Combining Tensor Networks and Monte Carlo for Lattice Gauge Theories

14th of June | *Patrick Emonts*, E. Zohar, M. C. Bañuls, I. Cirac | MPI for Quantum Optics

MPQ

Why do we need Hamiltonian Lattice Gauge theories?

Time Evolution

The Wick rotation eliminates the notion of time when going to a Eucledian action

$$t \to -i\tau$$

Sign Problem

$$\begin{aligned}\langle O[U]\rangle &= \frac{\int \mathcal{D}U\mathcal{D}\Psi\mathcal{D}\bar{\Psi}O[U]e^{-S_E[U]-\bar{\Psi}M[U]\Psi}}{\int \mathcal{D}U\mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{-S_E[U]-\bar{\Psi}M[U]\Psi}} \\ &= \frac{\int \mathcal{D}U O[U]\det(M[U])e^{-S_E[U]}}{\int \mathcal{D}U\det(M[U])e^{-S_E[U]}}\end{aligned}$$

For $\mu > 0$: $\det(M[U]) \in \mathbb{C}$

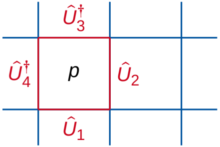
Kogut-Susskind Hamiltonian

KS Hamiltonian

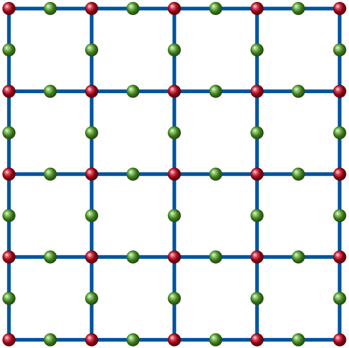
$$H_{KS} = -\frac{1}{2g^2}\sum_p \left(\hat{U}(p) + \hat{U}^\dagger(p)\right) + \frac{g}{2}\sum_{\mathbf{x},k} \hat{E}_k^2(\mathbf{x}) \quad \text{with} \quad \hat{U}(p) = \hat{U}_1\hat{U}_2\hat{U}_3^\dagger\hat{U}_4^\dagger$$

$$\hat{U}(p) + \hat{U}^\dagger(p) = \cos\left(\theta_1 + \theta_2 - \theta_3 - \theta_4\right) = \cos\left(a^2 B^2\right)$$

We recover the standard free field Hamiltonian of electrodynamics ($H \approx E^2 + B^2$) in the classic continuum limit.



Lattice Systems



Hilbert space

$$\mathcal{H} \subset \mathcal{H}_{\text{gauge fields}} \otimes \mathcal{H}_{\text{fermions}}$$

A general state

$$|\Psi\rangle = \int \mathcal{DG}\,|\mathcal{G}\rangle\,|\Psi(\mathcal{G})\rangle$$

with $\mathcal{DG} = \prod_{\mathbf{x},k} dg(\mathbf{x},k)$

Presentation follows Erez Zohar and J. Ignacio Cirac, 2018, *Physical Review D*

Expectation value of an Observable

Assume that O acts only on the gauge field and is diagonal in the group element basis:

$$
\begin{aligned}
\langle O \rangle &= \frac{\langle \Psi | O | \Psi \rangle}{\langle \Psi | \Psi \rangle} \\
&= \frac{\int \mathcal{DG} \int \mathcal{DG}' \, \langle \Psi(\mathcal{G}') | \, \langle \mathcal{G}' | \, O \, | \mathcal{G} \rangle \, | \Psi(\mathcal{G}) \rangle}{\int \mathcal{DG} \int \mathcal{DG}' \, \langle \Psi(\mathcal{G}') | \, \langle \mathcal{G}' | \mathcal{G} \rangle \, | \Psi(\mathcal{G}) \rangle} \\
&= \frac{\int \mathcal{DG} \, \langle \mathcal{G} | \, O \, | \mathcal{G} \rangle \, \langle \Psi(\mathcal{G}) | \Psi(\mathcal{G}) \rangle}{\int \mathcal{DG}' \, \langle \Psi(\mathcal{G}') | \Psi(\mathcal{G}') \rangle} \\
&= \int \mathcal{DG} \mathcal{F}_O(\mathcal{G}) p(\mathcal{G})
\end{aligned}
$$

with $p(\mathcal{G}) = \frac{\langle \Psi(\mathcal{G}) | \Psi(\mathcal{G}) \rangle}{\int \mathcal{DG}' \langle \Psi(\mathcal{G}') | \Psi(\mathcal{G}') \rangle} = \frac{\langle \psi(\mathcal{G}) | \psi(\mathcal{G}) \rangle}{Z}$

Open Questions

Expectation value

$$\langle O \rangle = \int \mathcal{D}\mathcal{G}\mathcal{F}_O(\mathcal{G})p(\mathcal{G})$$

$$\text{with } p(\mathcal{G}) = \frac{\langle \Psi(\mathcal{G})|\Psi(\mathcal{G})\rangle}{Z}$$

- How do we construct $|\Psi(\mathcal{G})\rangle$?
- How do we efficiently calculate $p(\mathcal{G})$?
- What can we do with this new formalism?

Creation of the fermionic state

Desirable properties

- $|\Psi\rangle$ fulfills the Gauss law
- $|\Psi(\mathcal{G})\rangle$ allows efficient calculations of
  - the norm
  - expectation values

Definition of $|\Psi\rangle$

$$|\Psi\rangle = \int \mathcal{DG}\,|\mathcal{G}\rangle\,|\Psi(\mathcal{G})\rangle$$

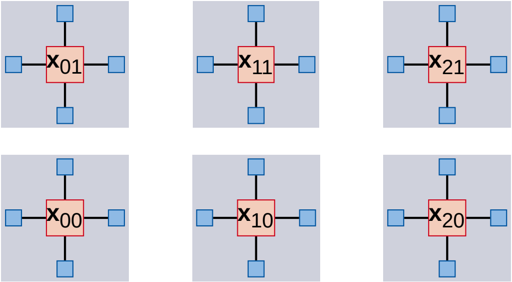
Choice for $|\Psi(\mathcal{G})\rangle$

We pick $|\Psi(\mathcal{G})\rangle$ to be a gaussian state and construct it with a tensor network.

Details about the construction may be found in Erez Zohar, 2018, *arXiv:1807.01294 [cond-mat, physics:hep-lat, physics:hep-th, physics:quant-ph]*

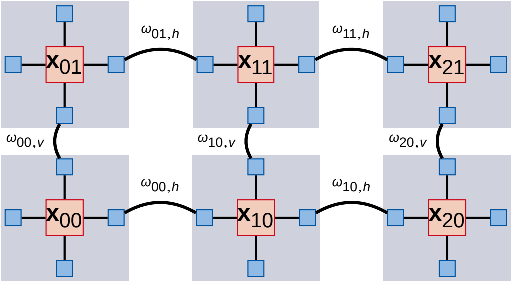
Creating a fermionic state

$$|\psi_0\rangle = \langle \Omega_V| \qquad \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) \, |\Omega\rangle$$

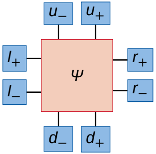


Creating a fermionic state

$$|\psi_0\rangle = \langle\Omega_V| \prod_{\mathbf{x},k} \omega(\mathbf{x},k) \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) |\Omega\rangle$$



Fiducial state in detail



$$\mathcal{A}(\mathbf{x}) = \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) a_j^\dagger(\mathbf{x})\right)$$

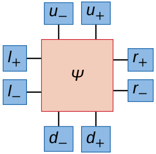
Symmetries

1. Translational invariance
2. Rotational invariance
3. Global U(1) invariance

Definition of Modes

Gauss law in terms of our modes

$$G_0 = E_r - E_l + E_u - E_d$$
$$= r_+^\dagger r_+ - r_-^\dagger r_- - l_+^\dagger l_+ + l_-^\dagger l_- + u_+^\dagger u_+ - u_-^\dagger u_- - d_+^\dagger d_+ + d_-^\dagger d_-$$



Erez Zohar et al., 2015, *Annals of Physics*

Definition of pos. and neg. modes

a: $\{l_+, r_-, u_-, d_+\}$ (neg. modes)
b: $\{l_-, r_+, u_+, d_-\}$ (pos. modes)

Fiducial operator

$$\mathcal{A}(\mathbf{x}) = \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) b_j^\dagger(\mathbf{x})\right)$$

Coupling different sites

Coupling in the x and y direction

$$\omega_0(\mathbf{x}) = \exp\Big(l_+^\dagger(\mathbf{x}+\mathbf{e}_1)r_-^\dagger(\mathbf{x})\Big)\exp\Big(l_-^\dagger(\mathbf{x}+\mathbf{e}_1)r_+^\dagger(\mathbf{x})\Big)$$
$$\omega_1(\mathbf{x}) = \exp\Big(d_+^\dagger(\mathbf{x}+\mathbf{e}_2)u_-^\dagger(\mathbf{x})\Big)\exp\Big(d_-^\dagger(\mathbf{x}+\mathbf{e}_2)u_+^\dagger(\mathbf{x})\Big)$$

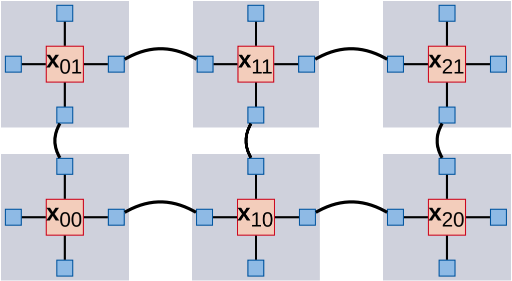
Unnormalized projector

$$\omega(x) = \omega_0(\mathbf{x})\omega_1(\mathbf{x})\Omega(\mathbf{x})\omega_1^\dagger(\mathbf{x})\omega_0^\dagger(\mathbf{x})$$

Fermionic state with global symmetry

Globally invariant state

$$|\psi_0(T)\rangle = \langle\Omega_V| \prod_{\mathbf{x}} \omega(\mathbf{x}) \prod_{\mathbf{x}} \mathcal{A}(\mathbf{x}) |\Omega\rangle$$

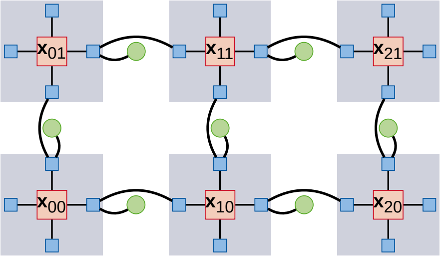


Moving towards local symmetry

Lattice Gauge theory

We demand a local symmetry

$$\sum_{\mathbf{x}} G(\mathbf{x})\,|\Psi\rangle = 0 \rightarrow G(\mathbf{x})\,|\Psi\rangle = 0$$

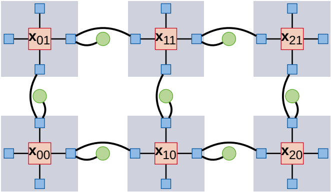


Erez Zohar et al., 2015, *Annals of Physics*
Erez Zohar and Michele Burrello, 2016, *New Journal of Physics*

Local symmetry – The state

Substitution

$$r_{\pm}^{\dagger}(x) \to e^{\pm i\theta(\mathbf{x})} r_{\pm}^{\dagger}(x)$$



Fermionic state

Fermionic state

$$|\psi(\mathcal{G})\rangle = \langle\Omega_V| \prod_x \omega(x) \prod_x \mathcal{U}_{\Phi(x)} \prod_x A(x) |\Omega\rangle$$

- ✔ Gauge invariance of $|\Psi\rangle$ by constructing $\Psi(\mathcal{G})$
- ✔ Obeys all demanded symmetries
- ? Efficient to calculate with

Is $|\Psi(\mathcal{G})\rangle$ special?

$$|\psi(\mathcal{G})\rangle = \langle\Omega_v| \prod_x \omega(x) \prod_x \mathcal{U}_{\Phi(x)} \prod_x \mathcal{A}(x) |\Omega\rangle$$

$$\mathcal{A}(\mathbf{x}) = \begin{cases} \exp\left(\sum_{ij} T_{ij} a_i^\dagger(\mathbf{x}) b_j^\dagger(\mathbf{x})\right) & \mathbf{x} \text{ even} \\ \exp\left(\sum_{ij} T_{ij} b_i^\dagger(\mathbf{x}) a_j^\dagger(\mathbf{x})\right) & \mathbf{x} \text{ odd.} \end{cases}$$

$$\omega(x) = \omega_0(\mathbf{x})\omega_1(\mathbf{x})\Omega(\mathbf{x})\omega_1^\dagger(\mathbf{x})\omega_0^\dagger(\mathbf{x})$$

$$\omega_0(\mathbf{x}) = \exp\left(l_+^\dagger(\mathbf{x}+\mathbf{e}_1) r_-^\dagger(\mathbf{x})\right) \exp\left(l_-^\dagger(\mathbf{x}+\mathbf{e}_1) r_+^\dagger(\mathbf{x})\right)$$

$$\omega_1(\mathbf{x}) = \exp\left(d_+^\dagger(\mathbf{x}+\mathbf{e}_2) u_-^\dagger(\mathbf{x})\right) \exp\left(d_-^\dagger(\mathbf{x}+\mathbf{e}_2) u_+^\dagger(\mathbf{x})\right)$$

Gaussian States

Definition

Fermionic Gaussian states are represented by density operators that are exponentials of a quadratic form in Majorana operators.

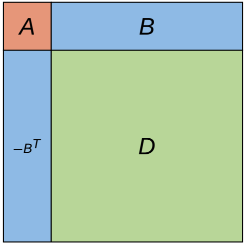
$$\rho = K \exp\left(-\frac{i}{4}\gamma^T G \gamma\right)$$

Covariance matrix

Covariance matrix for a state Φ:

$$\Gamma_{ab} = \frac{i}{2}\left\langle [\gamma_a, \gamma_b] \right\rangle = \frac{i}{2}\frac{\langle \Phi | [\gamma_a, \gamma_b] | \Phi \rangle}{\langle \Phi | \Phi \rangle}$$

Covariance matrices



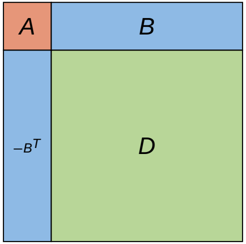
Majorana Fermions

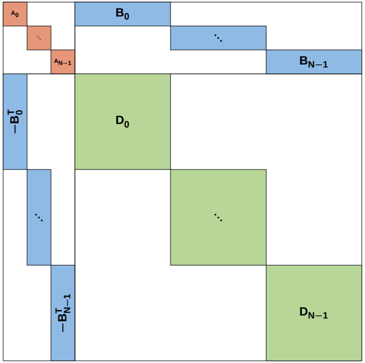
$$\gamma_i^{(1)} = \left(c_i + c_i^\dagger\right)$$
$$\gamma_i^{(2)} = i\left(c_i - c_i^\dagger\right).$$

$$\Gamma_{i,j}^{M} = \frac{i}{2}\left\langle\left[\gamma_i, \gamma_j\right]\right\rangle.$$

Covariance matrices

Single Site	Full system





Calculating the Norm and the Observables

$$|\psi(\mathcal{G})\rangle = \langle\Omega_V| \underbrace{\prod_{\mathbf{x}} \omega(x) \prod_{\mathbf{x}} \mathcal{U}_{\Phi(\mathbf{x})}}_{\rightsquigarrow \Gamma_{\text{in}}(\mathcal{G})} \underbrace{\prod_{\mathbf{x}} A(\mathbf{x})}_{\rightsquigarrow \Gamma_M} |\Omega\rangle$$

$$\Gamma^M_{i,j} = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}$$

A Physical-Physical correlations

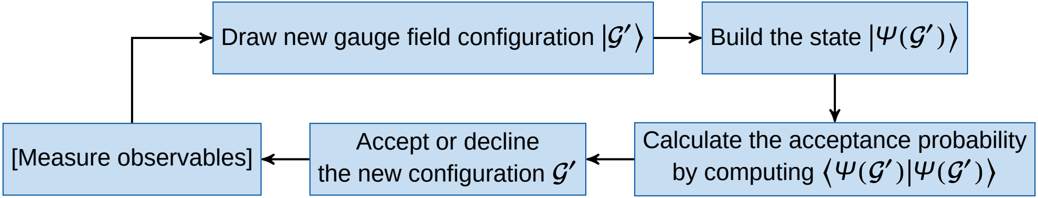
B Physical-Virtual correlations

C Virtual-Virtual correlations

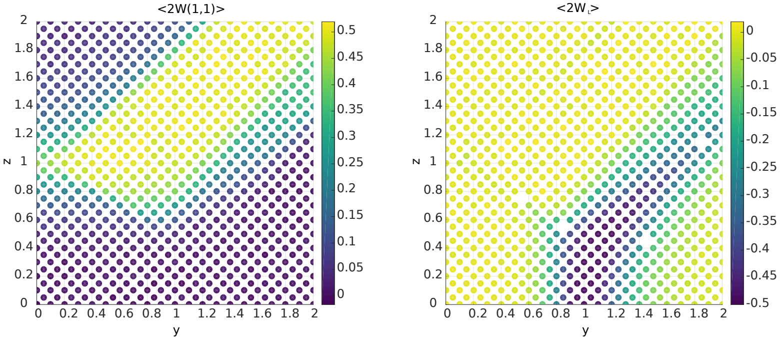
Norm

$$\langle\psi(\mathcal{G})|\psi(\mathcal{G})\rangle = \sqrt{\det\left(\frac{1 - \Gamma_{\text{in}}(\mathcal{G})M_D}{2}\right)}$$

The whole framework



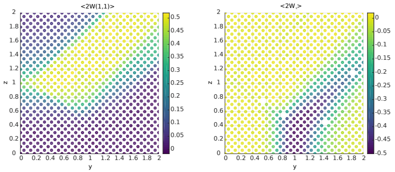
Results for $\mathbb{Z}_3$



Different phases

We can model different phases with our variational Ansatz for the state.

Translationally invariant system



Phase diagram

4 phases that are confining or deconfining static charges

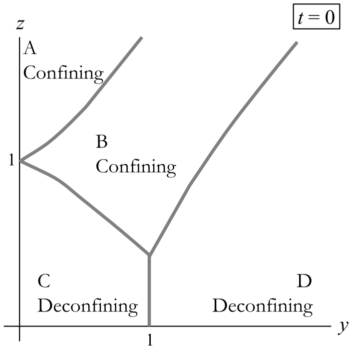
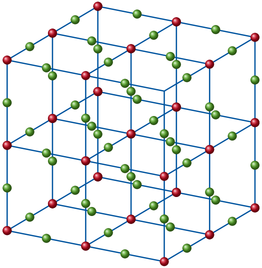
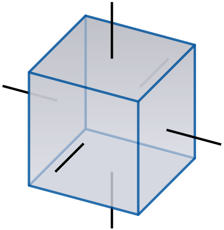


Image taken from Erez Zohar et al., 2015, *Annals of Physics*

Extension to 3 spatial dimensions





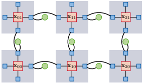
Fermionic state

$$|\psi(\mathcal{G})\rangle = \langle \Omega_V | \prod_x \omega(x) \prod_x \mathcal{U}_{\Phi(x)} \prod_x \mathcal{A}(x) \, |\Omega\rangle$$

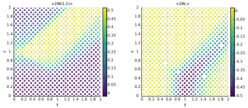
Summary

- A Hamiltonian approach shows promising possibilities (time evolution, finite μ)

- We can construct a gauged gaussian PEPS (GGPEPS) with local building blocks such that the state obeys the gauge symmetry



- The GGPEPS Ansatz shows confined and non-confined phases



- Extension to 3 spatial dimension follows a clear roadmap

Outlook

- Generalization of the Ansatz to 3 spatial dimensions
- Variational minimization of the energy in two and three spatial dimensions
- Optimization of the Monte Carlo procedure for the sampling

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References I

Erez Zohar. "Gauss law, Minimal Coupling and Fermionic PEPS for Lattice Gauge Theories". In: *arXiv:1807.01294 [cond-mat, physics:hep-lat, physics:hep-th, physics:quant-ph]* (July 3, 2018).

Erez Zohar and Michele Burrello. "Building projected entangled pair states with a local gauge symmetry". In: *New Journal of Physics* 18.4 (Apr. 8, 2016), p. 043008. DOI: `10.1088/1367-2630/18/4/043008`.

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