The *parton* Tales.

From LFWFs to Parton Distributions and Form Factors (part II)

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In collaboration with: … K. Raya, C.D. Roberts,...

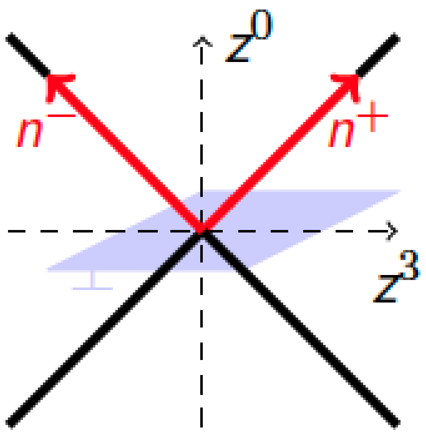
Continuum Functional Methods in QCD at New Generation Facilities,
7-10 May 2019, ECT*-Vila Tambosi, Trento.

Antecedents:

GPD definition:

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



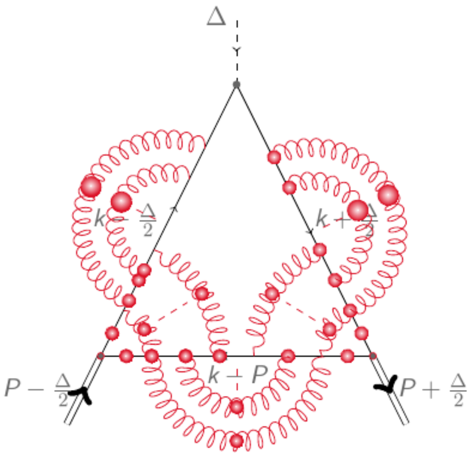
References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

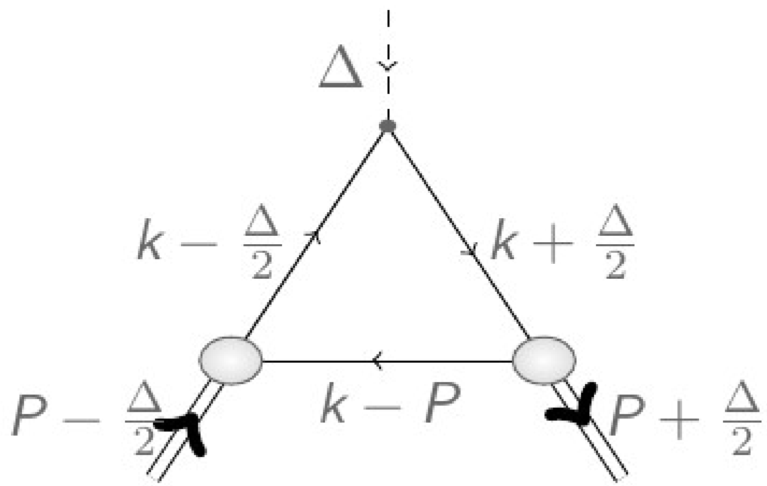
- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

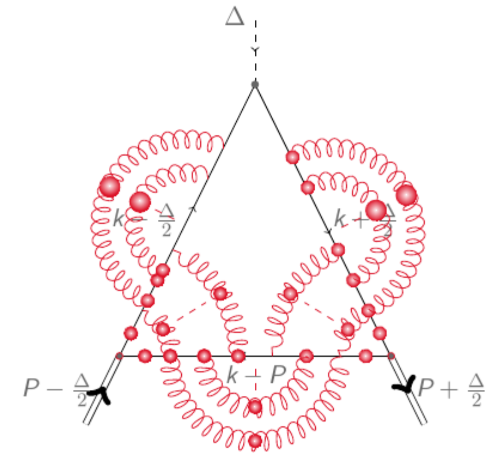


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P-\frac{\Delta}{2} \right\rangle$$



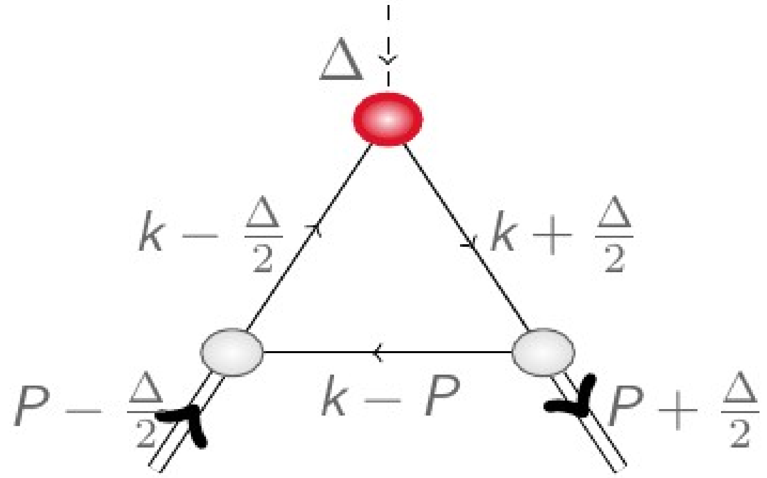
- Compute **Mellin moments** of the pion GPD H.

Antecedents:



GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

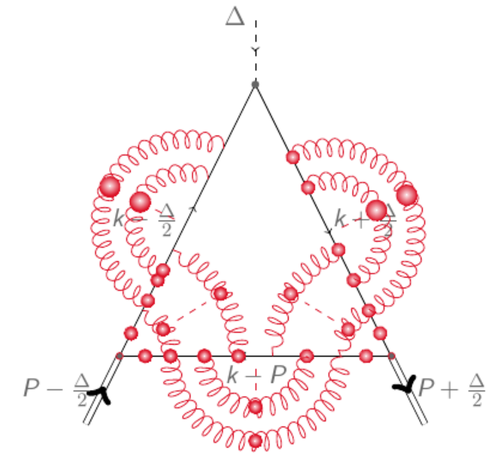
$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$



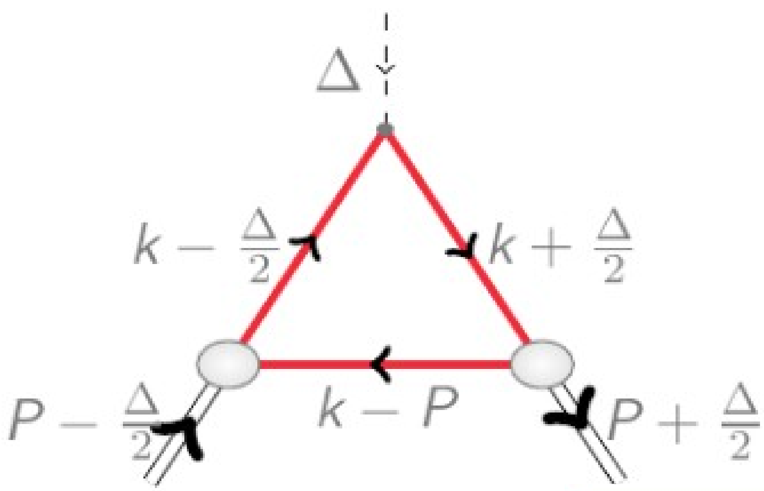
- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

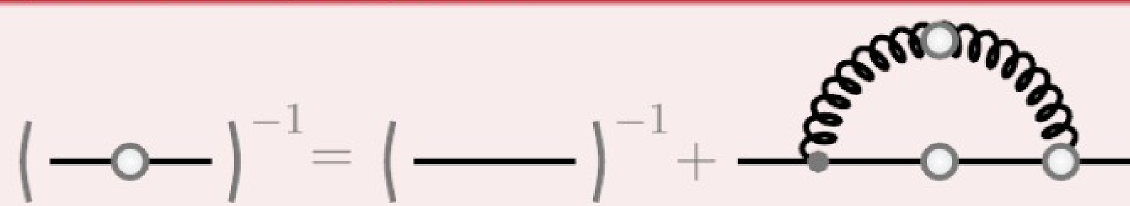


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



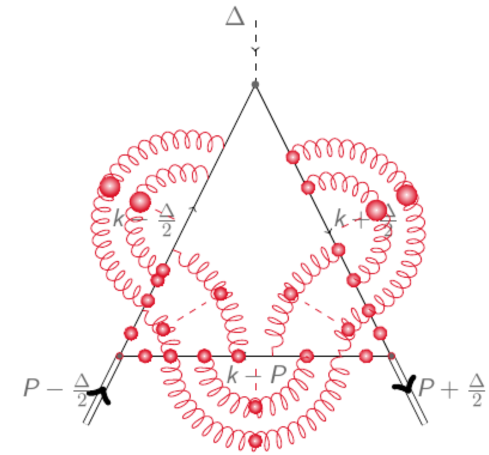
- Compute **Mellin moments** of the pion GPD *H*.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Dyson - Schwinger equation

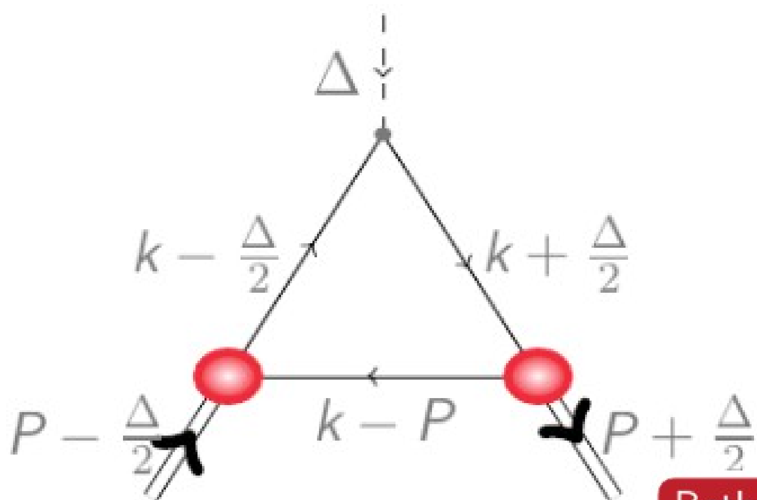


Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

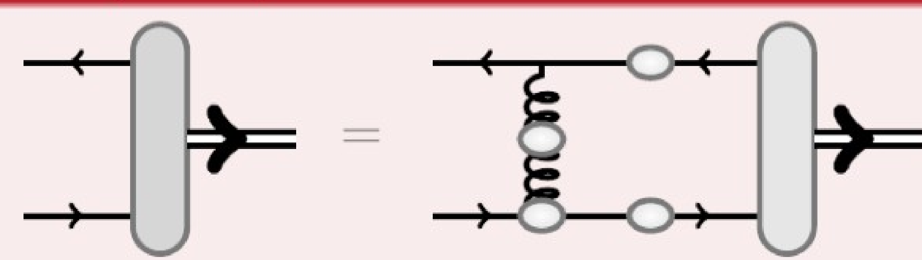


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Bethe - Salpeter equation



Antecedents:

GPD asymptotic algebraic model:

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[-i\gamma \cdot p + M \right] \Delta_M(p^2) \\
\Delta_M(s) &= \frac{1}{s + M^2} \\
\Gamma_\pi(k,p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z \, \rho_\nu(z) \, \left[\Delta_M(k_{+z}^2) \right]^\nu \\
\rho_\nu(z) &= R_\nu (1 - z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Antecedents:

GPD asymptotic algebraic model:

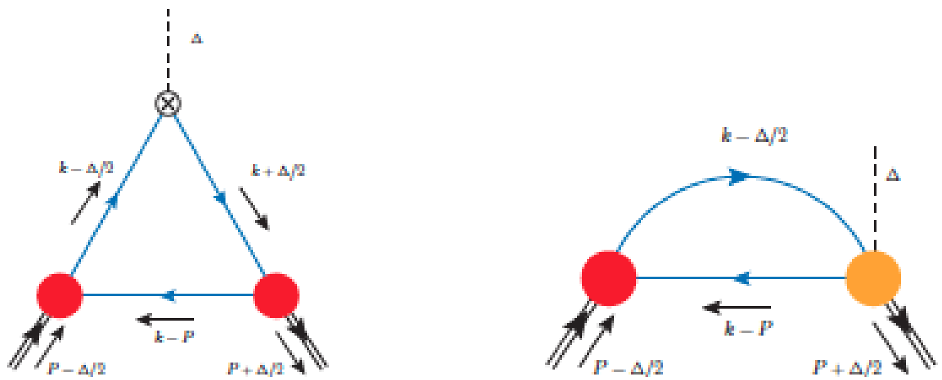
- **Analytic expression** in the DGLAP region.

$$
\begin{aligned}
H^{u}_{x\geq\xi}(x,\xi,0) \quad = \quad & \frac{48}{5}\Bigg\{ \frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3} \\
& \frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(\left(15(2x(x+5)+5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right)}{20\left(\xi^2-1\right)^3} \\
& + \frac{3\left(2x(5x(x+2)-6)+15\xi^6-5\xi^2+3\right)\log\left(1-\xi^2\right)}{20\left(\xi^2-1\right)^3}\Bigg\}
\end{aligned}
$$

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:

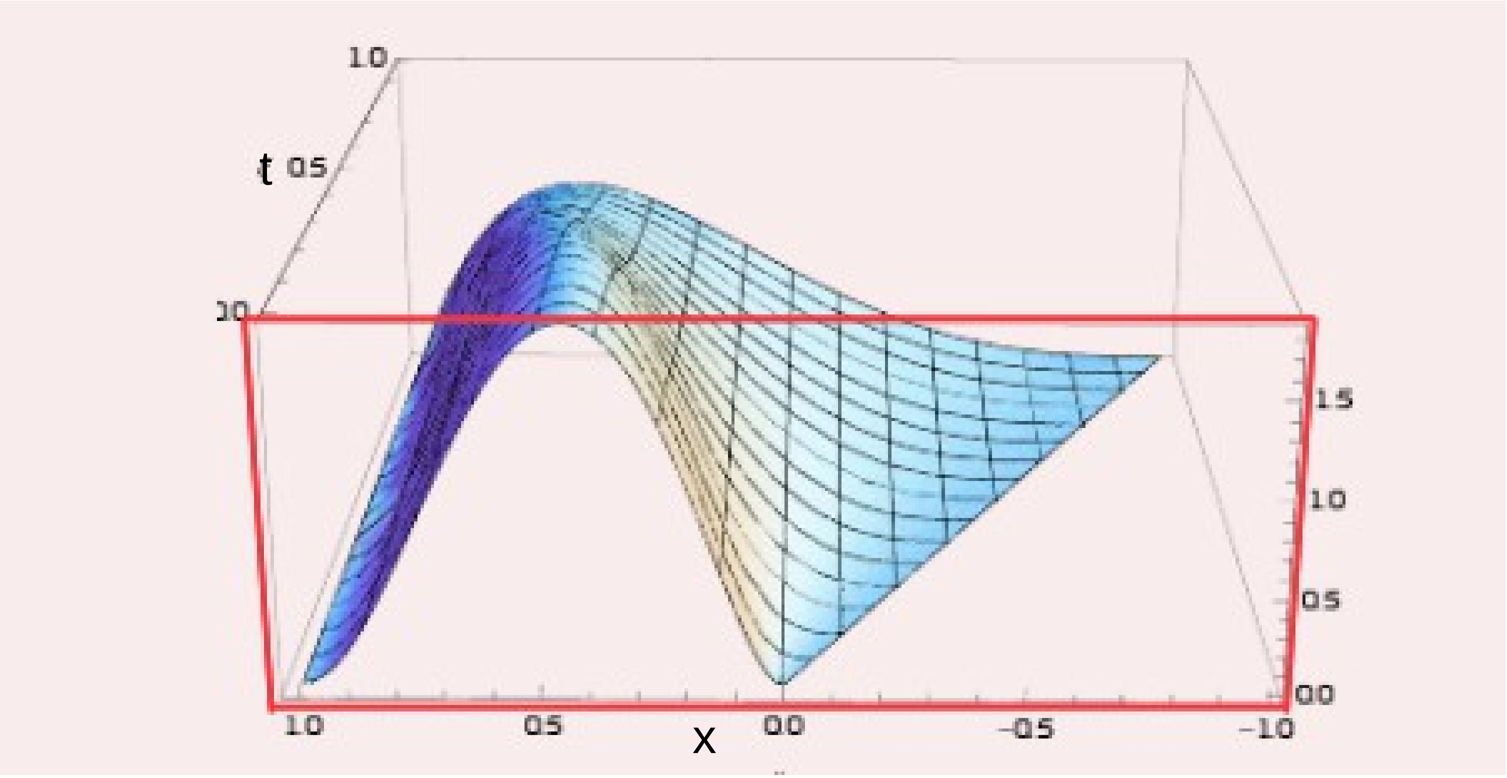


$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\; i\gamma\cdot n\; S(k+\frac{\Delta}{2})$$
$$\tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),$$

$$2(P\cdot n)^{m+1}\langle x^m\rangle^u = \mathrm{tr}_{CFD}\int\frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\tau_-\;\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)$$

Antecedents:

GPD asymptotic algebraic model (completion):

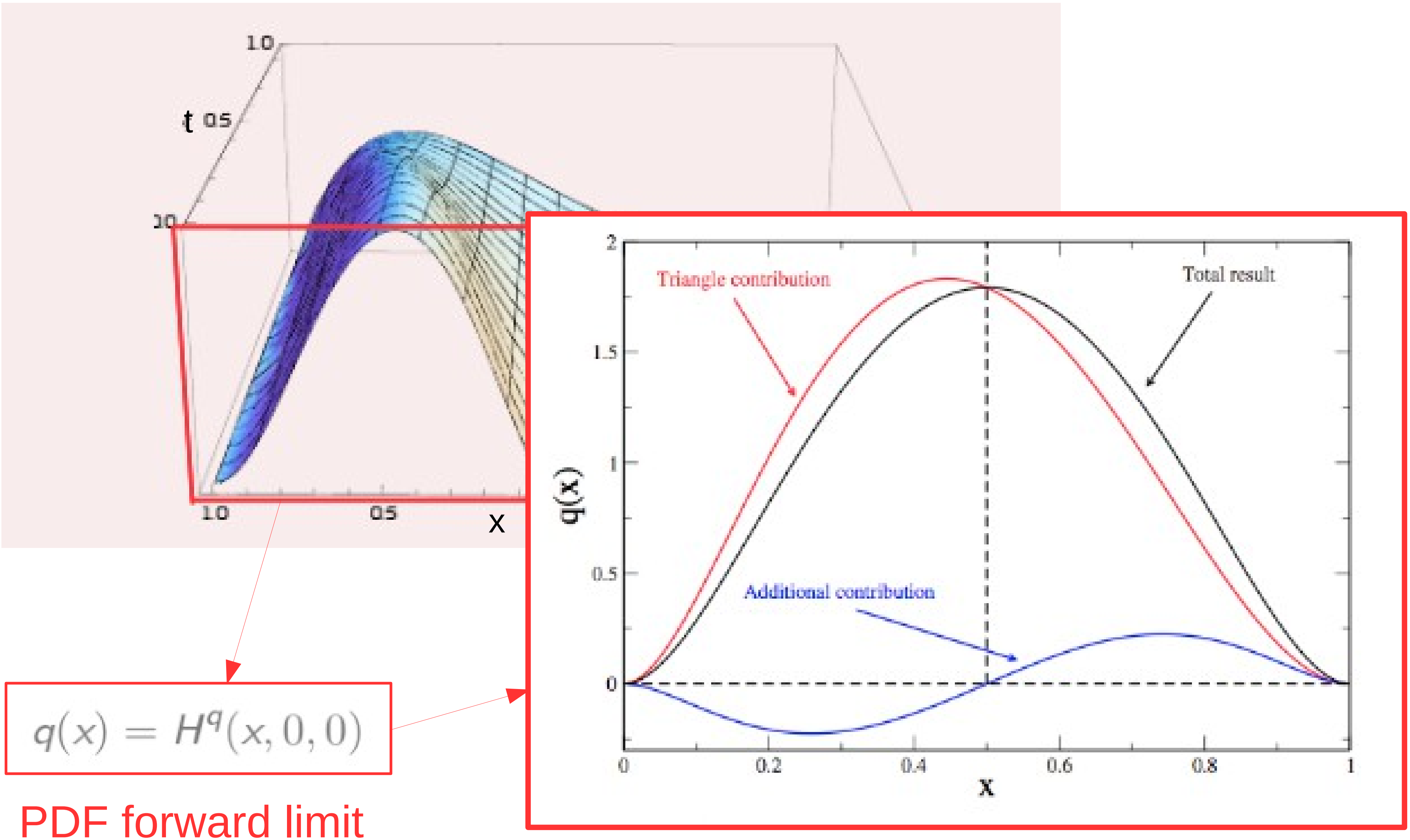


$$q(x) = H^q(x, 0, 0)$$

PDF forward limit

Antecedents:

GPD asymptotic algebraic model (completion):



$$q(x) = H^q(x, 0, 0)$$

PDF forward limit

Antecedents:

GPD overlap approach: The pion light front wave function

$$|H;P,\lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N \Psi^{\lambda}_{N,\beta}(\Omega) |N,\beta,k_1 \cdots k_N\rangle$$

$$\Omega = (x_1, \mathbf{k}_{\perp 1}, \cdots, x_N, \mathbf{k}_{\perp N})$$

$$[\mathrm{d}x]_N = \prod_{i=1}^{N} \mathrm{d}x_i\, \delta\left(1 - \sum_{i=1}^{N} x_i\right),$$

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^{N} \mathrm{d}^2\mathbf{k}_{\perp i}\, \delta^2\left(\sum_{i=1}^{N} \mathbf{k}_{\perp i} - \mathbf{P}_\perp\right)$$

N-partons LCWF for the hadron H

$$\sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N |\Psi^{\lambda}_{N,\beta}(\Omega)|^2 = 1.$$

Let's consider the two-body pion LCWF:

$$|\pi^+, P\rangle|^{\text{2-body}}_{\uparrow\downarrow} = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^3} \frac{\mathrm{d}x}{\sqrt{x(1-x)}} \Psi_{\uparrow\downarrow}(k^+, \mathbf{k}_\perp) \Big[b^\dagger_{u\uparrow}(x,\mathbf{k}_\perp) d^\dagger_{d\downarrow}(1-x,-\mathbf{k}_\perp) + b^\dagger_{u\downarrow}(x,\mathbf{k}_\perp) d^\dagger_{d\uparrow}(1-x,-\mathbf{k}_\perp) \Big] |0\rangle\,,$$

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\, \chi(k,P)\, S^{-1}(k_1).$$

BS wave function

$$2P^+ \Psi_{\uparrow\downarrow}(k^+, \mathbf{k}_\perp) = \int \frac{\mathrm{d}k^-}{2\pi} \mathrm{Tr}\left[\gamma^+ \gamma_5 \chi(k,P)\right]$$

Antecedents:

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$$2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int \frac{dk^-}{2\pi}\mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]$$

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- Expressions for vertices and propagators:

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$$\Delta_M(s) = \frac{1}{s+M^2}$$

$$\Gamma_\pi(k,p) = i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}dz\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu$$

$$\rho_\nu(z) = R_\nu(1-z^2)^\nu$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

Keeping so contact with the previous "covariant" approach" based on DSE and BSE.

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \sqrt{2}\sum_{N,N'}\sum_{\beta,\beta'}\int [\mathrm{d}\hat{x}']_{N'}[\mathrm{d}^2\hat{\mathbf{k}}'_\perp]_{N'}[\mathrm{d}\tilde{x}]_N[\mathrm{d}^2\tilde{\mathbf{k}}_\perp]_N\Psi^*_{N',\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

$$\times\int\frac{\mathrm{d}z^-}{2\pi}e^{iP^+z^-}\langle N',\beta,k'_1\cdots k'_N|\,\phi^{q\dagger}\left(-\frac{z}{2}\right)\phi^q\left(\frac{z}{2}\right)|N,\beta,k_1\cdots k_N\rangle$$

$$=\sum_N\sqrt{1-\xi}^{\,2-N}\sqrt{1+\xi}^{\,2-N}\sum_{\beta=\beta'}\sum_j\delta_{s_jq}$$

$$\times\int[\mathrm{d}\bar{x}]_N[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_N\delta(x-\bar{x}_j)\Psi^*_{N,\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

In DGLAP kinematics: $\zeta\leqslant x\leqslant 1$

$$=\int[\mathrm{d}\bar{x}]_2[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_2\delta(x-\bar{x}_j)\Psi^*_{\uparrow\downarrow}(\hat{\Omega}')\Psi_{\uparrow\downarrow}(\tilde{\Omega})$$

In the pion 2-body case

$+$ Helicity-1 component

$$\hat{x} \to \frac{x-\zeta}{1-\zeta}, \qquad \tilde{x} \to \frac{x+\zeta}{1+\zeta}$$

$$=\frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\; u^\nu v^\nu\delta(1-u-v)\,\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2\hat{x}^\nu(1-\hat{x})^\nu\tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}.$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}} x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\, u^\nu v^\nu \delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2 \hat{x}^\nu(1-\hat{x})^\nu \tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}},$$

$\xi \leqslant x \leqslant 1$

$\hat{x} = \frac{x-\xi}{1-\xi}$, $\tilde{x} = \frac{x+\xi}{1+\xi}$

$$= 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\,\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\,\frac{1-2z}{1+z}\,\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu .$$

GPD in the overlap approach:

$$H(x,\xi,t) = 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\,\frac{\text{arct}\ldots\sqrt{z\ldots}}{\ldots\sqrt{1+z}}\right)$$

1

$0 \leqslant x \leqslant 1$

$\xi \to 0$, $t \to 0$

Forward limit

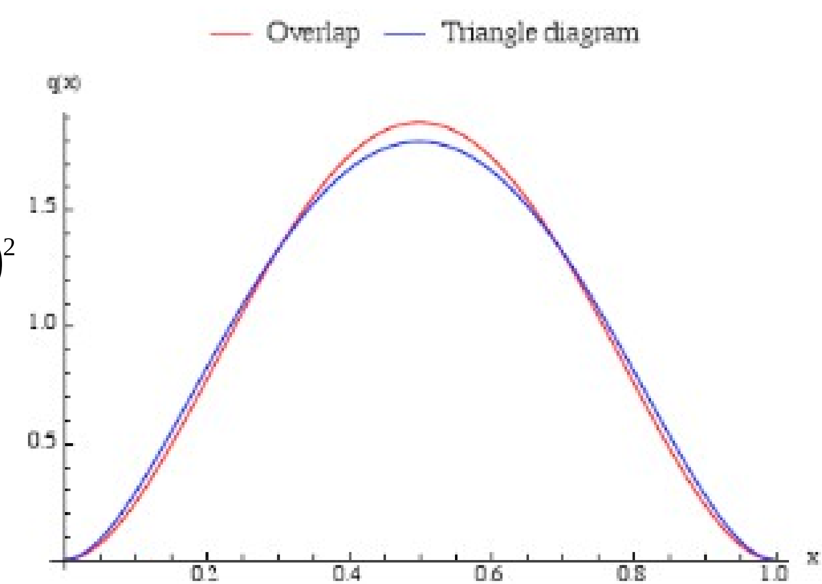
$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

PDF:

$$H(x,0,0) = q(x) = 30\,x^2(1-x)^2$$

Compares numerically very well with the results obtained from the Triangle diagram!!!



Consistent descriptions from both approaches!!!
(tested with a simple model)

Pion (kaon maybe) realistic picture:

- The pseudoscalar LFWF can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \mathrm{tr}_{CD} \int_{dk_{||}} \delta(n\cdot k - x n\cdot P_K)\gamma_5\gamma\cdot n \chi_K^{(2)}(k_-^K;P_K) \ .$$

- The moments of the distribution are given by:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{1}{f_K n\cdot P}\int_{dk_{||}} \left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n\chi_K^{(2)}(k_-^K;P_K)$$

$$\int_0^1 d\alpha\alpha^m \left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right] \ , \quad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha)+M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2) \ .$$

Uniqueness of Mellin moments ➡

$$\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)$$

$$\chi_K(\alpha;\sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^1 dv + \int_{1-2\alpha}^1 d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^1 dv\right] \frac{\rho_K(\omega)}{n_K}\frac{\Lambda_K^2}{\sigma^3} \ .$$

The spectral density $\rho_K(z)$ can be modelled...
...Or taken with BSE solutions as an input!

$$\Rightarrow \psi_K^{\uparrow\downarrow}(x,k_\perp^2) \sim \int d\omega \ \cdots \rho_K(\omega) \cdots$$

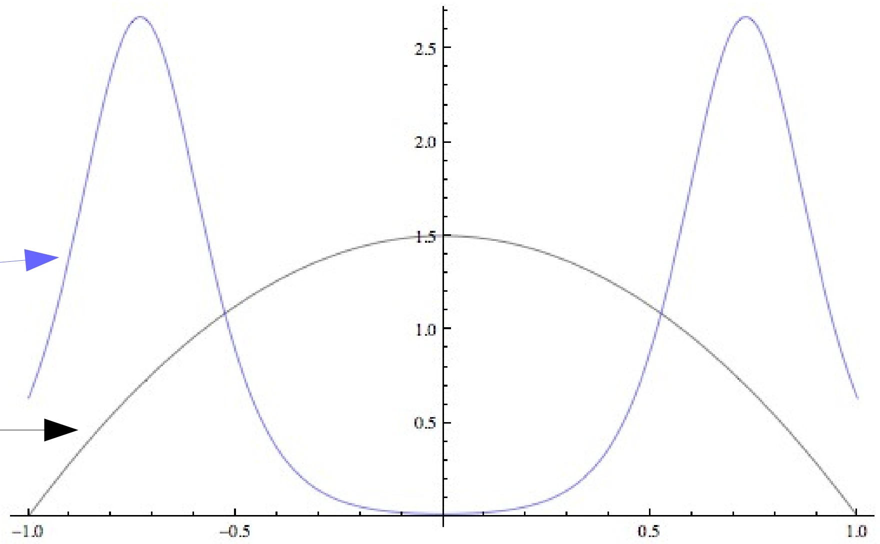
Pion realistic picture:

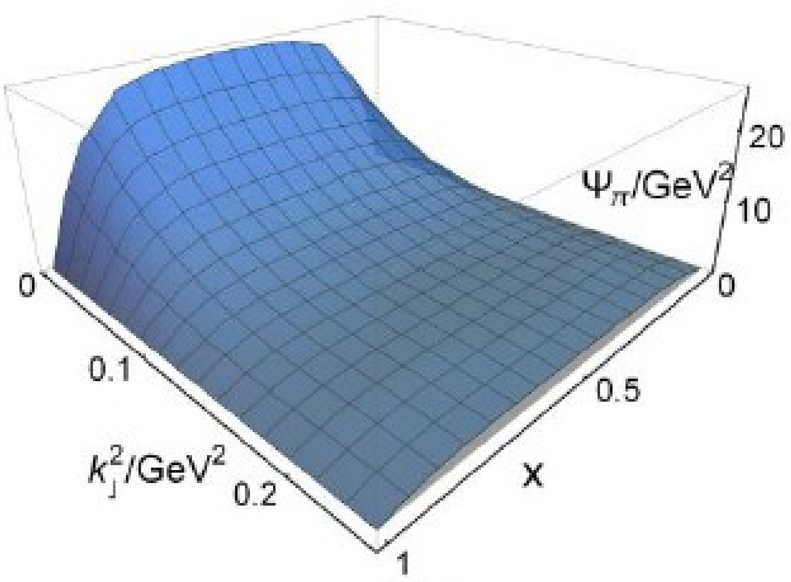
- Spectral density is chosen as:

$$u_G\rho_G(\omega) = \frac{1}{2b_0^G}\left[\operatorname{sech}^2\left(\frac{\omega-\omega_0^G}{2b_0^G}\right)+\operatorname{sech}^2\left(\frac{\omega+\omega_0^G}{2b_0^G}\right)\right]$$

Phenomelogical model: $b_0^\pi = 0.1, b_0^\pi = 0.73$;

Asymptotic case: $\rho(\omega;\nu) \sim (1-\omega^2)^\nu$

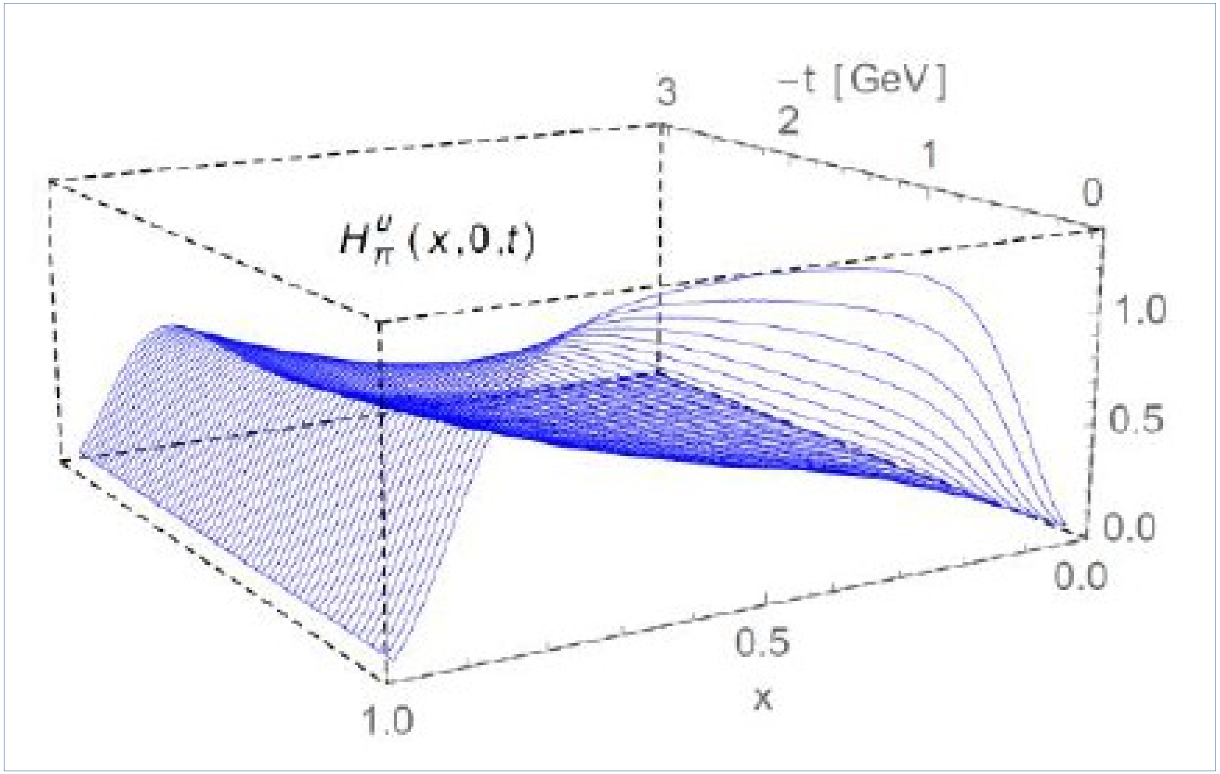




Pion realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

Phenomenological model



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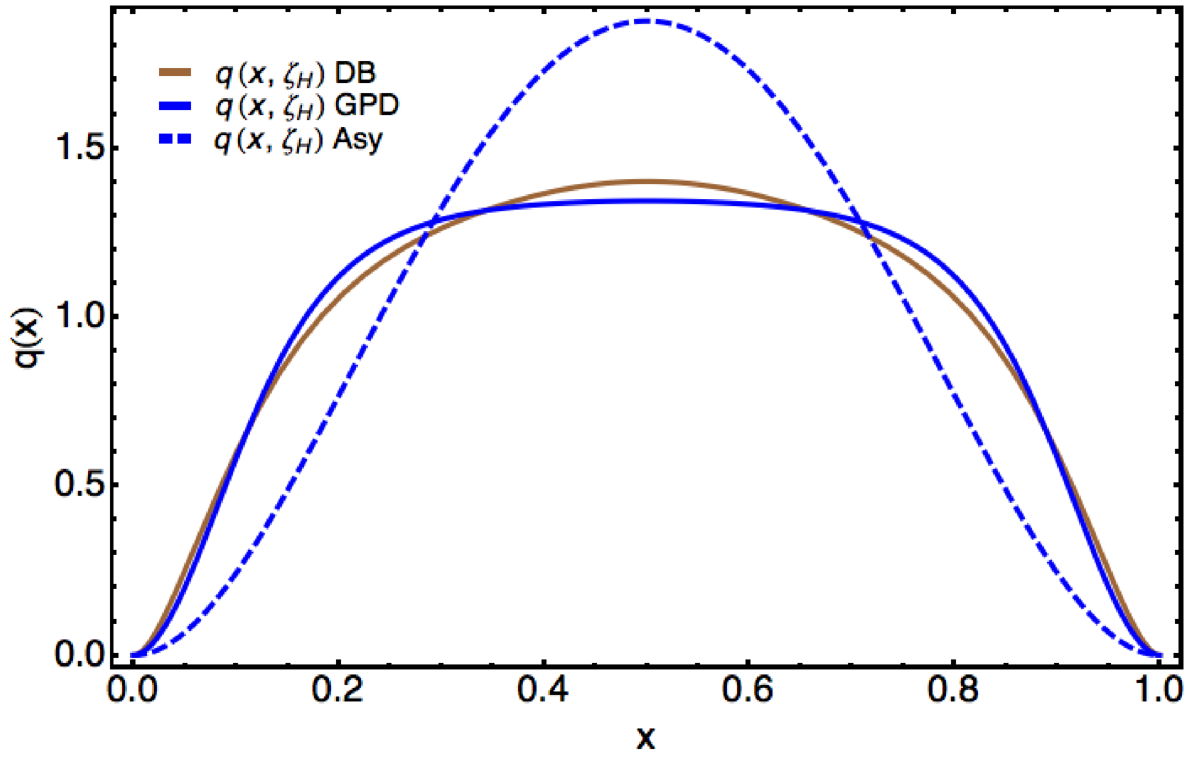
$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$



One should focus on the forward limit: PDF (benchmark) case

Pion realistic picture:

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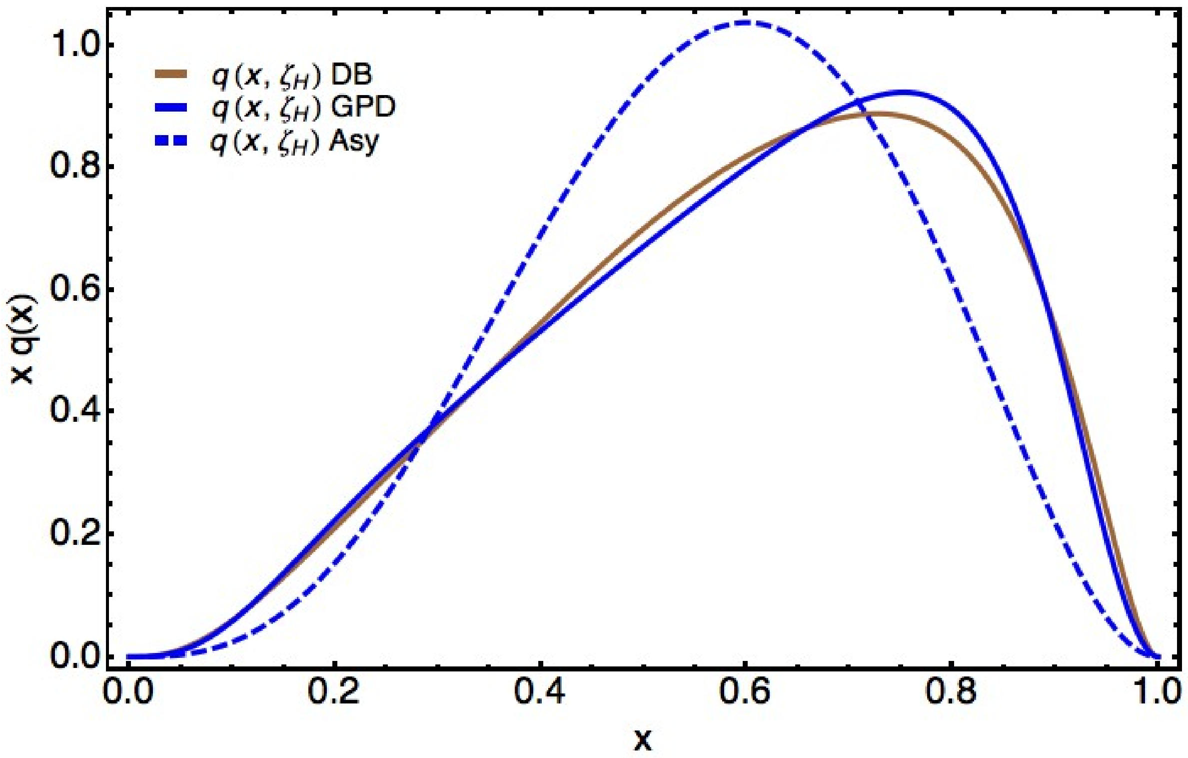
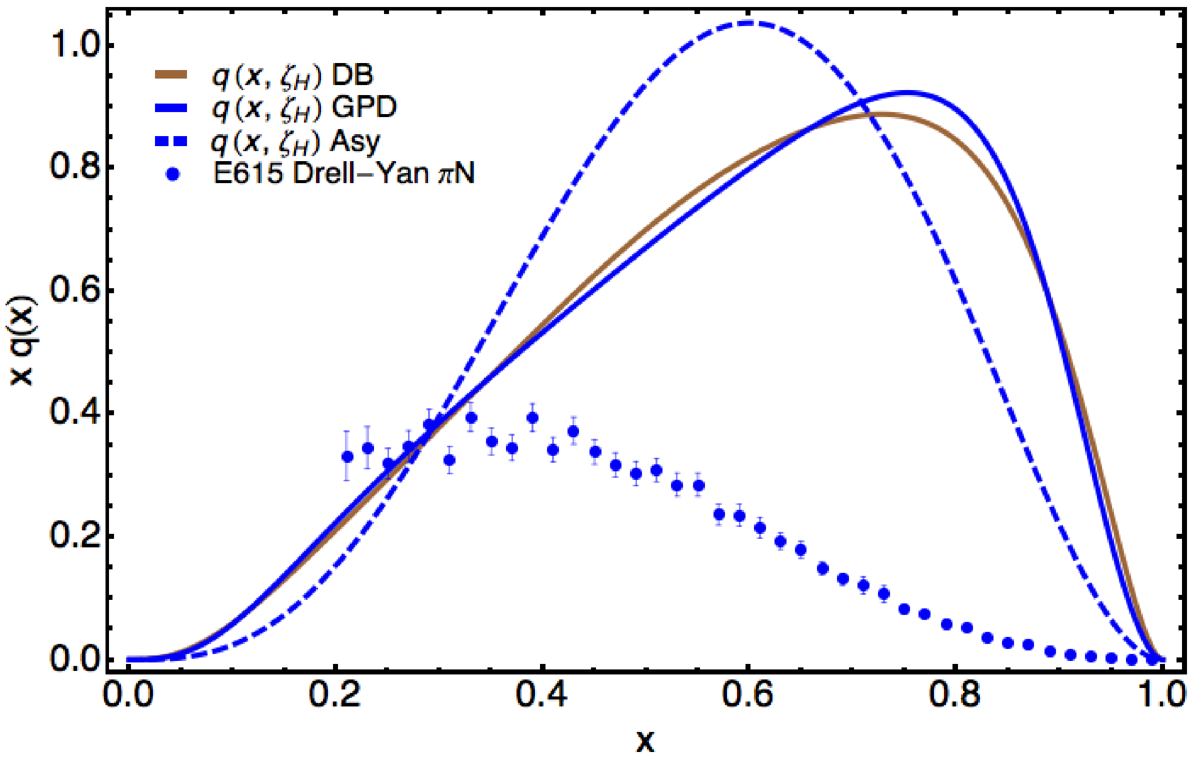


One should focus on the forward limit: PDF (benchmark) case

Pion realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$



One should focus on the forward limit: PDF (benchmark) case

Pion realistic picture: PDF DGLAP evolution

$$M_n(t)=\int_0^1 dx\, x^n q(x,t)$$

$$t=\ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

Moments' evolution (1-loop):

$$\frac{d}{dt}M_n(t)=-\frac{\alpha(t)}{4\pi}\gamma_0^n M_n(t)+\ldots$$

Pion realistic picture: PDF DGLAP evolution

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

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$$\gamma_n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

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$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots$$

$$t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

Pion realistic picture: PDF DGLAP evolution

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$$t_\Lambda = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$$M_n(t) = M_n(t_0)\left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}$$

Pion realistic picture: Coupling and effective charge

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

Which value of Lambda? I

Pion realistic picture: Coupling and effective charge

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Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

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The evolution will thus depend on the scheme because of the perturbative truncation

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The evolution will thus depend on the scheme because of the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

PDG2018: [PRD98(2018)030001]

$$\Lambda^{(5)}_{\overline{MS}} = (210 \pm 14)\ \text{MeV}, \qquad (9.24b)$$

$$\Lambda^{(4)}_{\overline{MS}} = (292 \pm 16)\ \text{MeV}, \qquad (9.24c)$$

$$\Lambda^{(3)}_{\overline{MS}} = (332 \pm 17)\ \text{MeV}, \qquad (9.24d)$$

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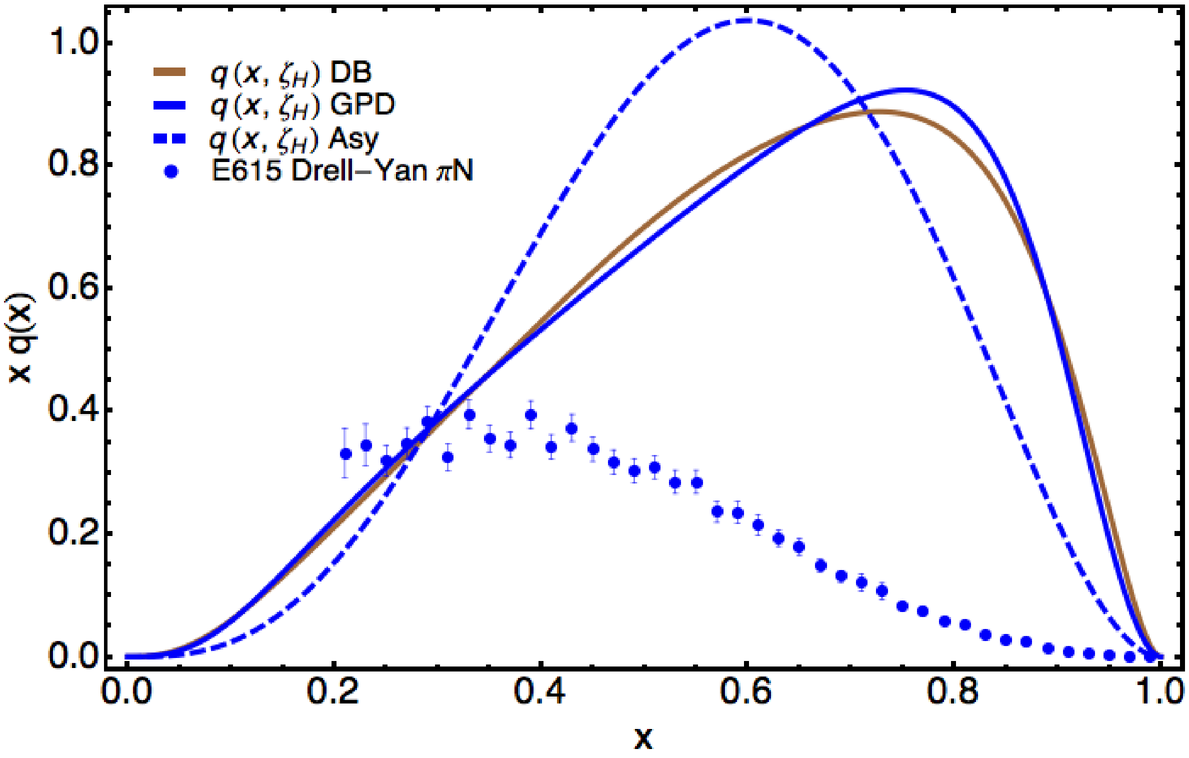
$\zeta_0 \rightarrow \zeta_2 = 5.2$ GeV



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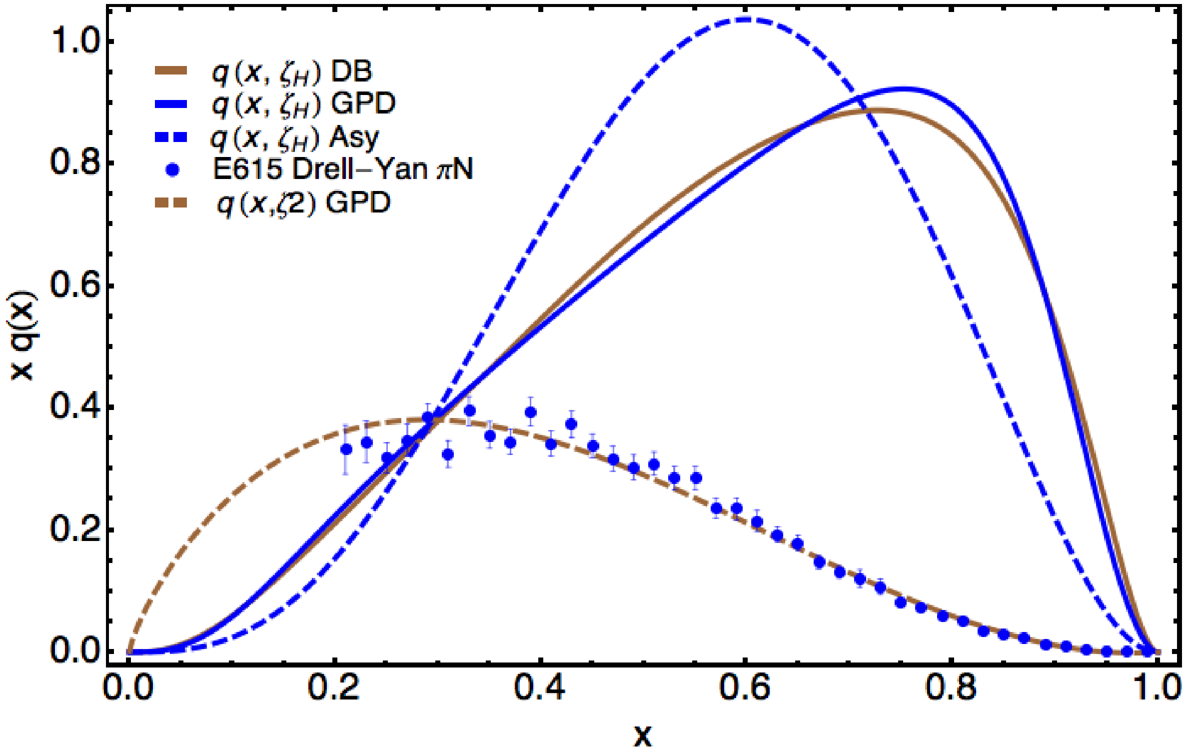

$\Lambda_{\overline{MS}} = 0.234\,;\,\zeta_0 = 0.349$ [GeV
]

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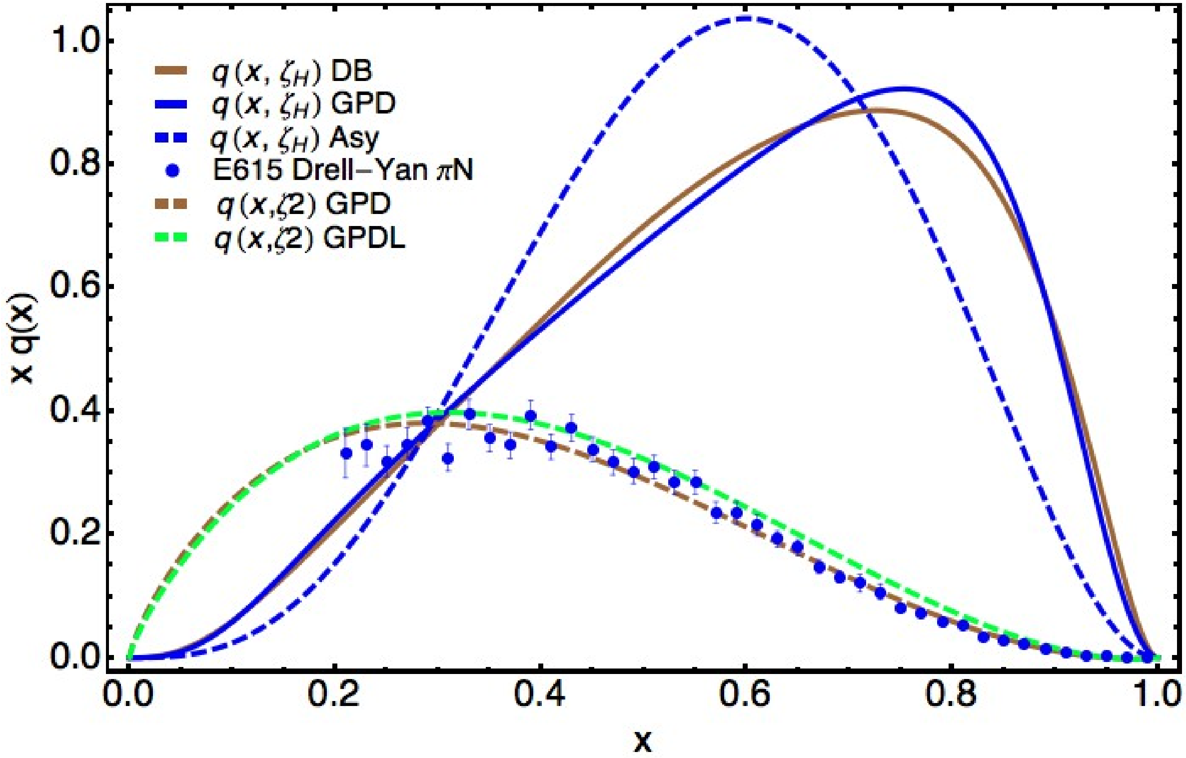
$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

$\zeta_0 \rightarrow \zeta_L = 2\,\text{GeV} \rightarrow \zeta_2 = 5.2\,\text{GeV}$



$\Lambda_{\overline{MS}} = 0.234\,;\,\zeta_0 = 0.349$ [GeV]

$\Lambda_{\overline{MS}} = 0.234\,;\,\zeta_0 = 0.374$ [GeV]

Pion realistic picture: Coupling and effective charge

$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln(\frac{\zeta^2}{\Lambda^2})} + \ldots$$

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\ \overline{\alpha}(t) + \ldots\right)$$

$$\ln(\frac{\Lambda^2}{\overline{\Lambda}^2}) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

$$\frac{d}{dt} M_n(t) = -\frac{\overline{\alpha}(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$\frac{d}{dt} \overline{\alpha}(t) = -\frac{\overline{\alpha}^2(t)}{4\pi} \beta_0 + \ldots$$

The evolution will thus depend on the scheme because of the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

The use of $\Lambda_{\overline{MS}} = 0.234$ GeV can be interpreted as the choice of new scheme, differing from MS.

Pion realistic picture: Coupling and effective charge

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_a^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

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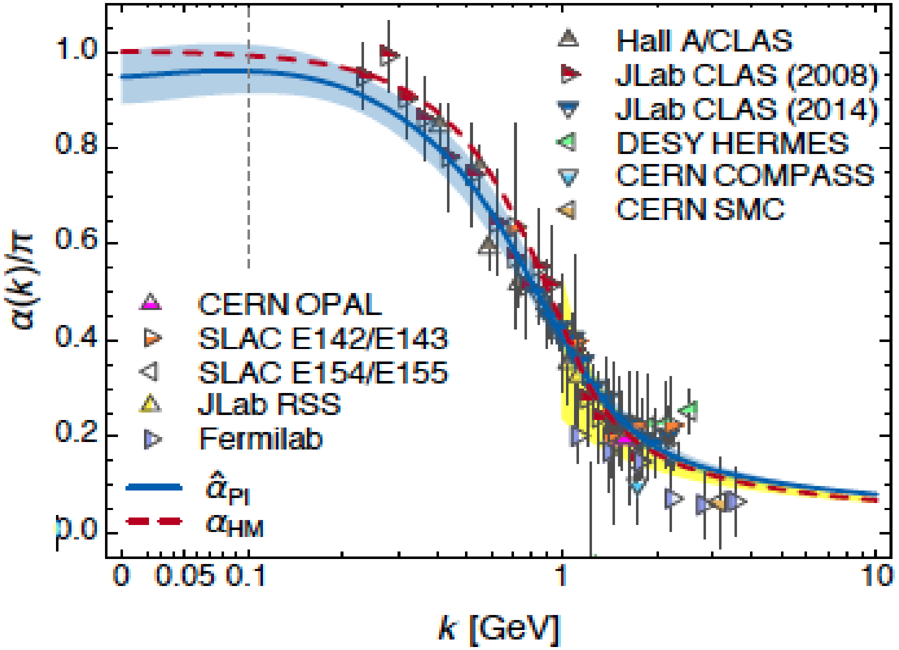
The evolution will thus depend on the scheme because of the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

The use of $\Lambda_{\overline{MS}} = 0.234$ GeV can be interpreted as the choice of new scheme, differing from MS. And it can be furthermore defined in such a way that one-loop DGLAP is exact (Grunberg's effective charge).

Pion realistic picture: Coupling and effective charge



- **Equivalence in the perturbative domain**
  reasonable definitions of the charge

$$\alpha_{g_1}(k^2) = \alpha_{\overline{MS}}(k^2)[1 + 1.14\alpha_{\overline{MS}}(k^2) + \cdots]$$
$$\widehat{\alpha}_{PI}(k^2) = \alpha_{\overline{MS}}(k^2)[1 + 1.09\alpha_{\overline{MS}}(k^2) + \cdots]$$

- **Equivalence in the non-perturbative domain**
  highly non-trivial (ghost-gluon interactions)

- **Process dependent effective charges**
  fixed by the leading-order term in the expansion of a given observable
  Grunberg, PRD 29 (1984)

- **Bjorken sum rule**
  defines such a charge
  Bjorken, PR 148 (1966); PRD 1 (1970)

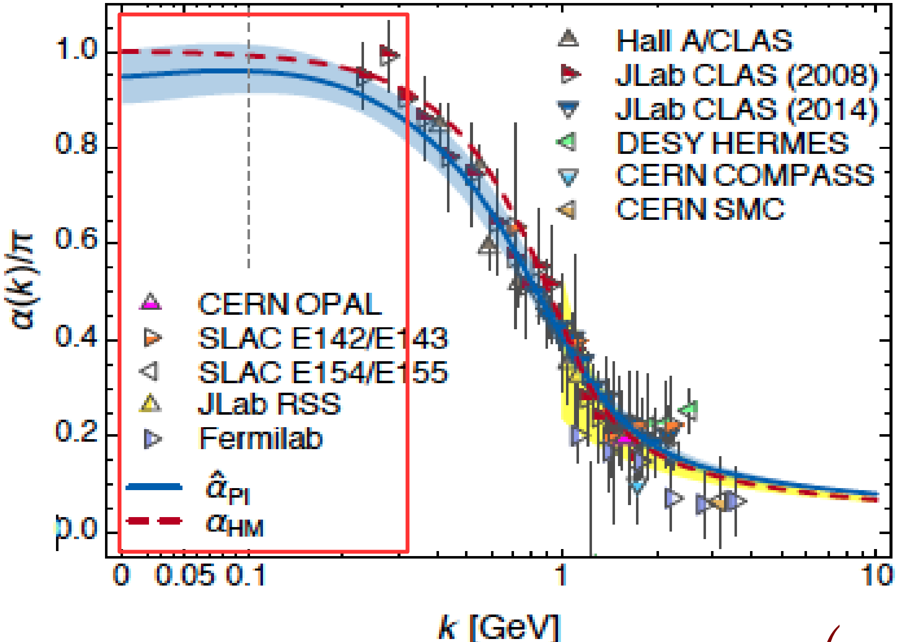
$$\int_0^1 dx\,[g_1^p(x,k^2) - g_1^n(x,k^2)] = \frac{g_A}{6}[1 - \alpha_{g_1}(k^2)/\pi]$$

  - $g_1^{p,n}$ **spin dependent p/n structure functions**
    extracted from measurements using unpolarized targets
  - g^A **nucleon flavour-singlet axial charge**

- **Many merits**
  - **Existence of data**
    for a wide momentum range
  - **Tight sum rules constraints on the integral**
    at IR and UV extremes
  - **Isospin non-singlet**
    suppress contributions from hard-to-compute processes

D. Binosi, C. Mezrag, J. Papavassiliou, J.R-Q, C.D. Roberts, arXiv:1612.04835

Pion realistic picture: Coupling and effective charge



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…re functions
…g unpolarized targets

…charge

$$\alpha(0) = \alpha_{PI}(0)$$

$$\zeta_0 = \zeta_H = m_a = 0.300\ \text{GeV}$$

- **Equivalence in the perturbat…**
  reasonable definitions of the c…

$$\alpha_{g_1}(k^2) = \alpha_{\overline{MS}}(k^2)[1 + 1.14\alpha_{\overline{M}}$$
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- **Equivalence in the non-perturbative domain**
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Pion realistic picture: PDF DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_a^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)}$$

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i}\right)$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$

Numerical integration with the effective charge

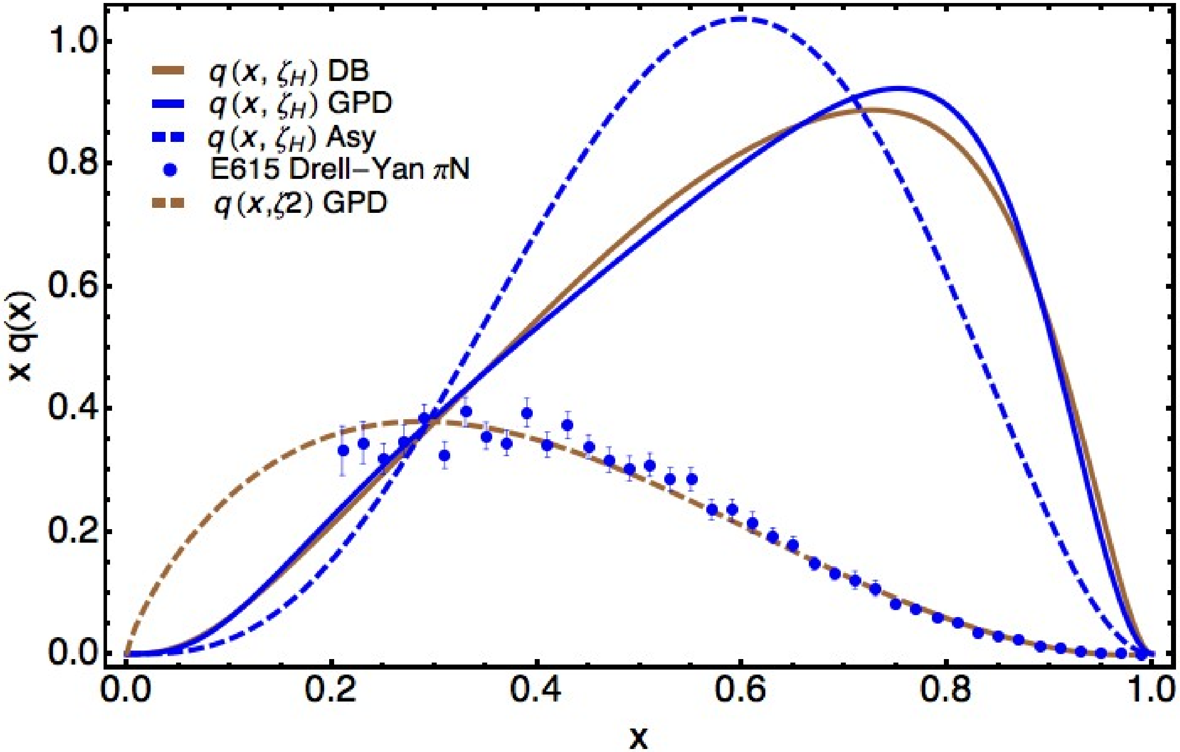
$$M_n(t) = M_n(t_0) \exp\left(-\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz\, \alpha(z)\right)$$

No free parameter to be fitted. All the scales (and the evolution between them) appear fixed.

Pion realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

$\zeta_0 = \zeta_H = 0.3$ GeV → $\zeta_2 = 5.2$ GeV



No free parameter to be fitted. All the scales (and the evolution between them) appear fixed. And the agreement with the Aicher et al. reanalysis of E615 data is perfect!!!

Pion realistic picture: PDF DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_a^2+\zeta_0^2 \exp(t)}{\Lambda^2}\right)}$$

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i}\right)$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$

$$\int_0^1 dx\, P(x) = \gamma_0^n$$

$$\frac{d}{dt} q(x,t) = -\frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right)$$

$$P(x) = \frac{8}{3}\left(\frac{1+z^2}{(1-x)_+} + \frac{3}{2}\delta(x-1)\right)$$

Numerical integration with the effective charge for the master equation. No need for a reconstruction with evolved Mellin moments!

No free parameter to be fitted. All the scales (and the evolution between them) appear fixed.

Pion realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

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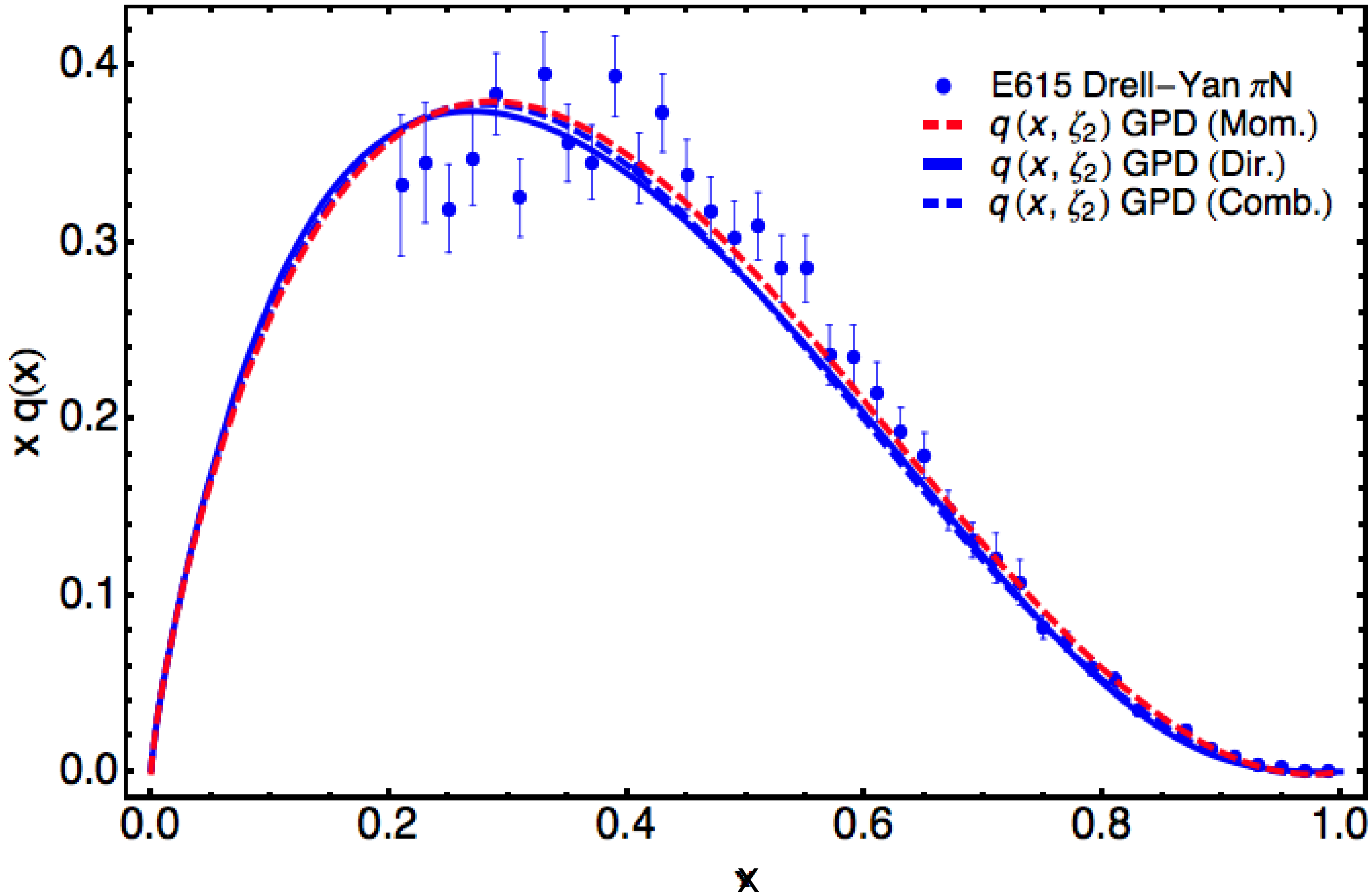


Pion (more) realistic picture:

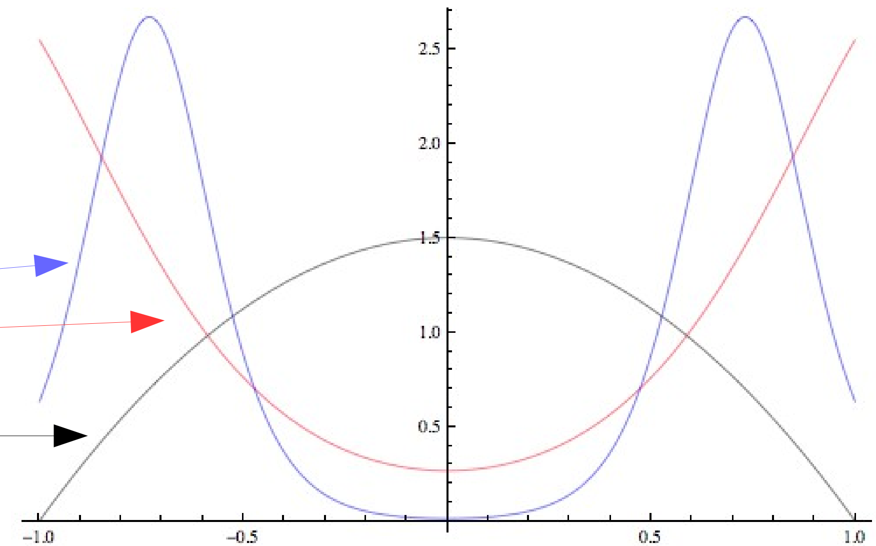
- Spectral density is chosen as:

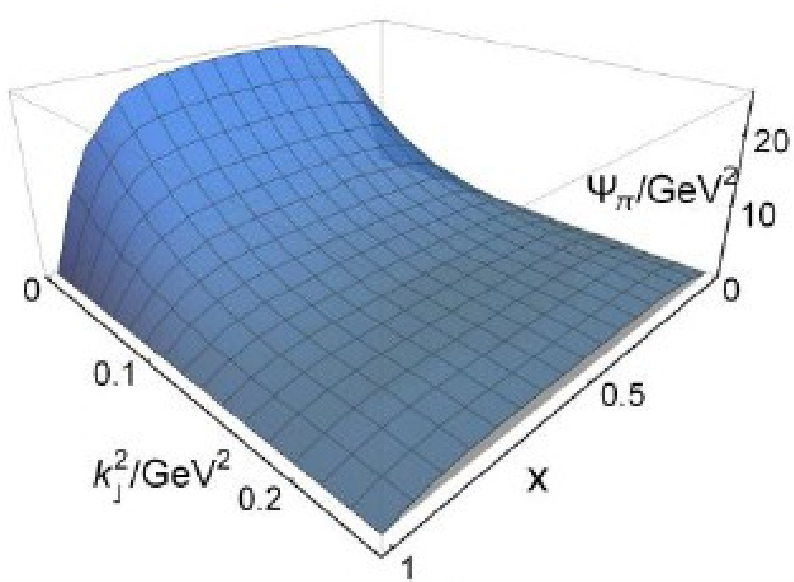
$$u_G \rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega - \omega_0^G}{2b_0^G}\right) + \mathrm{sech}^2\left(\frac{\omega + \omega_0^G}{2b_0^G}\right)\right]$$

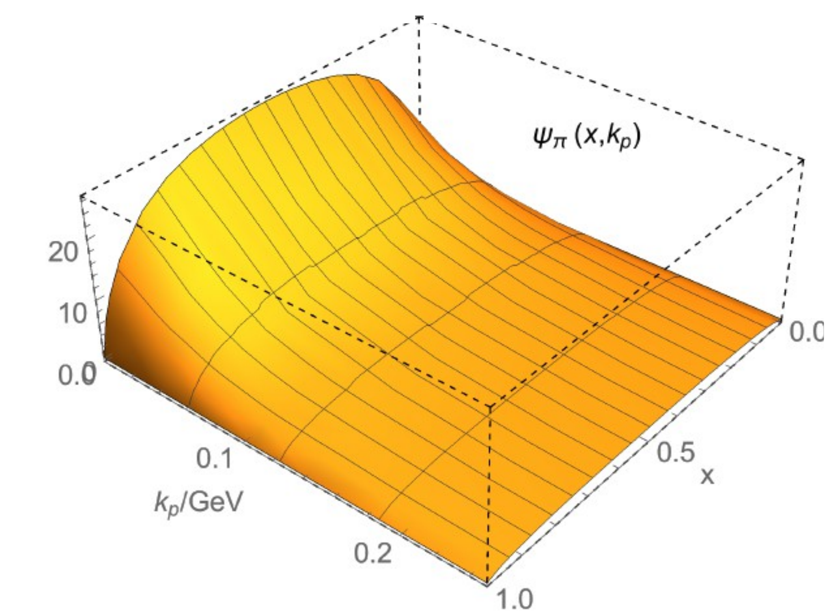
Phenomelogical model: $b_0^\pi = 0.1, w_0^\pi = 0.73$;

Realistic case: $b_0^\pi = 0.275, b_0^\pi = 1.23$;

Asymptotic case: $\rho(\omega;\nu) \sim (1-\omega^2)^\nu$







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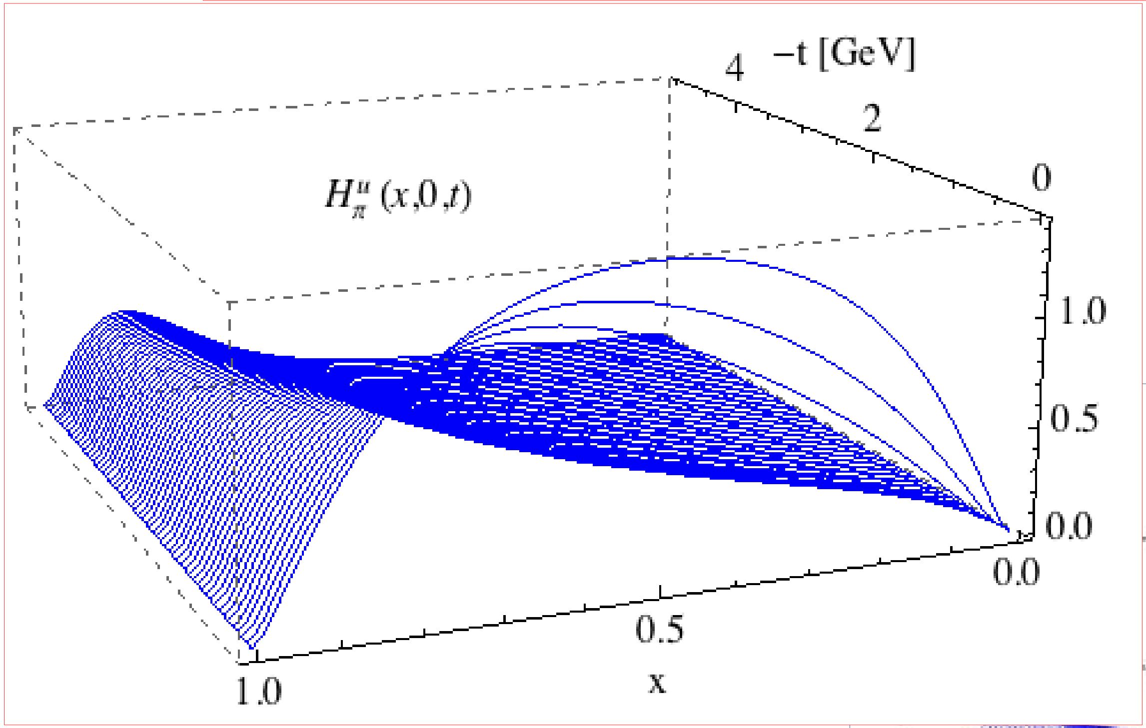
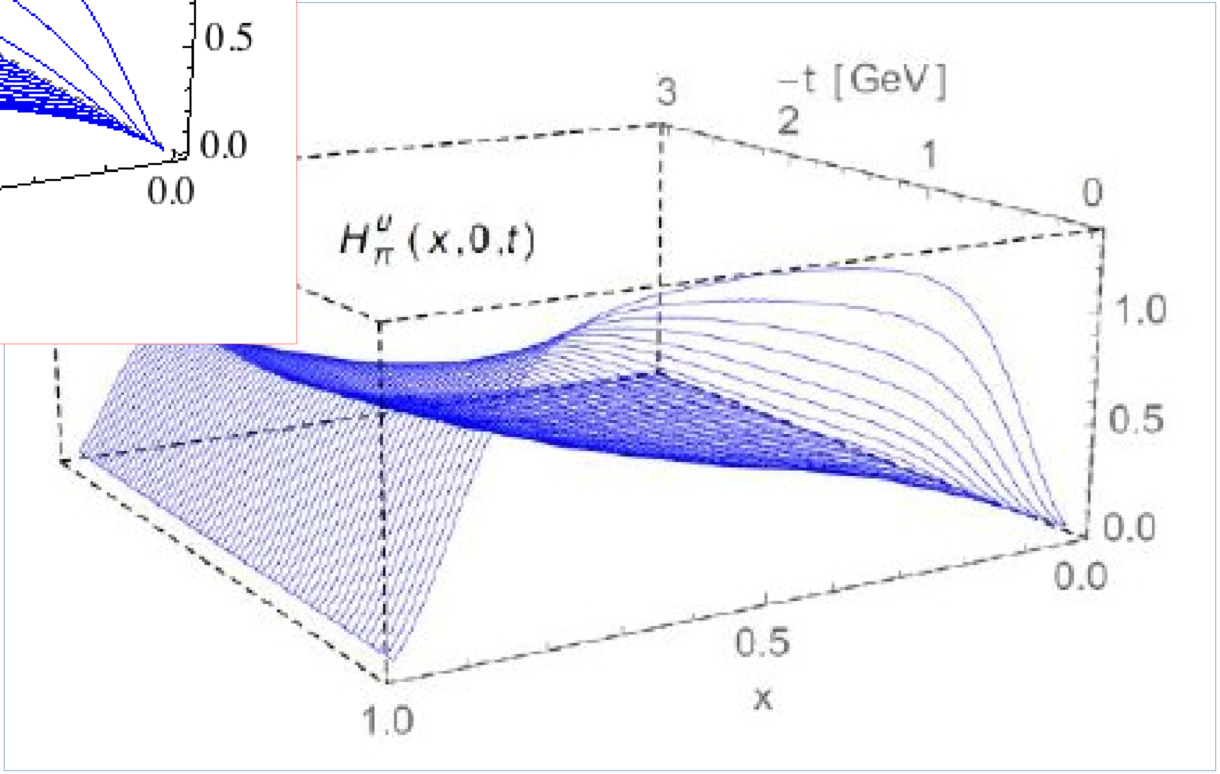


Realistic case

Phenomenological model



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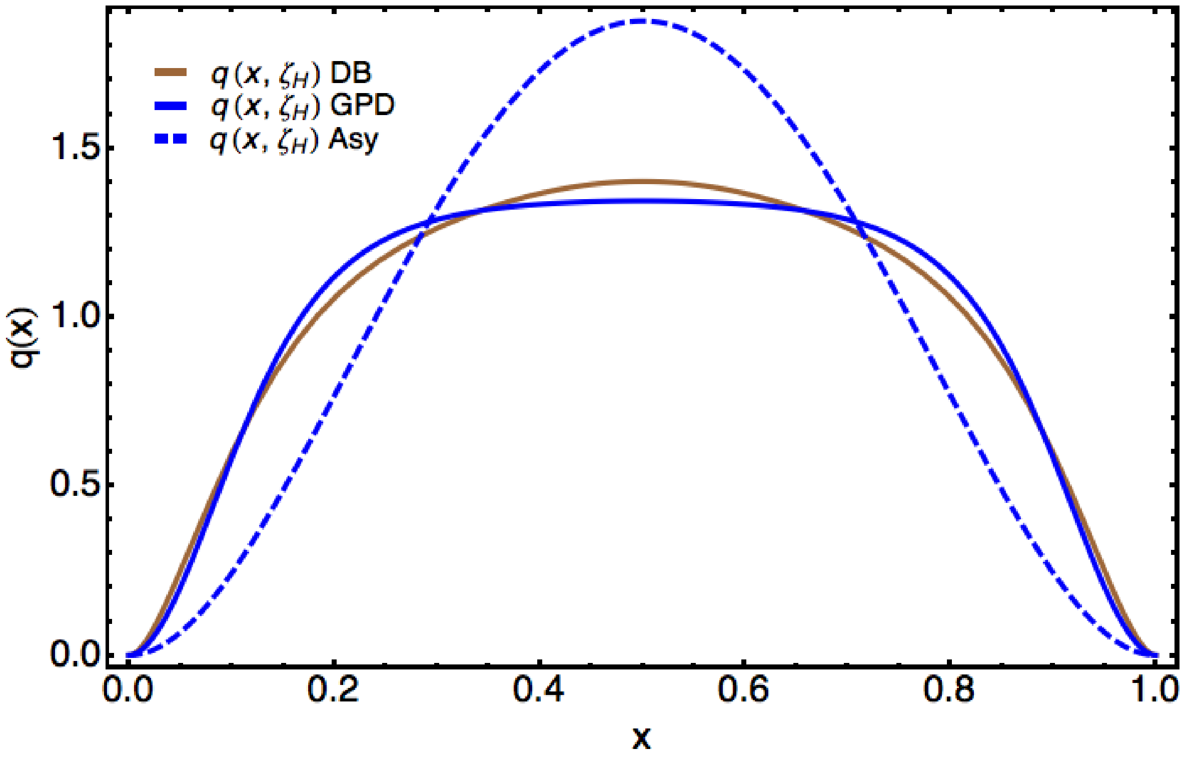
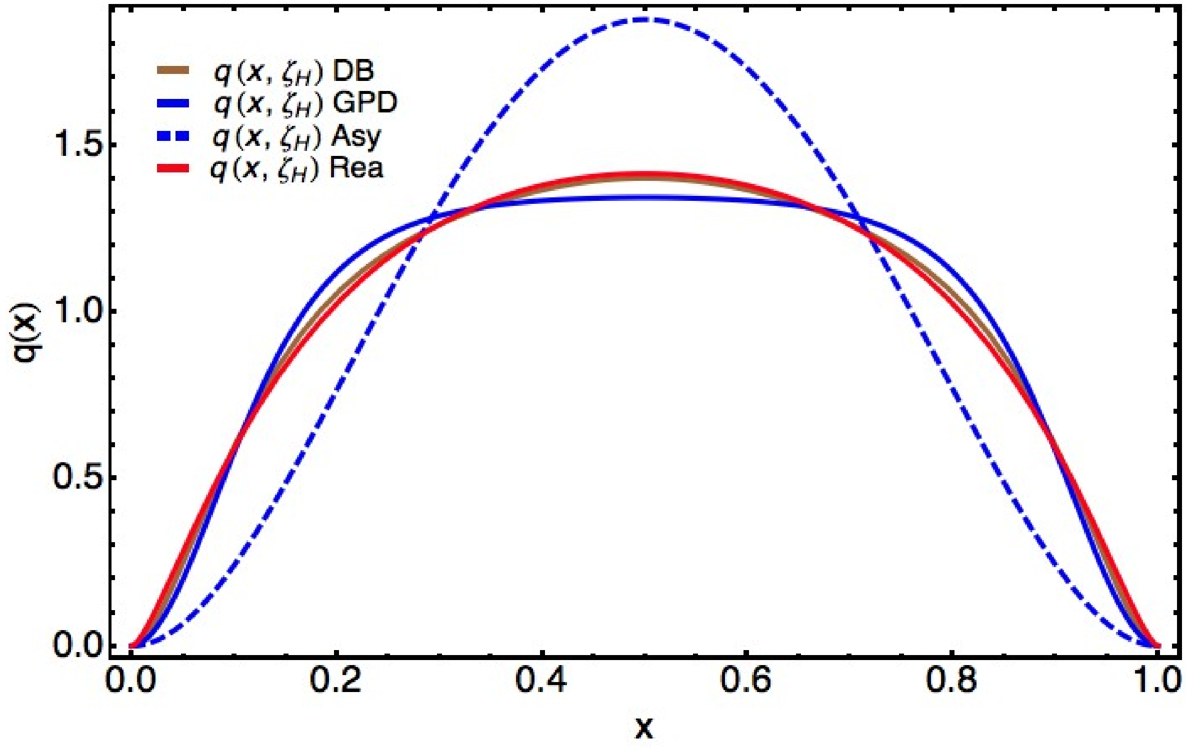


PDF (benchmark) case

Pion (more) realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

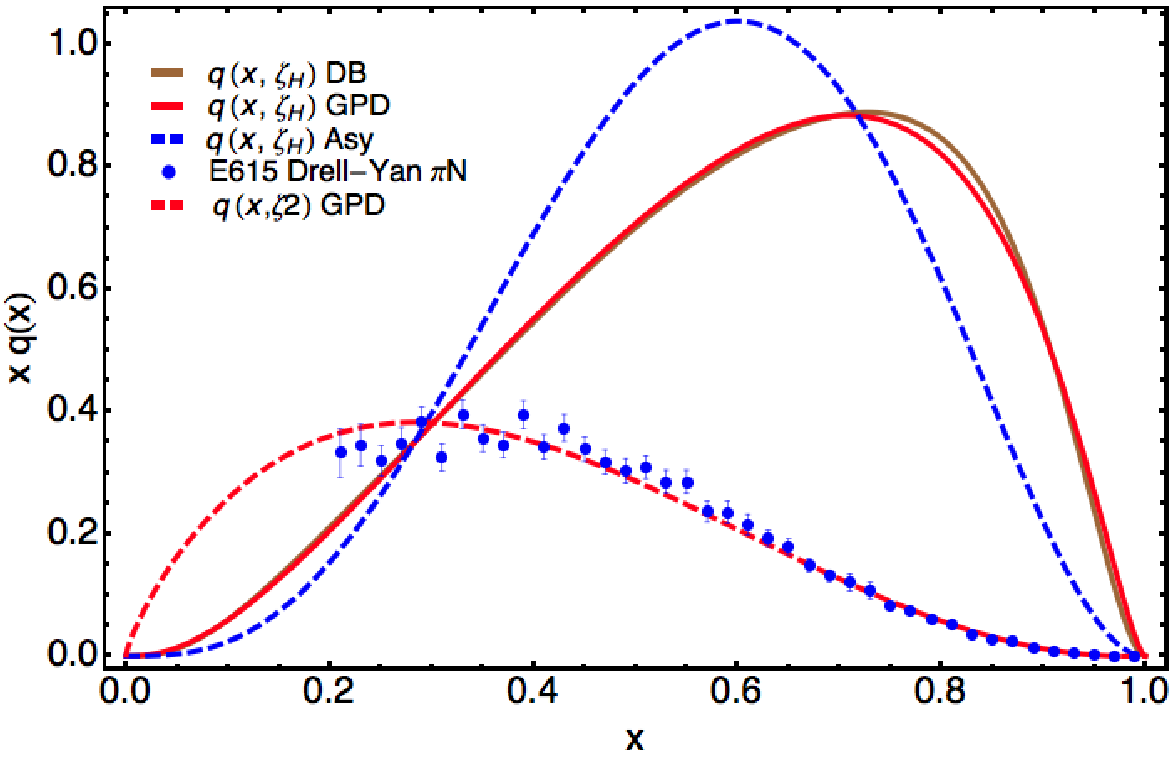


PDF (benchmark) case

Pion (more) realistic picture:

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3}\Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi},\mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right)\Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi},\mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

$\zeta_0 = \zeta_H = 0.3\,\text{GeV} \rightarrow \zeta_2 = 5.2\,\text{GeV}$



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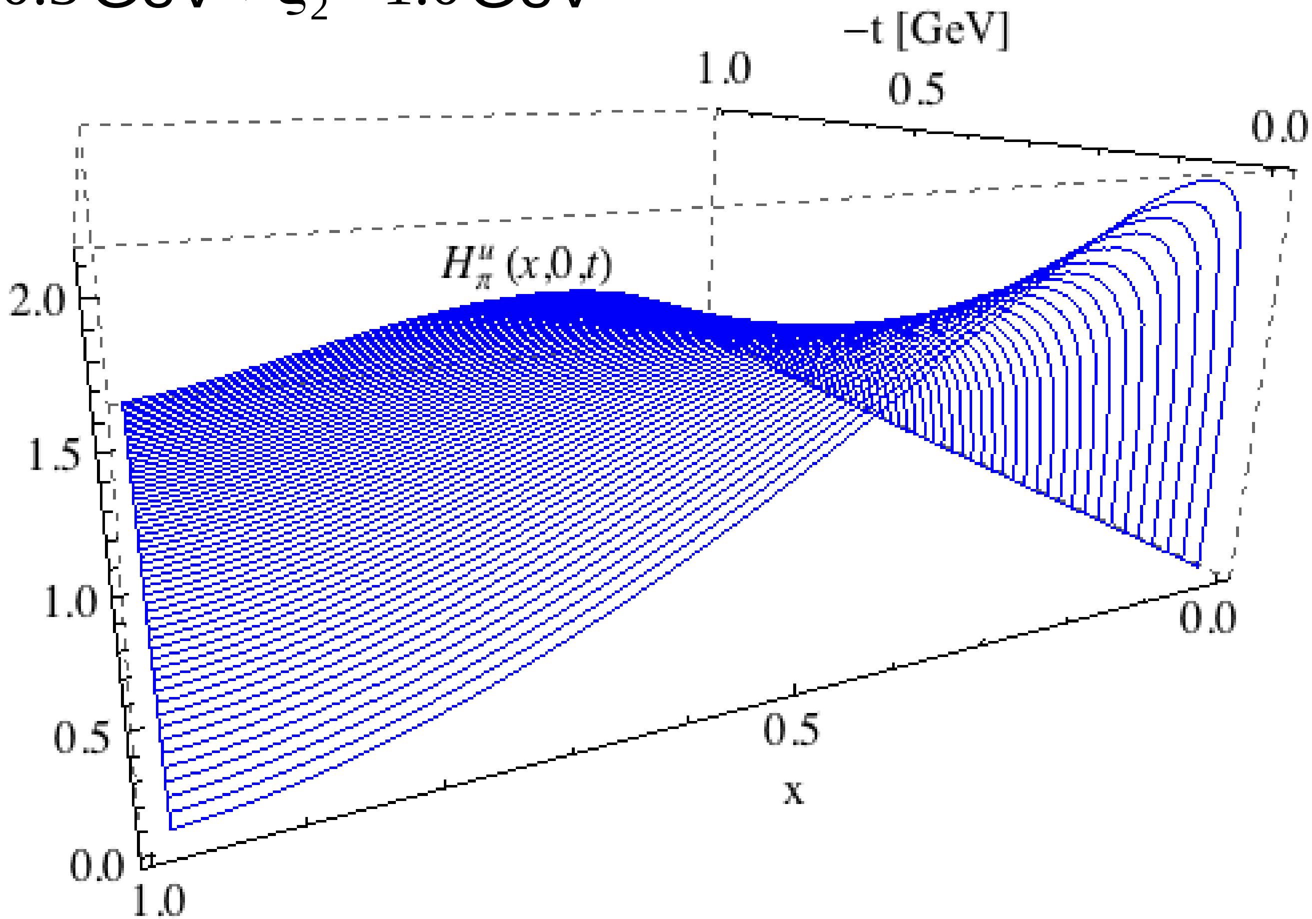
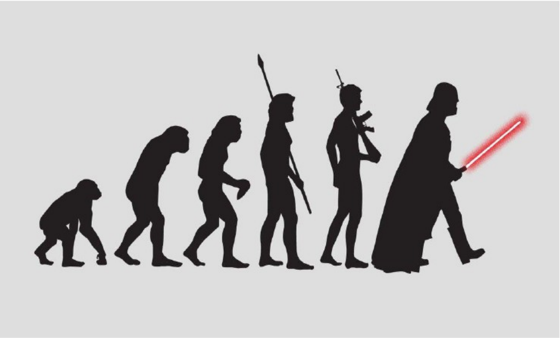


About PDA and LFWF evolution



Standard PDA evolution:

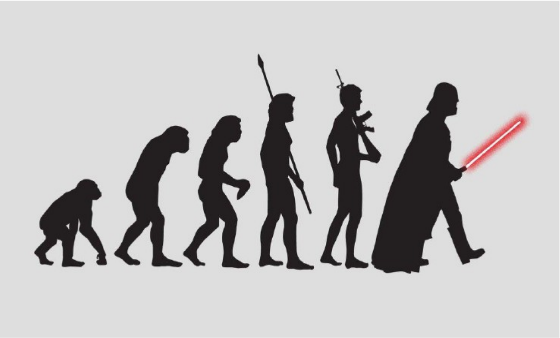
- We project **PDA** onto a 3/2-Gegenbauer polynomial basis. Such that it **evolves**, from an initial scale ζ_0 to a final scale ζ, **according to** the corresponding **ERBL equations**:

$$\phi(x;\zeta) = 6x(1-x)\left[1+\sum_{n=1} a_n(\zeta) C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) = a_n(\zeta_0)\left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0} , \quad \gamma_0^n = -\frac{4}{3}\left[3+\frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right] .$$

- Thus, any PDA at hadronic scale evolves logarithmically towards its conformal distribution, $\phi(x)=6x(1-x)$.
  - ➢ Quark mass and flavor become irrelevant. Broad PDA becomes narrower, skewed PDA becomes symmetric.

About PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3} \int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)$$

- We look for a way to evolve the LFWF.
- First, let's assume that the LFWF admits a similar Gegenbauer expansion. That is:

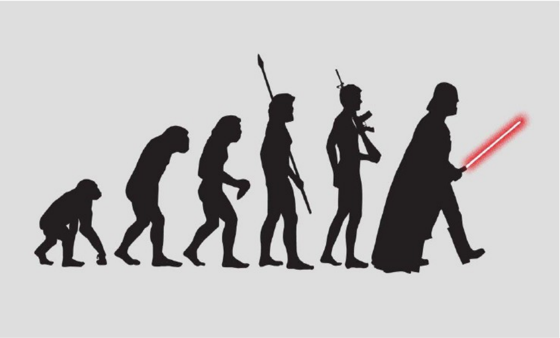
$$\psi(x, k_\perp^2; \zeta) = 6x(1-x) \left[\sum_{n=0} b_n(k_\perp^2; \zeta)\, C_n^{3/2}(2x-1) \right] ,$$

$$a_n(\zeta) = \frac{1}{16\pi^3} \int d^2\vec{k}_\perp\, b_n(k_\perp^2; \zeta) \text{ (for } n \geq 1) , \quad \frac{1}{16\pi^3} \int d^2\vec{k}_\perp\, b_0(k_\perp^2; \zeta) = 1 .$$

- 1-loop ERBL evolution of $a_n(\zeta)$ implies:

$$\frac{1}{a_n(\zeta)} \frac{d}{d\ln\zeta^2} a_n(\zeta) = \frac{\int d^2\vec{k}_\perp \frac{d}{d\ln\zeta^2} b_n(k_\perp^2; \zeta)}{\int d^2\vec{k}_\perp\, b_n(k_\perp^2; \zeta)} ,$$

About PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3} \int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)$$

- Now, if we take a factorization assumtion, we arrive at:

$$\frac{b_n(k_\perp^2;\zeta)}{b_n(k_\perp^2;\zeta_0)} = \frac{\widehat{b}_n(\zeta)}{\widehat{b}_n(\zeta_0)} = \left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0}, \; b_n(k_\perp^2;\zeta) \equiv \widehat{b}_n(\zeta)\chi_n(k_\perp^2) \; .$$

- Suplemented by the condition $\chi_n(k_\perp^2) \equiv \chi(k_\perp^2)$, one gets $\widehat{b}_n(\zeta) \equiv a_n(\zeta)$.

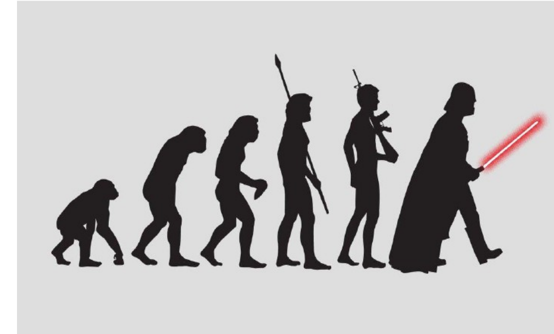
- Such that, the followiong factorised form is obtained:

$$\psi(x, k_\perp^2;\zeta) \; \equiv \; \phi(x;\zeta) \, \chi(k_\perp^2)$$

→ **LFWF Evolves like PDA**

- Which is far from being a general result, but an useful approximation instead.

About PDA and LFWF evolution

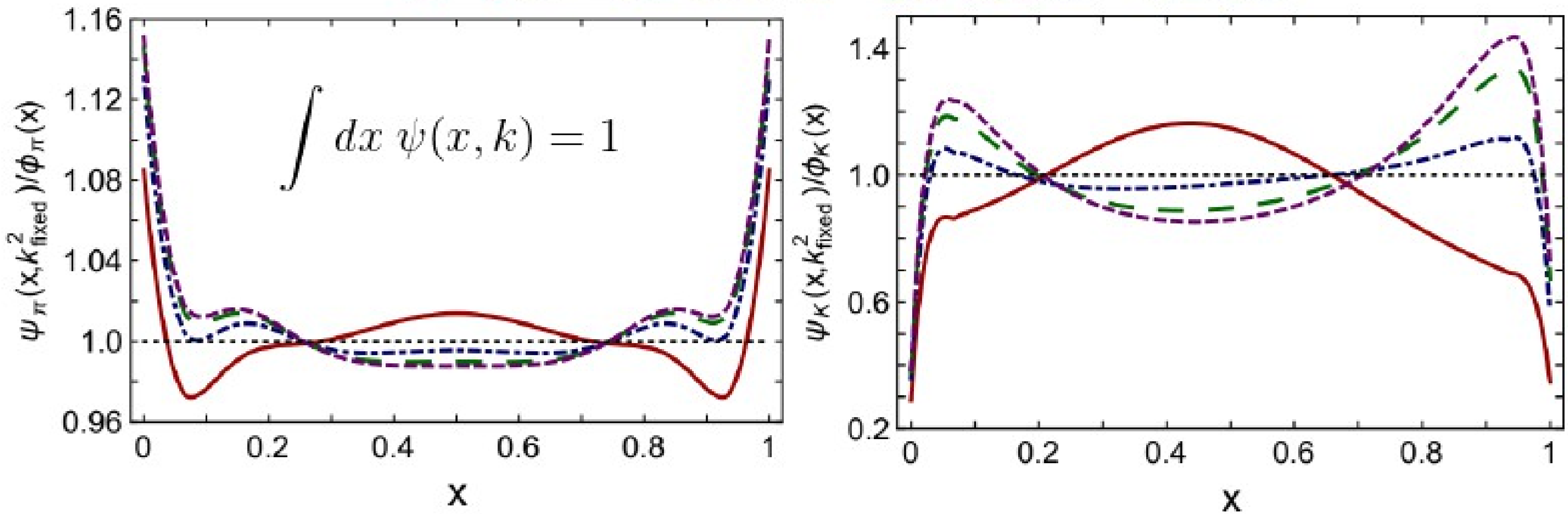


Testing the factorization ansatz:

$$\psi(x, k_\perp^2; \zeta) \;\equiv\; \phi(x;\zeta)\,\chi(k_\perp^2)$$

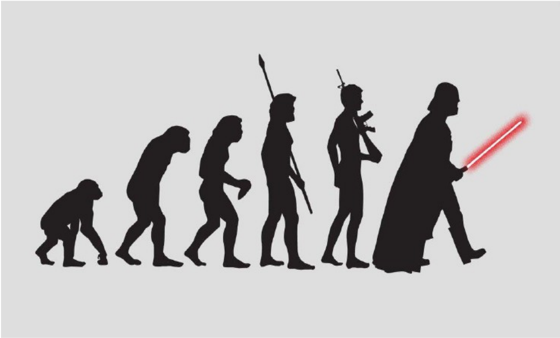
- A first validation of the factorized ansätz is addressed in **Phys.Rev. D97 (2018) no.9, 094014**:

k^2=0, k^2=0.2 GeV, k^2=0.8 GeV, k^2=3.2 GeV



- If the factorized ansatz is a good approximation, then the plotted ratio must be 1. For the pion, it slightly deviates from 1; for the kaon, the deviation is much larger.

About PDA and LFWF evolution



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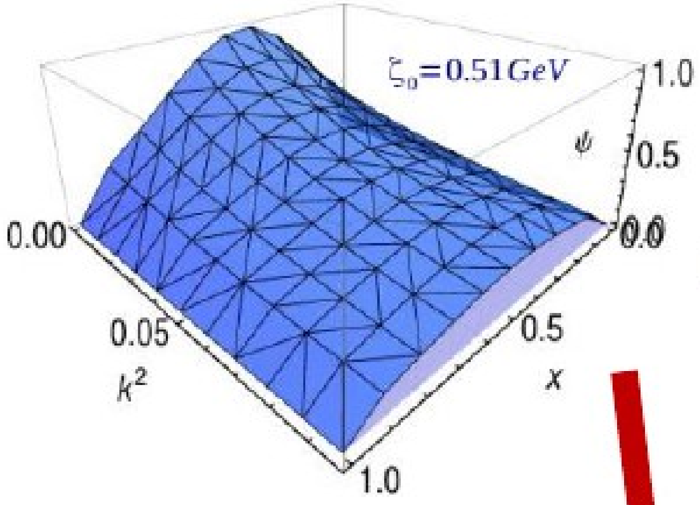
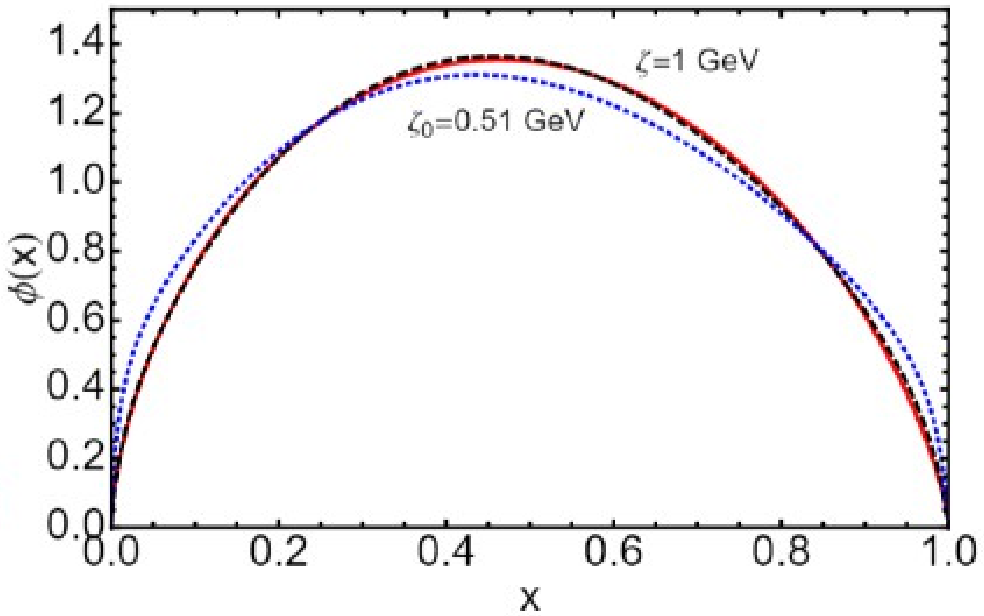
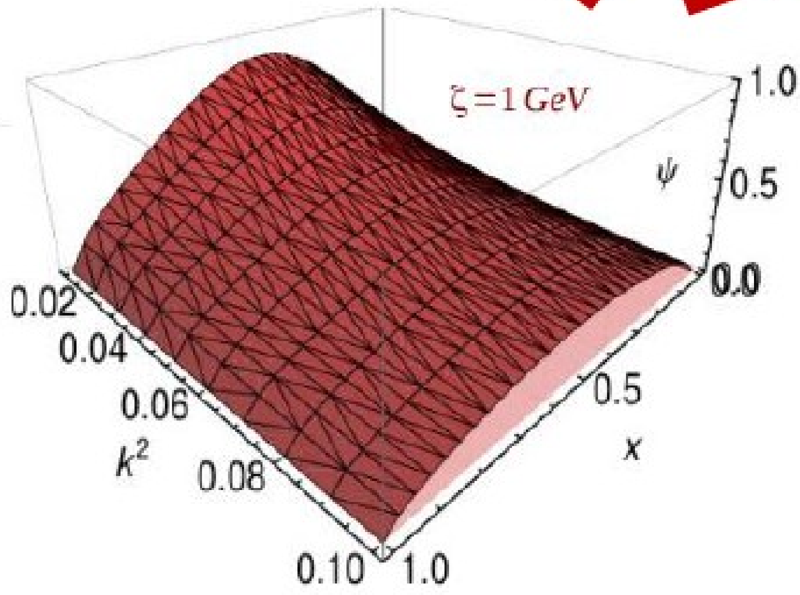




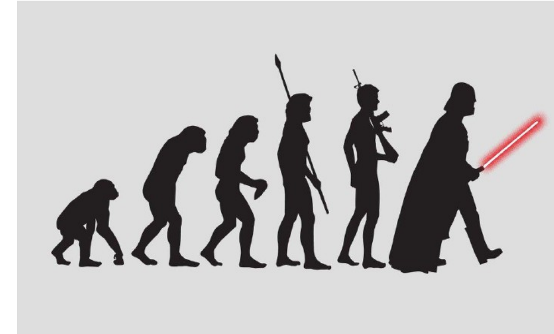


1) Compute LFWF and ERBL running of PDA

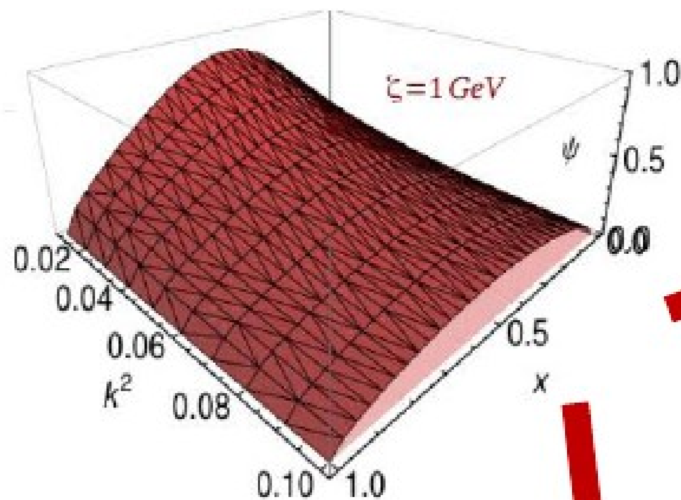
2) ERBL running of LFWF and compute PDA

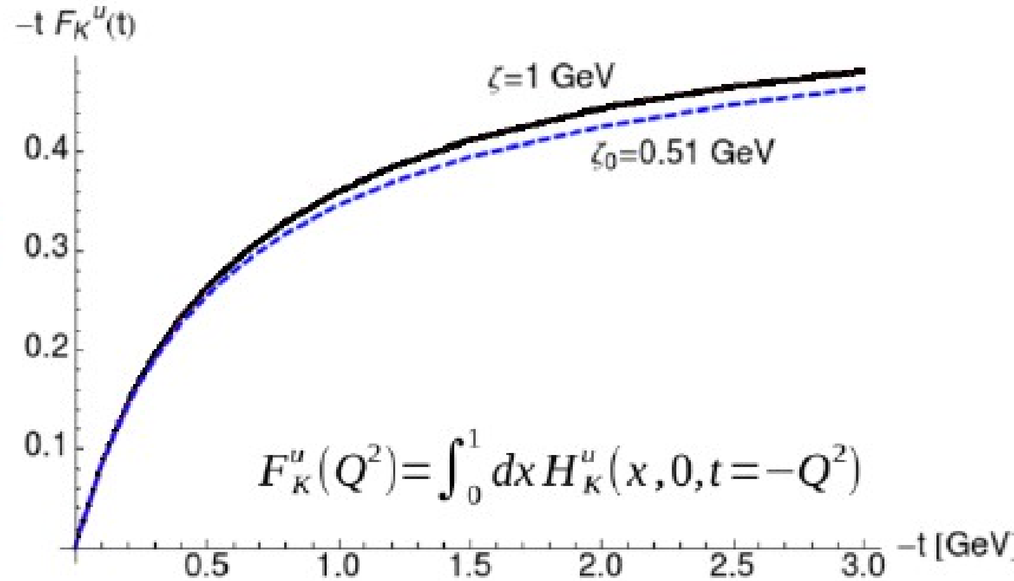
Notably, **1)** and **2)** are **equivalent**. Factorization assumption and evolution seem reasonable.

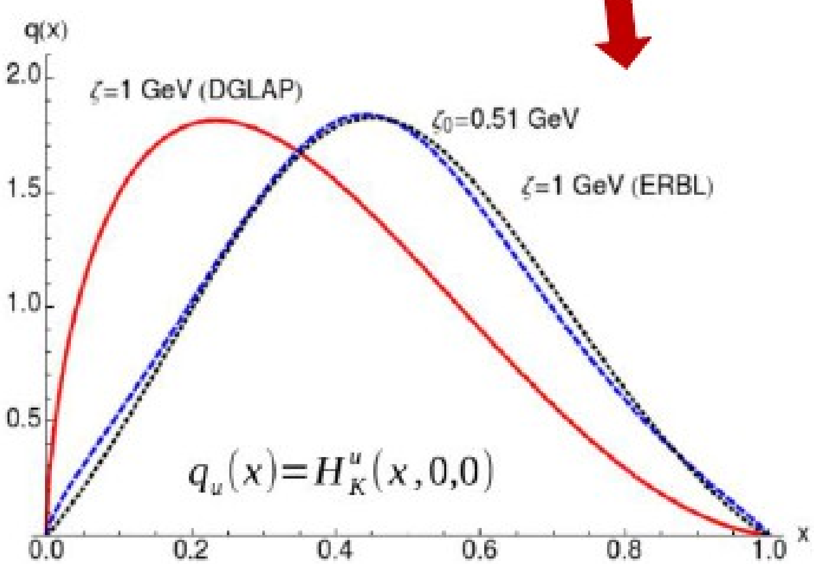
About PDA and LFWF evolution



How ERBL and DGLAP evolutions make contact:





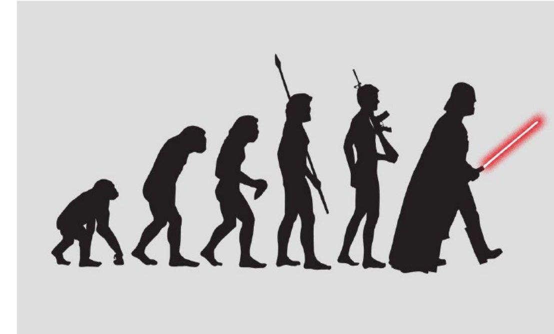


1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

About PDA and LFWF evolution



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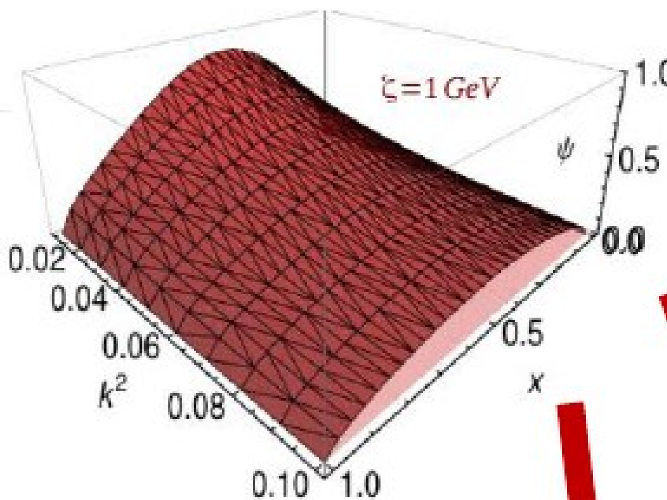

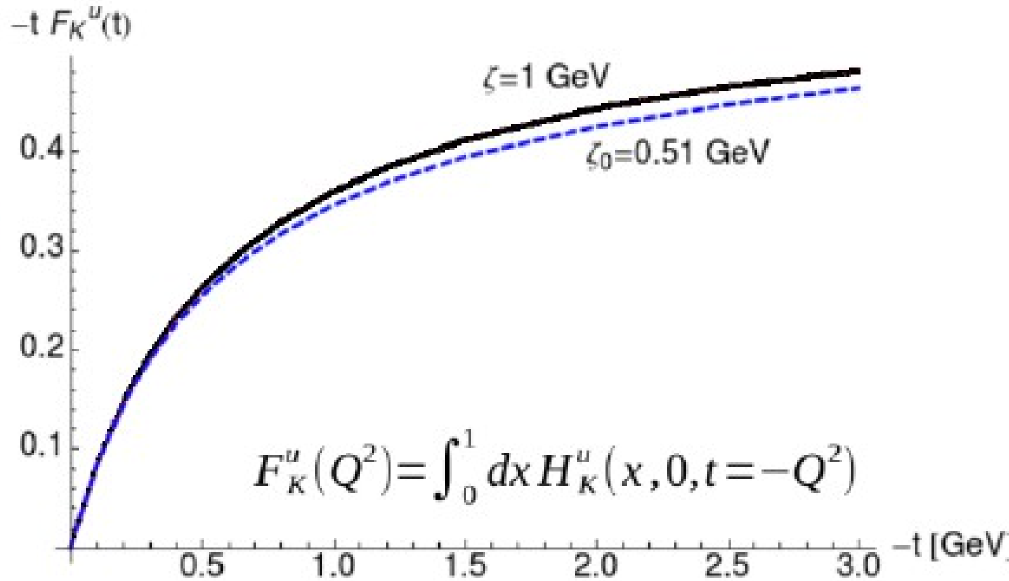

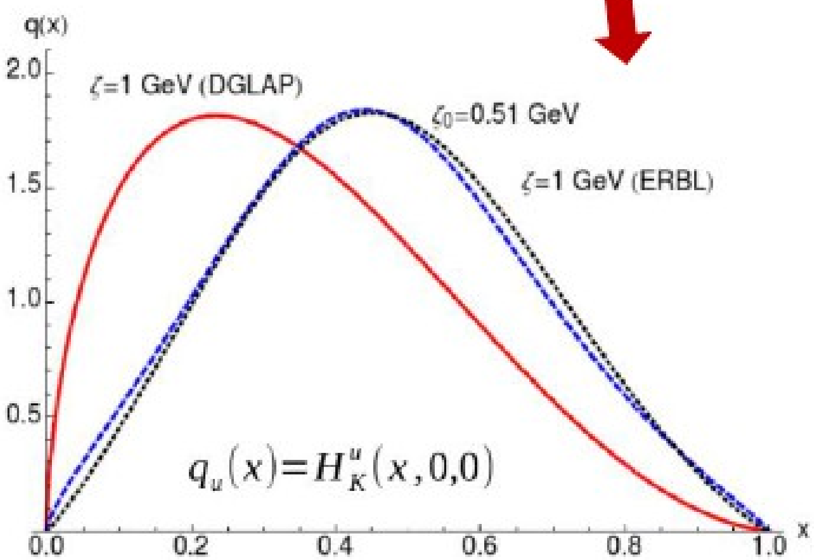

1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

Sea-quark and gluon content incorporated to the parton distribution by DGLAP are obviously not present in the valence-quark PDF from LFWFs!!!

About gravitational Form Factors

A word about GPD polinomiality first:

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

$$\text{linear combination of } (P^+)^{m+1-k}(\Delta^+)^k \text{ for } 0 \leq k \leq m+1$$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) \; .$$

About gravitational Form Factors

Definition and evaluation:

Polinomiality!

- Pion gravitational form factors are defined through*:

$$J_{\pi^+}(-t,\xi) \equiv \int_{-1}^{1} dx\; x H_{\pi^+}(x,\xi,t) = \Theta_2(t) - \Theta_1(t)\xi^2 \;.$$

- Taking ξ=0 + isospin symmetric limit, one can readily compute:

$$\Theta_2(t) = \int_0^1 dx\; x[H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(x,0,t)] = \int_0^1 dx\; 2x H^u_{\pi^+}(x,0,t) \;.$$

- To obtain $\Theta_1(t)$, we need to take a non zero value of ξ; hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate $\Theta_1(t)$, by estimating the derivative of $J_{\pi^+}(-t,\xi)$ with respect to ξ^2 as:

$$D(\xi+\Delta/2) \equiv \frac{J(\xi+\Delta) - J(\xi)}{2(\xi+\Delta/2)\Delta} \;, \quad \Delta \to 0 \;.$$

*Phys.Rev. D78 (2008) 094011.

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***Phys.Rev. D78 (2008) 094011.**

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About gravitational Form Factors

Definition and evaluation:

- To get a clearer picture, let's split $J(-t,\xi)$ as follows:

$$J(-t,\xi) = \int_{-\xi}^{1} dx\; 2xH(x,\xi,t) = \left[\int_{-\xi}^{\xi} dx + \int_{\xi}^{1} dx\right] 2xH(x,\xi,t)$$

$$\Rightarrow J(-t,\xi) = J^{\mathrm{ERBL}}(-t,\xi) + J^{\mathrm{DGLAP}}(-t,\xi) \;,$$

- Notice that, because of the polinomiality of the *complete* GPD:

$$J^{\mathrm{DGLAP}}(-t,\xi) = \Theta_2(t) - \xi^2\Theta_1(t)^{\mathrm{DGLAP}} + \sum_{i=1}^{\infty} c_i(t)\xi^{2+i} \;,$$

$$J^{\mathrm{ERBL}}(-t,\xi) = -\xi^2\Theta_1(t)^{\mathrm{ERBL}} - \sum_{i=1}^{\infty} c_i(t)\xi^{2+i}$$

- Thus, since so far we can only access DGLAP region: (overlap approximation)

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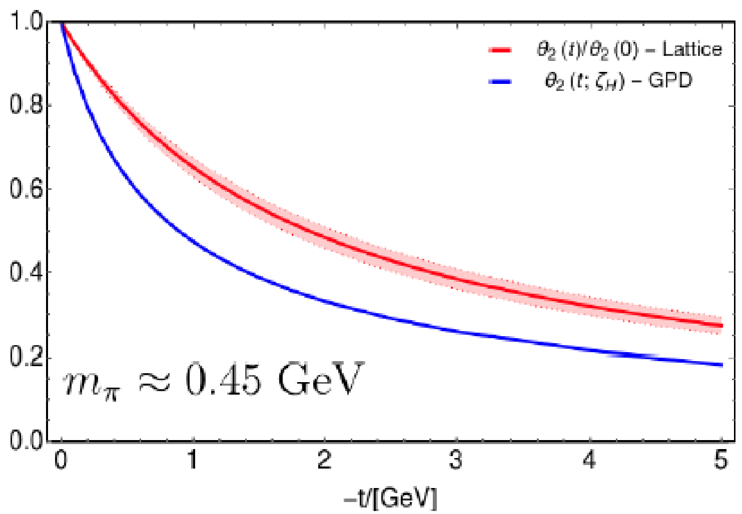
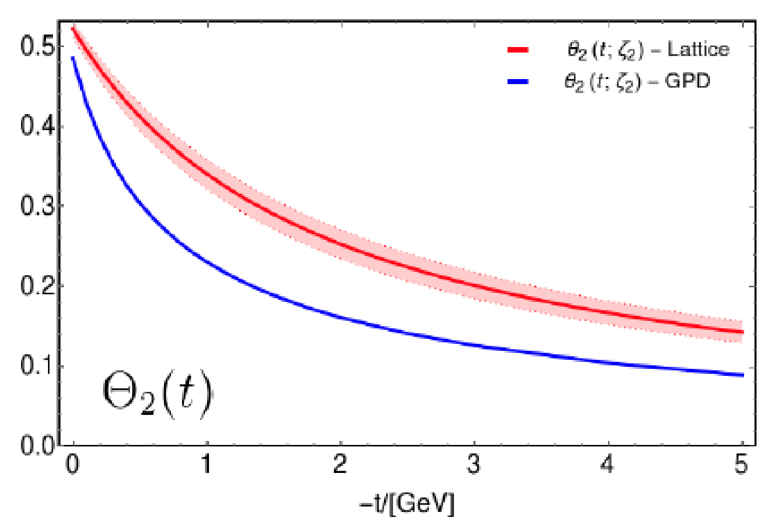
Definition and evaluation:

- The extensión to **ERBL region** is then **needed**. Taking advantage of the soft-pion theorem, one can conect PDA with $J(-t,\xi)^{ERBL}$ and thus with $\Theta_1(t)^{ERBL}$.

- Nonetheless, polinomiality of GPD is not fulfilled without the ERBL región. Such extension is necessary to provide a more reliable computation of Θ_1.





Lattice: (2007) Brömmel's dissertation. $\Theta_2(0)/2 =< x >= 0.261(5)$

GPD + Ding et al. $\Theta_2(0)/2 =< x >= 0.242(20)$

Latt.: D. Brommel, Ph.D. thesis, University of Regensburg, Regensburg, Germany (2007), DESY-THESIS-2007-023

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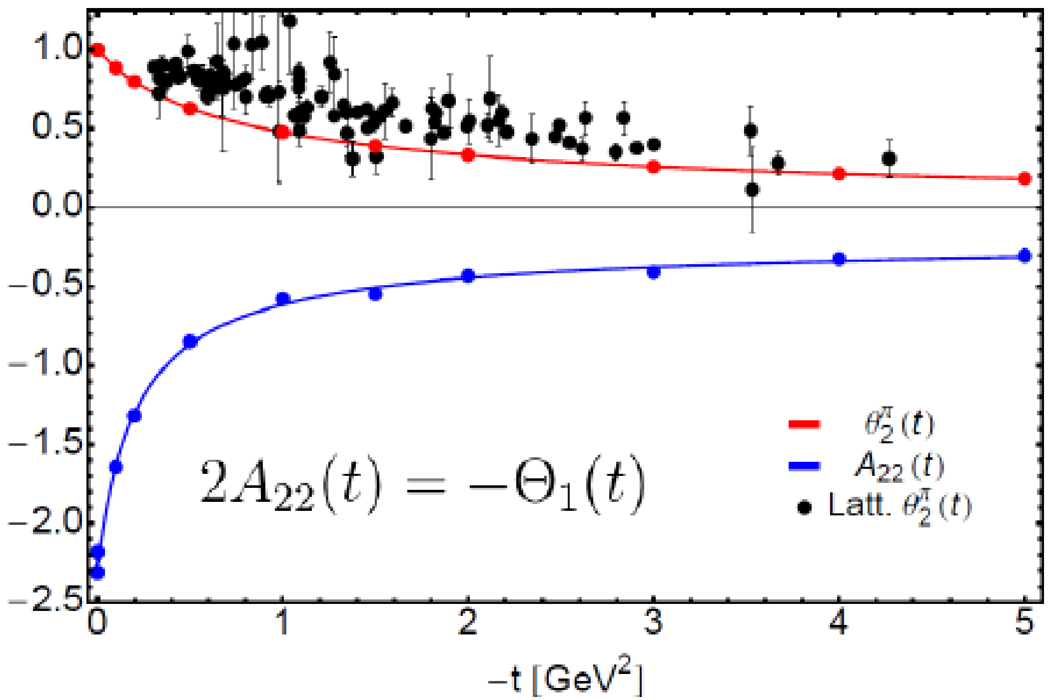

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Conclusions

- Khépani's previous conclusions & ...
- A good choice for the scheme of the coupling or, furthermore, the definition of a particular effective charge, makes possible a successful DGLAP evolution of the PDF's results, from an unambigous hadronic scale, to the scale of available experimental data. This effective charge is intimately connected to the PI one.
- The comparison of the valence-quark PDF directly obtained from LFWFs at any non-hadronic scale and the evolved one might result insightful.
- Gravitational form factors can be obtained from the overlap GPD, only after some modelling in the case of $\theta_1(t)$.