Continuum Functional Methods for QCD at New Generation Facilities,
7-10 May 2019, ECT*, Trento, Italy

Study on Heavy-Light Meson Beyond Rainbow-ladder Truncation

Pianpian Qin

Supervisor: Prof. Yuxin Liu

8, May, 2019

- Introduction
- Theoretical Framework
- Numerical Details
- Results
- Summary

- Introduction
- Theoretical Framework
- Numerical Details
- Results
- Summary

Bound-State problem

Experiment (Interaction) → **Bound-State** ← **Theory (Dynamics)**

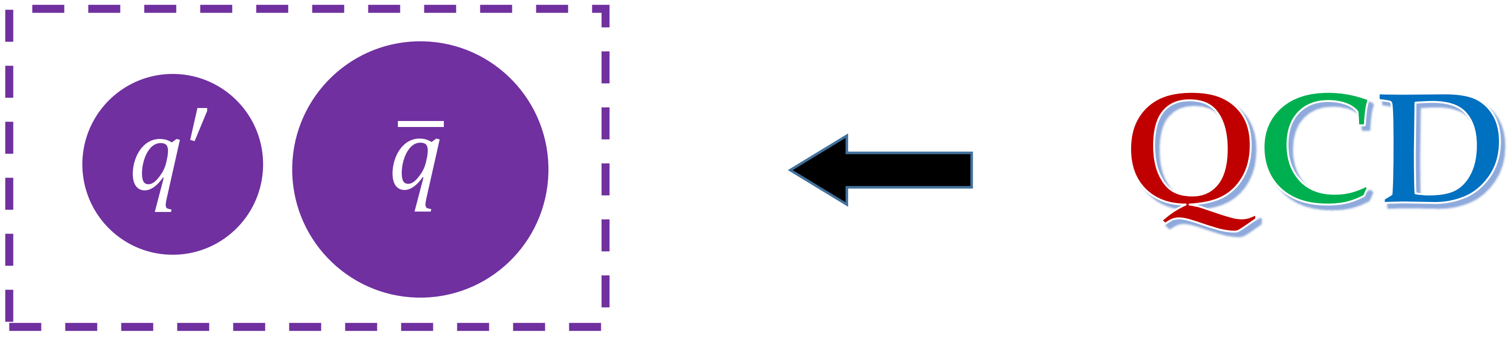
Gravitation	The Earth-moon system	Newtonian mechanics
Electromagnetic force	Atom (nucleus-electron system)	QM/QED
Strong force	Meson (quark-quark system)	QCD

Bound-State problem

Experiment (Interaction)	→ **Bound-State** ←	**Theory (Dynamics)**
Gravitation	The Earth-moon system	Newtonian mechanics
Electromagnetic force	Atom (nucleus-electron system)	QM/QED
Strong force	**Meson (quark-quark system)**	QCD

Meson problem

- Meson is 2-body bound-state system of strong interaction.



- Non-perturbative methods
  - Lattice QCD
  - Functional Renormalization Group (FRG)
  - **Dyson-Schwinger Equation (DSE) Method**

- Bethe-Salpeter equation (BSE) can be used for 2-body bound-state problem.

Dyson-Schwinger-Bethe-Salpeter Equation (DSBSE)

Gluon model

- Contact Model
- Maries-Tandy Model
- **Qin-Chang Model**
- ……

Truncation scheme

- Rainbow-Ladder (RL) truncation

C. D. Roberts, A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994)

P. Maris, C. D. Roberts, P. C. Tandy, Phys. Lett. B. **420**, 267 (1998)

P. Maris and P. C. Tandy, Phys. Rev. C. **60**, 055214 (1999)

Si-xue Qin, et al. Phys. Rev. C **85**, 035202 (2012)

Light-light meson

Heavy-heavy meson

Chiral Limit → Heavy quark Limit

Dyson-Schwinger-Bethe-Salpeter Equation (DSBSE)

Gluon model

- Contact Model
- Maries-Tandy Model
- **Qin-Chang Model**
- ……

Truncation scheme

- Rainbow-Ladder (RL) truncation

C. D. Roberts, A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994)

P. Maris, C. D. Roberts, P. C. Tandy, Phys. Lett. B. **420**, 267 (1998)

P. Maris and P. C. Tandy, Phys. Rev. C. **60**, 055214 (1999)

Si-xue Qin, et al. Phys. Rev. C **85**, 035202 (2012)

Light-light meson | **Heavy-light meson** | Heavy-heavy meson

Chiral Limit → Heavy quark Limit

Heavy-Light meson

- Lots of efforts have been devoted to this topic.

M. Gómez-Rocha, et al., Phys. Rev. D **92**, 054030 (2015)

D. Binosi, et al., Phys. Lett. B 790, 257 (2019)

M. A. Bedolla, et al., EPJ Web Conf. **192** 00039 (2018)

Muyang Chen and Lei Chang, arXiv: 1903.07808.

- Heavy-light meson has been explored with Rainbow-Ladder truncation and QC model
  - Light-light parameters, underestimate
  - Heavy-heavy parameters, overestimate

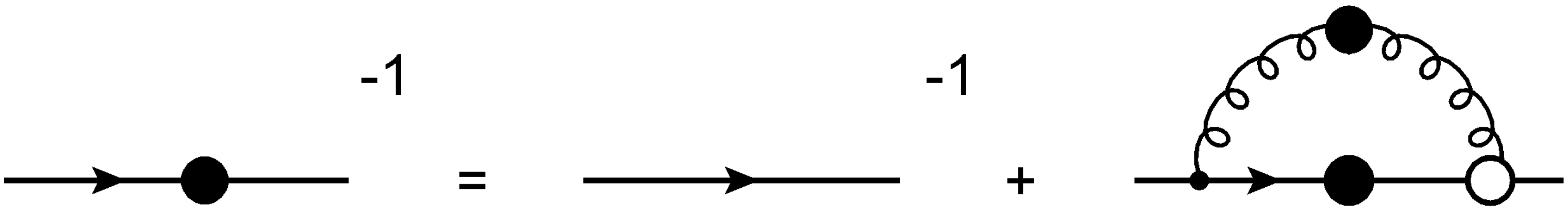
In this work

We propose a modified kernel and study the heavy-light meson with the Rainbow truncation and QC model.

- Introduction
- **Theoretical Framework**
- Numerical Details
- Results
- Summary

Dyson-Schwinger Equation

$T = 0, \mu = 0$



$$S^{-1}(k) = Z_2(i\not{k} + Z_m m) + g^2 Z_{1F} \int \frac{d^4q}{(2\pi)^4} t^a \gamma_\mu S(q) \Gamma^b_\nu(q,k) D^{ab}_{\mu\nu}(k-q),$$

Truncation Scheme   Gluon Model

- Renormalization condition
- Decomposing with Lorentz structures

$$S^{-1}(\zeta) = i\not{\zeta} + m$$

$$S(k) = \frac{1}{i\not{k}A(k^2) + B(k^2)}$$
$$= -i\not{k}\sigma_v(k^2) + \sigma_s(k^2)$$

Gluon Model

$$g^2 D^{ab}_{\mu\nu}(k) = \delta^{ab}\mathcal{G}(k^2)D^{free}_{\mu\nu}(k), \qquad \mathcal{G}(k^2) = \mathcal{G}_{IR}(k^2) + 4\pi\alpha_{pQCD}(k^2)$$

Infrared Part: Model

- Maris-Tandy Model

$$\frac{\mathcal{G}_{IR}(k^2)}{k^2} = \frac{4\pi^2}{\omega^6}Dk^2 e^{-k^2/w^2},$$

- **Qin-Chang Model**

$$\frac{\mathcal{G}_{IR}(k^2)}{k^2} = \frac{8\pi^2}{\omega^4}De^{-k^2/w^2},$$

$$\varsigma^3 := D\omega$$

Ultra-violet Part: pert.

$$\alpha_{pQCD}(k^2) = \frac{2\pi\gamma_m(1-e^{-k^2/4m_t^2})}{\ln[\tau + (1+k^2/\Lambda^2_{QCD})^2]}$$

$$\gamma_m = \frac{12}{33-2N_f},$$

$$N_f = 4,$$

$$\Lambda_{QCD} = 0.234GeV,$$

$$\tau = e^2 - 1,$$

$$m_t = 0.5GeV.$$

P. Maris, P. C. Tandy, Phys. Rev. C 60,055214 (1999)
S. X. Qin, L. Chang, Y. X. Liu, C. D. Roberts and D. J. Wilson, Phys. Rev. C 84, 042202 (2011)

Truncation Scheme

- BC vertex

$$\Gamma_\mu^{BC}(k_+,k_-) = \gamma_\mu\left(\frac{A(k_+^2)+A(k_-^2)}{2}\right)+(k_+ + k_-)_\mu\left[\frac{\not k_+ + \not k_-}{2}\frac{A(k_+^2)-A(k_-^2)}{k_+^2-k_-^2} - i\frac{B(k_+^2)-B(k_-^2)}{k_+^2-k_-^2}\right]$$

- CLR vertex

$$\Gamma_\mu(k,P) = \Gamma_\mu^{BCL}(k_+,k_-) + \Gamma_\mu^{ACM}(k,P),$$

$$\Gamma_\mu^{ACM}(k_+,k_-) = \left(\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right)[l_\nu \not P + i\gamma_\nu \sigma^{l,P}]\tau_4(k_+,k_-) + \sigma_{\mu\nu}P_\nu \tau_5(k_+,k_-),$$

$$\tau_5(k_+,k_-) = \eta\frac{B(k_+^2)-B(k_-^2)}{k_+^2-k_-^2}, \qquad \tau_4(k_+,k_-) = \frac{4\tau_5(k_+,\ k_-)[M(k_+^2)+M(k_-^2)]}{k_+^2+k_-^2+M^2(k_+^2)+M^2(k_-^2)},$$

- **Bare vertex**

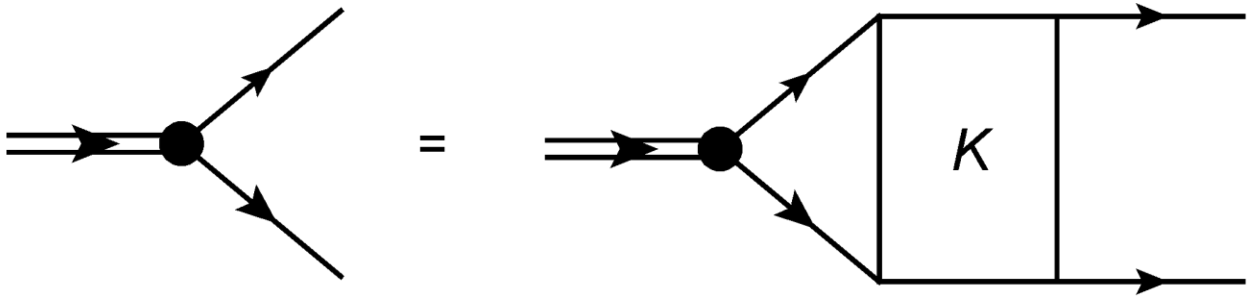
$$\Gamma_\mu(k;P) = \gamma_\mu,$$

"Ranbow"

J. Ball, T. W. Chiu, Phys. Rev. D **22**, 2542 (1980)
S. X. Qin, L.Chang, Y. X. Liu, C. D. Roberts, Phys. Rev. Lett **106**, 172301 (2011)
L.Chang, Y. X. Liu, C. D. Roberts, Phys. Rev. Lett **106**, 072001 (2011)

Bethe-Salpeter Equation (BSE)

Homogenous



$$\Gamma(k,P) = \int_q^{\Lambda} S_f(q+)\Gamma(q;P)S_g(q-)K(k,q;P)$$

- Symmetry analysis

  - **AV-WTI (2 flavor)**

$$P_\mu \Gamma_{5\mu}^{fg}(k,P) + i(m_f+m_g)\Gamma_5^{fg}(k,P) = S_f^{-1}(k_+)i\gamma_5 + i\gamma_5 S_g^{-1}(k_-)$$

  - **Truncation Scheme** “Ladder”

$$K_{tu}^{rs}(q,k;P) \to -\mathcal{G}((k-q)^2)D_{\mu\nu}^{free}(k-q)(\gamma_\mu \frac{\lambda^a}{2})_{tr}(\gamma_\nu \frac{\lambda^a}{2})_{su}$$

Beyond Rainbow-Ladder

Details in Appendix A

- Inspired from the AV-WTI, we propose a modified kernel:

"Rainbow" + "Beyond Ladder"

$$K = -\frac{1}{2}\gamma \otimes \Sigma^{-1} \cdot D_\Sigma \cdot \gamma - \frac{1}{2}\gamma \cdot D_\Sigma \cdot \Sigma^{-1} \otimes \gamma$$

$$D_\Sigma = (D^f S_f(q_+) + D^g S_g(q_-))\gamma_5$$

$$\Sigma = (S_f(q_+) + S_g(q_-))\gamma_5$$

Infrared behavior

$$K_{IR} = -\gamma \otimes (\frac{D^f \sigma_{fs}(q_+^2) + D^g \sigma_{fs}(q_-^2)}{\sigma_{fs}(q_+^2) + \sigma_{fs}(q_-^2)})\gamma$$

Ultraviolet behavior

$$K_{UV} = -\gamma \otimes (\frac{D^f + D^g}{2})\gamma$$

- Introduction
- Theoretical Framework
- **Numerical Details**
- Results
- Summary

Solve the DSE on Complex Plane

$$P^2 = -M^2$$

- Momentum partition

$$q_+ = q + \alpha P = (\vec{q}, q_4 + i\alpha M)$$

$$q_- = q - (1-\alpha)P = (\vec{q}, q_4 - i(1-\alpha)M)$$

Meson rest frame

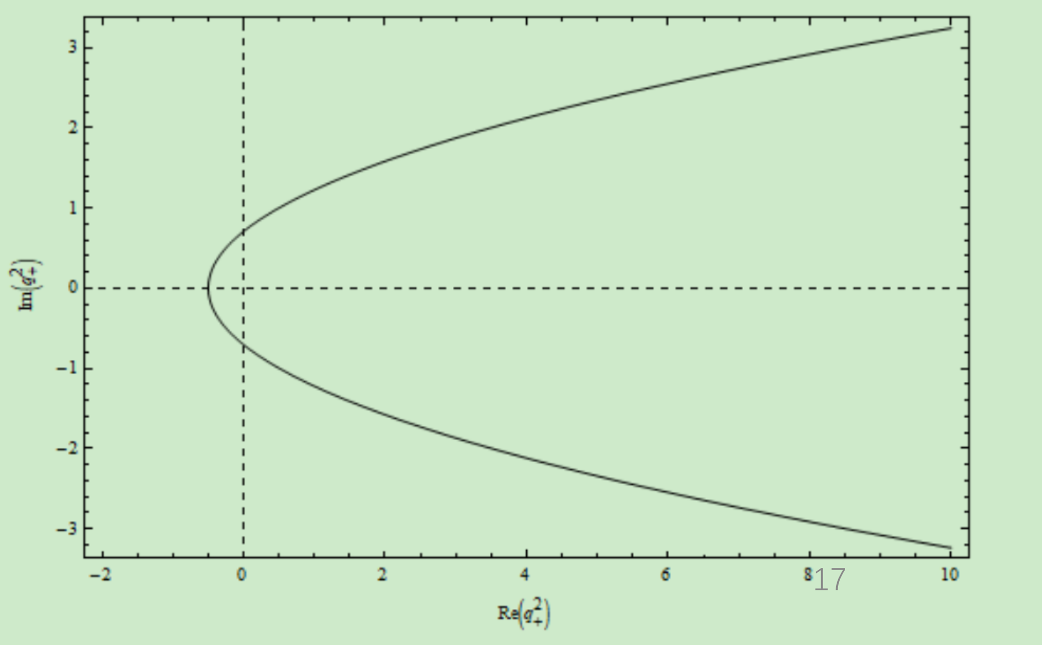
$$P = (0, 0, 0, iM)$$

$$q_+^2 = (q + \alpha i M)^2 = q^2 + 2q_4\alpha i M - \alpha^2 M^2$$

$$q_-^2 = (q - (1-\alpha)iM)^2 = q^2 - 2q_4(1-\alpha)iM - (1-\alpha)^2 M^2$$

$$L_+ : \quad \mathrm{Re}(q_+^2) + \alpha^2 M^2 \geq \frac{\mathrm{Im}(q_+)^2}{M^2}$$

$$L_- : \quad \mathrm{Re}(q_-^2) + (1-\alpha)^2 M^2 \geq \frac{\mathrm{Im}(q_-)^2}{M^2}$$



Solve the DSE on Complex Plane

Choice of the momentum partition

Corresponding vertex	Maximum vertex
$L_+ : \quad (-\alpha^2 M^2, 0)$	$(-\Delta_1^2, 0)$
$L_- : \quad (-(1-\alpha)^2 M^2, 0)$	$(-\Delta_2^2, 0)$

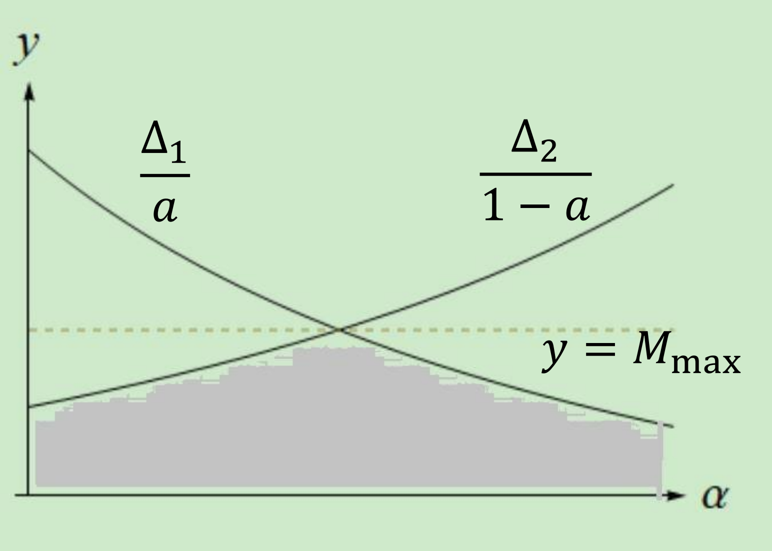
"Rainbow Ladder"

↓

an unphysical singularity structure

Two quarks are required to avoid the singularity.

$$\begin{cases} \Delta_1 \geq \alpha M \\ \Delta_2 \geq (1-\alpha) M \end{cases} \Rightarrow \begin{cases} M \leq \dfrac{\Delta_1}{\alpha} \\ M \leq \dfrac{\Delta_2}{1-\alpha} \end{cases}$$



Thus $$1 - \frac{\Delta_2}{M} \leq \alpha \leq \frac{\Delta_1}{M}$$

Best alpha $$\frac{\Delta_1}{\alpha} = \frac{\Delta_2}{1-\alpha} \Rightarrow \alpha = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

Solve the DSE on Complex Plane

☐ Procedure

1. Initialize the $A(p^2), B(p^2)$ **on the parabola**

2. Cauchy integral to get $A(q_+), B(q_+)$ **within the parabola**

$$S(q_0^2) = \oint_C dq^2 \frac{S(q^2)}{q^2 - q_0^2} \Big/ \oint_C dq^2 \frac{1}{q^2 - q_0^2}$$

3. Obtaining $A(p^2), B(p^2)$ from $A(q_+), B(q_+)$ with the DSE

4. Repeat step 2, 3 until the relevant error is smaller than 10^{-6}

Solve the homogenous BSE

$$\Gamma(k;P) = \int_q^\Lambda S(q_+)\Gamma(q;P)S(q_-)K(k,q;P)$$

Bethe-Salpeter Amplitude (BSA) Expansion: $\Gamma(k;P) = \sum_{i=1}^{N} \tau_i(k;P)\mathcal{F}_i(k^2, z_k; P^2)$

Pseudo-Scalar Meson ($J^P = 0^-$)

$$\tau_1 = i\gamma_5,$$
$$\tau_2 = \gamma_5 \not{P},$$
$$\tau_3 = \gamma_5 \not{k}(k \cdot P),$$
$$\tau_4 = i\gamma_5 \sigma^{P,k}.$$

Vector Meson ($J^P = 1^-$)

$$\tau_1 = i\gamma_\mu^T, \quad \tau_2 = ik_\mu^T \not{k},$$
$$\tau_3 = ik_\mu^T \not{P}(k \cdot P), \quad \tau_4 = \gamma_5 \epsilon_{\mu\nu\alpha\beta}^T \gamma_\nu k_\alpha P_\beta,$$
$$\tau_5 = k_\mu^T, \quad \tau_6 = \sigma_{\mu\nu}^T k_\nu (k \cdot P),$$
$$\tau_7 = i\gamma_5 \sigma^{P,k} \quad \tau_8 = k_\mu^T \sigma_{\mu\nu}^T k_\alpha P_\beta.$$

with

$$\sigma^{P,k} = \sigma_{\mu\nu} P_\mu k_\nu, \sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

$$l_\mu^T = P_{\mu\nu} l_\nu, P_{\mu\nu} = \delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}$$

P, C Transformation of the BSA

$$\Gamma(k;P) = \sum_{i=1}^{N} \tau_i(k;P)\mathcal{F}_i(k^2, z_k; P^2) \qquad \mathcal{F}^{JP}(k^2, k\cdot P) = \sum_{j=0} a_j(k^2) U_j(k\cdot P/\sqrt{k^2P^2})$$

$$\Gamma(k;P) \to \hat{P}\Gamma(\tilde{k};\tilde{P})\hat{P}^{-1}$$ P Transformation

$$\Gamma(k;P) \to \bar{\Gamma}(k;P) = \hat{C}\Gamma^t(-k;P)\hat{C}^{-1}$$ C Transformation

with $\tilde{k} = (k_4, -\vec{k}), \hat{P} = \gamma_4, \hat{C} = \gamma_2\gamma_4$

Basis(τ_i)	P	C
pseudo-scalar	-	+
vector	-	-

$\mathcal{F}_i(k^2, k\cdot P)$	P	C
Even terms	+	+
Odd terms	+	-

Solve the homogenous BSE

$$tr[\tau_i(k;P)LHS] = tr[\tau_i(k;P)RHS] \Rightarrow K(P^2)\Gamma_i = \lambda_i(P^2)\Gamma_i$$

$$\lambda_i(P^2) = 1$$

Solve the homogenous BSE as an Eigenvalue Problem:

1. Set the Meson Mass as M, and parametrize in the rest frame;
2. Rearrange the BSE to an eigenvalue equation;
3. Solve the eigenvalue equation.
4. If the eigenvalue equals to 1, then M is the meson mass. Else, change the M and repeat step 1, 2.

Decay Constant

$$f_{0^-} P_\mu = Z_2 N_c tr \int \frac{1}{\sqrt{2}} i\gamma_5 \gamma_\mu S_f(k_+) \Gamma_{0^-}(k,P) S_g(k_-)$$

$$f_{1^-} M = \frac{Z_2 N_c}{3} tr \int \frac{1}{\sqrt{2}} \gamma_\mu S_f(k_+) \Gamma^\mu_{1^-}(k,P) S_g(k_-)$$

Nakanishi Normalization:

$$(\frac{\partial \ln(\lambda)}{\partial P^2})^{-1} = \frac{N_c}{N_J} tr \int_k^\Lambda \bar{\Gamma}(k;-P) S(k_+) \Gamma(k;P) S(k_-)$$

Thus

$$N_{norm} = (\frac{\partial \ln(\lambda)}{\partial P^2}) \cdot \frac{N_c}{N_J} tr \int_k^\Lambda \bar{\Gamma}(k;-P) S(k_+) \Gamma(k;P) S(k_-)$$

$$\Gamma_{norm} = \Gamma / \sqrt{N_{norm}}$$

Parameters in Detail

- Renormalization Point:

$$\zeta \quad = \; 19 \text{ GeV}$$

- IR cut : $\Lambda_{IR} \; = \; 0.01 \text{ GeV}$
- UV cut : $\Lambda_{UV} = \; 100 \text{ GeV}$
- α-independent

"Rainbow" + " Ladder"

"Rainbow" + "Beyond Ladder"

Bring in no New Parameter!

Parameters for light quark

Gluon Para:

$$w_q = 0.5\text{GeV}$$

$$\varsigma_q = 0.8 \, GeV$$

Quark Para:

$$m_\pi = 0.138 \text{ GeV} \Longrightarrow m_u = 0.0033 \text{ GeV}$$

$$m_K = 0.495 \text{ GeV} \Longrightarrow m_s = 0.0746 \text{ GeV}$$

Parameters for heavy quark

Gluon Para:

$$w_Q = 0.8 \text{ GeV}$$

$$\varsigma_Q = 0.6 \text{ GeV}$$

Quark Para:

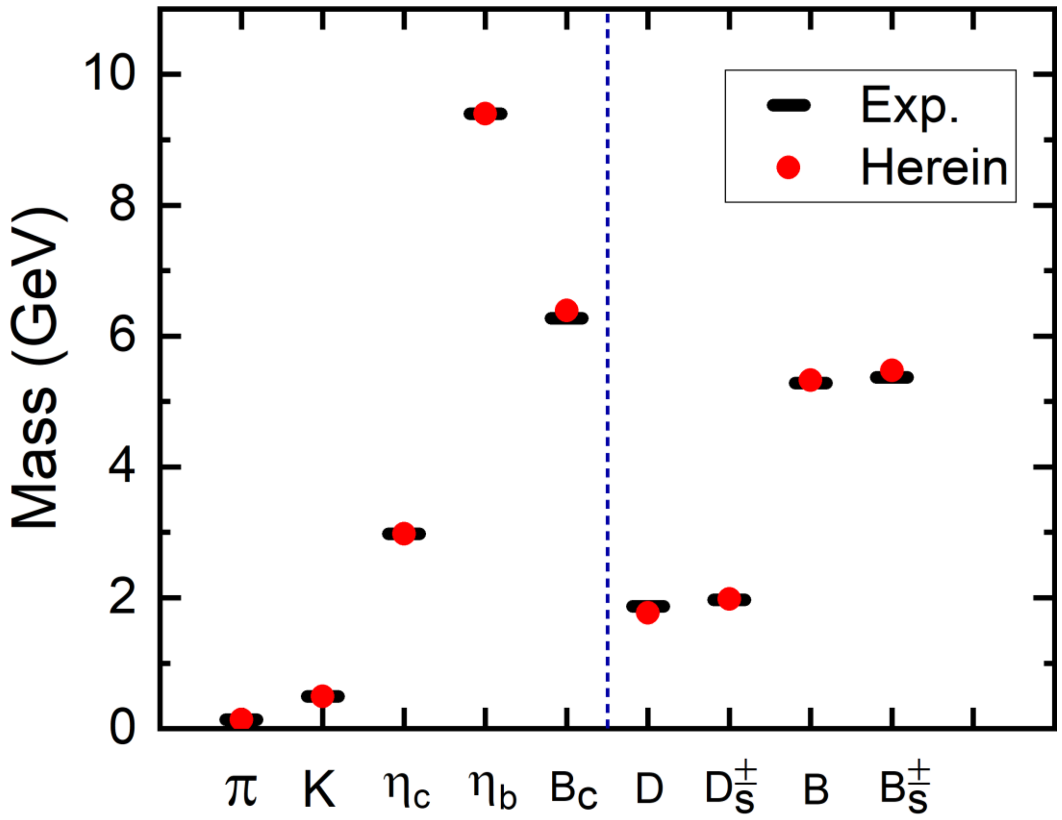
$$m_{\eta_c} = 2.98 \text{ GeV} \Longrightarrow m_c = 0.817 \text{ GeV}$$

$$m_{\eta_b} = 9.40 \text{ GeV} \Longrightarrow m_b = 3.587 \text{ GeV}$$

- Introduction
- Theoretical Framework
- Numerical Details
- Results
- Summary

The mass of the pseudo-scalar meson

- π, K and η_c, η_b, B_c — Rainbow - Ladder
- $D, D_s^{\pm}, B, B_s^{\pm}$ — Rainbow – beyond Ladder

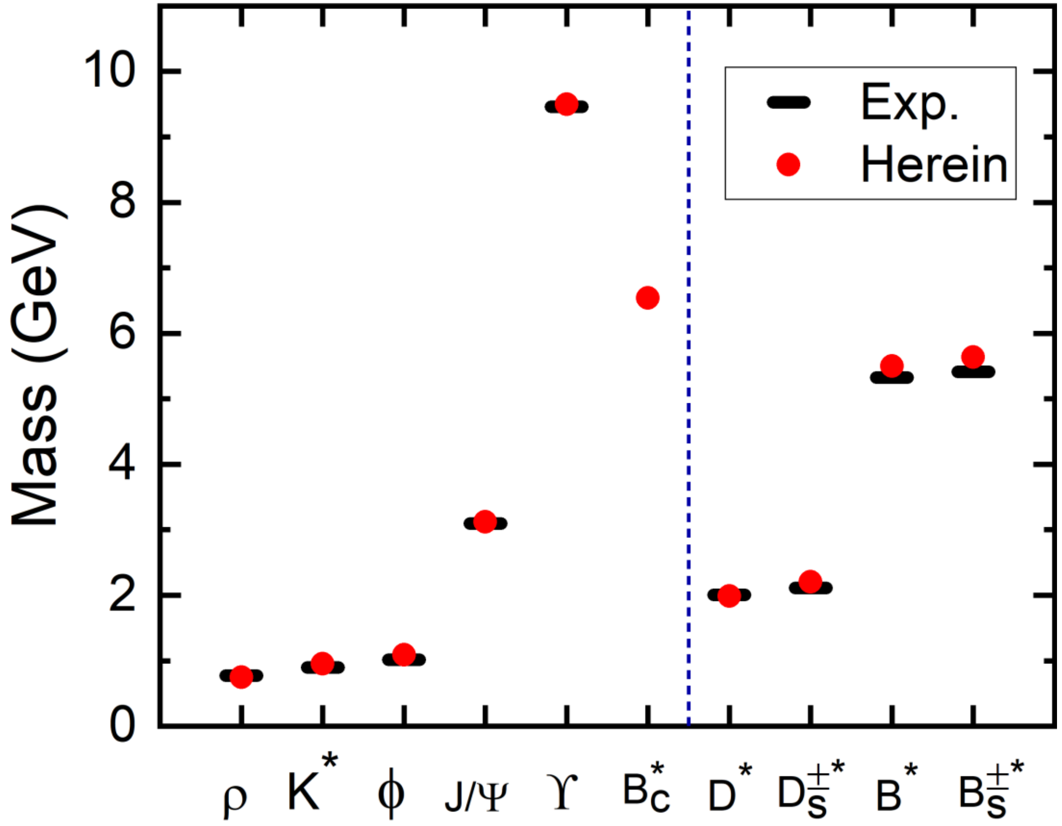


Meson	Herein	lQCD	Exp.	error(%)
π	0.138	/	0.138	0
K	0.495	/	0.495	0
η_c	2.98	/	2.98	0
η_b	9.4	/	9.4	0
B_c	6.388	6.276	6.275	1.8
D	1.771	1.865	1.868	-5.2
$D_s^{\pm}$	1.981	1.968	1.968	0.66
B	5.324	5.283	5.279	0.85
$B_s^{\pm}$	5.478	5.366	5.367	2.1

The relevant error is smaller than 6%.

The mass of the vector meson

- ρ, K^*, ϕ and $J/\Psi, \Upsilon, B_c^*$ — Rainbow - Ladder
- $D^*, D_s^{\pm *}, B^*, B_s^{\pm *}$ — Rainbow - beyond Ladder



Meson	Herein	lQCD	Exp.	error(%)
ρ	0.749	0.780	0.775	3.4
$K*$	0.953	0.993	0.896	6.8
ϕ	1.089	1.032	1.019	6.9
J/Ψ	3.122	3.098	3.097	0.81
Υ	9.497	/	9.460	0.39
B_c*	6.542	6.331	/	/
$D*$	1.988	2.013	2.009	−1.0
$D_s^{\pm}*$	2.206	2.116	2.112	4.5
$B*$	5.501	5.321	5.325	3.3
$B_s^{\pm}*$	5.635	5.411	5.415	4.1

The relevant error is smaller than 7%.

Equal spacing rule（ESR）and ESR comparison

M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

For baryons with experimental data:

$$1.38 = m_{\Sigma^*} \approx m_{\Sigma^*}^{\text{interp}} := \frac{2}{3}m_\Delta + \frac{1}{3}m_\Omega = 1.36$$

$$1.53 = m_{\Xi^*} \approx m_{\Xi^*}^{\text{interp}} := \frac{1}{3}m_\Delta + \frac{2}{3}m_\Omega = 1.52$$

(uus), (ssu), (uuu), (sss)

Exp. ↑ ... ↑ ESR Exp.

For vector mesons

$$0.896 = m_{K*} \approx m_{K*}^{interp} := \frac{1}{2}m_\rho + \frac{1}{2}m_\phi = 0.897$$

(su), (uu), (ss)

Exp. ↑ ... ↑ ESR Exp.

$$0.953 = m_{K*} \approx m_{K*}^{interp} := \frac{1}{2}m_\rho + \frac{1}{2}m_\phi = 0.919$$

Herein ↑ ... ↑ ESR Herein

6.8% **Relevant error** **2.4%**

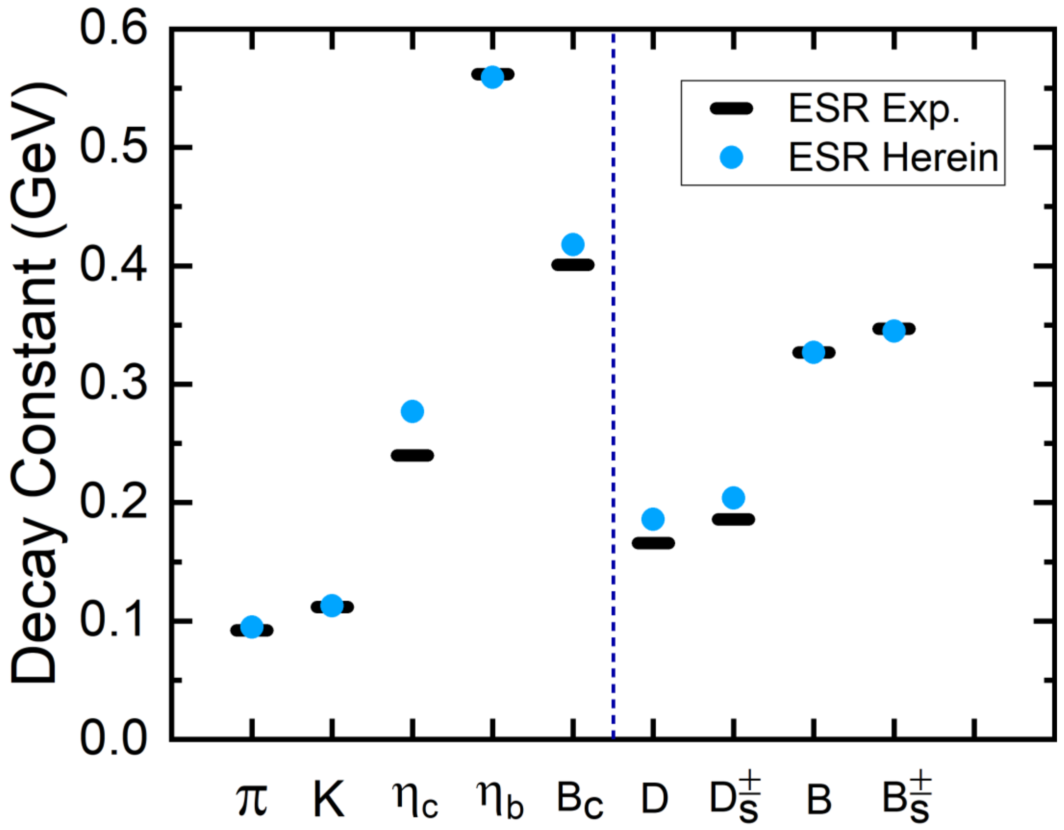
Pseudo-scalar mesons

Meson	ESR Herein	ESR Exp.	error(%)
π	0.138	0.138	0
K	0.495	0.495	0
η_c	2.98	2.98	0
η_b	9.4	9.4	0
B_c	6.19	6.19	0
D	1.559	1.559	0
$D_s^\pm$	1.916	1.916	0
B	4.769	4.769	0
$B_s^\pm$	5.126	5.126	0

Vector meson

Meson	ESR Herein	ESR Exp.	error(%)
ρ	0.749	0.775	−3.4
$K*$	0.919	0.897	2.4
ϕ	1.089	1.019	6.9
J/Ψ	3.122	3.097	0.81
Υ	9.497	9.460	0.39
B_c*	6.310	6.279	0.49
$D*$	1.936	1.936	0
$D_s^\pm*$	2.106	2.058	0.98
$B*$	5.123	5.118	0.11
$B_s^\pm*$	5.293	5.240	1.0

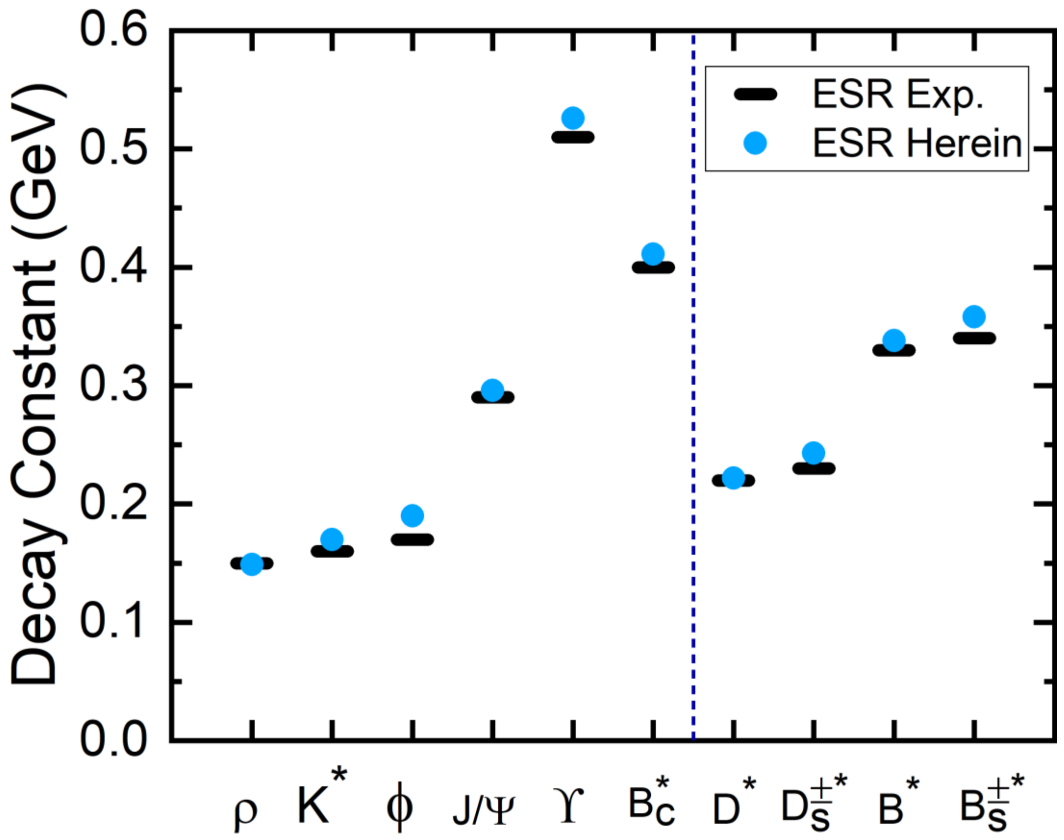
The decay constant of the pseudo-scalar meson



Meson	Herein	ESR Herein	ESR Exp.	lQCD	Exp.
π	0.095	0.095	0.092	0.093	0.092
K	0.113	0.113	0.112	0.111	0.11
η_c	0.277	0.277	0.24	0.278	0.24
η_b	0.559	0.559	0.562	0.472	/
B_c	0.429	0.418	0.401	0.307	/
D	0.169	0.186	0.166	0.150	/
$D_s^{\pm}$	0.212	0.204	0.186	0.177	/
B	0.212	0.327	0.327	0.134	/
$B_s^{\pm}$	0.248	0.345	0.347	0.163	/

The ESR comparison shows much smaller error than the direct comparison.

The decay constant of the vector meson



Meson	Herein	ESR Herein	ESR Exp.	lQCD	Exp.
ρ	0.149	0.149	0.15	/	0.15
$K*$	0.179	0.170	0.16	/	0.16
ϕ	0.190	0.190	0.17	0.170	0.17
J/Ψ	0.296	0.296	0.29	0.286	0.29
Υ	0.526	0.526	0.51	0.459	0.51
B_c*	0.483	0.411	0.40	0.298	/
$D*$	0.199	0.222	0.22	0.158	/
$D_s^{\pm}*$	0.256	0.243	0.23	0.190	/
$B*$	0.246	0.338	0.33	0.131	/
$B_s^{\pm}*$	0.283	0.358	0.34	0.158	/

The ESR comparison for the vector meson shows smaller difference than the pseudo scalar meson.

- Introduction
- Theoretical Framework
- Numerical Details
- Results
- **Summary**

Summary

- We studied the pseudo-scalar and vector heavy-light meson with the bare vertex truncation and a modified kernel.
  - The modified kernel is **inspired from the AV-WTI**
  - In this kernel the **heavy and light gluon propagators** are involved
  - The modified kernel **can come back to the RL truncation** in the light-light or heavy-heavy system
  - The computation of **the meson mass shows small difference** with the experimental data
  - **ESR comparison** of the decay constant with experiment is reasonable, especially for the vector meson.
  - **Our modified kernel captures the features of QCD** and it is a tool for heavy-light problem.

Thanks for Your Attention!

Appendix

Appendix A： AV-WTI

Two flavored meson

Two flavored AV-WTI:

$$P_\mu \Gamma_{5\mu}^{fg}(k,P) + i(m_f + m_g)\Gamma_5^{fg}(k,P) = S_f^{-1}(k_+) i\gamma_5 + i\gamma_5 S_g^{-1}(k_-)$$

Note: color-singlet.

Rainbow-Ladder respect this identity.The proof is as follows:

$$\Gamma_{5\mu}^{fg}(k,P) = \gamma_5\gamma_\mu + \int_q K S_f(q_+)\Gamma_{5\mu}^{fg} S_g(q_-)$$

$$\Gamma_5^{fg}(k,P) = \gamma_5 + \int_q K S_f(q_+)\Gamma_5^{fg} S_g(q_-)$$

$$S^{-1}(k) = S_0^{-1}(k) + \int_q D_{\mu\nu}(q-k)\gamma_\mu S(q)\Gamma_\nu(q,k)$$

Tree level

$$P_\mu\gamma_5\gamma_\mu + i(m_f + m_g)\gamma_5 = (i\not{k}_+ + m_f)i\gamma_5 + i\gamma_5(i\not{k}_- + m_g)$$

$$\text{with} \quad k_+ = k + \frac{P}{2} \quad k_- = k - \frac{P}{2}$$

√

Loop level:

$$P_\mu \int_q K S_f(q_+)\Gamma^{fg}_{5\mu} S_g(q_-) + i(m_f+m_g)\int_q K S_f(q_+)\Gamma^{fg}_5 S_g(q_-)$$
$$= \int_q D_{\mu\nu}\gamma_\mu S_f(q_+)\Gamma_\nu(q,P) i\gamma_5 + i\gamma_5 \int_q D_{\mu\nu}\gamma_\mu S_g(q_-)\Gamma_\nu(q,P)$$

thus

$$\int_q K S_f(q_+)[P_\mu \Gamma^{fg}_{5\mu}(q,P) + i(m_f+m_g)\Gamma^{fg}_5] S_g(q_-)$$
$$= \int_q D_{\mu\nu}\gamma_\mu [S_f(q_+)\Gamma_\nu(q,P) i\gamma_5 - i\gamma_5 S_g(q_-)\Gamma_\nu(q,P)]$$

apply AV-WTI again

$$\int_q K S_f(q_+)[S_f^{-1}(q_+) i\gamma_5 + i\gamma_5 S_g^{-1}(q_-)] S_g(q_-)$$
$$= \int_q D_{\mu\nu}\gamma_\mu [S_f(q_+)\Gamma_\nu(q,P) i\gamma_5 - i\gamma_5 S_g(q_-)\Gamma_\nu(q,P)]$$

substitute the vertex with $\Gamma_\nu = \gamma_\nu$, then

$$\int_q K[i\gamma_5 S_g(q_-) + S_f(q_+) i\gamma_5] = \int_q D_{\mu\nu}\gamma_\mu [S_f(q_+)\gamma_\nu i\gamma_5 - i\gamma_5 S_g(q_-)\gamma_\nu]$$

thus

$$\int K i\gamma_5 S_g(q_-) = -\int D_{\mu\nu}\gamma_\mu i\gamma_5 S_g(q_-)\gamma_\nu$$
$$\int K S_f(q_+) i\gamma_5 = -\int D_{\mu\nu}\gamma_\mu S_f(q_+) i\gamma_5 \gamma_\nu$$

$$\therefore \quad K = -D\gamma \otimes \gamma$$

$\surd$

Different flavor with different gluon

$$\int_q K[S_f(q_+)i\gamma_5 + i\gamma_5 S_g(q_-)] = \int_q [D_f \gamma S_f(q_+)\gamma i\gamma_5 + D_g i\gamma_5 \gamma S_g(q_-)\gamma]$$
$$= -\int_q \gamma[D_f(S_f i\gamma_5) + D_g(i\gamma_5 S_g)]\gamma$$

if $D_f = D_g$, then $\quad K = -D\gamma \otimes \gamma$

if $D_f \neq D_g$, then $\quad K = -\gamma \otimes [\frac{D_f S_f \gamma_5 + D_g \gamma_5 S_g}{S_f \gamma_5 + \gamma_5 S_g}]\gamma$

Thus $\quad K = -\gamma \otimes [\frac{D_f S_f \gamma_5 + D_g \gamma_5 S_g}{S_f \gamma_5 + \gamma_5 S_g}]\gamma = -\gamma \otimes [\frac{D_f S_f(q_+) + D_g S_g(-q_-)}{S_f(q_+) + S_g(-q_-)}]\gamma$

【Discussion】

1. infrared behavior ($q \to 0$),

   As $S(q) = -i\not{q}\sigma_v(q^2) + \sigma_s(q^2)$,

   then $\frac{D_f S_f(q_+) + D_g S_g(-q_-)}{S_f(q_+) + S_g(-q_-)} \sim \frac{D_f \sigma_{fs}(q_+^2) + D_g \sigma_{gs}(q_-^2)}{\sigma_{fs}(q_+^2) + \sigma_{gs}(q_-^2)}$.

2. ultra-violet behavior ($q \to \infty$),

   then $\frac{D_f S_f(q_+) + D_g S_g(-q_-)}{S_f(q_+) + S_g(-q_-)}$ singularity

To Dismiss the Singularity

$q_- \to -q_-$

$$K = -\gamma \otimes [\frac{D_f S_f(q_+) + D_g S_g(q_-)}{S_f(q_+) + S_g(q_-)}]\gamma$$

1. infrared behavior ($q \to 0$),

   As $S(q) = -i\not{q}\sigma_v(q^2) + \sigma_s(q^2)$,

   then $\frac{D_f S_f(q_+)+D_g S_g(q_-)}{S_f(q_+)+S_g(q_-)} \sim \frac{D_f \sigma_{fs}(q_+^2)+D_g \sigma_{gs}(q_-^2)}{\sigma_{fs}(q_+^2)+\sigma_{gs}(q_-^2)}$

2. ultra-violet behavior ($q \to \infty$),

   then $\frac{D_f S_f(q_+)+D_g S_g(q_-)}{S_f(q_+)+S_g(q_-)} \sim \frac{D_f+D_g}{2}$

Rearrange to flavor symmetric

$$K = -\frac{1}{2}\gamma \otimes \Sigma^{-1} \cdot D_\Sigma \cdot \gamma - \frac{1}{2}\gamma \cdot D_\Sigma \cdot \Sigma^{-1} \otimes \gamma$$

$$D_\Sigma = (D^f S_f(q_+) + D^g S_g(q_-))\gamma_5 \quad \Sigma = (S_f(q_+) + S_g(q_-))\gamma 5$$

$$K_{IR} = -\gamma \otimes (\frac{D^f \sigma_{fs}(q_+^2) + D^g \sigma_{fs}(q_-^2)}{\sigma_{fs}(q_+^2) + \sigma_{fs}(q_-^2)})\gamma$$

$$K_{UV} = -\gamma \otimes (\frac{D^f + D^g}{2})\gamma$$

If $D_f = D_g$, this "beyond ladder" kernel is reduced to "Ladder" kernel.

Similar Derivation in Loop level from WTI

$$i(p-q)_\mu \Gamma_\mu(p,q) = S^{-1}(p) - S^{-1}(q)$$

$$K = -\gamma \otimes [\frac{D_f S_f(q_+) - D_g S_g(q_-)}{S_f(q_+) - S_g(q_-)}]\gamma$$