Rotor formulation of 3D $U(1)$ gauge theory and quantum simulation with ultra cold atoms in an optical lattice

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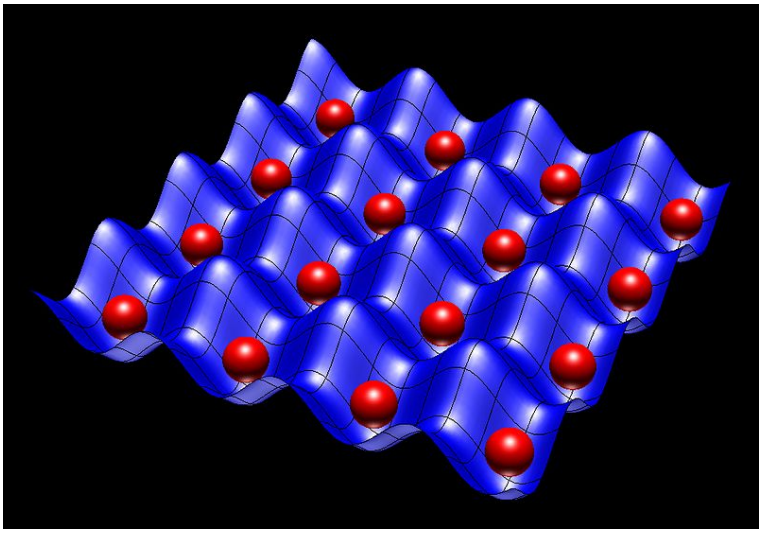
arXiv:1803.11166, 1807.09186, 1811.05884

High energy physics at ultra cold temperatures: June 13th, 2019

Motivation

The situation:

- quantum computing (QC) will solve many issues with classical computing for lattice gauge theory.
- The scaling for digital QC is **slow** right now for number of qubits.
- Other quantum simulation techniques could give answers faster.
- I'll focus on **optical lattice** simulations.



Plan

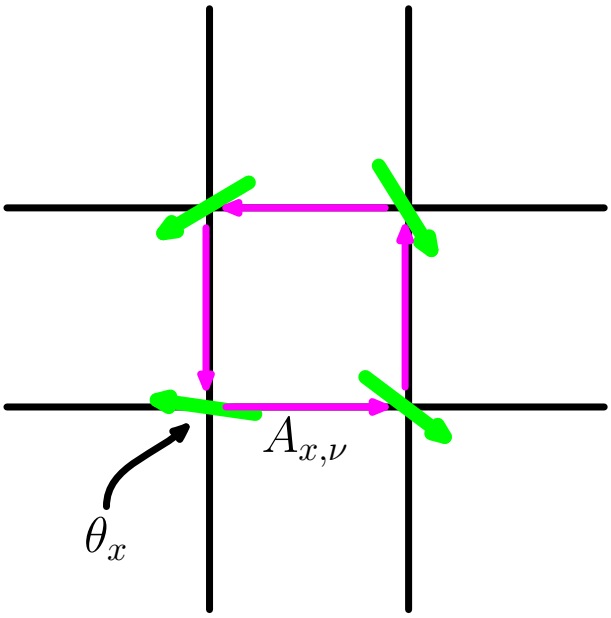
- 3D $U(1)$ is what I'm excited about
- 2D Abelian Higgs model as a step
- Tensor & continuous-time limit
- Optical lattice set-up
- back to 3D $U(1)$

the Abelian Higgs model

The Abelian Higgs model in **two Euclidean space-time dimensions**: This model

- is the Schwinger model with the fermion replaced by a complex scalar field.
- is believed to be confining, in the sense that there is a linear potential.
- has topological solutions.
- Here the Higgs mode is taken infinitely massive.

$$S = -\beta_{pl} \sum_x \sum_{\nu<\mu} \cos((A_{x,\mu} - A_{x+\nu,\mu}) - (A_{x,\nu} - A_{x+\mu,\nu}))$$
$$- 2\kappa \sum_x \sum_{\nu=1}^{2} \cos(\theta_{x+\nu} - \theta_x + A_{x,\nu})$$



The Abelian Higgs model

- The original partition function is a sum over the **compact** fields

$$Z = \int \mathcal{D}[A_{x,\mu}]\mathcal{D}[\theta_x]e^{-S}$$

- The Boltzmann weights can be **Fourier expanded**

$$e^{\beta_{pl}\cos(F_{x,\mu\nu})} = \sum_{m=-\infty}^{\infty} I_m(\beta_{pl})e^{imF_{x,\mu\nu}}$$

$$e^{2\kappa\cos(\theta_{x+\nu}-\theta_x+A_{x,\nu})} = \sum_{n=-\infty}^{\infty} I_n(2\kappa)e^{in(\theta_{x+\nu}-\theta_x+A_{x,\nu})}$$

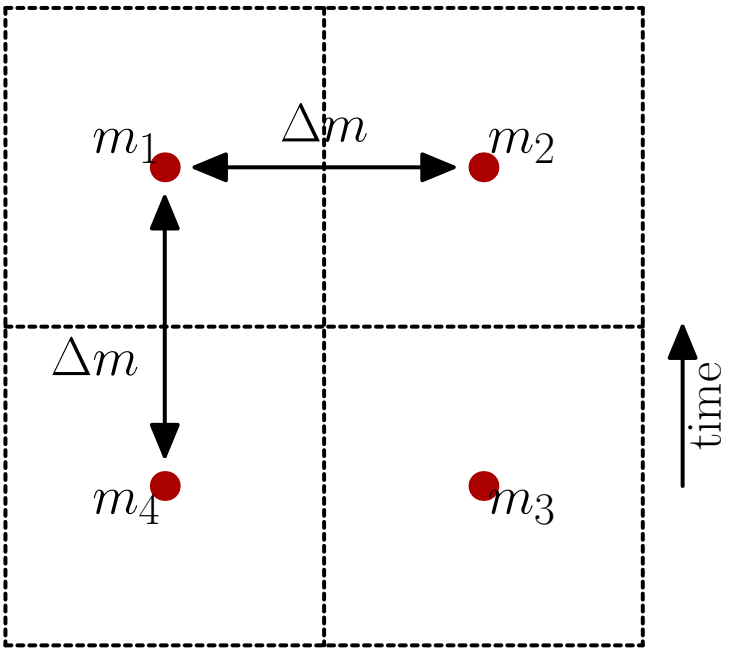
- Integrating over θs gives $\delta_{\Delta_\mu n_\mu,0}$ which is solved by $n_\mu = \epsilon_{\mu\nu}\Delta_\nu m$.
- But $\delta_{n_\mu-\epsilon_{\mu\nu}\Delta_\nu m,0}$ is the constraint from A integration,
  giving

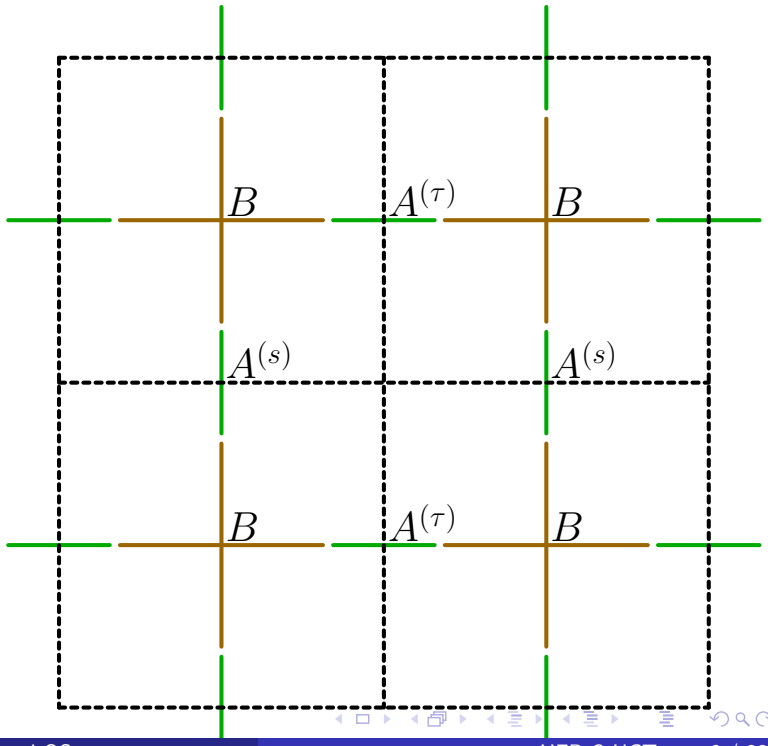
$$Z = \sum_{\{m\}}\left(\prod_{x,\mu\nu} I_m(\beta_{pl})\right)\left(\prod_{x,\mu} I_{m-m'}(2\kappa)\right)$$

  The ms are associated with the **plaquettes** (dual sites).

- **No residual gauge freedom.**

The Abelian Higgs model





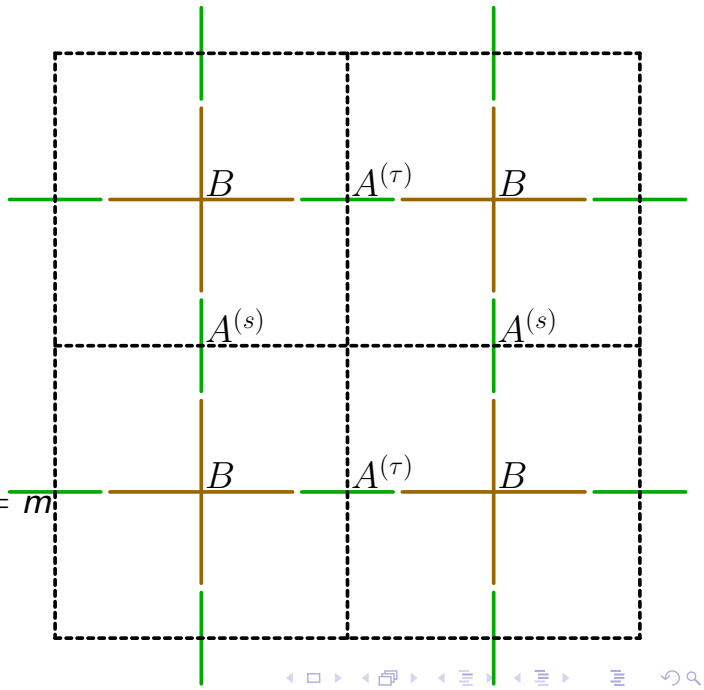
The Abelian Higgs model

The tensors

$$Z = \sum_{\{m\}} \left(\prod_{x,\mu\nu} I_m(\beta_{pl}) \right) \left(\prod_{x,\mu} I_{m-m'}(2\kappa) \right)$$

$$A_{mm'} \equiv I_{m-m'}(2\kappa)$$

$$B_{m_1 m_2 m_3 m_4} \equiv \begin{cases} I_m(\beta) & \text{if } m_1 = m_2 = m_3 = m_4 = m \\ 0 & \text{otherwise.} \end{cases}$$



The Polyakov loop

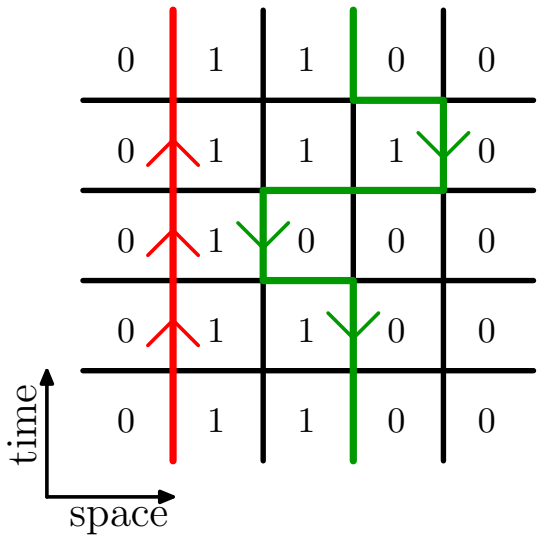
We looked at the **Polyakov loop**: A Wilson loop wrapped around the temporal direction of the lattice. This operator

- is a product of gauge fields in the time direction.
- is an order parameter for confinement in gauge theories.

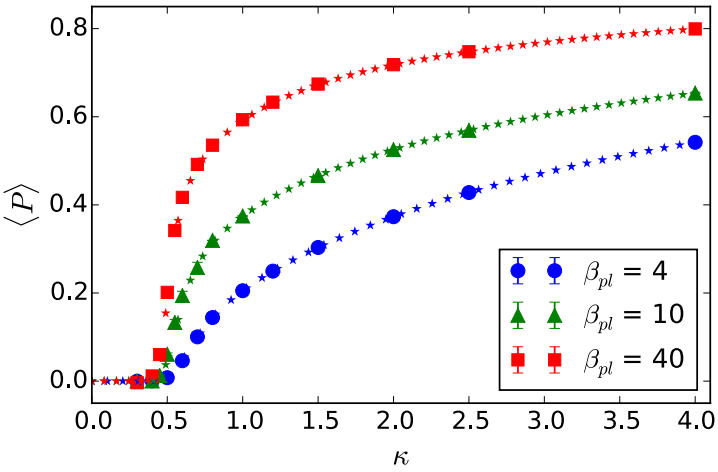
$$P = \prod_{n=1}^{N_\tau} U_{x^*+n\hat{\tau},\tau}.$$

$$\langle P \rangle = \frac{1}{Z} \int \mathcal{D}[A_{x,\mu}]\mathcal{D}[\theta_x] e^{-S} P$$

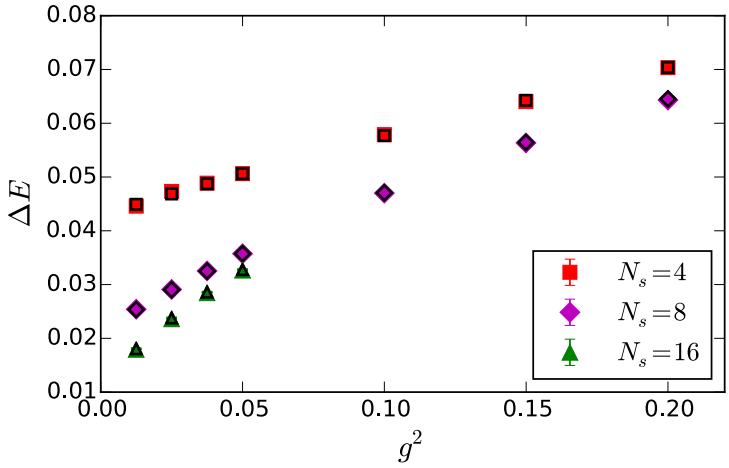
$$= \frac{1}{Z} \sum_{\{m\}} \left(\prod_{x,\mu\nu} I_m(\beta_{pl}) \right) \left(\prod_{x,\mu} I_{m-m'}(2\kappa) \right) \left(\prod_{n=1}^{N_\tau} \frac{I_{m-m'-1}(2\kappa)}{I_{m-m'}(2\kappa)} \right)$$



TRG & MC comparison

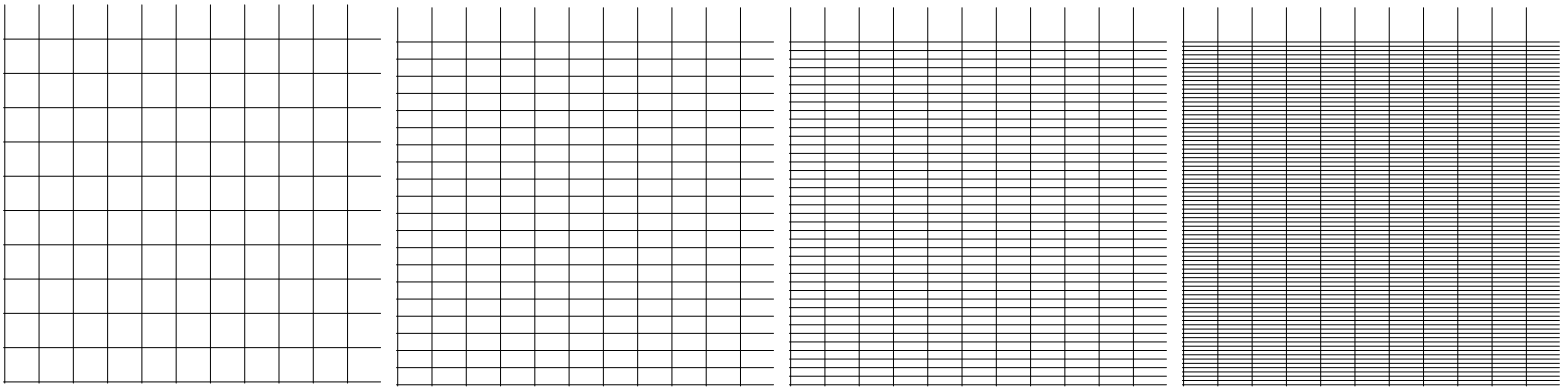


Varying β_{pl} keeping volume fixed. 16×16.



Comparison between MC and TRG.

Continuous time limit



original lattice $\rightarrow$

a, κ_s smaller &
β_{pl}, κ_τ larger

a, κ_s smaller &
β_{pl}, κ_τ larger

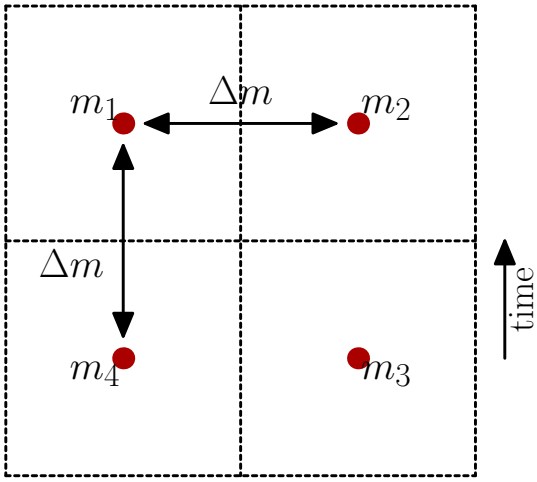
a, κ_s smaller &
β_{pl}, κ_τ larger

The quantum Hamiltonian

- This model has a **continuous-time limit** which has **no residual gauge freedom**.
- The continuous-time limit: taking $\beta_{pl}, \kappa_\tau \to \infty$, and $\kappa_s, a \to 0$, such that

$$U \equiv \frac{1}{\beta_{pl} a} = \frac{g^2}{a}, \quad Y \equiv \frac{1}{2\kappa_\tau a}, \quad X \equiv \frac{2\kappa_s}{a}$$

are held constant.



- $$H = \frac{U}{2}\sum_{i=1}^{N_s}(L_i^z)^2 + \frac{Y}{2}\sum_i{}'(L_{i+1}^z - L_i^z)^2 - X\sum_{i=1}^{N_s} U_i^x$$

$$L^z|m\rangle = m|m\rangle, \quad U^x = \frac{1}{2}(U^+ + U^-), \quad U^\pm|m\rangle = |m \pm 1\rangle.$$

The Polaykov loop

- The Polyakov loop has a continuous-time limit:

$$P = \prod_{n=1}^{N_\tau} U_{x^*+n\hat{\tau},\tau} \to \prod_{n=1}^{N_\tau} \frac{I_{m-m'-1}(2\kappa)}{I_{m-m'}(2\kappa)} \mapsto -\frac{Y}{2}(2(L^z_{i^*+1} - L^z_{i^*}) - 1)$$

  This gets put into the quantum Hamiltonian.
- The Hamiltonian with the Polyakov loop inserted:

$$\tilde{H} = H - \frac{Y}{2}(2(L^z_{i^*+1} - L^z_{i^*}) - 1)$$

- In this form ΔE comes from the difference in the ground states of the two Hamiltonians.

Collapse across limits

- The energy gap between a system with a Polyakov loop, and one without:

$$\Delta E = E^{(0)}_{\mathrm{PL}} - E^{(0)},$$

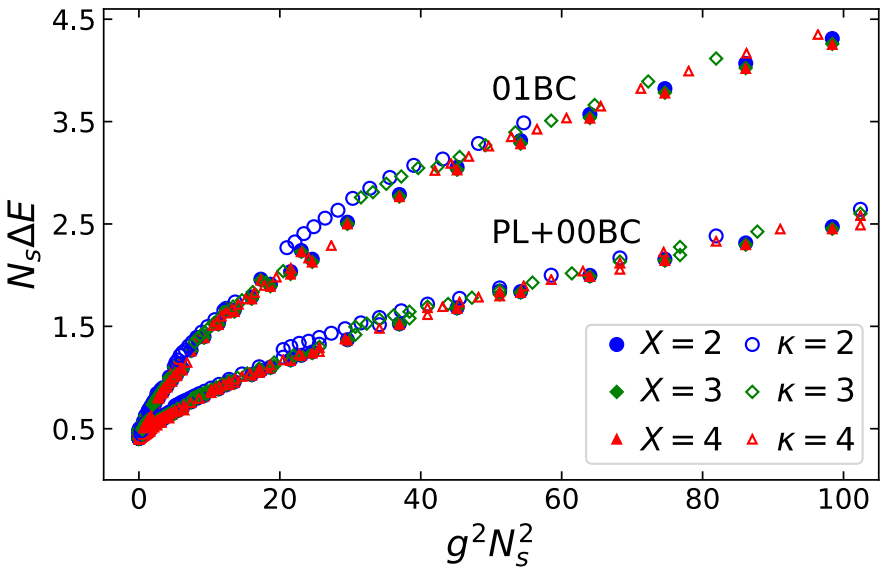
and a system with an external field, and one without:

$$\Delta E = E^{(0)}_{\mathrm{01BC}} - E^{(0)}.$$

- We found for sufficiently small $(gN_s)^2$

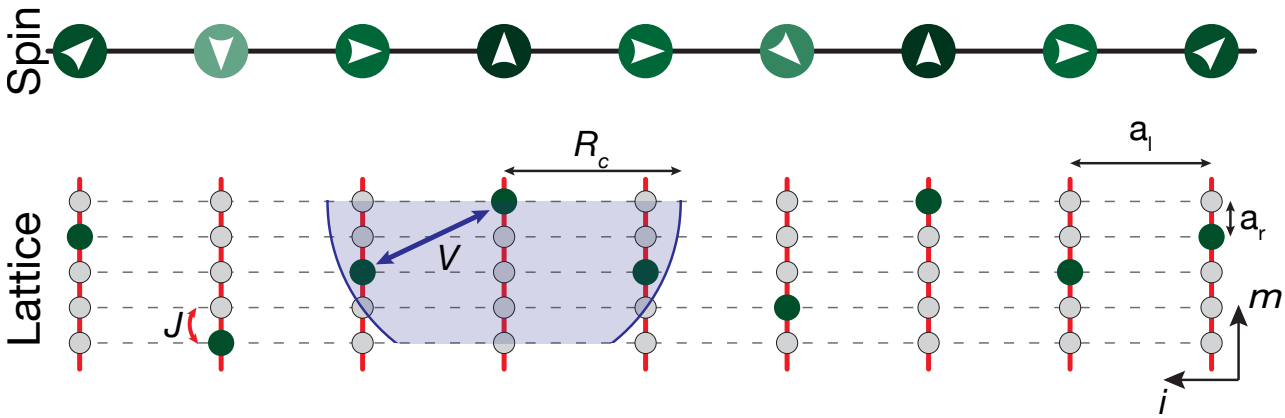
$$N_s\,\Delta E = f(g^2 N_s^2)$$

Furthermore, **this collapse survives the continuous time limit!**



Ladder system Hamiltonian

A 5-state truncation



$$\hat{H} = -\frac{J}{2}\sum_{i=1}^{N_s}\sum_{m=-s}^{s-1}(\hat{a}^{\dagger}_{m,i}\hat{a}_{m+1,i} + \text{h.c.}) - \sum_{i=1}^{N_s}\sum_{m=-s}^{s}\epsilon_{m,i}\hat{n}_{m,i} + \sum_{i,i'=1}^{N_s}\sum_{m,m'=-s}^{s} V_{m,m',i,i'}\hat{n}_{m,i},\hat{n}_{m',i'}$$

Hamiltonian mapping

$$H_{\text{rotor}} \rightarrow H_{\text{boson}} \tag{1}$$

$$-\frac{X}{2}\sum_{i=1}^{N_s}(U_i^+ + U_i^-) \rightarrow -\frac{J}{2}\sum_{i=1}^{N_s}\sum_{m=-s}^{s-1}(\hat{a}^\dagger_{i,m+1}\hat{a}_{i,m} + \hat{a}^\dagger_{i,m}\hat{a}_{i,m+1}) \tag{2}$$

$$\frac{U}{2}\sum_{i=1}^{N_s}(L_i^z)^2 \rightarrow \sum_{i=1}^{N_s}\sum_{m=-s}^{s}\epsilon_{m,i}\hat{n}_{m,i} \tag{3}$$

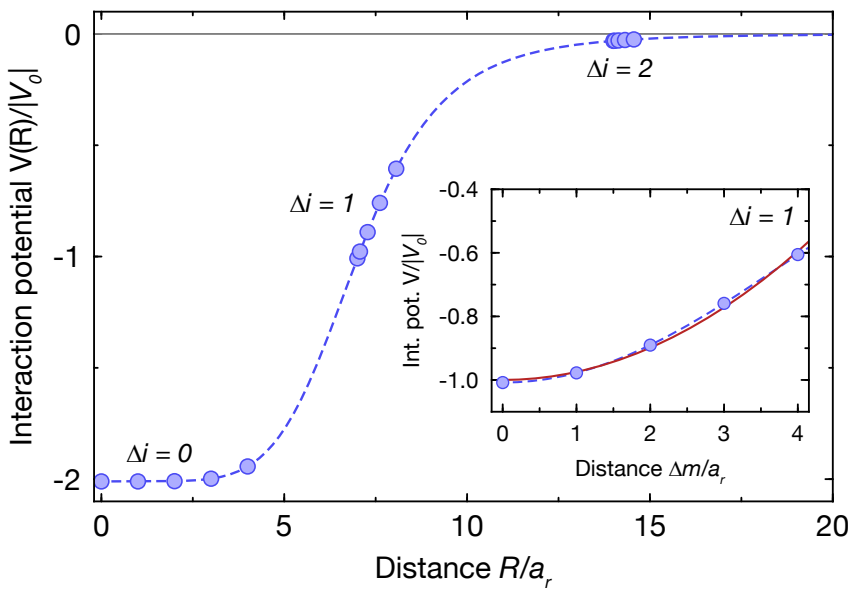
$$\frac{Y}{2}\sum_{\langle ij\rangle}(L_i^z - L_j^z)^2 \rightarrow \sum_{\langle ij\rangle}\sum_{m,m'=-s}^{s} V_{m,m',i,j}\hat{n}_{m,i}, \hat{n}_{m',j} \tag{4}$$

The quadratic potential

- The local potentials and hopping map straightforwardly.
- The nearest-neighbor rung interactions:

$$V_{m,m',i,i'} = V_{m,m'}\delta_{i',i+1}$$
$$= (-|V_0| + \frac{Y}{2}(m-m')^2)\delta_{i',i+1}$$

can be accomplished using an **asymmetric ladder** and a **Rydberg-dressed potential**.



3D $U(1)$ gauge theory

- Free electrodynamics in 3D (2+1D):

$$S = -\beta \sum_{x,\mu\nu} \cos(F_{x,\mu\nu})$$

$$\rightarrow Z = \int \mathcal{D}[A_{x,\mu}] e^{-S}$$

- One can expand just as before:

$$e^{\beta \cos(F_{x,\mu\nu})} = \sum_{m=-\infty}^{\infty} I_m(\beta_{pl}) e^{imF_{x,\mu\nu}}$$

- $$\int_0^{2\pi} \frac{dA_{x,\mu}}{2\pi} e^{iA_{x,\mu}(n_1+n_2-n_3-n_4)} = \delta_{n_1+n_2,n_3+n_4}$$

  associated with each link. The ns are **associated with the plaquettes**.
- The Kronecker delta is solved by a **curl**

$$\Delta_\mu n_{x,\mu\nu} = 0$$

$$\implies n_{x,\mu\nu} = \epsilon_{\mu\nu\rho} \Delta_\rho m_{x^*}$$

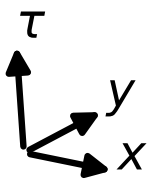
3D $U(1)$ gauge theory

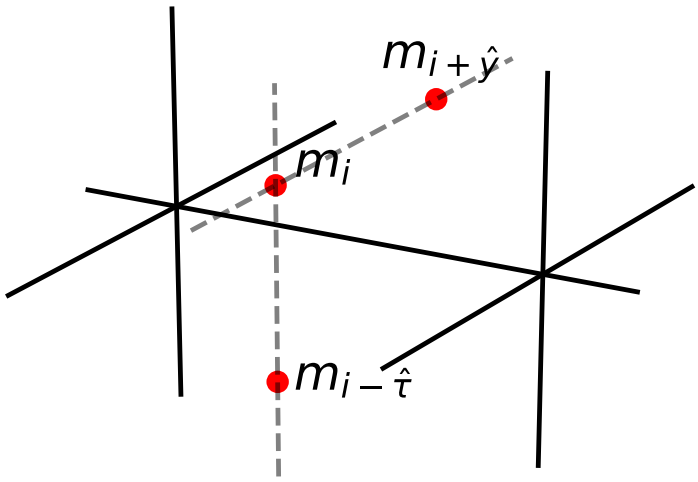
- Z can now be written as

$$Z = \sum_{\{m\}} \left(\prod_{i^*,\mu} I_{\Delta_\mu m_{i^*}}(\beta) \right)$$

- Separating space and time in anticipation

$$Z = \sum_{\{m\}} \left(\prod_{i^*,\tau} I_{\Delta_\tau m_{i^*}}(\beta_s) \right) \times \left(\prod_{i^*,a} I_{\Delta_a m_{i^*}}(\beta_\tau) \right)$$





The quantum Hamiltonian

- Separating the temporal and spatial (dual) links and taking the continuous-time limit: $\beta_\tau \to \infty$, β_s, $a \to 0$ and keep these ratios finite

$$U \equiv \frac{1}{\beta_\tau a}, \quad X \equiv \frac{\beta_s}{a}$$

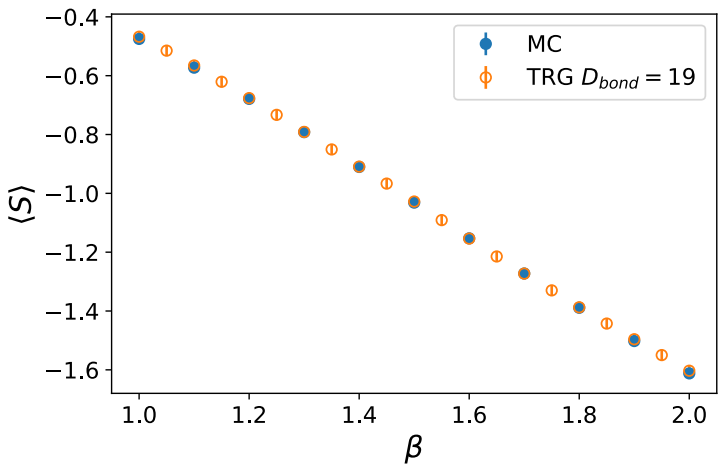
- $$H = \frac{U}{2} \sum_{\langle ij \rangle}{}' (L_i^z - L_j^z)^2 - X \sum_i U_i^x$$

  with

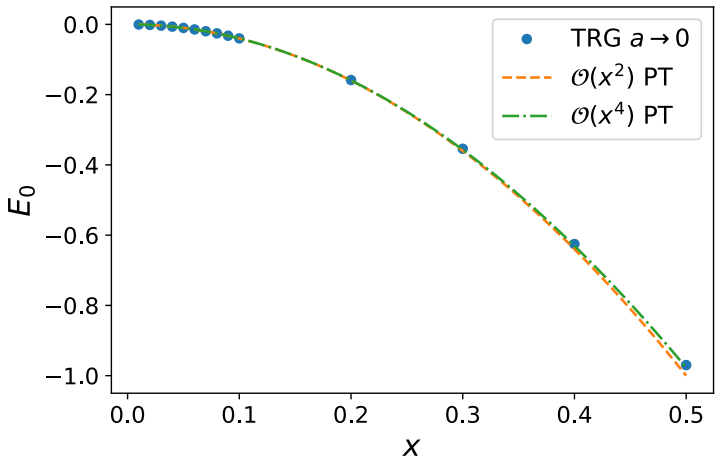
  $$L^z|m\rangle = m|m\rangle, \; U^x = \frac{1}{2}(U^+ + U^-), \; U^\pm|m\rangle = |m \pm 1\rangle$$

- The L^z variables are **unconstrained**.
- **Local** (nearest neighbor)
- Similar to Abelian Higgs model
- **Discrete spectrum** amiable to simulation

MC check



Average action computed compared between the tensor renormalization group (TRG) and Monte Carlo methods. $16 \times 16 \times 16$



Ground state energy of the Hamiltonian computed using perturbation theory and compared with TRG calculations. 4×4

Hamiltonian mapping

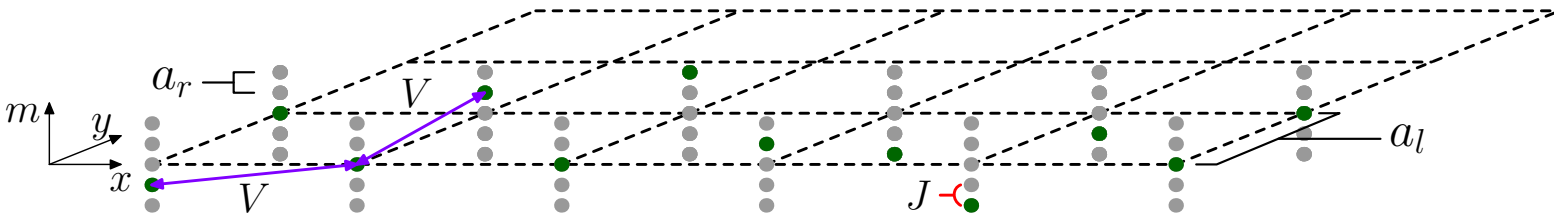
$$H_{\text{rotor}} \rightarrow H_{\text{boson}} \tag{5}$$

$$-\frac{X}{2}\sum_{i=1}^{N_s}(U_i^+ + U_i^-) \rightarrow -\frac{J}{2}\sum_{i=1}^{N_s}\sum_{m=-s}^{s-1}(\hat{a}^\dagger_{i,m+1}\hat{a}_{i,m} + \hat{a}^\dagger_{i,m}\hat{a}_{i,m+1}) \tag{6}$$

$$\frac{U}{2}\sum_{\partial}(L_i^z)^2 \rightarrow \sum_{i=1}^{N_s}\sum_{m=-s}^{s}\epsilon_{m,i}\hat{n}_{m,i} \tag{7}$$

$$\frac{U}{2}\sum_{\langle ij\rangle}(L_i^z - L_j^z)^2 \rightarrow \sum_{\langle ij\rangle}\sum_{m,m'=-s}^{s} V_{m,m',i,j}\hat{n}_{m,i}, \hat{n}_{m',j} \tag{8}$$

Ladder system Hamiltonian



$$\hat{H} = -\frac{J}{2}\sum_{i=1}^{N_s}\sum_{m=-s}^{s-1}(\hat{a}^\dagger_{m,i}\hat{a}_{m+1,i} + \text{h.c.}) - \sum_{i=1}^{N_s}\sum_{m=-s}^{s}\epsilon_{m,i}\hat{n}_{m,i} + \sum_{\langle ij\rangle}\sum_{m,m'=-s}^{s} V_{m,m'}\hat{n}_{m,i},\hat{n}_{m',j}$$

s here indicates the state truncation.

Pro:

- ϵ, and V are very similar to before.

Con:

- Next-nearest neighbors are now $\sqrt{2}a_l$ away (diagonal) instead of $2a_l$.

In conclusion

- The Abelian Higgs model has local continuous-time limit with no residual gauge freedom.
- We propose a physical, multi-leg, optical-lattice ladder to quantum simulate the Abelian Higgs model in 2D.
- We can achieve the desired interactions for the lattice model using an asymmetric lattice and a Rydberg-dressed potential.
- **arXiv:1803.11166**, **1807.09186**
- $U(1)$ gauge theory has a similar, continuous-time, rotor limit.
- Can be mapped to a similar boson Hamiltonian.
- **1811.05884**
- Looking for simulation.

Thank you!