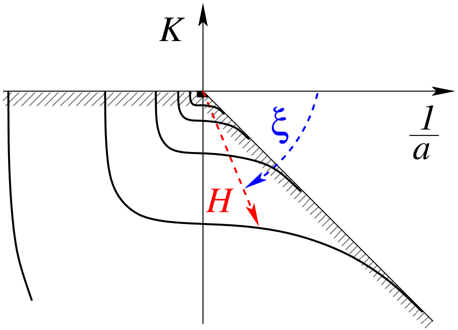
Zero-Range Equations: Efimov spectrum

$$E_3^n/(\hbar^2/ma^2) = \tan^2\xi$$

$$\kappa_* a = \mathrm{e}^{\pi(n-n_*)/s_0}\,\mathrm{e}^{-\Delta(\xi)/2s_0}/\cos\xi$$

- The ratio E_3^n/E_2 defines the angle ξ
- The three-body parameter κ_* defines the energy of the system at the unitary limit $E_u = \hbar^2\kappa_*^2/m$
- The product $\kappa_* a$ is a function of ξ governed by the universal function $\Delta(\xi)$
- The universal function $\Delta(\xi)$ is obtained by solving the STM equations (Faddeev equation in the zero-range limit) and is the same for all levels n

Three boson spectrum in the zero-range limit



The three-boson system

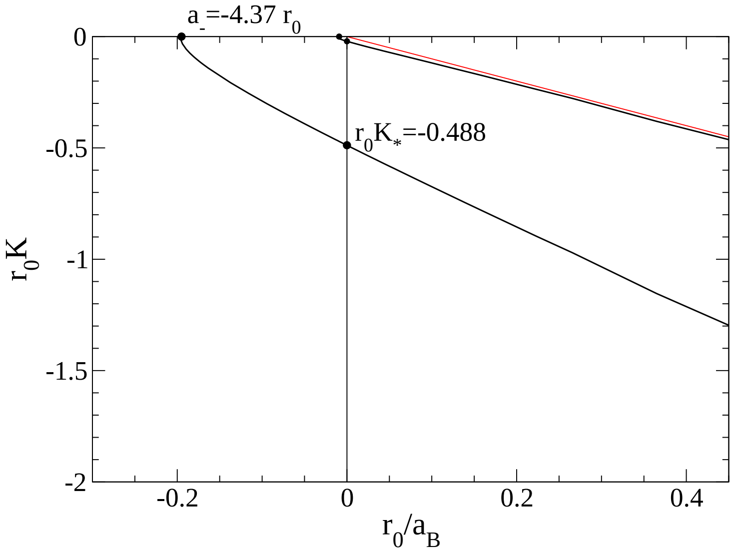
Finite-Range Equations: Gaussian spectrum

$$E_3^n/E_2 = \tan^2 \xi_n$$

$$\kappa_*^n a_B = \mathrm{e}^{\pi(n-n_*)/s_0} \mathrm{e}^{-\widetilde{\Delta}_n(\xi_n)/2s_0} / \cos \xi$$

- The ratio E_3^n/E_2 defines the angle ξ_n
- The three-body parameter κ_*^n defines the energy of the system at the unitary limit $E_u = \hbar^2(\kappa_*^n)^2/m$
- The product $\kappa_*^n a_B$ is a function of ξ_n governed by the level function: $\widetilde{\Delta}(\xi_n) = s_0 \ln\left(\frac{E_3^n + E_2}{E_u^n}\right)$
- The level function $\widetilde{\Delta}(\xi)$ is obtained by solving the Schrödinger equation.
- For $n > 1$ $\widetilde{\Delta}(\xi) \to \Delta(\xi)$

Three boson spectrum with a Gaussian potential



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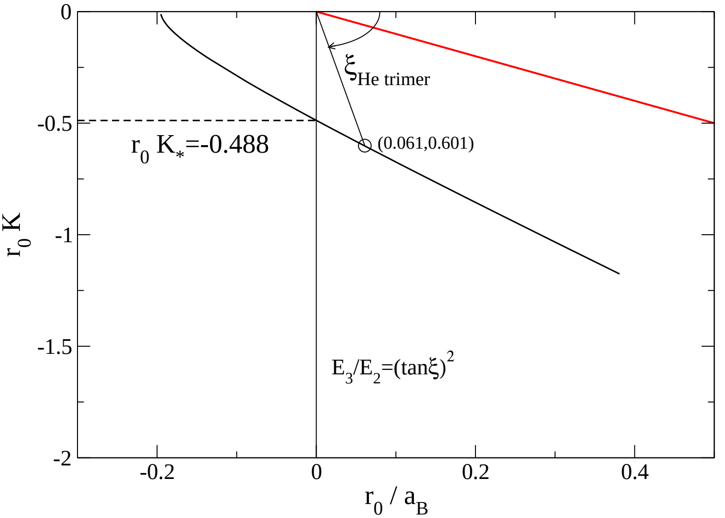

Scale invariance

The three-body parameter κ_* for the helium trimer

- The quantity $\kappa_* a_B = f(\xi)$ is a function of ξ, where $E_3/E_2 = \tan^2\xi$
- To determine ξ we use experimental results $E_3 = 126$mK and $E_2 = 1.3$mK. Accordingly $\tan^2\xi = 97.0$

$$r_0/a_B = 0.061 \Rightarrow r_0 = 11.13 a_0 \Rightarrow \kappa_* = 0.488/11.13\, a.u. = 0.044\, a.u.$$

$$[E_*]^{He} = \hbar^2\kappa_*^2/m \approx 83mK$$

- Different results given in the literature agree with this prediction!
- The helium trimer is on the gaussian trajectory

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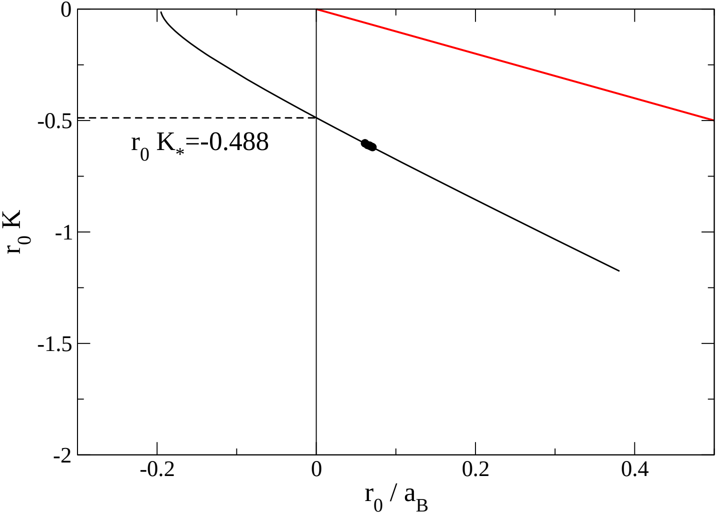
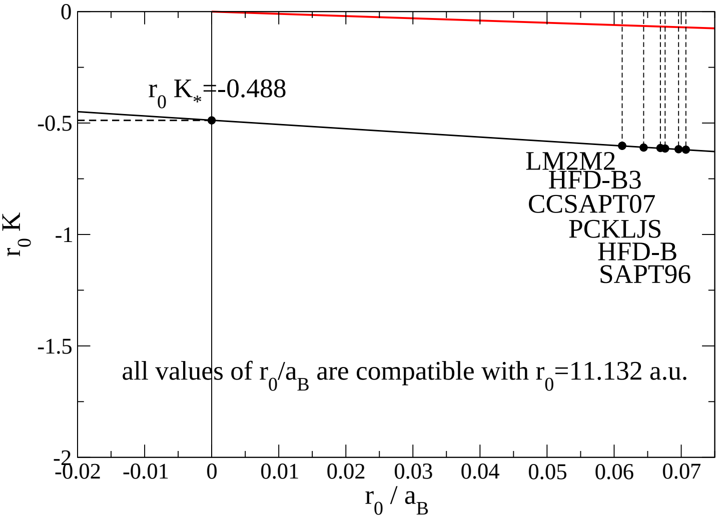
Different He-He potentials



Different He-He potentials



Moving on the Gaussian trajectory

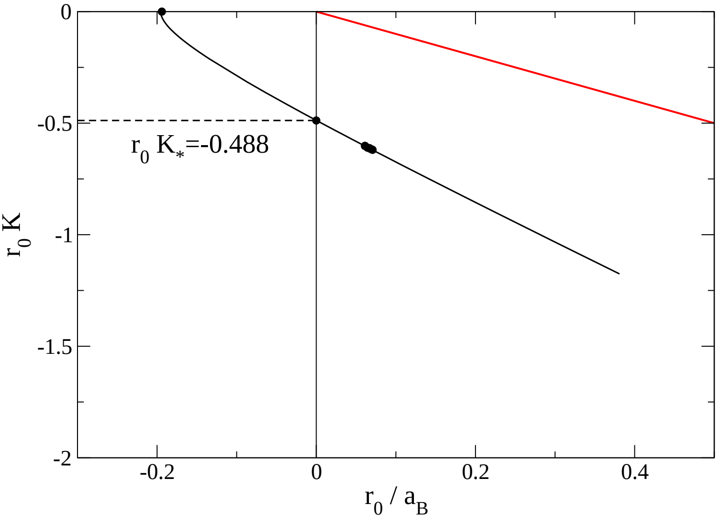
Effective low-energy soft potential

- All He-He potentials are on the Gaussian curve
- In all cases $r_0 = 11.132\, a.u.$
- Accordingly the potential energy for three bosons can be represented as

$$\sum_{ij} V_{He-He}(r_{ij}) = V_0 \sum_{ij} \mathrm{e}^{-r_{ij}^2/r_0^2}$$

- Varying V_0 the helium trimer moves on trajectory first going on κ_* and then climbing to the a_- value
- We predict $a_- = -4.37 r_0 = -48.6 a.u.$ or $a_- = -9.8 r_{vdW}$
- In the literature the value $a_- = -48.2 a.u.$ is reported
- Along the trajectory the correlation length r_B is almost constant

Moving on the Gaussian trajectory



Collapsing on the Gaussian trajectory

- Calculate the gaussian trajectory using any gaussian
- From one experimental three-body point calculate the angle: $E_3/E_2 = \tan^2\xi$
- Crossing the gaussian trajectory it determines the value $x = r_0/a_B$
- The gaussian range is $r_0 = x\, a_B$
- $\kappa_* = 0.488 r_0$
- $a_- = -4.37 r_0$
- The potential energy for three bosons can be represented as

$$V_0 \sum_{ij} \mathrm{e}^{-r_{ij}^2/r_0^2}$$

- Let us analyse $N > 3$

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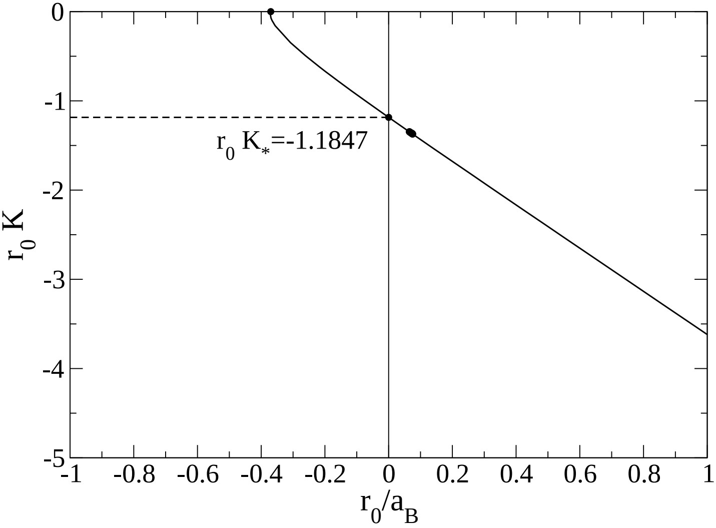
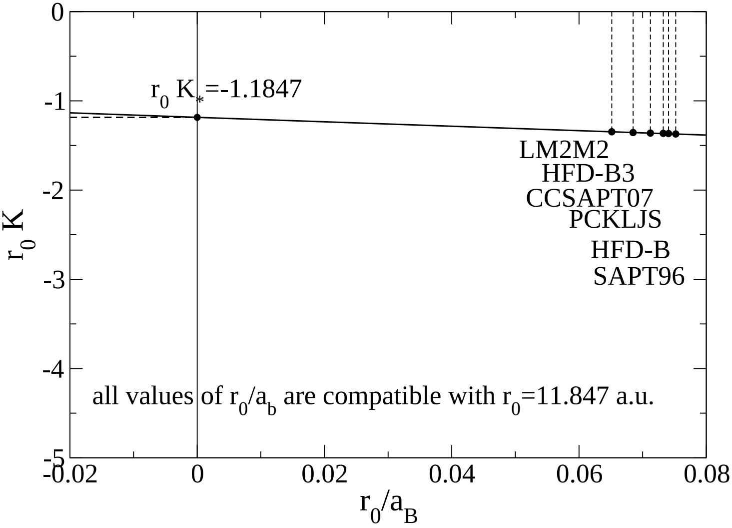
$N = 4$: different He-He potentials



$N = 4$: different He-He potentials



Moving on the Gaussian trajectory

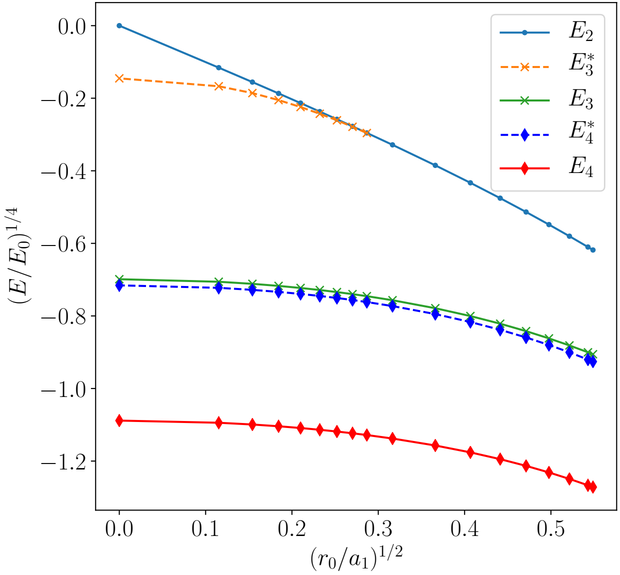
Effective low-energy soft potential

- As in the $N = 3$, all He-He potentials are on the Gaussian curve
- In all cases $r_0 = 11.847\, a.u.$
- Accordingly the potential energy for four bosons can be represented as

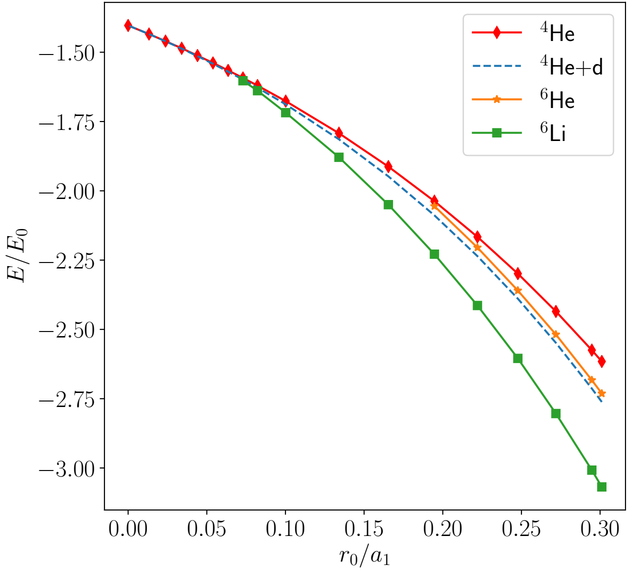
$$\sum_{ij} V_{He-He}(r_{ij}) = V_0 \sum_{ij} \mathrm{e}^{-r_{ij}^2/r_0^2}$$

- Varying V_0 the helium tetramer moves on trajectory first going on κ_* and then climbing to the a_- value
- We predict $a_- = -1.965 r_0 \approx -23 a.u.$
- In the literature the value $a_- = -22.7 a.u.$ is reported
- At the unitary limit $\kappa_* = 1.1847/r_0 = 0.1 \Rightarrow E_* = 432$ mK
- in the literature the value $E_* = 439$ mK is quoted

Gaussian trajectories for nuclear systems: $A \leq 4$



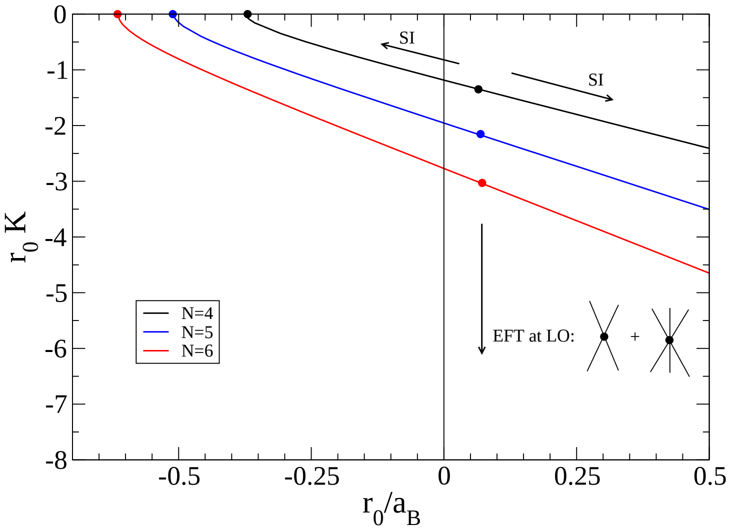
Gaussian trajectories for nuclear systems: $A \lesssim 6$



Comments on the $A \leq 6$ Efimov plot

- For $A = 3$ only one state survive at the physical point
- The excited state disappears at $r_0/a_1 \approx 0.09$
- The nuclear physics point is $r_0/a_1 \approx 0.3$
- Resembling the bosonic case, there is a shallow four-body excited state
- At the physical point $E_4/E_3 \approx 3.9$, very close to the experimental ratio $E(^4\mathrm{He})/E(^3\mathrm{He}) \approx 3.7$
- At unitary no five-body and six-body bound state appear.
- The ^{6}Li state appears quite close to the unitary limit
- The ^{6}He state appears a little bit further
- The energy values at unitary can be estimated: $r_0/a_B = 0.4573 \Rightarrow r_0 = 1.974 fm \Rightarrow \kappa_* = 0.4883/r_0 \Rightarrow E_3 = 2.54 MeV$
- With similar procedure $E_3 = 13.5 MeV$

Increasing N

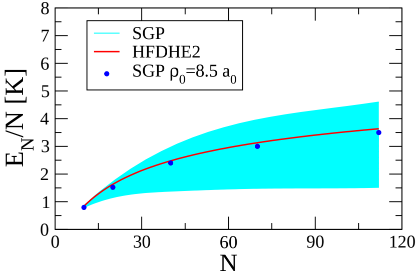


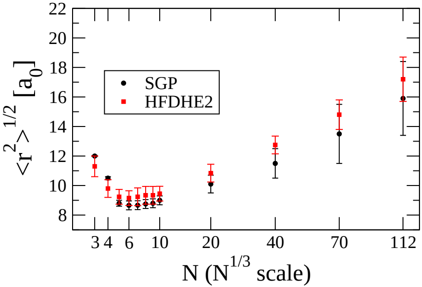
Propagation of universal behavior with N

Saturation properties of helium drops

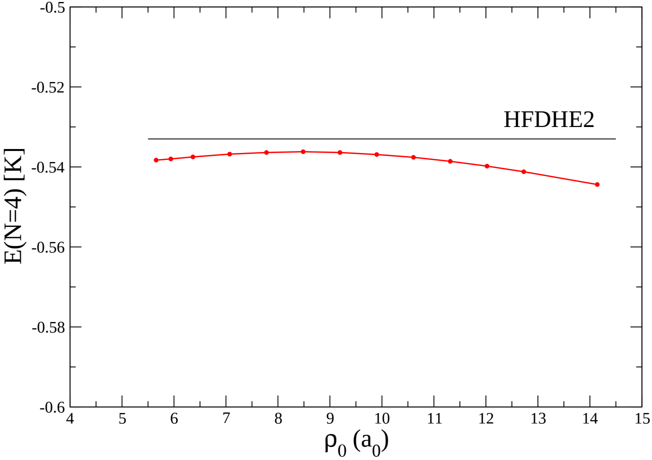
- We define a soft potential model to describe E_3
- It consists in a two- plus a three-body term $V = V(i,j) + W(i,j,k)$
- $V(i,j) = V_0 e^{-r_{ij}^2/r_0^2}$
- $W(i,j,k) = W_0 e^{-\rho_{ijk}^2/\rho_0^2}$
- W_0 is determined from E_3
- ρ_0 is taken as a parameter
- E/N is calculated for increasing values of N as a function the ρ_0
- the saturation properties are determined from a liquid drop formula:
  $E_N/N = E_v + E_s x + E_c x^2$ with $x = N^{-1/3}$
- In general drops with N around 100 is sufficient to determine E_v and E_s

drops with $N \leq 112$





drops with $N = 4$



Propagation of universal behavior with N

Saturation properties of helium drops

- Using the appropriate value of ρ_0
- we obtain (in K):
  $E_N/N = 6.79 - 18.0x + 9.98x^2$
- To be compared to the results of the HFDHE2 potential:
  $E_N/N = 7.02 - 18.8x + 11.2x^2$
- The experimental result is 7.14 K
- for the surface tension $t = E_s/4\pi r_0^2(\infty)$
  the experimental value is 0.29 KA^2
  with the gaussian soft potential the result is 0.27 KA^2
- Since the potential model is determined only from the two, three and four body sector, we can conjeture that the saturation energy can be extracted from E_2, E_3 and E_4
- What happens in nuclear physics?

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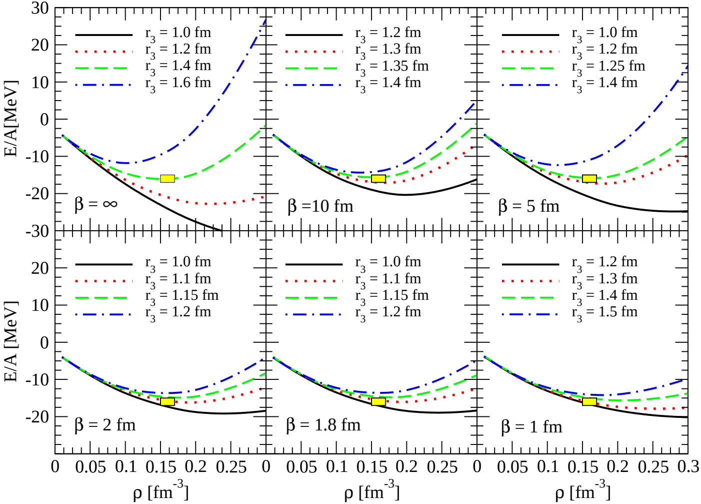
- We define a soft potential model to describe E_2, E_3, E_4
- It consists in a two- plus a three-body term $V = V(i,j) + W(i,j,k)$
- $V(i,j) = V_0 e^{-r_{ij}^2/r_0^2}\mathcal{P}_0 + V_1 e^{-r_{ij}^2/r_1^2}\mathcal{P}_1 + V_\beta(OPEP)$
- V_S, r_S are determined by the a_S^{np}, r_S^{np} in spin channel S
- $W(i,j,k) = W_0 e^{-\rho_{ij}^2/\boldsymbol{r}_3^2} e^{-\rho_{ik}^2/\boldsymbol{r}_3^2}$
- W_0 is determined from E_3
- $\boldsymbol{r}_3$ is taken as a parameter looking at E_4
- $V_\beta(OPEP) \to 0$ as $\beta \to \infty$

$$V_\beta(OPEP) = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 Y_\beta(r) + S_{12} T_\beta(r)\right]$$

- E/N and the saturation density ρ_0 are calculated by the BHF method

Saturation properties of Nucleat Matter



Conclusions

- Weakly bound systems can be made collapse on gaussian trajectories
- At each value of N a gaussian representation of the potential energy can be constructed
- The range of the gaussian is used to assign values to the notable points of the trajectory, κ_* and a_-.
- Concepts of EFT has been used to study propagation of the universal behaviour as N increases.
- A soft potential with an attractive two-body term and a repulsive three-body term has benn constructed
- The range of the three-body interaction has been taken as a parameter
- Essentially the soft potential has been determined by low-energy parameters as E_2, E_3 and E_4
- Many properties can be predicted using gaussian trajectories