

Scale anomalies and crossovers in low-dimensional fermions

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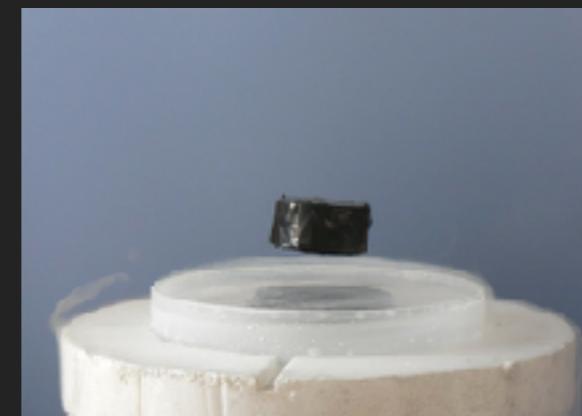
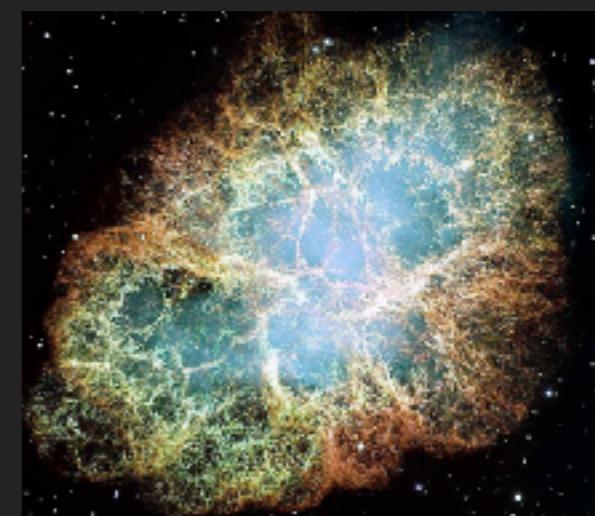
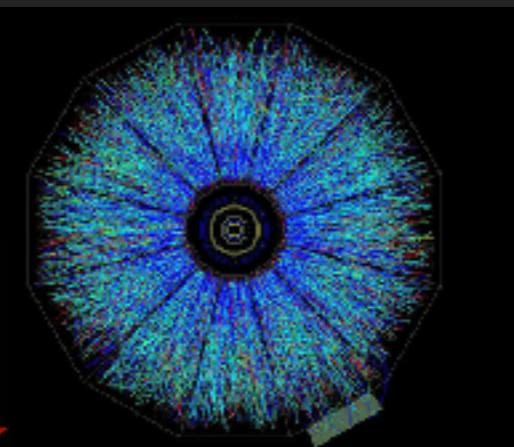
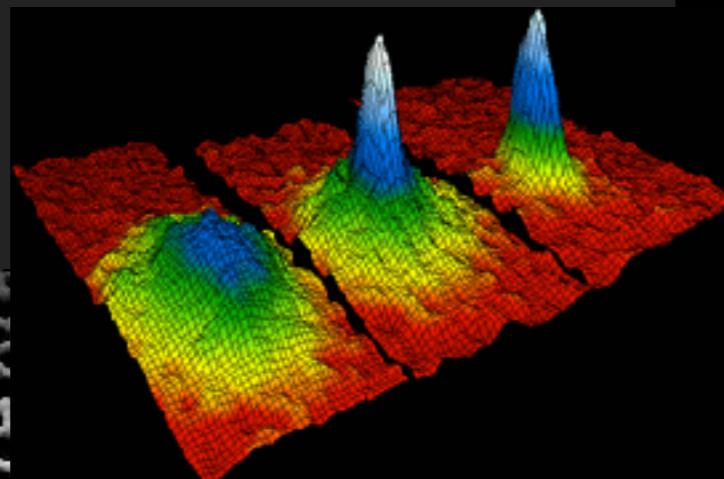
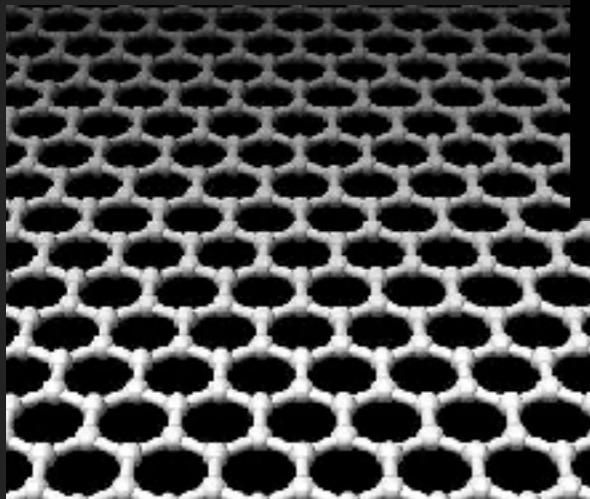
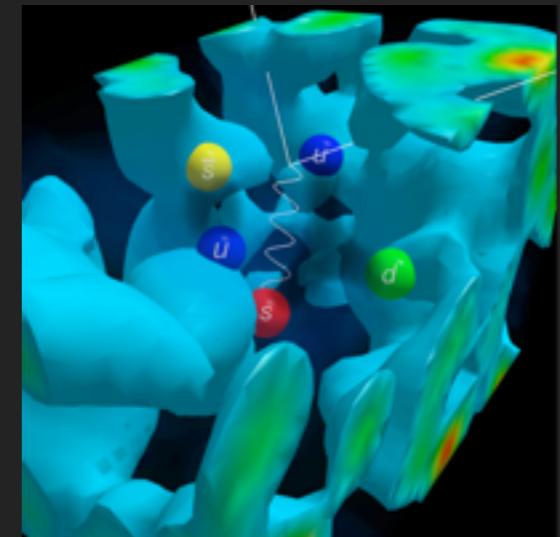
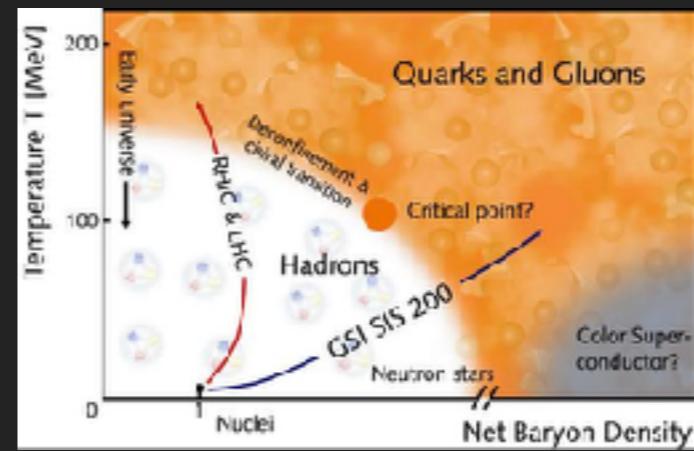
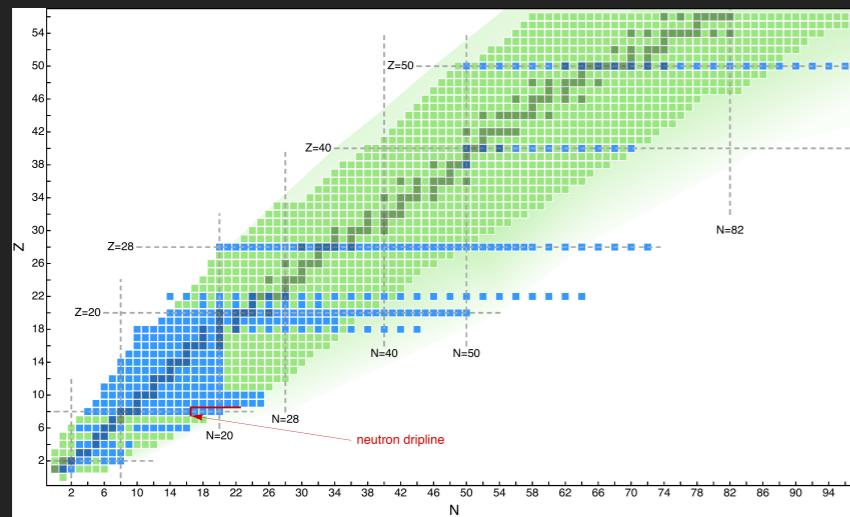


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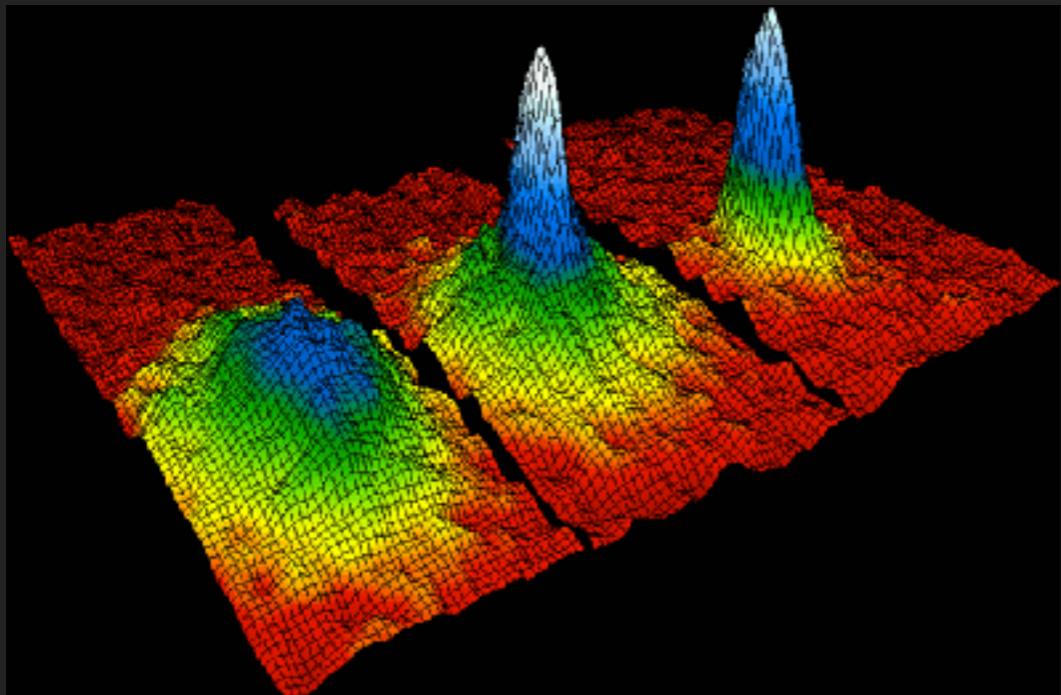
ECT*, Trento, June 2019

Quantum matter...

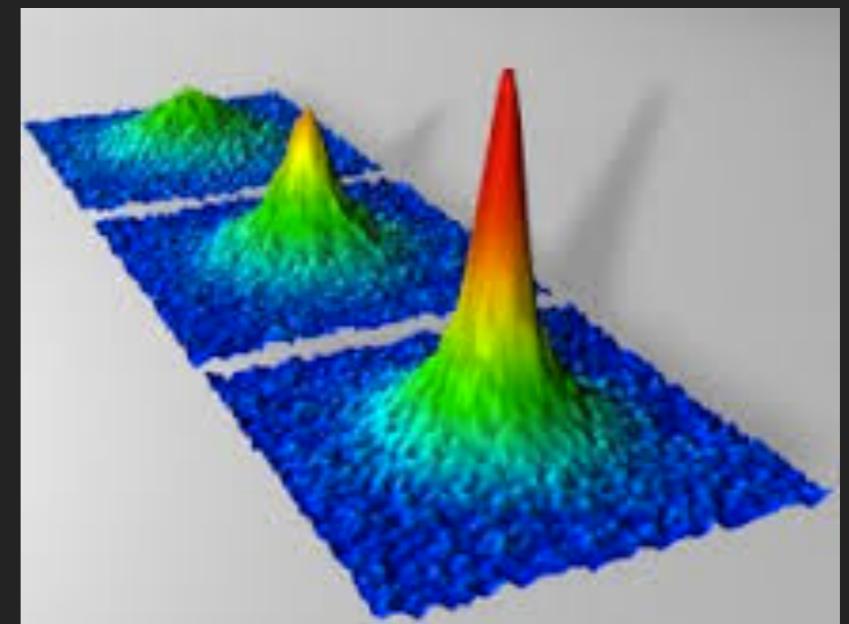


...is everywhere, but...

Ultracold atoms



Bose-Einstein
condensates
(1995)



Fermionic condensates
(2004)

Ultracold atoms

Astonishing degree of control...

- **Temperature** (Superfluid transitions)
- Polarization (LOFF-type phases, polarons)
- **Coupling** (BEC-BCS crossover)
- Shape of external trapping potential
- Mass imbalance (different isotopes)
- **Dimension** (highly anisotropic traps & lattices)
- Bosons, fermions, mixtures: Li, K, Sr, Yb, Dy, Er,...

... and astonishing degree of measurement/detection...

- Thermodynamics
- Phase transitions
- Collective modes
- Spin response
- Hydrodynamic response
- Entanglement
- Time-dependent dynamics
- ...

Many experimental groups around the world!

Outline

- Scale anomalies
 - In **2D** fermions
 - Selected results (from lattice MC calculations):
Thermodynamics, virial coefficients
 - In **1D** fermions
 - Exact mappings
 - Few-body physics & thermodynamics
- Summary and Conclusions

3D Fermions: Hamiltonian & scales

Two species of fermions with a contact two-body force

$$\hat{H} = \int d^3x \left[\sum_{s=\uparrow,\downarrow} \hat{\psi}_s^\dagger(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\psi}_s(\mathbf{x}) - g \hat{n}_\uparrow(\mathbf{x}) \hat{n}_\downarrow(\mathbf{x}) \right]$$

Coupling is dimensionful $[g] = L$ $[m] = 1$

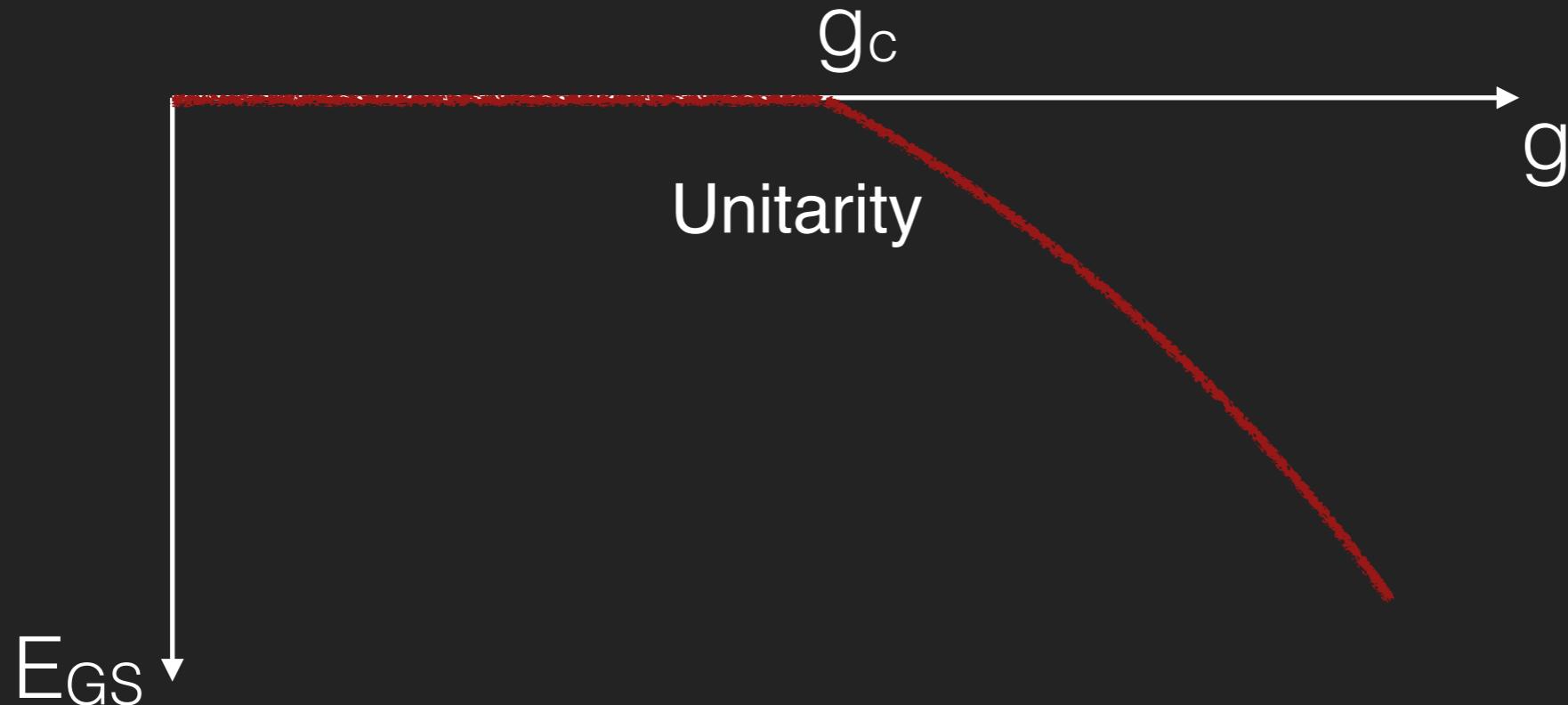
Renormalize by solving the two-body problem and
relating bare coupling to scattering length

$$\frac{1}{g} = \frac{1}{L^3} \sum_k \frac{1}{2\epsilon_k - E}$$

$$p \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} p^2 + \dots$$

Two-body problem

Bound state appears at a critical attractive coupling

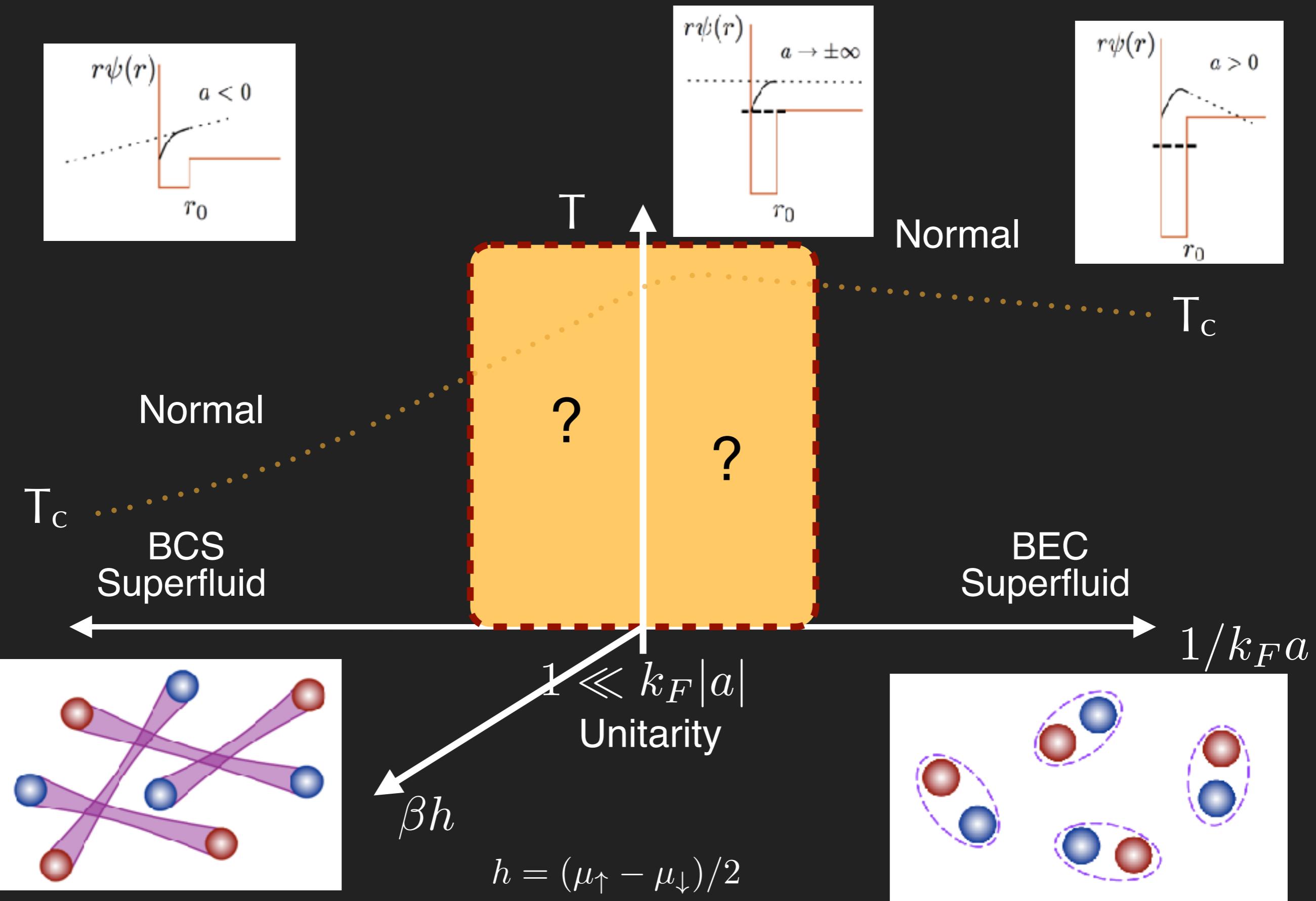


Scattering length and density determine the
physical dimensionless coupling for the
many-body problem

$$1/(k_F a)$$

$$k_F = (3\pi^2 n)^{1/3}$$

The 3D BCS-BEC crossover



Scale invariance & universality

Lack of scales other than the density

$$0 \leftarrow r_0 \ll k_F^{-1} \ll a \rightarrow \infty$$

Dimensionful observables must get their units from powers of k_F : Universality!

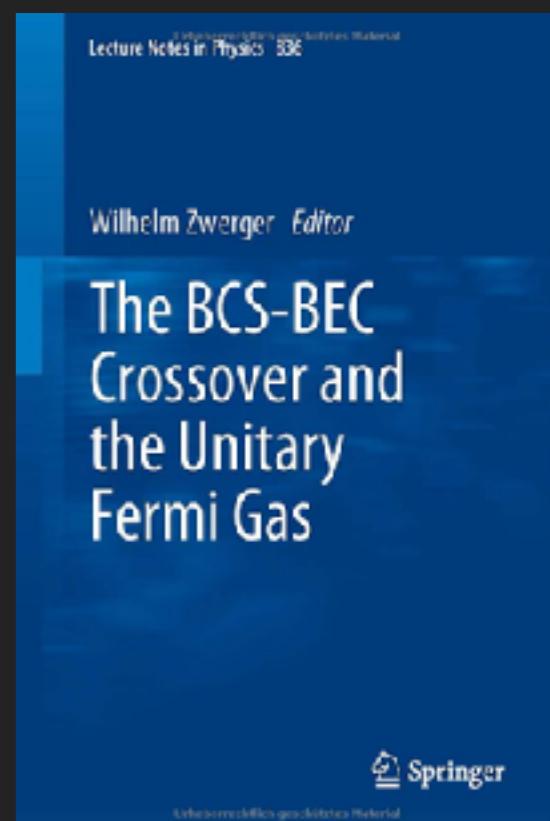
E.g. the ground-state energy

$$E = \xi E_{\text{FG}}$$

$$E_{\text{FG}} = \frac{3}{5} N \epsilon_F$$

Bertsch parameter

$$\epsilon_F = \frac{k_F^2}{2}$$



Scale anomalies



2D Fermions: Hamiltonian & scales

Two species of fermions with a contact two-body force

$$\hat{H} = \int d^2x \left[\sum_{s=\uparrow,\downarrow} \hat{\psi}_s^\dagger(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\psi}_s(\mathbf{x}) - g \hat{n}_\uparrow(\mathbf{x}) \hat{n}_\downarrow(\mathbf{x}) \right]$$

Coupling is dimensionless

$$[g] = 1$$

$$[m] = 1$$

Classically scale invariant!

Again: renormalize by solving two-body system

Factor out center-of-mass motion and solve “relative” problem:

$$\left[\frac{-\nabla^2}{2\bar{m}} - g\delta(\mathbf{x}) \right] \phi(\mathbf{x}) = E_r \phi(\mathbf{x})$$

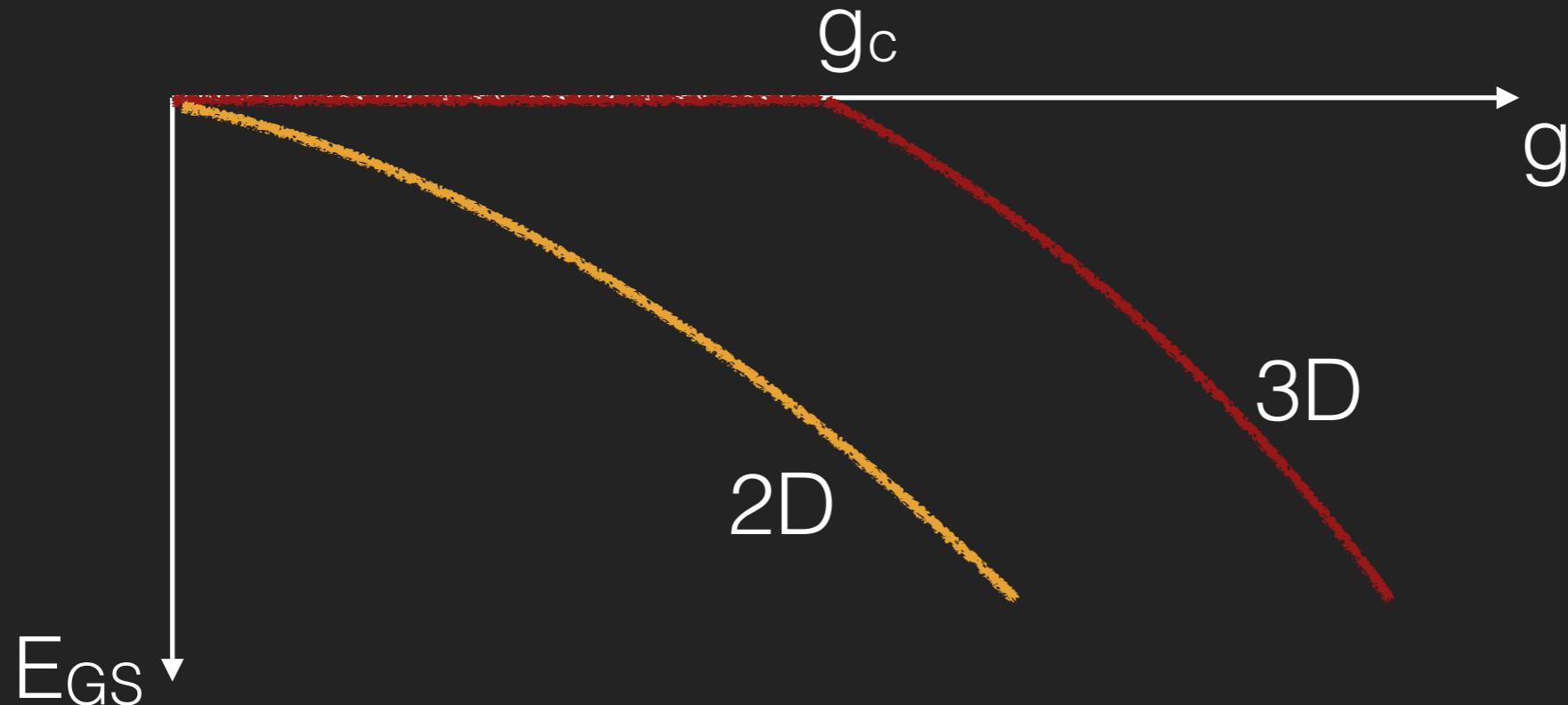
Bound-state energy

$$\epsilon_B = \Lambda e^{-4\pi/|g|}$$

Quantum mechanically **not** scale invariant

Two-body problem & anomaly

Cutoff required, bound state exists for all attractive couplings



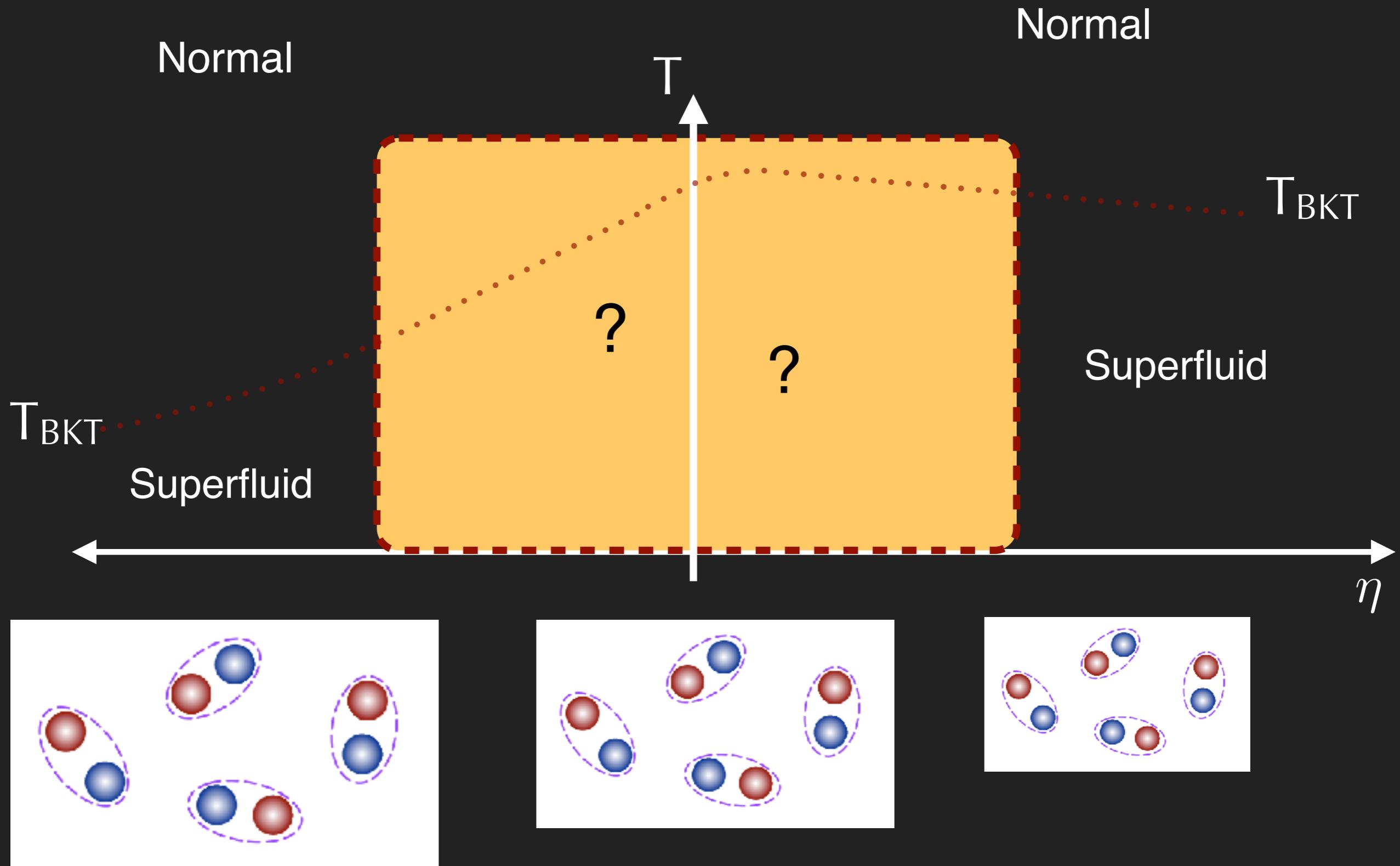
The binding energy represents a **scale anomaly**.

Binding energy and density determine the
dimensionless physical coupling

$$\epsilon_F \propto n$$

$$\eta = \frac{1}{2} \ln(2\epsilon_F/\epsilon_B)$$

The 2D BCS-BEC crossover



Selected results in 2D

Ground state, thermodynamics, contact

G. Bertaina and S. Giorgini,
Phys. Rev. Lett. **106**, 110403 (2011).

H. Shi, S. Chiesa, and S. Zhang,
Phys. Rev. A **92**, 033603 (2015).

A. Galea, H. Dawkins, S. Gandolfi, A. Gezerlis,
Phys. Rev. A **93**, 023602 (2016).

E. R. Anderson, J. E. Drut
Phys. Rev. Lett. **115**, 115301 (2015).

L. Rammelmüller, W. J. Porter, J. E. Drut
Phys. Rev. A **93**, 033639 (2016).

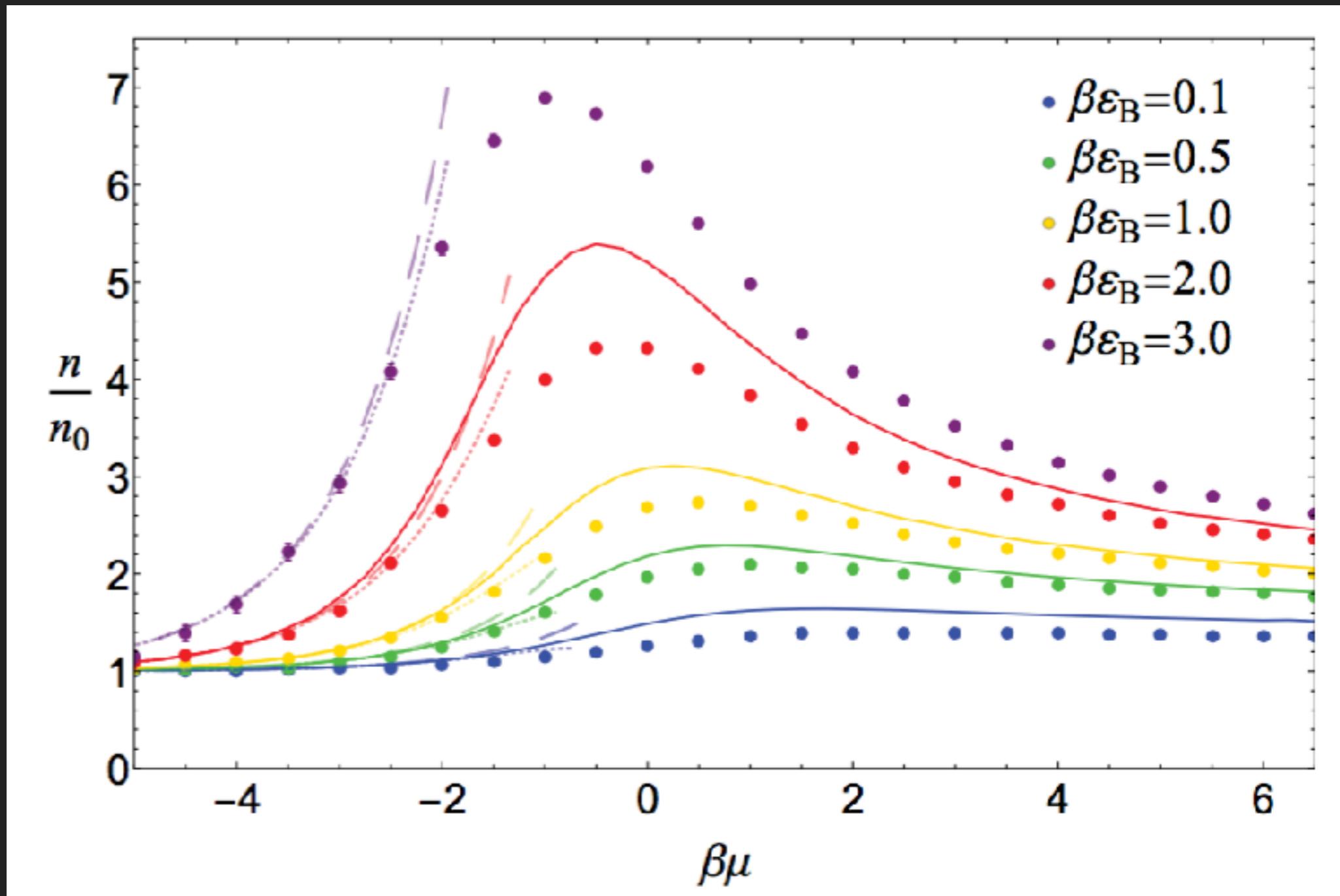
Z.-H. Luo, C. E. Berger, J. E. Drut
Phys. Rev. A **93**, 033604 (2016).

M. Bauer, M. M. Parish, and T. Enss,
Phys. Rev. Lett. **112**, 375 135302 (2014).

J. Hofmann,
Phys. Rev. Lett. **108**, 185303 (2012).

E. Taylor and M. Randeria,
Phys. Rev. Lett. **109**, 135301 (2012).

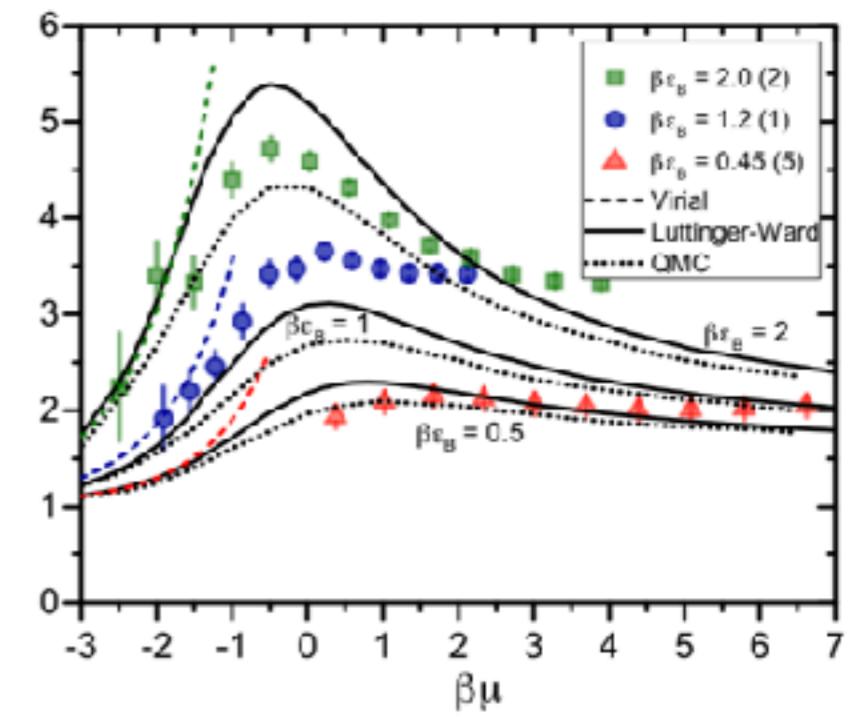
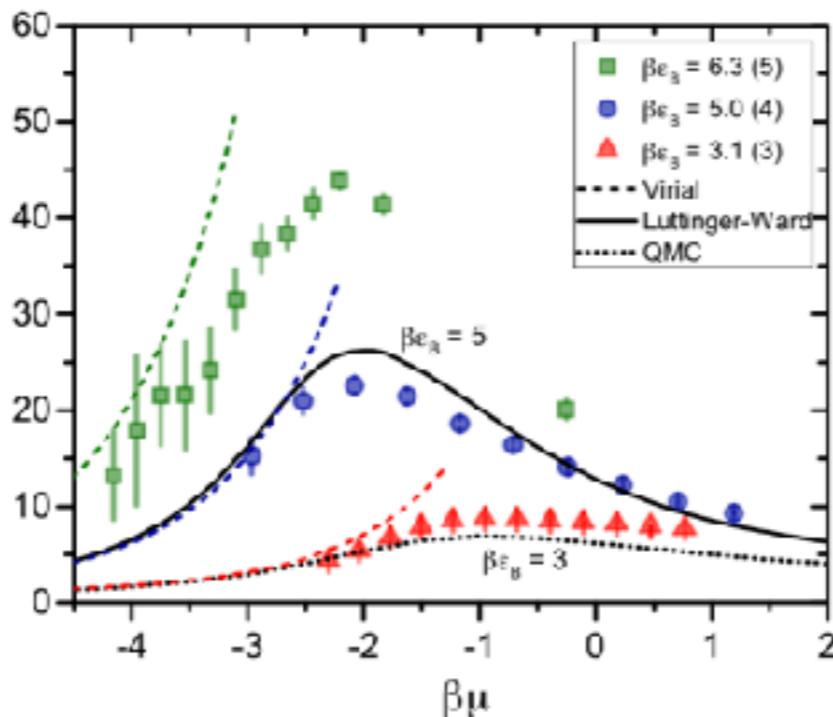
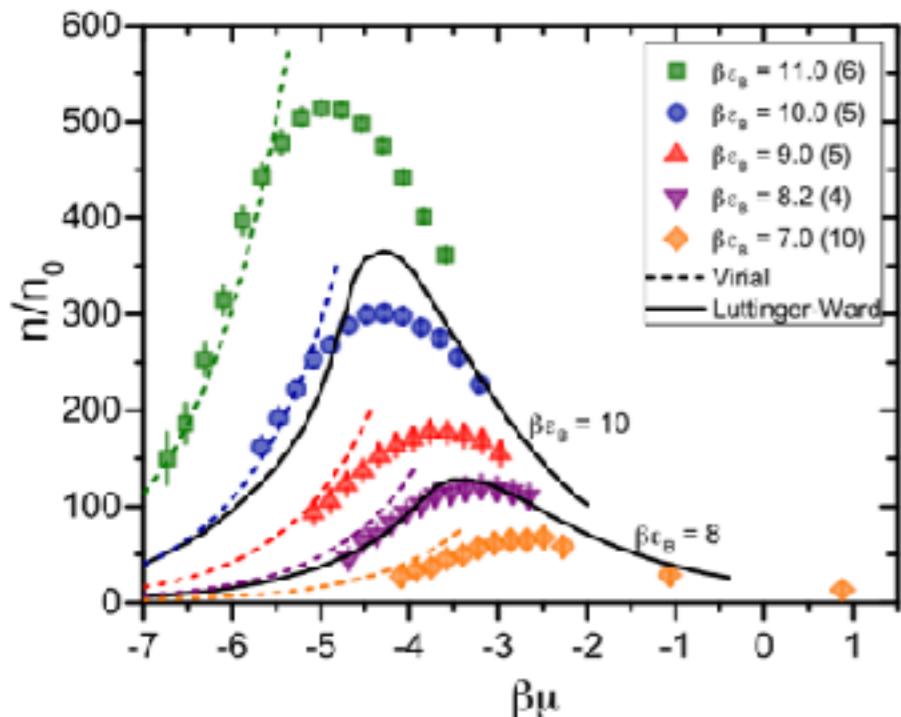
Results: Density EoS



E. R. Anderson, J. E. Drut
Phys. Rev. Lett. **115**, 115301 (2015).

Results: Density EoS

Aside: Experimental results

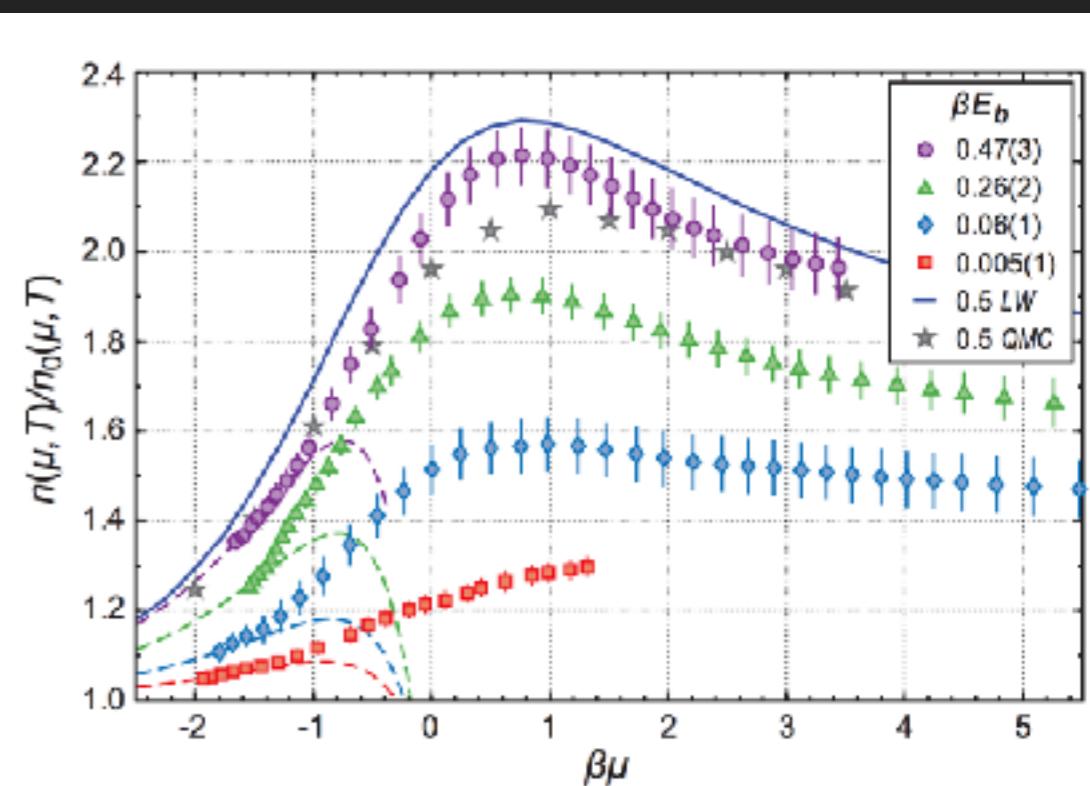


Phys. Rev. Lett. **116**, 045303 (2016)

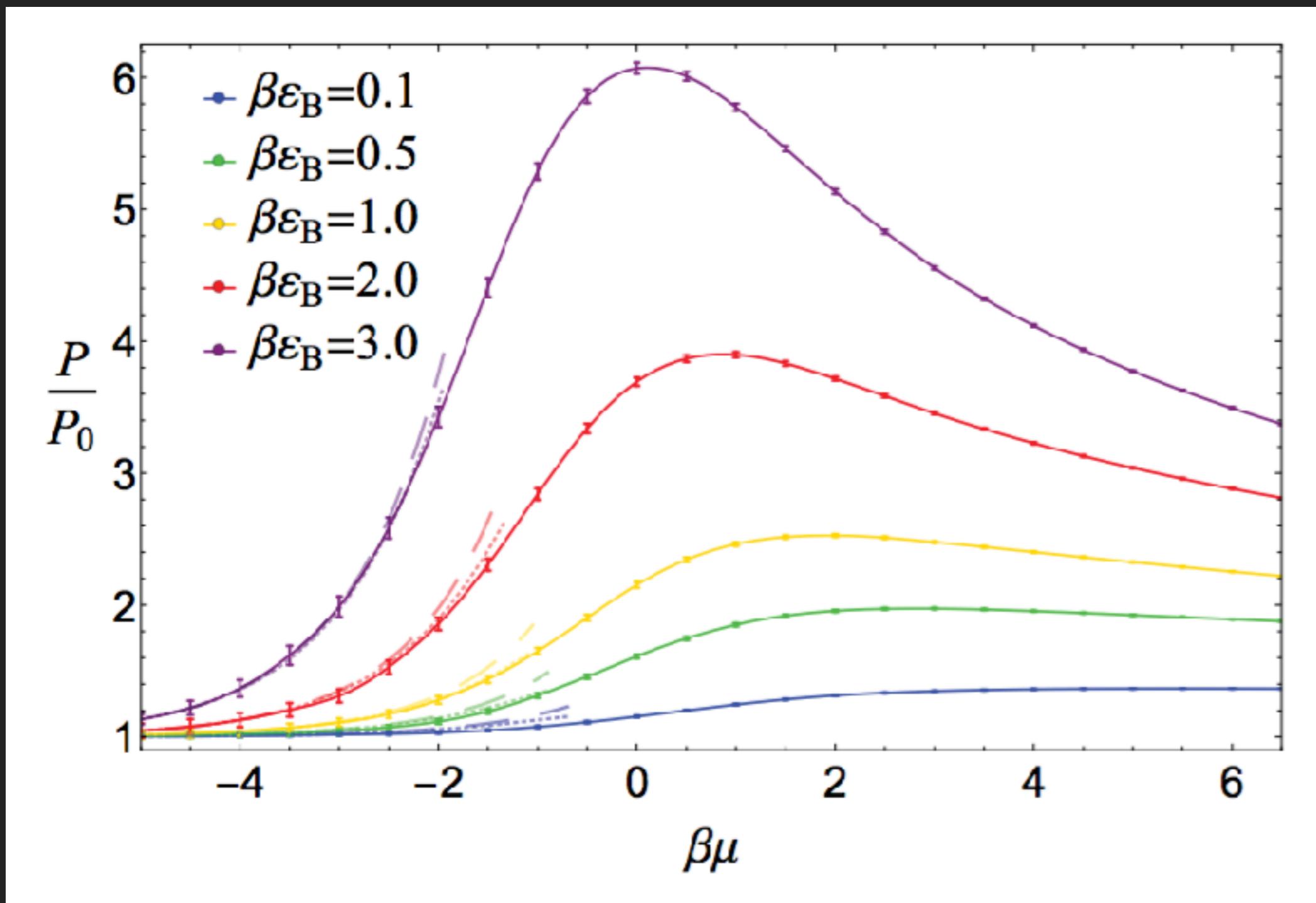
Jochim's group

Vale's group

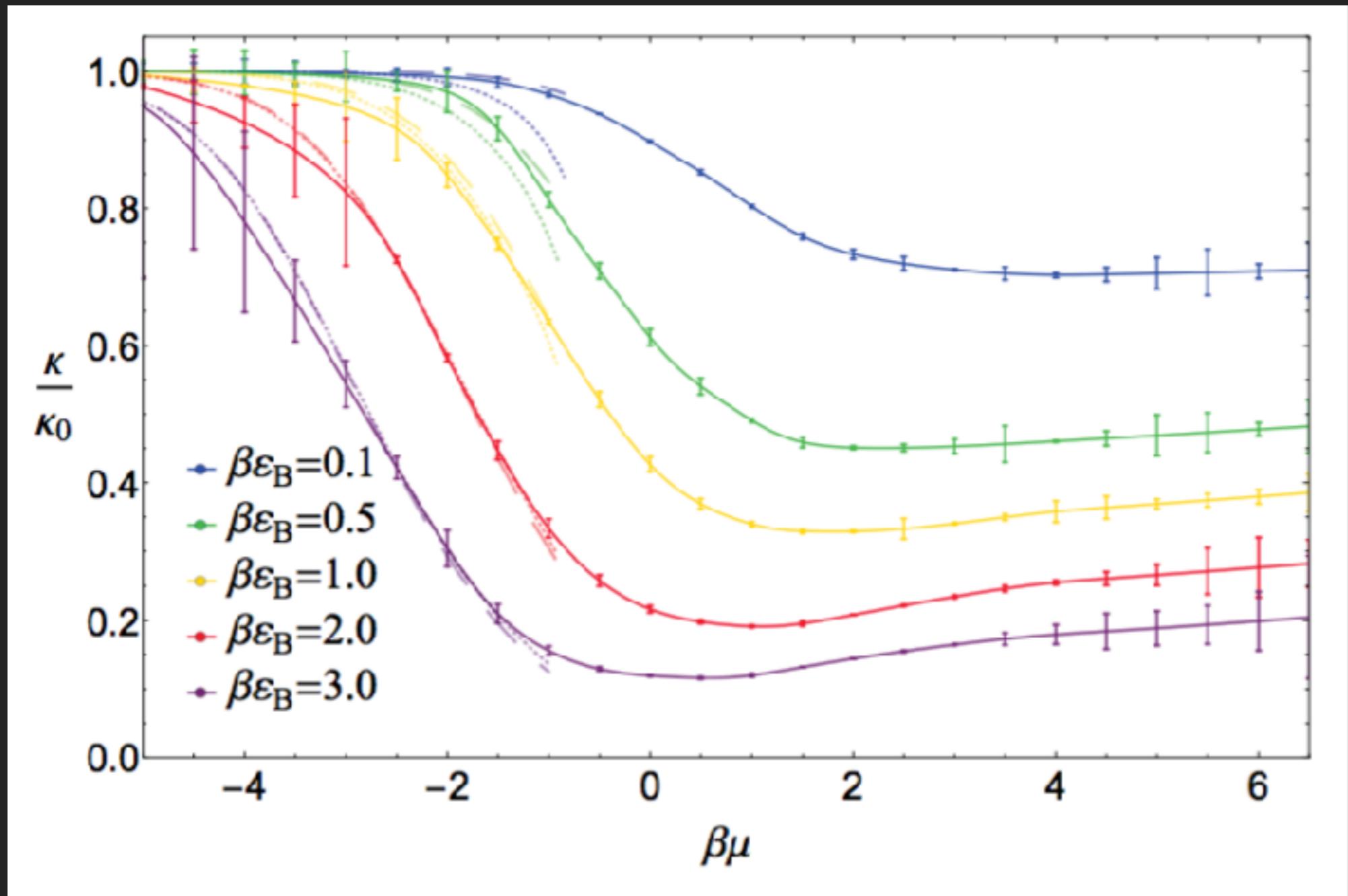
Phys. Rev. Lett. **116**, 045302 (2016)



Results: Pressure EoS



Results: Compressibility



$$\kappa = \frac{\beta}{n^2} \left. \frac{\partial n}{\partial(\beta\mu)} \right|_{\beta}$$

E. R. Anderson, J. E. Drut
Phys. Rev. Lett. **115**, 115301 (2015).

Results: Density EoS

Virial expansion (relative to noninteracting case)

$$-\beta\Delta\Omega = \ln(\mathcal{Z}/\mathcal{Z}_0) = Q_1 \sum_{n=2}^{\infty} \Delta b_n z^n$$

Determines the thermodynamics at low fugacity $z = e^{\beta\mu}$

$\Delta b_n = b_n - b_n^{(0)}$ are typically computed by solving the n-body problem

Δb_2 : Known from Beth-Uhlenbeck formula

Δb_3 : Determined numerically with exact methods

V. Ngampruetikorn, J. Levinsen, and M. M. Parish,
Phys. Rev. Lett. **111**, 265301 (2013).

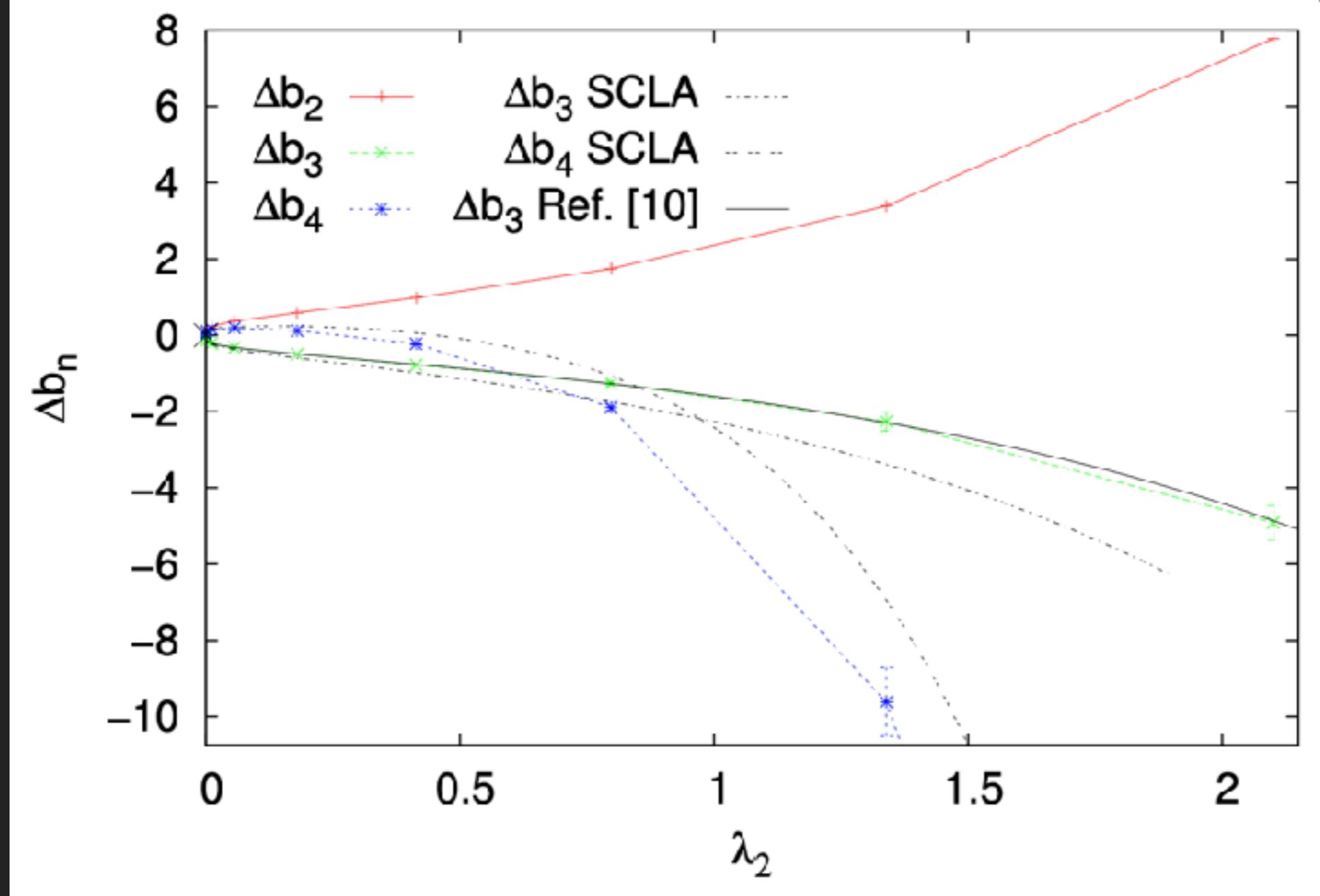
Results: Virial coefficients

Via complex Langevin and a semiclassical lattice approximation (SCLA)

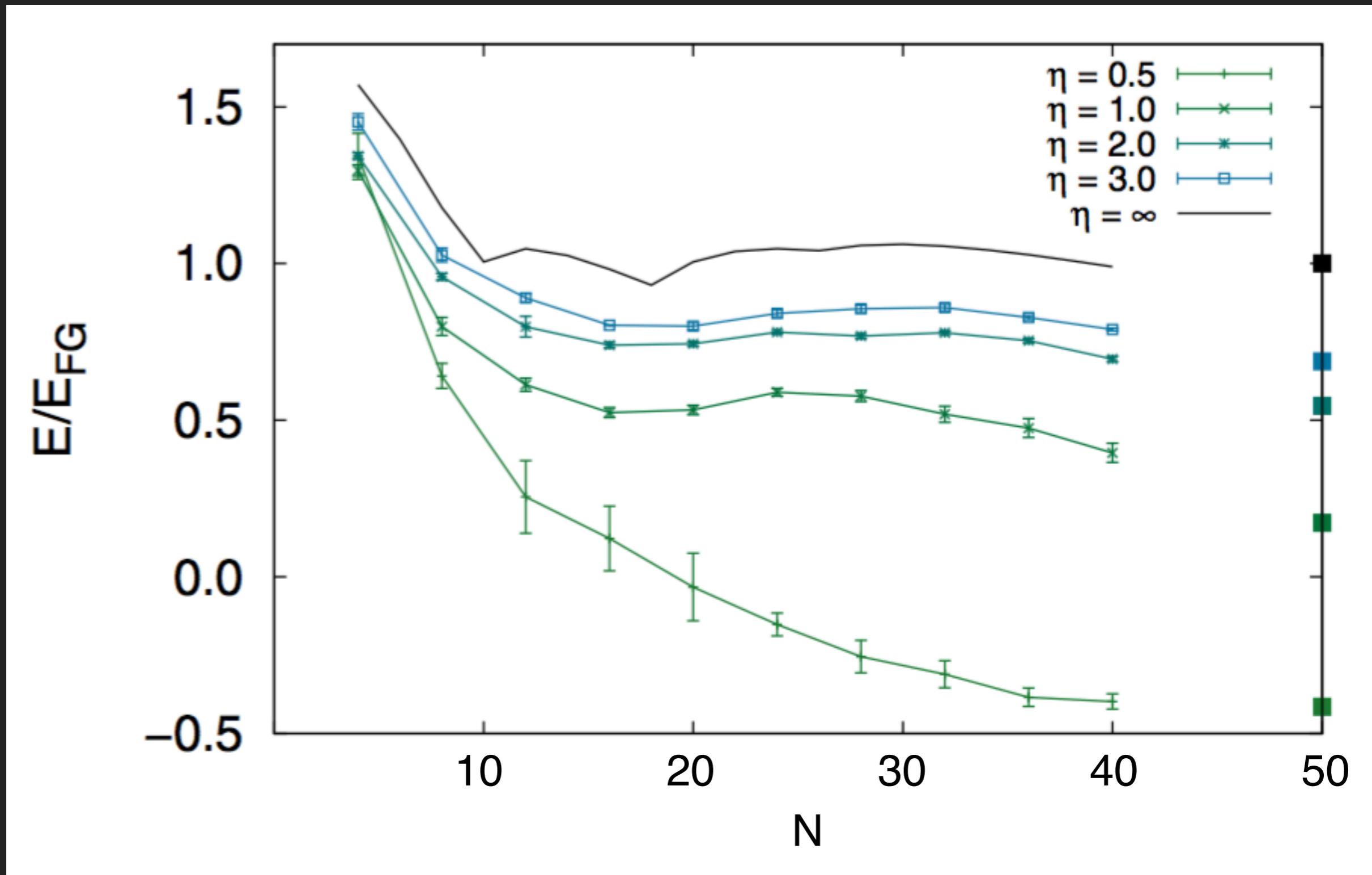
C. R. Shill, J. E. Drut
Phys. Rev. A **98**, 053615 (2018).

LO-SCLA

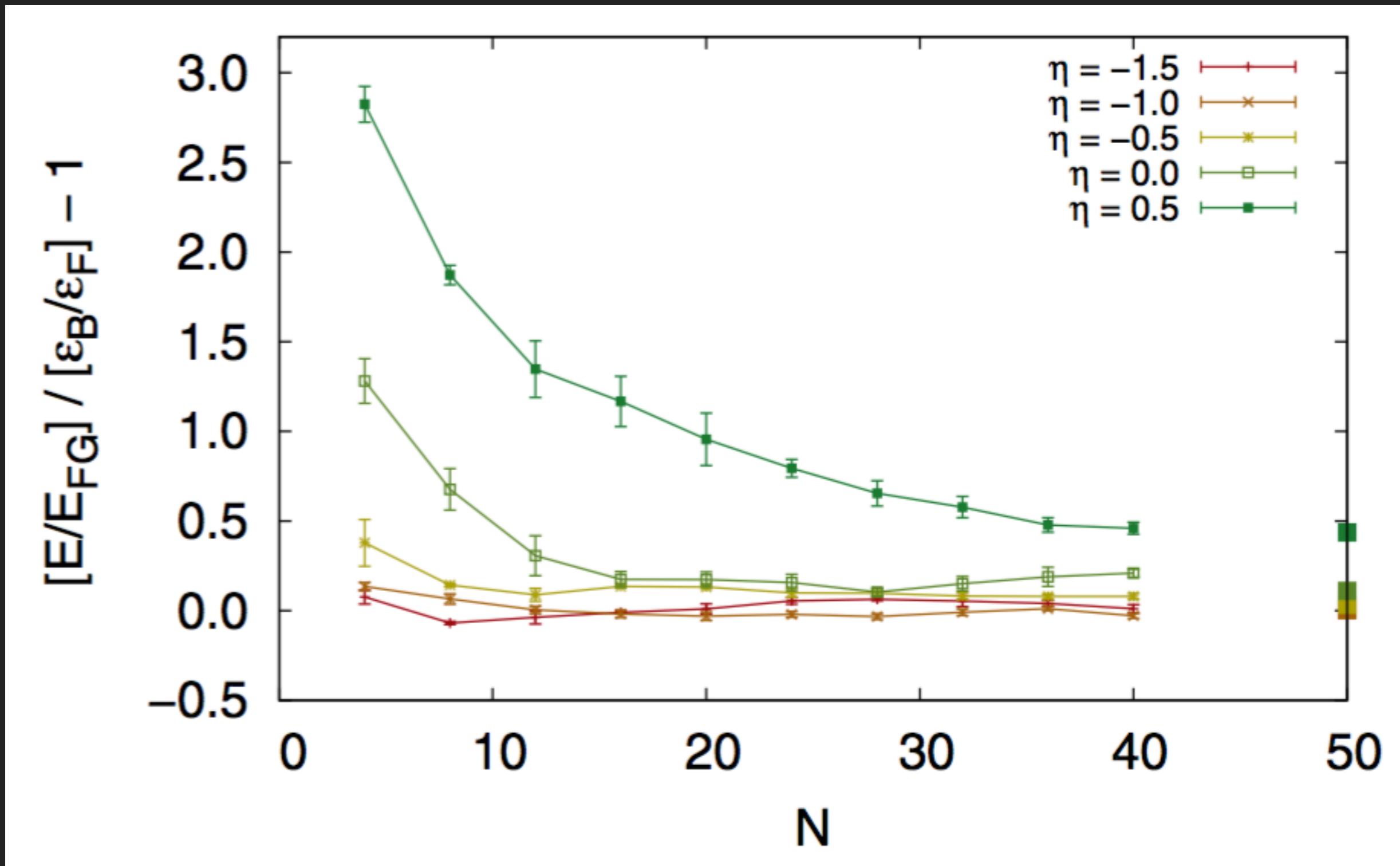
$$\begin{aligned}\Delta b_3 &= -2^{1-d/2} \Delta b_2, \\ \Delta b_4 &= 2(3^{-d/2} + 2^{-d-1}) \Delta b_2 \\ &\quad + 2^{1-d/2} (2^{-d-1} - 1) (\Delta b_2)^2\end{aligned}$$



Results: GS Energetics



Results: GS Energetics



An anomalous system in 1D

Work in collaboration with

Josh McKenney



THE UNIVERSITY
of NORTH CAROLINA
at CHAPEL HILL

W. Daza

C. Lin

C. Ordóñez

UNIVERSITY of
HOUSTON

J. E. Drut, J. R. McKenney, W. Daza, C. Lin, C. Ordóñez
Phys. Rev. Lett. **120**, 243002 (2018).

An anomalous system in 1D

Bosonic version studied by other groups: Droplets!

Y. Sekino and Y. Nishida,
Phys. Rev. A **97**, 011602R (2018).

Y. Nishida
Phys. Rev. A **97**, 061603R (2018).

L. Pricoupenko,
Phys. Rev. A **97**, 061604R (2018).

G. Guijarro, A. Pricoupenko, G. E. Astrakharchik, J. Boronat, and D. S. Petrov
Phys. Rev. A **97**, 061605R (2018).

Droplets in 2D and in 3D at unitarity also studied by many groups!

An anomalous system in 1D

Three species of fermions with a contact three-body force

$$\hat{H} = \int dx \left[\sum_{s=1,2,3} \hat{\psi}_s^\dagger(x) \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\psi}_s(x) - g \hat{n}_1(x) \hat{n}_2(x) \hat{n}_3(x) \right]$$

Three species: 1, 2, 3

It is 1D, but not amenable to Bethe Ansatz.

Only a contact three-body force (nothing else!)

Coupling is dimensionless!

$$[g] = 1$$

$$[m] = 1$$

Does a 3-body bound state form?

Is there an anomaly?

Three-body problem & anomaly

Mapping to two-dimensional one-body problem

$$\left[-\frac{\nabla_X^2}{2m} + g\delta(x_2 - x_1)\delta(x_3 - x_2) \right] \psi(X) = E\psi(X)$$

$$X = (x_1, x_2, x_3)$$

Separating center-of-mass and relative motion...

$$\left[-\frac{\nabla_q^2}{2\bar{m}} + \tilde{g}\delta(q_1)\delta(q_2) \right] \phi(q_1, q_2) = E_r\phi(q_1, q_2)$$

$$\tilde{g} = (2/\sqrt{3})g$$

$$Q = \frac{1}{3}(x_1 + x_2 + x_3)$$

Scale-anomalous 2D problem!

$$q_1 = x_2 - x_1$$

Trimers form!

$$q_2 = \frac{1}{\sqrt{3}}(x_1 + x_2 - 2x_3)$$

Three-body problem & anomaly

Other cases?

Coupling units (contact n-body, d spatial dimensions)

$$[g] = L^{d(n-1)-2}$$

When is it dimensionless?

$$d(n - 1) = 2 \quad \longrightarrow \quad \begin{aligned} d = 1 &\quad \& \quad n = 3 \\ d = 2 &\quad \& \quad n = 2 \end{aligned}$$

$$\epsilon_B \sim \Lambda e^{-4\pi/|g|}$$

Only two cases!

$$\frac{\partial g}{\partial \ln(\beta \epsilon_B)} \propto g^2$$



Side note: What if we allow derivatives?

At least one more possibility

Coupling units (contact n-body, d spatial dimensions, p derivatives)

$$[g] = L^{d(n-1)+p-2}$$

When is it dimensionless?

$$d(n - 1) = 2 - p \rightarrow \begin{cases} d = 1 & \& n = 2 & \& p = 1 \\ d = 1 & \& n = 3 & \& p = 0 \\ d = 2 & \& n = 2 & \& p = 0 \end{cases}$$

$$\epsilon_B \sim \Lambda g^4$$

1D two-body
derivative-delta
interaction

$$\epsilon_B \sim \Lambda e^{-4\pi/|g|}$$

(previous slides)

Thermodynamics and contact

Exact properties

Virial coefficients

No interaction unless 3 or more particles present...

$$\rightarrow \Delta b_2 = 0$$

Equivalence of the 1D 3-body and 2D 2-body problems...
(in relative coordinates)

$$\rightarrow \boxed{\Delta b_3 = \frac{1}{\sqrt{3}} \Delta b_2^{(2D)}}$$

Known from Beth-Uhlenbeck formula

Thermodynamics and contact

Exact properties

Truly scale invariant

$$P = \beta^\alpha f(\beta\mu)$$

$$\alpha = -d/2 - 1$$

$$P = \frac{2}{d} \frac{E}{V}$$

Anomalous

$$P = \beta^\alpha f(\beta\mu, \beta\epsilon_B)$$

$$P - \frac{2}{d} \frac{E}{V} = \frac{2}{d} \beta^\alpha \frac{\partial f}{\partial \ln(\beta\epsilon_B)}$$

$$\mathcal{C}_3 = - \frac{\partial g}{\partial \ln(\beta\epsilon_B)} \langle \hat{n}_1 \hat{n}_2 \hat{n}_3 \rangle$$

“Contact density”
of our 1D problem

Thermodynamics and contact

Exact properties in a harmonic trap

Truly scale invariant

$$\Omega = \omega f(\beta\mu, \beta\omega)$$

$$E = 2V_{\text{ext}}$$

Anomalous

$$\Omega = \omega f(\beta\mu, \beta\omega, \beta\epsilon_B)$$

$$E - 2V_{\text{ext}} = \omega \frac{\partial f}{\partial \ln(\beta\epsilon_B)}$$

$$C_3 = \frac{\partial g}{\partial \ln(\beta\epsilon_B)} \int dx \langle \hat{n}_1 \hat{n}_2 \hat{n}_3 \rangle$$

“Contact” of our 1D problem

Thermodynamics and contact

Toward the many-body problem

The path-integral representation of the partition function
requires a Hubbard-Stratonovich transformation

For the usual two-body force case:

$$e^{\tau g_2 \hat{n}_1(\mathbf{x}) \hat{n}_2(\mathbf{x})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma (1 + A \hat{n}_1(\mathbf{x}) \sin \sigma)(1 + A \hat{n}_2(\mathbf{x}) \sin \sigma),$$

interaction

auxiliary field

$$A = \sqrt{2(e^{\tau g_2} - 1)}$$

$$\mathcal{Z} = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} = \int \mathcal{D}\sigma \det^2 M[\sigma]$$

Thermodynamics and contact

Toward the many-body problem

The path-integral representation of the partition function
requires a Hubbard-Stratonovich transformation

For the three-body force case:

$$e^{\tau g_3 \hat{n}_1(\mathbf{x}) \hat{n}_2(\mathbf{x}) \hat{n}_3(\mathbf{x})} = \frac{1}{3\pi} \int d\sigma (1 + B\hat{n}_1(\mathbf{x})F(\sigma))(1 + B\hat{n}_2(\mathbf{x})F(\sigma))(1 + B\hat{n}_3(\mathbf{x})F(\sigma))$$

interaction

auxiliary field

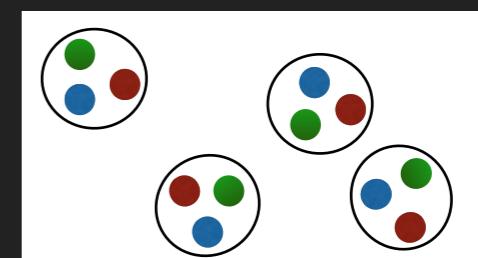
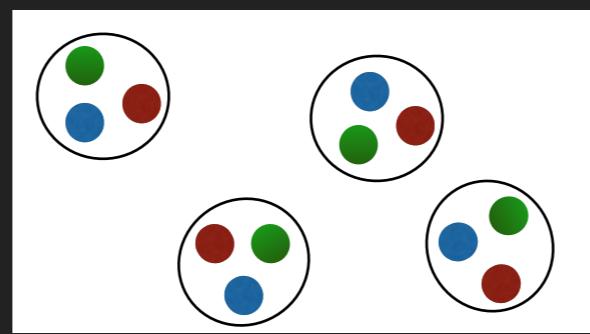
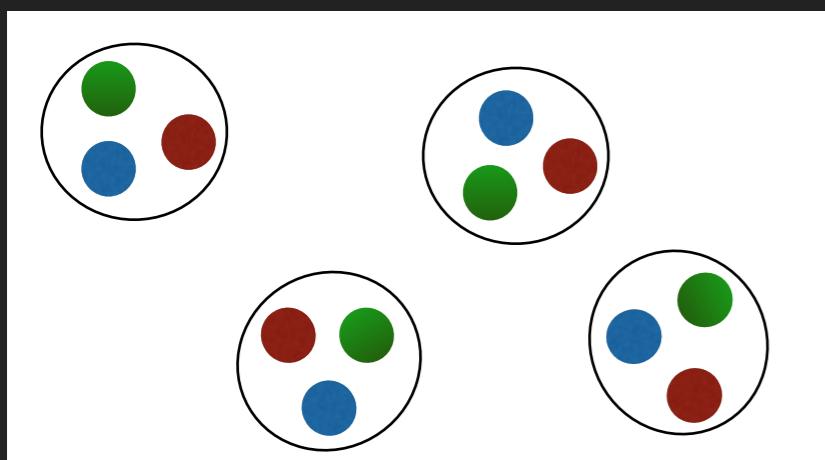
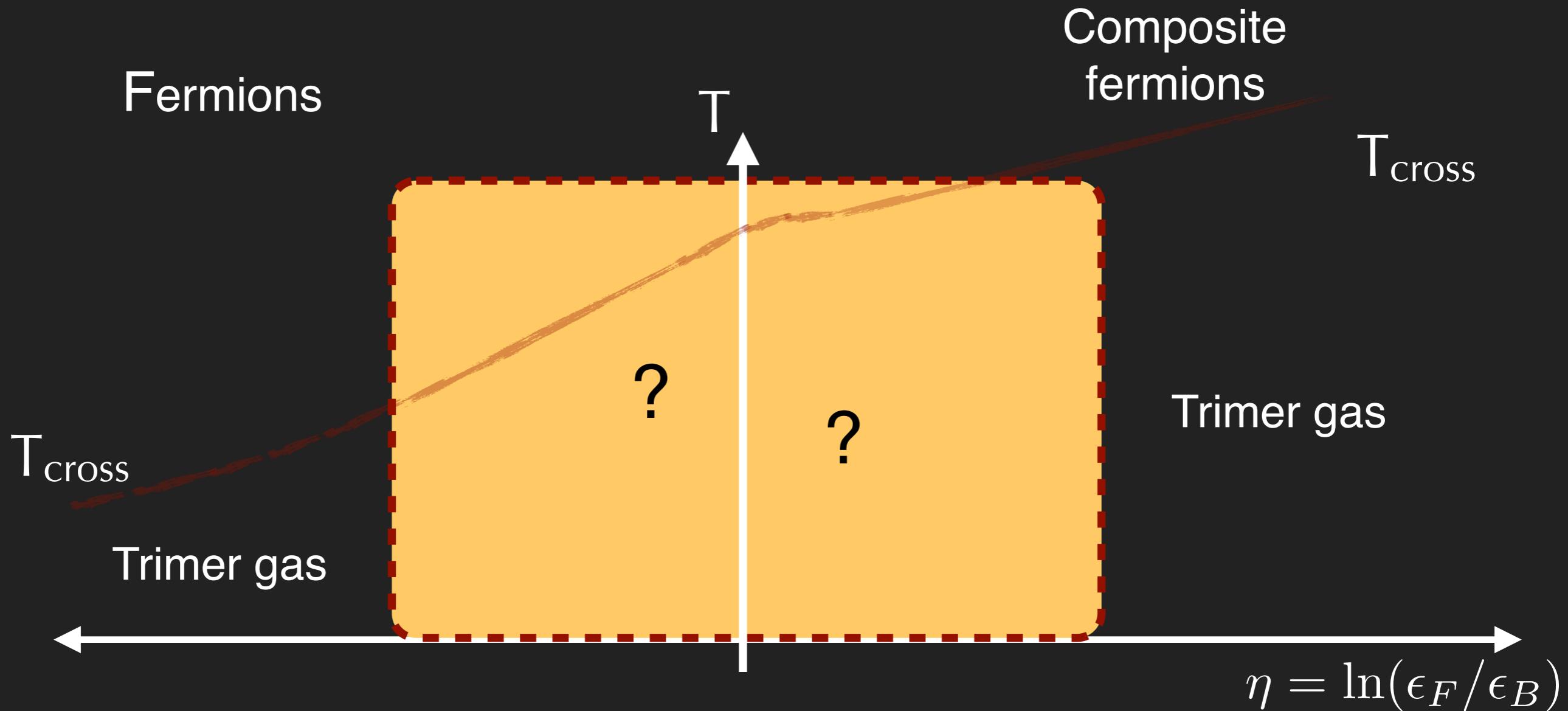
$$F(\sigma) = e^{i2\sigma/3} \cos^2 \sigma$$

$$B = 1.63\dots (e^{\tau g_3} - 1)^{1/3}$$

$$\mathcal{Z} = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} = \int \mathcal{D}\sigma \det^3 M[\sigma]$$

Straightforwardly generalized to n-body forces. But: sign problem.

The 1D trimer crossover

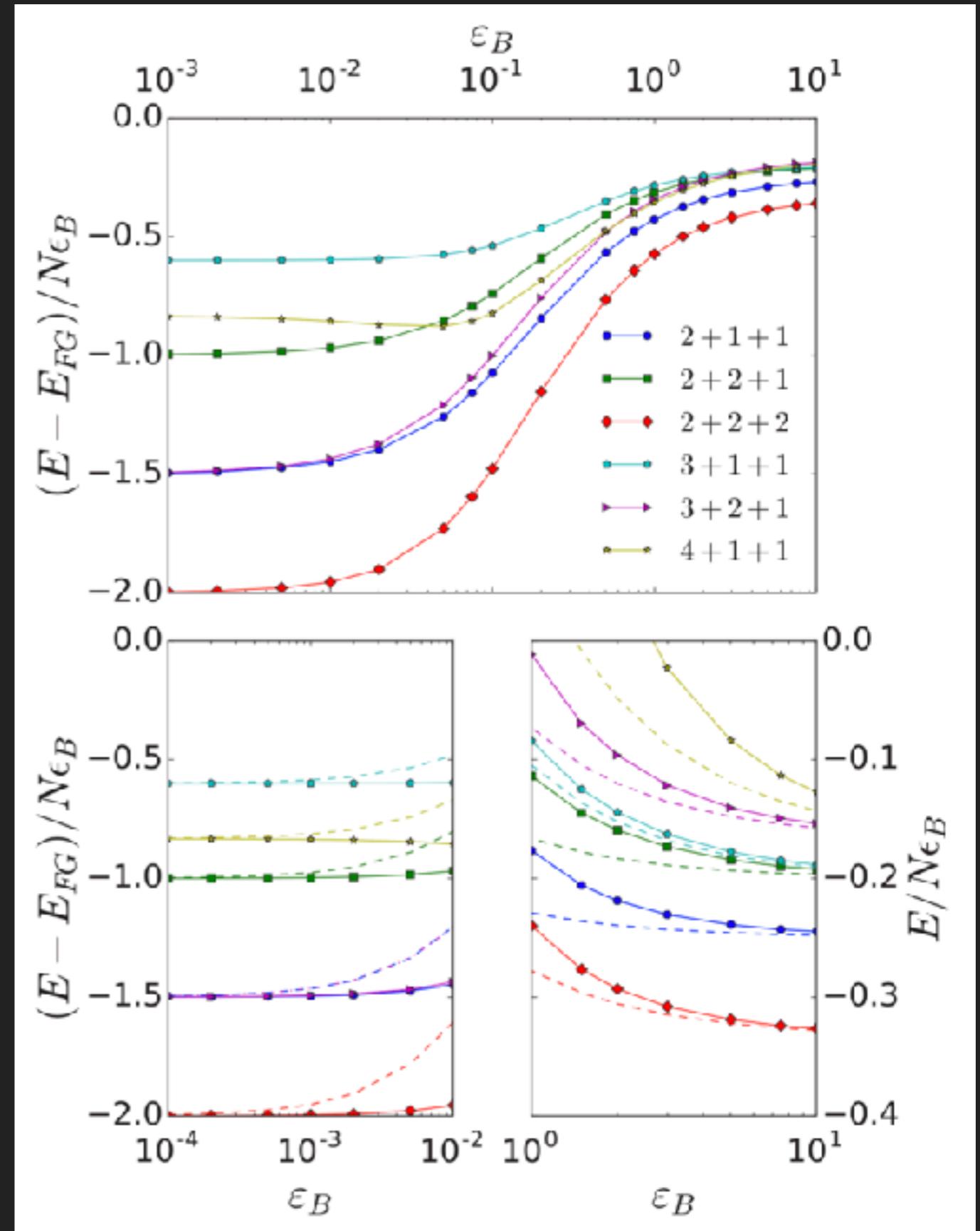


The 1D trimer crossover: few body case

J. R. McKenney and J. E. Drut,
Phys. Rev. A **99**, 013615 (2019).

Exact diagonalization
on the lattice

Computed the ground-state
properties of all 4-, 5- and 6-particle
systems



Open questions

How can we realize this system experimentally?

What is the effect of asymmetries?

Can we induce superfluid correlations?

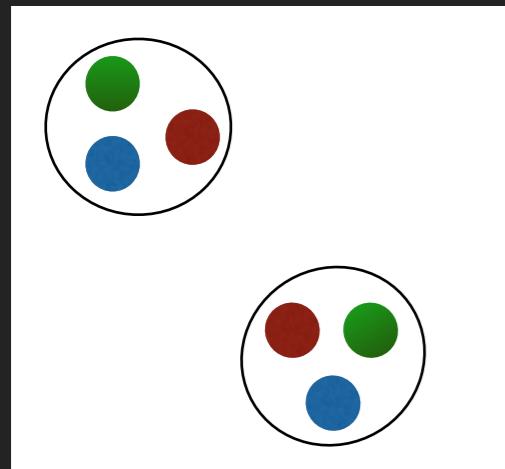
How to deal with sign problem?

What are the transport properties?

Increasing flavor number

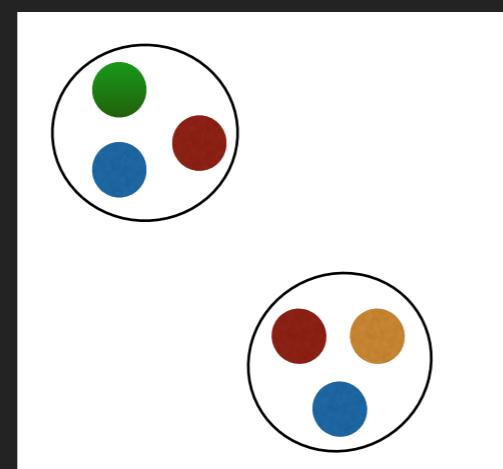
When do higher-body bound states begin to form?

We know trimers don't form hexamers with only **3 flavors**



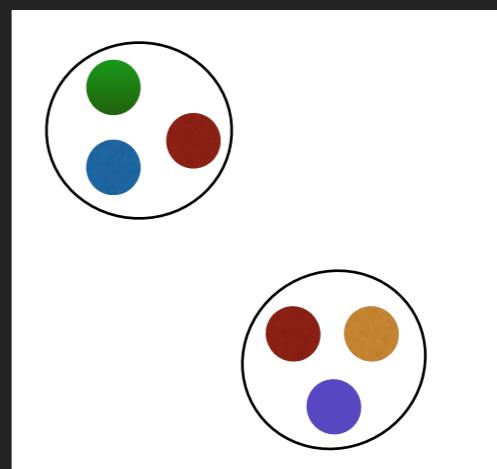
We know 6 bosons do form a bound state

What about with **4 flavors**?



Still no :(

What about with **5 flavors**?



Yes! ;-)

Summary

- There are two possible scale-anomalous non-relativistic systems with contact interactions: 2D with 2-body forces
1D with 3-body forces

Other cases are possible if we allow derivatives.
- We have a first a characterization of the thermodynamics and contact of these systems, both in the ground state and at finite temperature, complementing other approaches and comparing with experiments.
- There is room for improvement in the 2D case at finite temperature.
- Treating the 1D many-body case with 3-body forces in a non-perturbative way presents a sign problem that remains open.
- We have derived universal relations and virial theorems involving the 3-body contact, and determined b_3 .
- We have begun the characterization of the 1D few-body case.

Thank you!