Schlessinger Point Method: Theory and Applications

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Schlessinger Point Method (SPM)

- **SPM interpolating fraction**
  Given a finite set of N data points (x_i, f_i) the SPM constructs the rational interpolant $p(x)/q(x)$ given by
  Schlessinger, PR 167 (1968)

$$C_N(x) = \frac{p(x)}{q(x)} = \frac{f_1}{1+}\frac{a_1(x-x_1)}{1+}\frac{a_2(x-x_2)}{1+}\cdots\frac{a_{N-1}(x-x_{N-1})}{1}$$

- **Coefficients a_i obtained recursively**

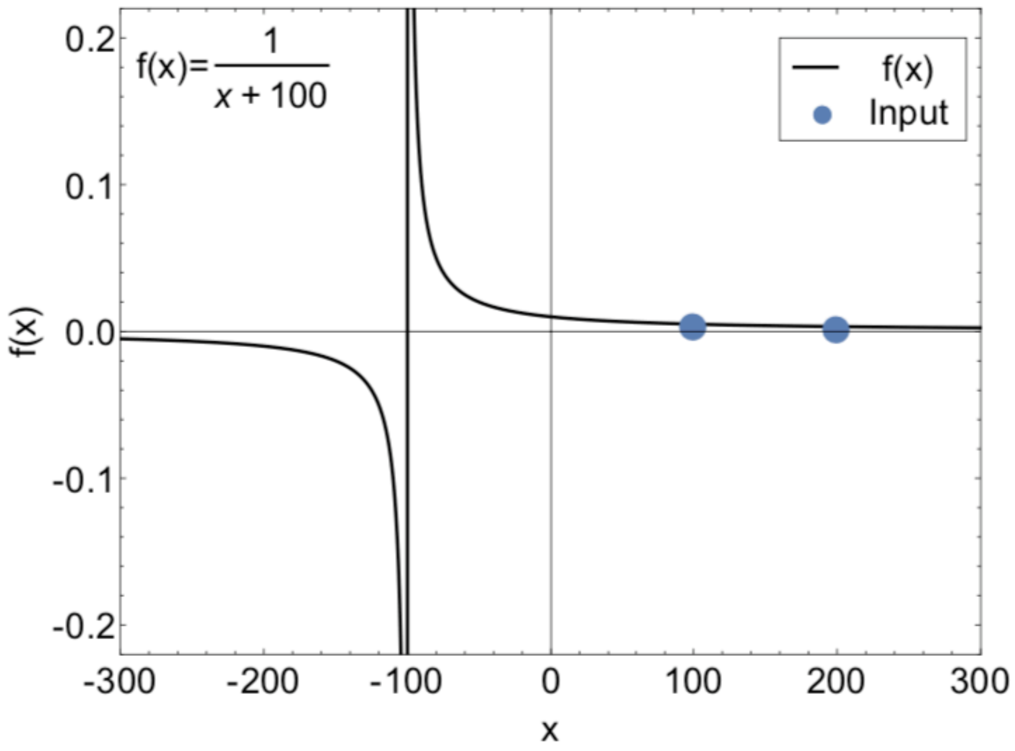
$$a_1 = \frac{f_1/f_2 - 1}{x_2 - x_1}$$

$$a_i = \frac{1}{x_i - x_{i+1}}\left[1 + \frac{a_{i+1}(x_{i+1}-x_{i-1})}{1+}\frac{a_{i-2}(x_{i+1}-x_{i-2})}{1+}\cdots\frac{a_1(x_{i+1}-x_1)}{1-f_1/f_{i+1}}\right]$$

- $[p(x), q(x)]$ **order**
  - $[N/2-1, N/2]$
    for an even number of input points
  - $[(N-1)/2, (N-1)/2]$
    for an odd number of input points

SPM examples

- $f(x) = 1/(x + \mu)$
  - **Only 2 points**
    needed to reproduce these curves
  - **“First guess”**
    of the method



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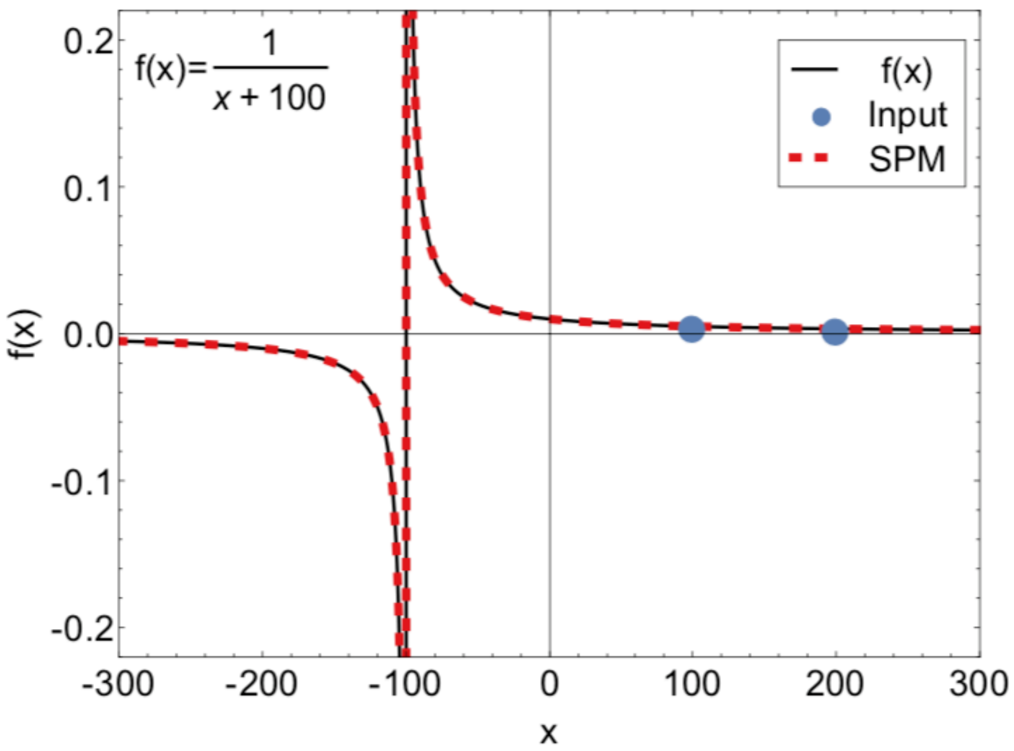

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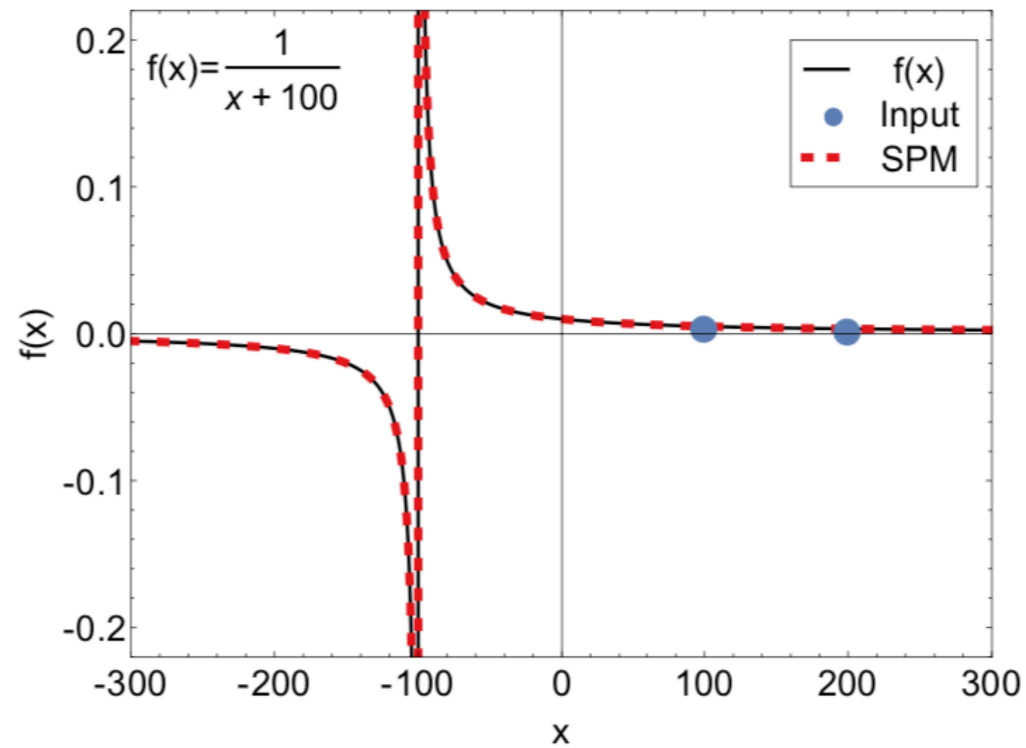
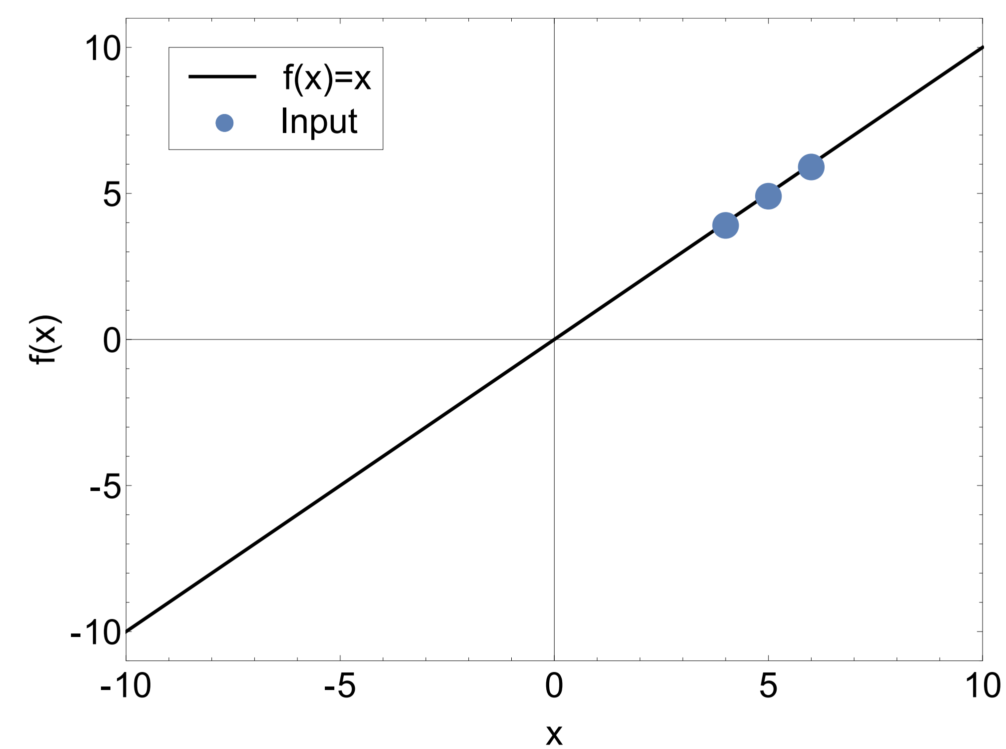


- $f(x) = x$
  - **Use 3 points**
    to reproduce a straight line
  - **15 digits precision**
    yields

$$f(x) = \frac{22 + 1.8 \times 10^{15} x}{1.8 \times 10^{15} - x} \approx x$$



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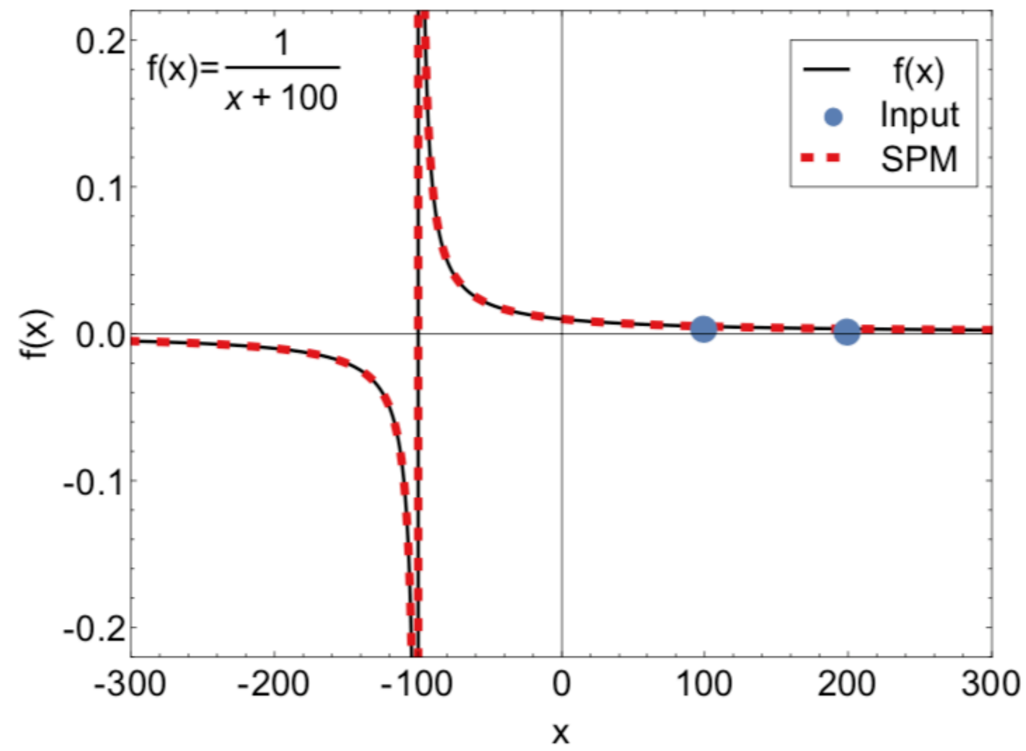

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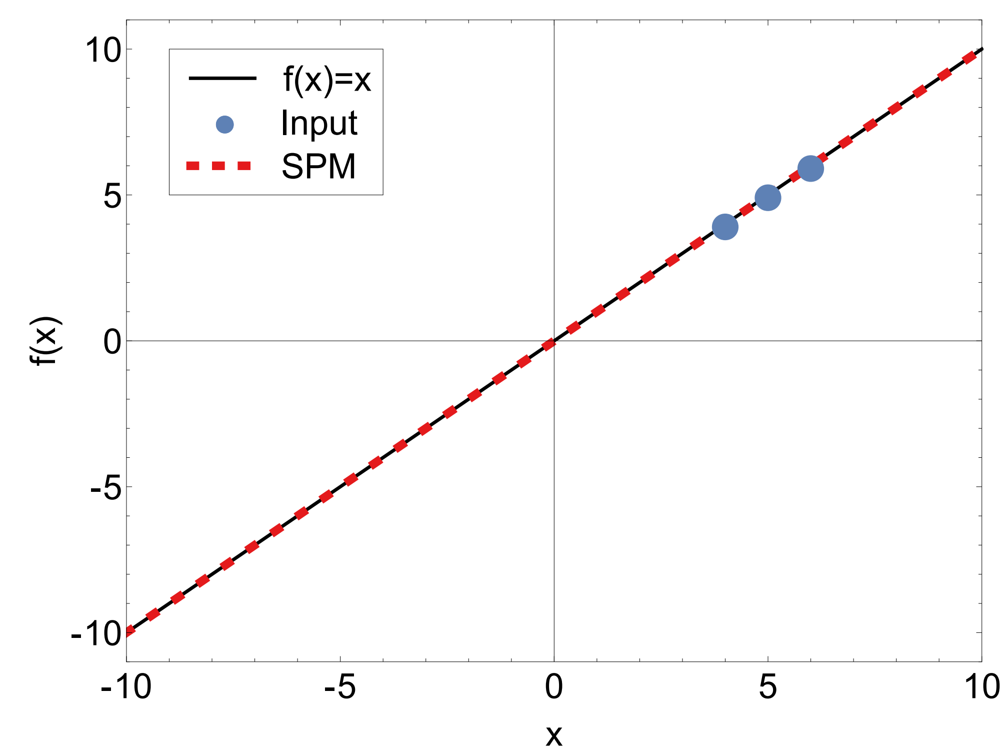
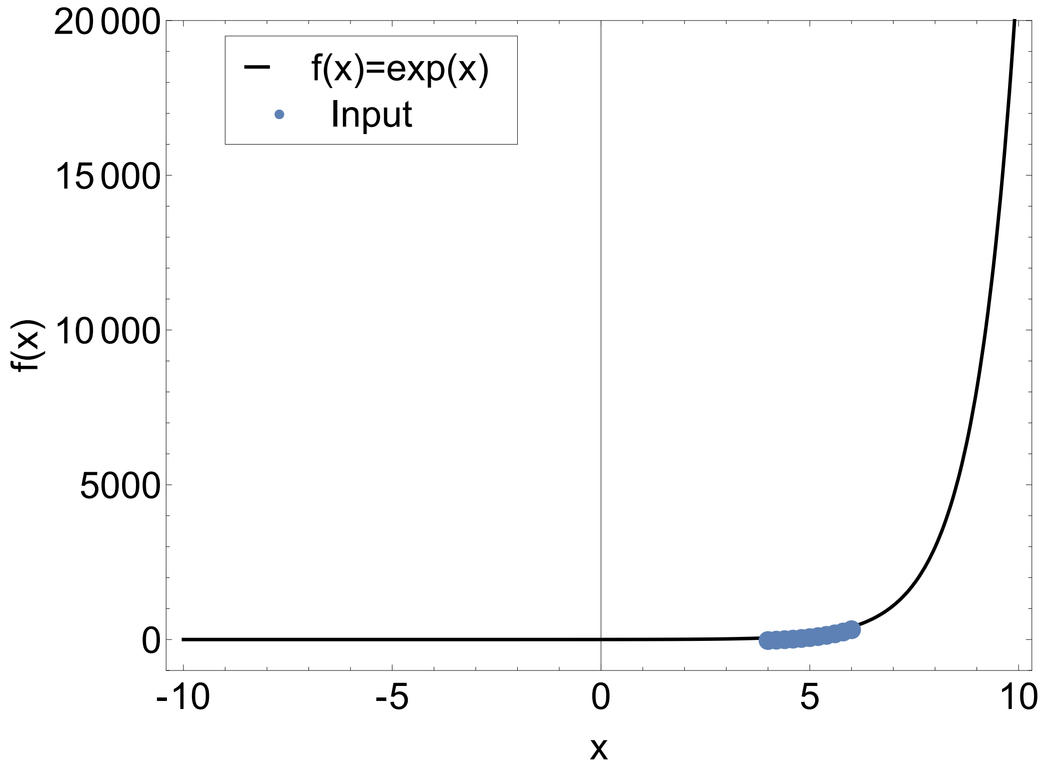


SPM examples

- $f(x) = e^x$
  - **Use 11 points**
  - **15 digits precision**
    yields

$$f(x) = \frac{263504 + 170536x + 46451x^2 + 10389x^3 + 756x^4 + 148x^5}{265568 - 98809x + 15473x^2 - 1274x^3 + 55x^4 - x^5}$$



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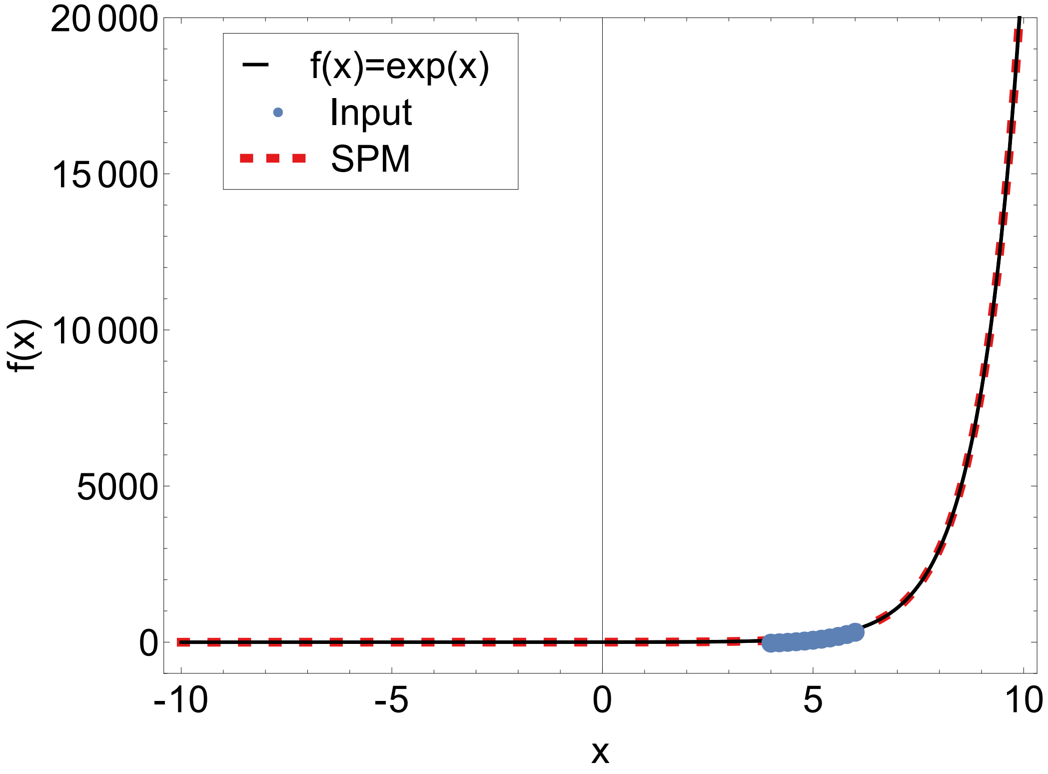

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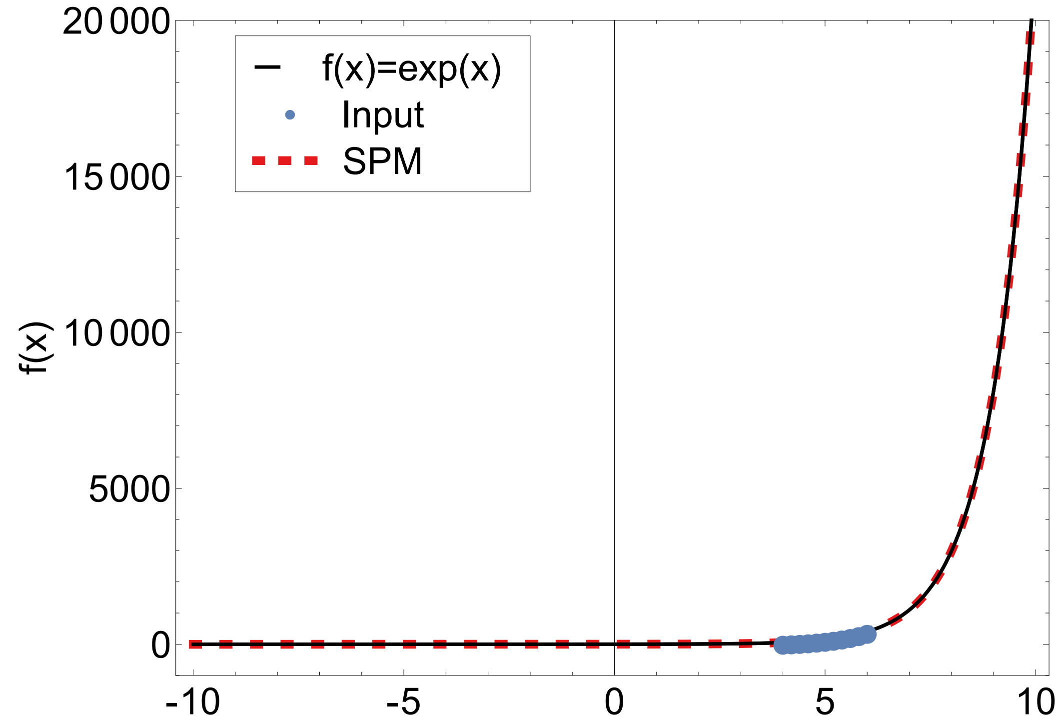
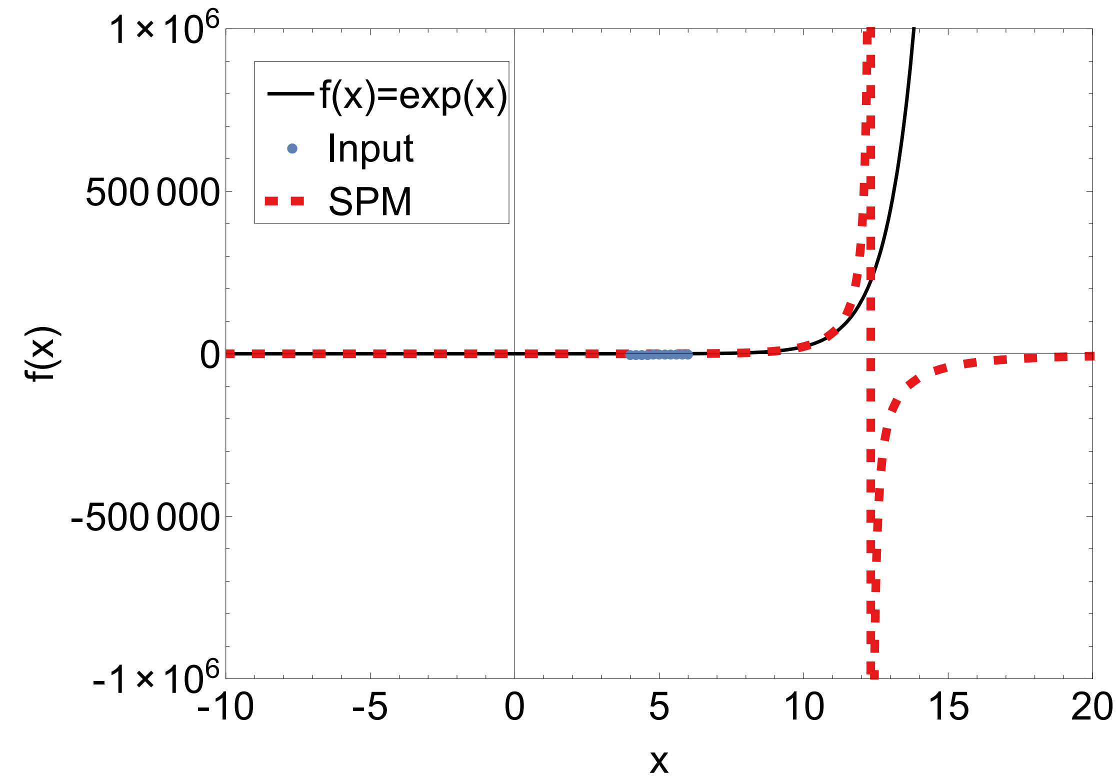
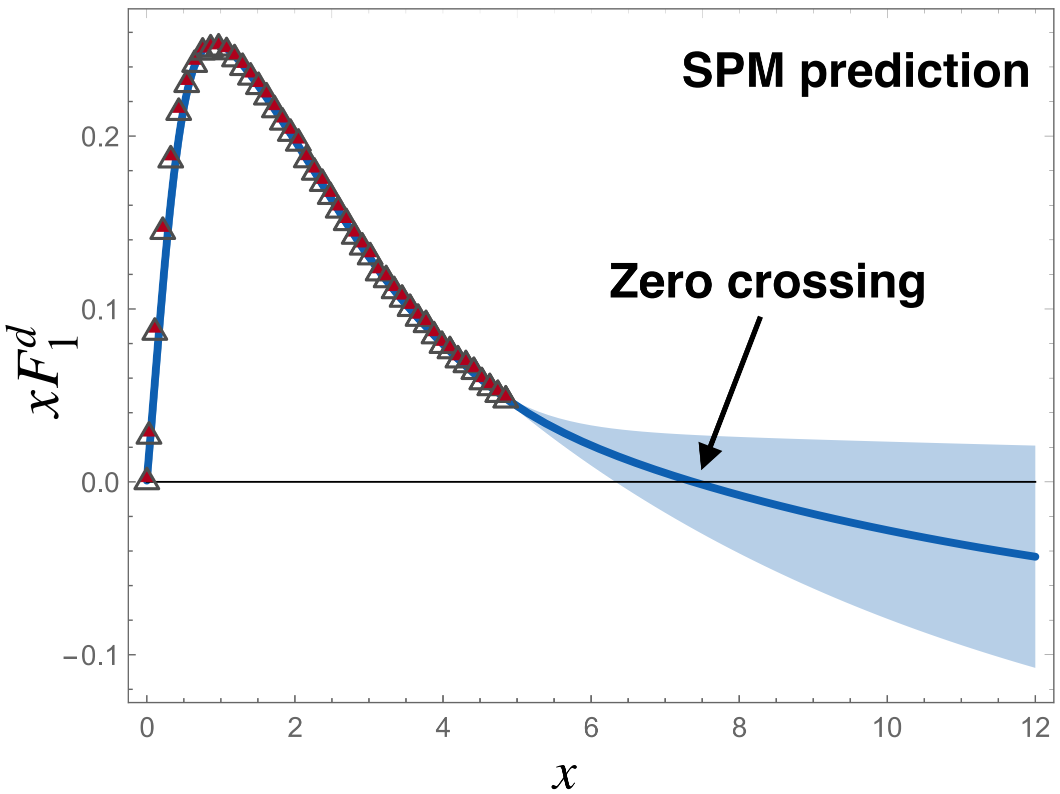
  - **Essential singularities**
    will not be reproduced everywhere





The beginning of a beautiful friendship

- F_1 **d quark proton form factor**
  results available up to $x = 8.6$ GeV but provided only up to $x = 4.9$ GeV



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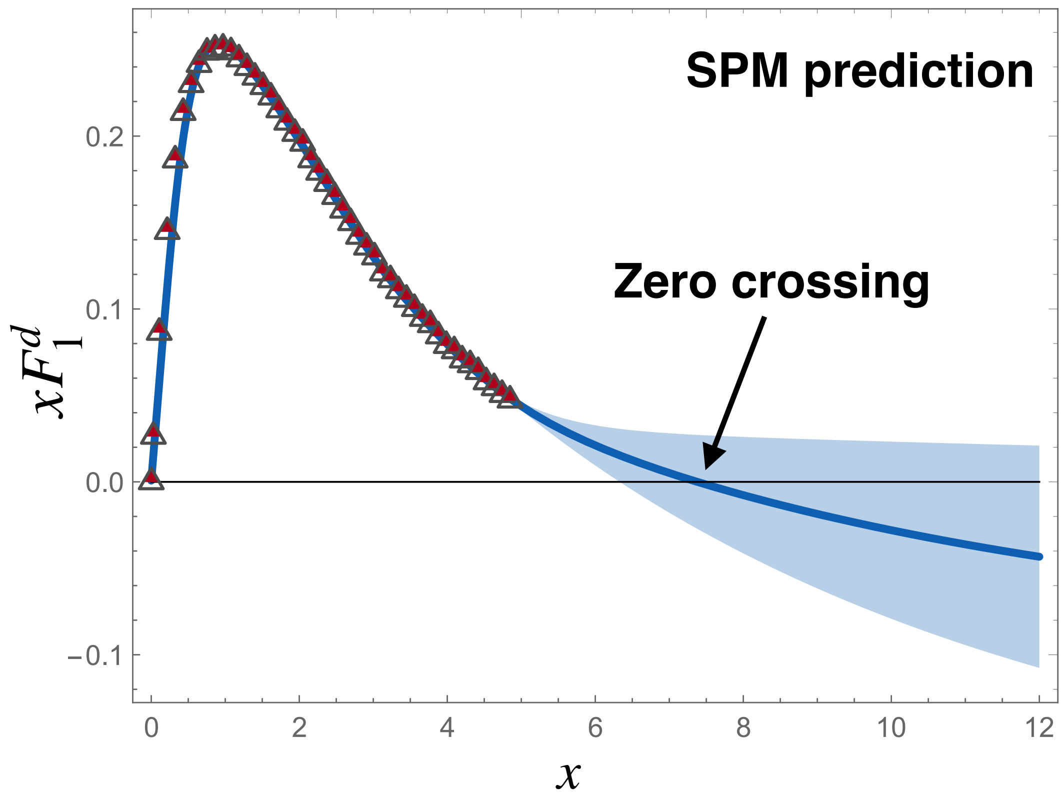

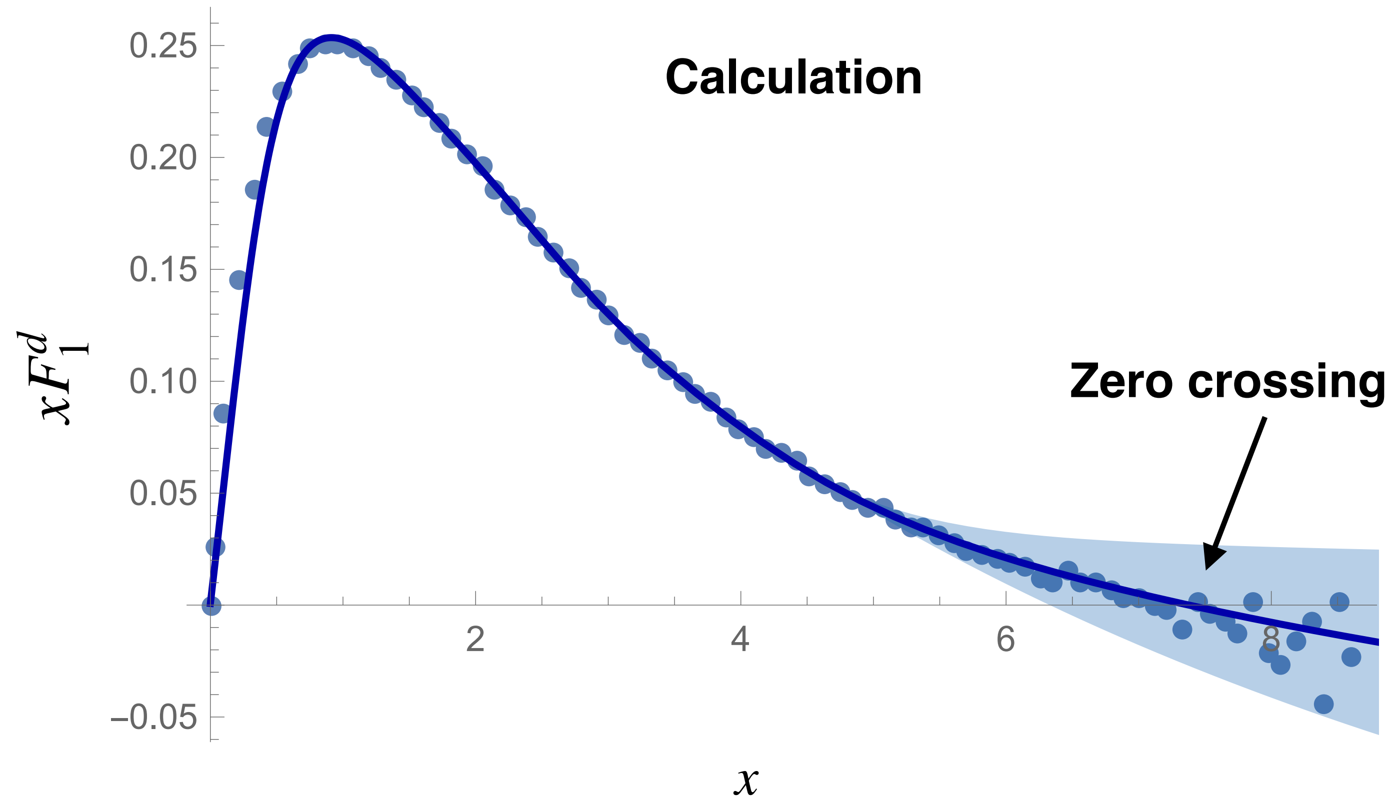

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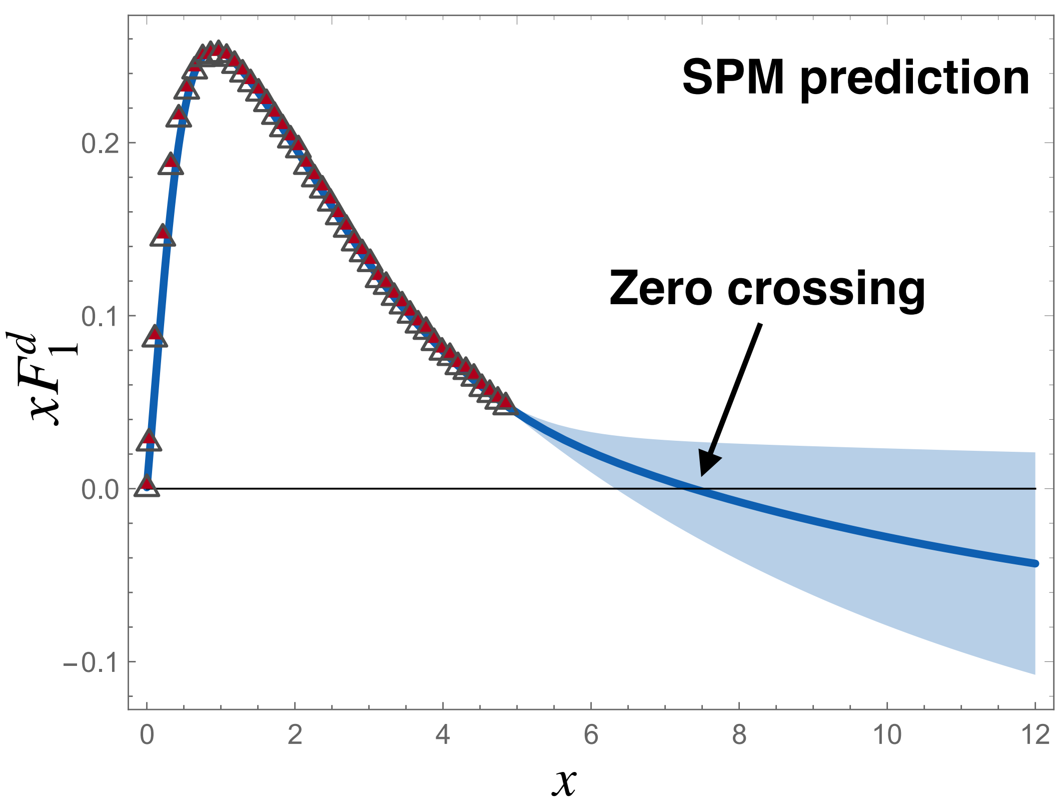
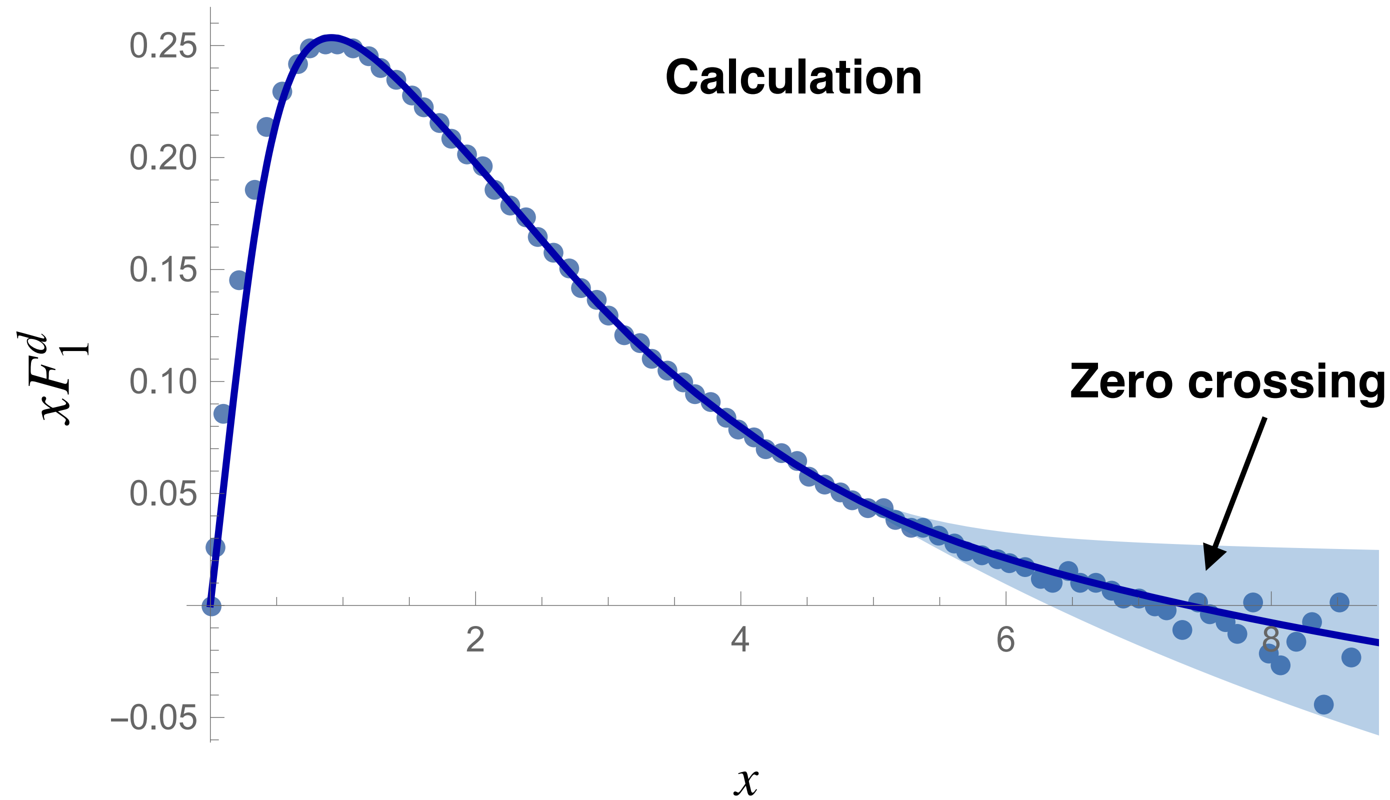
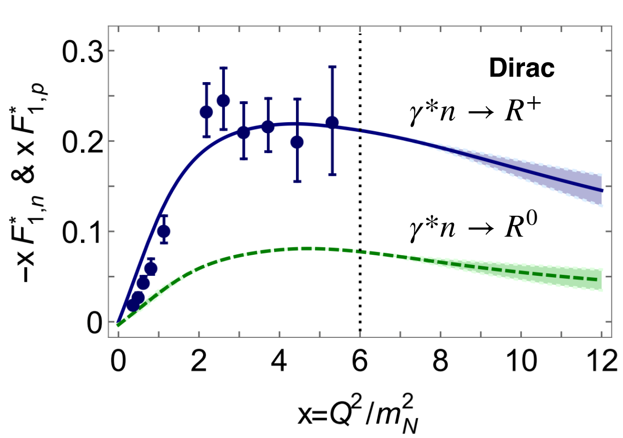
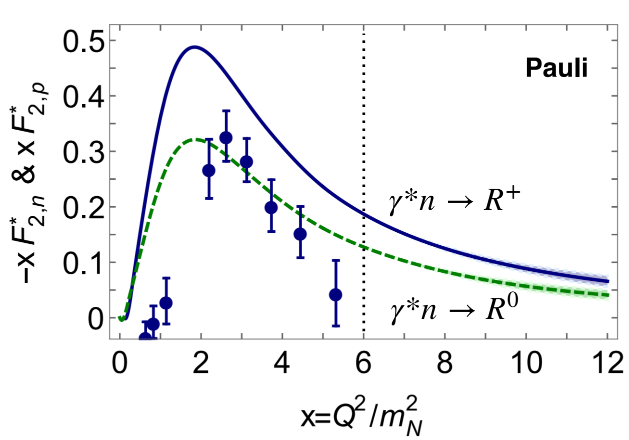


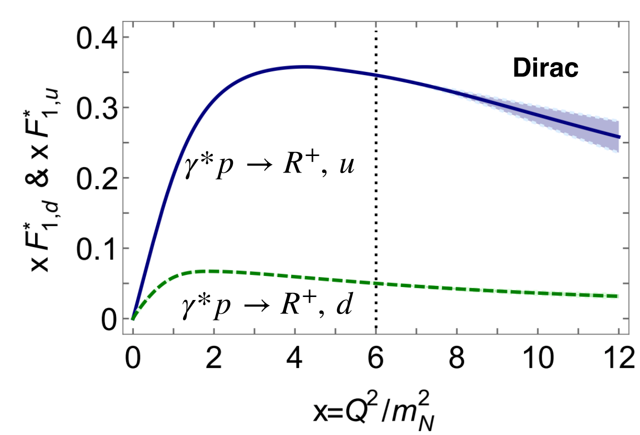


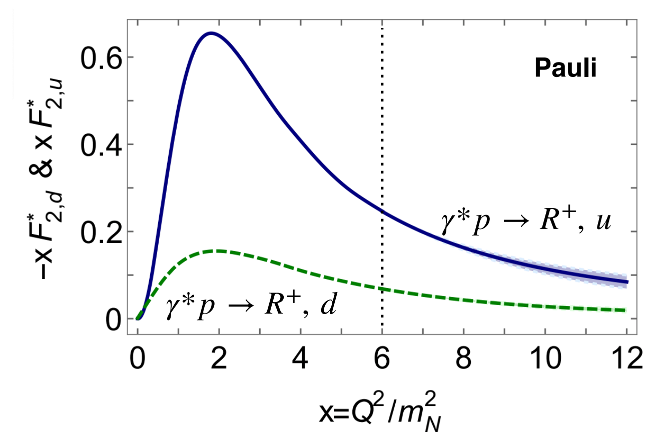
- **Extrapolation of nucleon to roper form factors**
  Chen, Lu, DB, Chang, Roberts, Rodríguez-Quintero, Segovia, PRD (2019)

- **Array of prediction for JLab12**
  G_p^E, G_n^M, G_n^E up to 14, 16 and 11 m_N^2



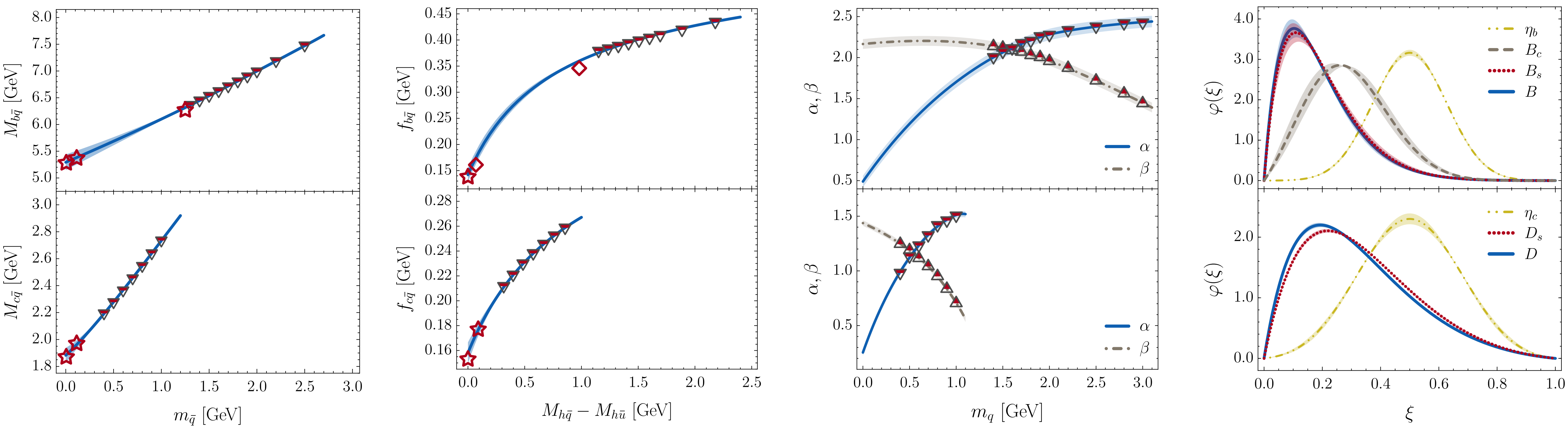




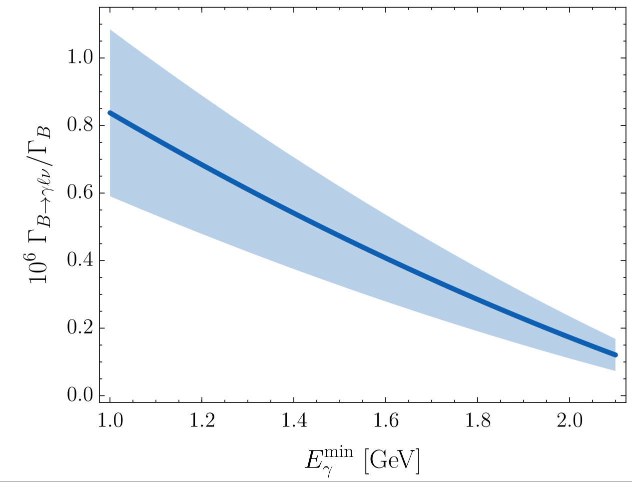


The beginning of a beautiful friendship

- **PDAs for heavy-light systems**
  DB, Chang, Ding, Gao, Papavassiliou, Roberts, PLB (2019)
  - **SPM extrapolation of**
    Masses
    Decay constants
    PDA parameters



- **Provides**
  Basic test of factorization approach to hard exclusive processes
  Branching ratio $B \to \gamma \ell \nu_\ell$

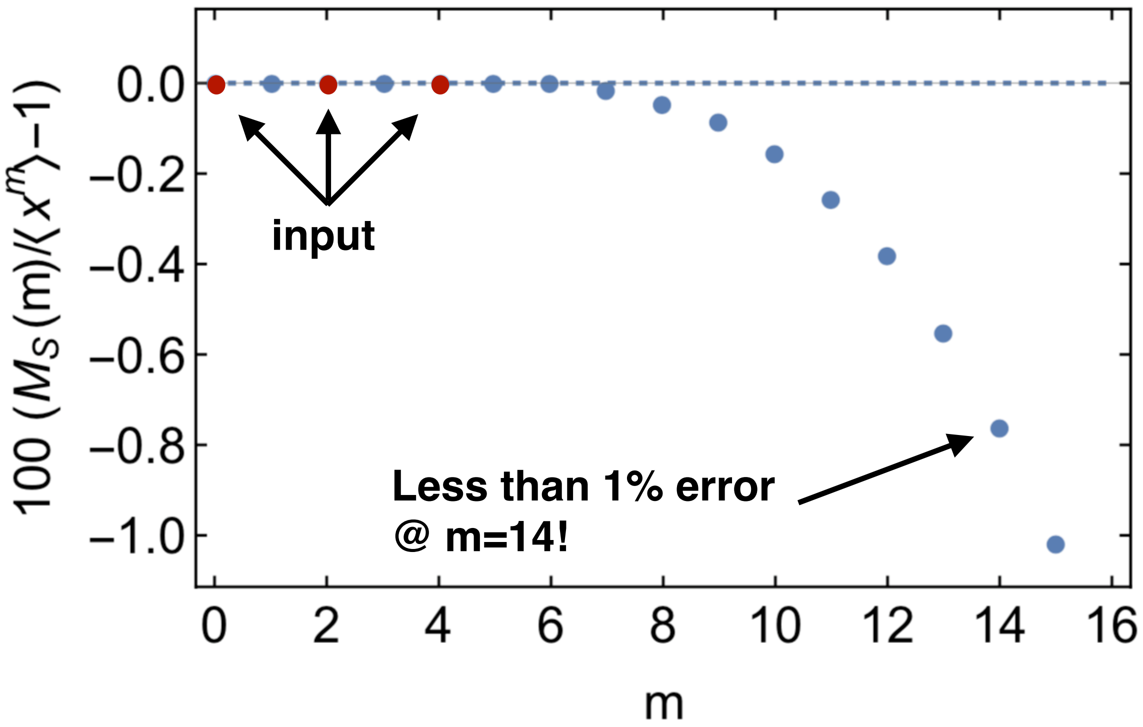


The beginning of a beautiful friendship

- **Pion PDF**

  Ding, Raya, DB, Chang, Roberts, Schmidt, in preparation

  - **SPM extrapolation of PDF moments**
    comparison with algebraic model



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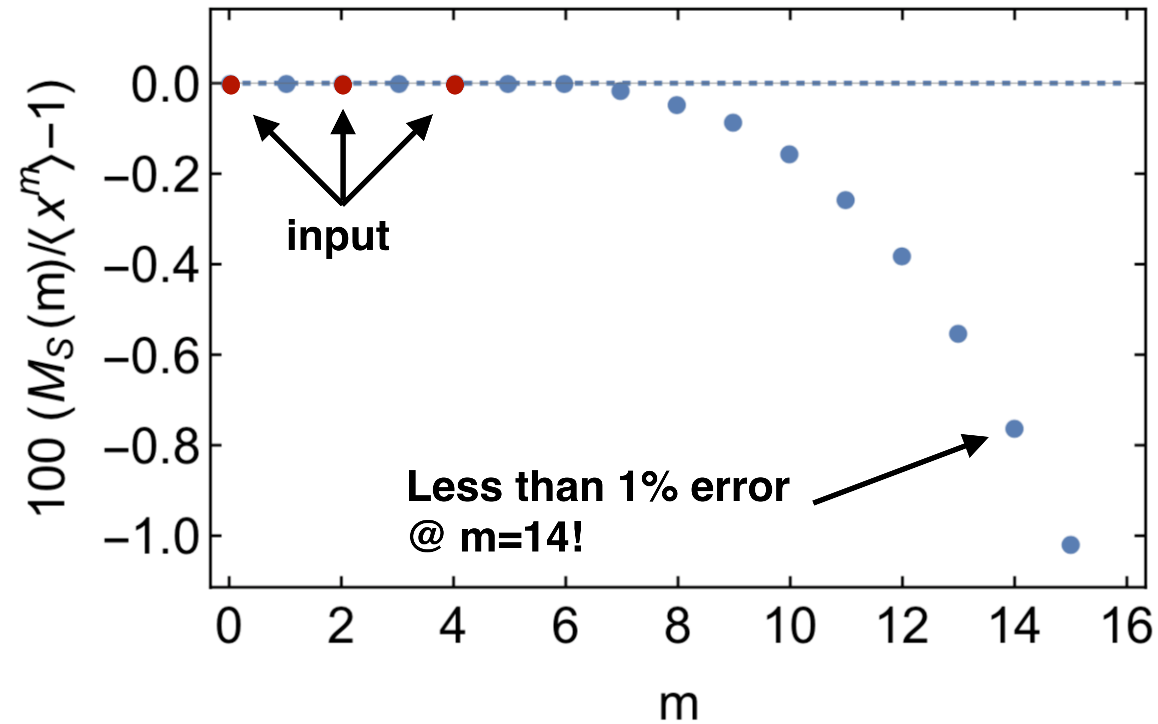
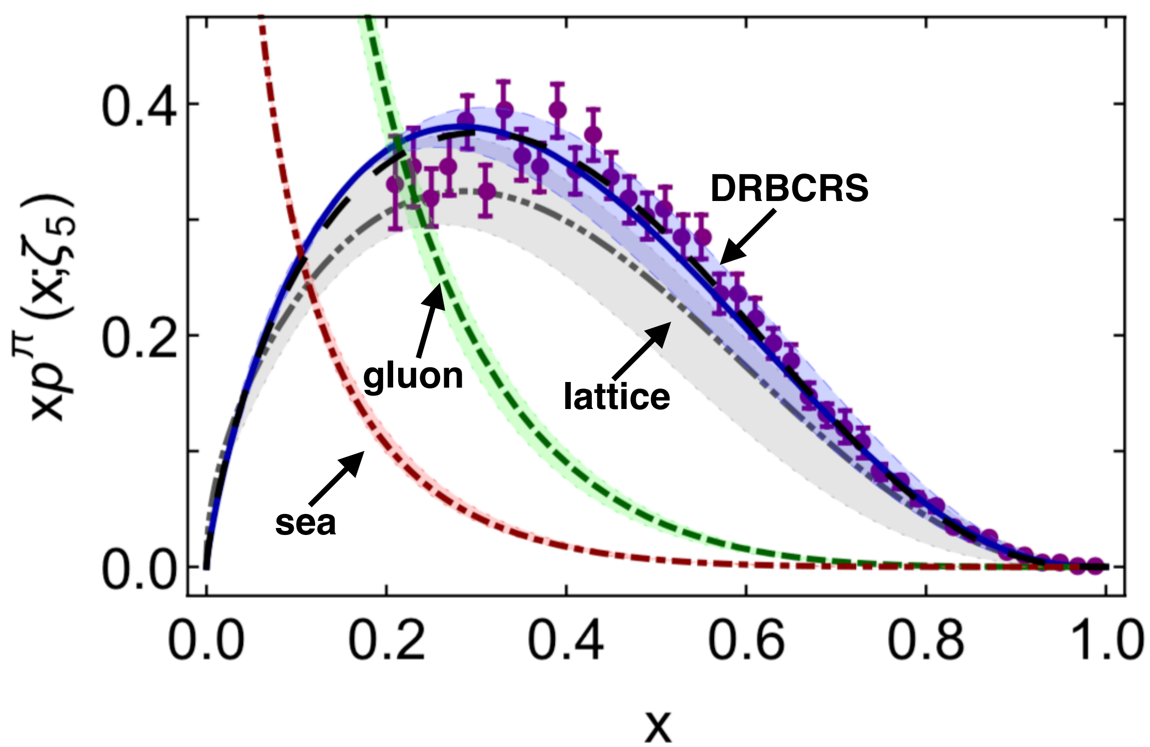


  - **Pion valence-quark momentum distribution**
    evolved @ 5.2 GeV



SPM and analytic continuation

- **Analytic continuation**
  Can be obtained by setting $x = \alpha e^{i\theta}$ in $C_N(x)$
  - **Pole singularities**
    can be exactly reproduced
  - **Branch cuts**
    can be approximately reproduced by a series of poles
  - **Rational fractions can have only one sheet**
    many sheeted functions can only be reconstructed on a single sheet

- **(Generalized) spectral functions**
  perfect application domain

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2} + \sum_{j=1}^{n} \frac{Z_j}{p_0^2 - z_j}$$

$$\rho(\omega) = 2 \, \mathrm{Im} \, D(p_0 \to -i(\omega + i0^+))$$

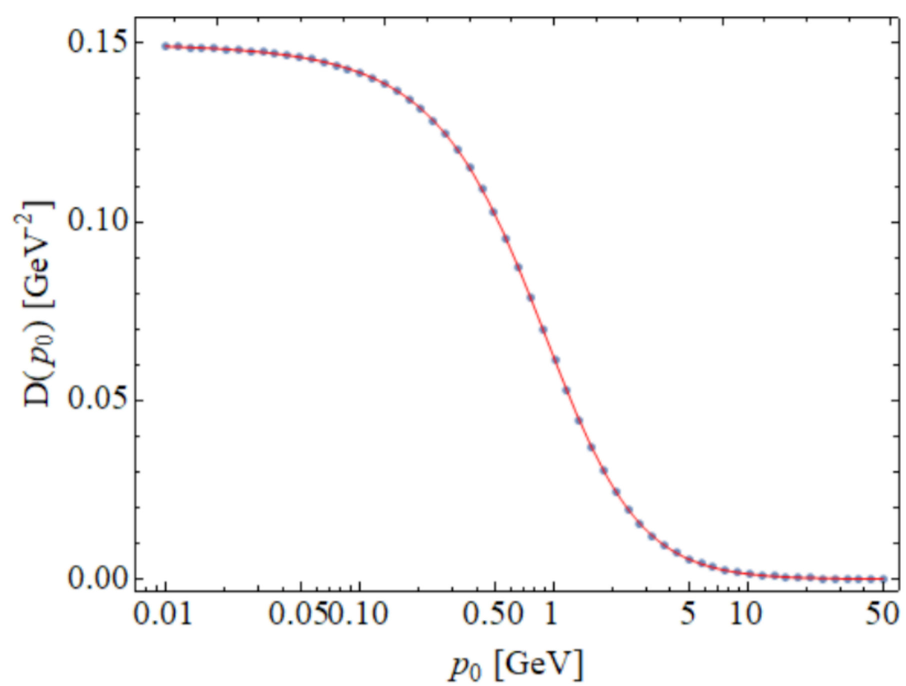
Breit-Wigner (BW) propagator

- **SPM test**
  Simple Breit Wigner propagator

$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+)) = \frac{1}{\pi}\frac{2\Gamma\omega}{(\omega^2 - \Gamma^2 - M^2)^2 + 4\Gamma^2\omega^2}$$

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2} = \frac{1}{2\pi}\frac{1}{(p_0 + \Gamma)^2 + M^2}$$

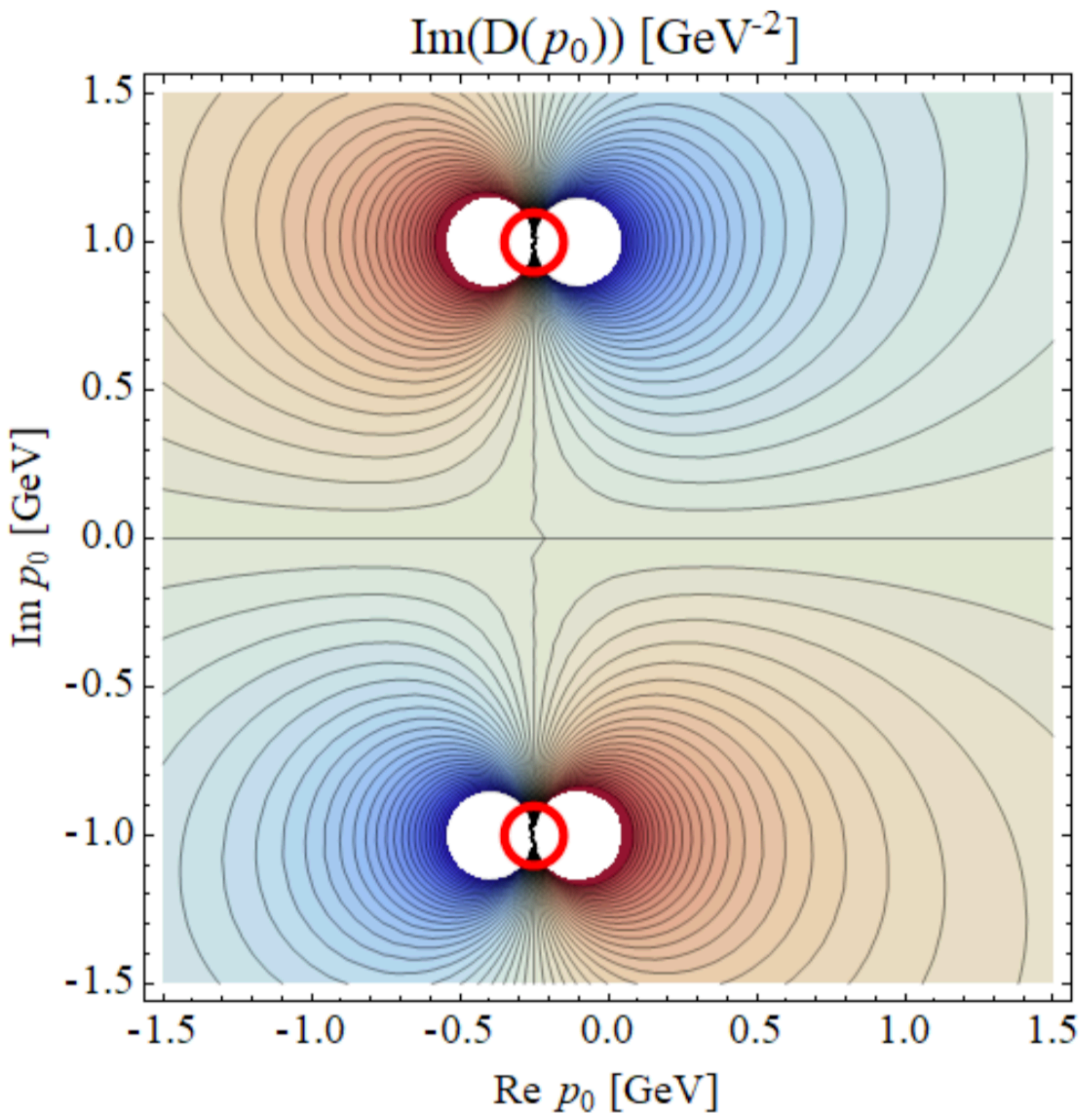
- **Procedure**
  - **Choose parameters**
    $M = 4\Gamma = 1\ \mathrm{GeV}$
  - **Generate Euclidean propagator data**
    100 points between 0.01 and 50 GeV
  - **Apply SPM**
    on your favourite 60 input points
  - **Compare exact/reconstructed analytic structure**
    in the p_0 and p_0^2 complex plane
  - **Construct the spectral function**
    compare it with the exact one
  - **Calculate the propagator**
    compare it with the exact one



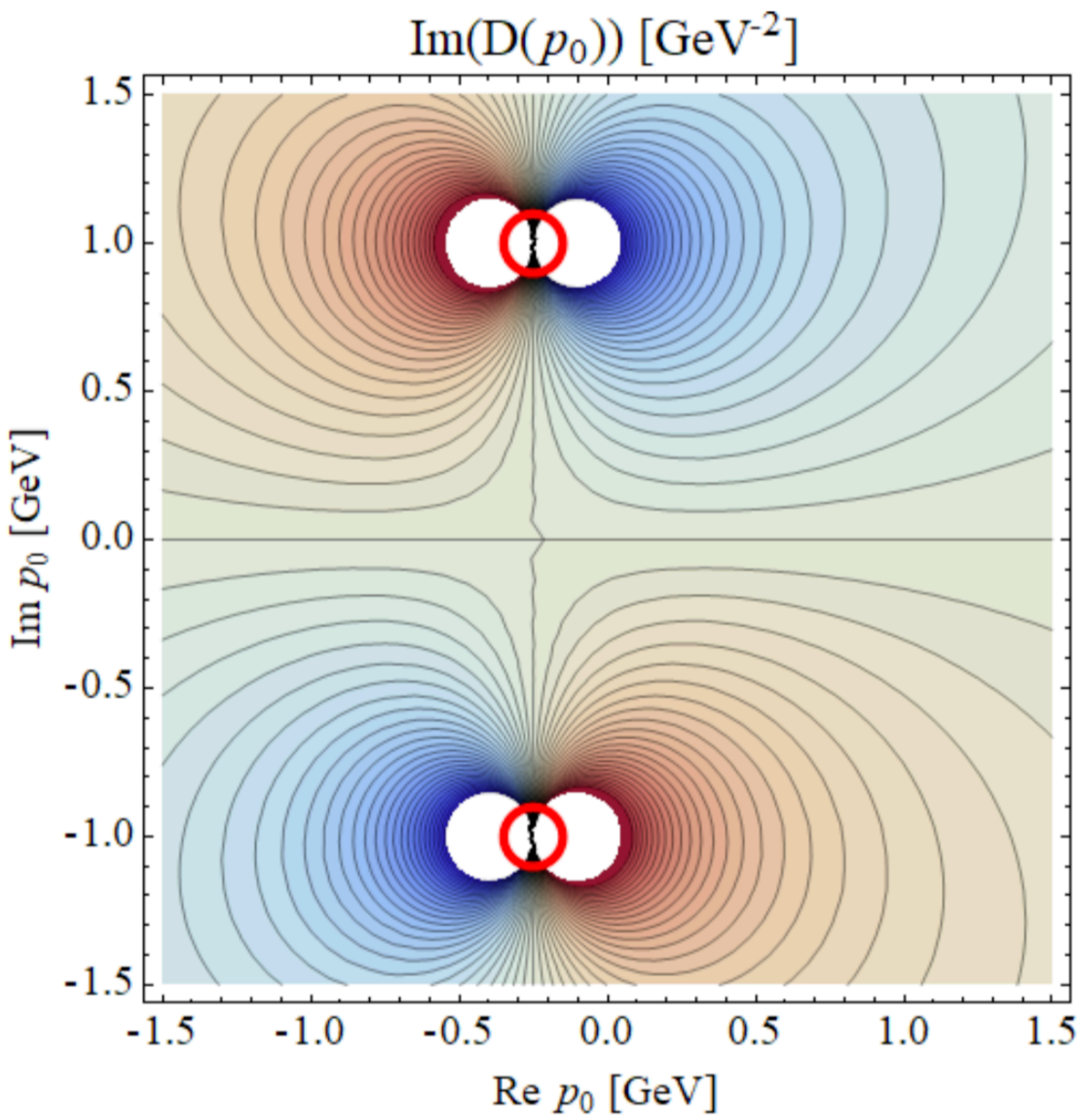
BW propagator I: p_0 plane

- **Poles:**
  perfectly reconstructed

  - **Exact**

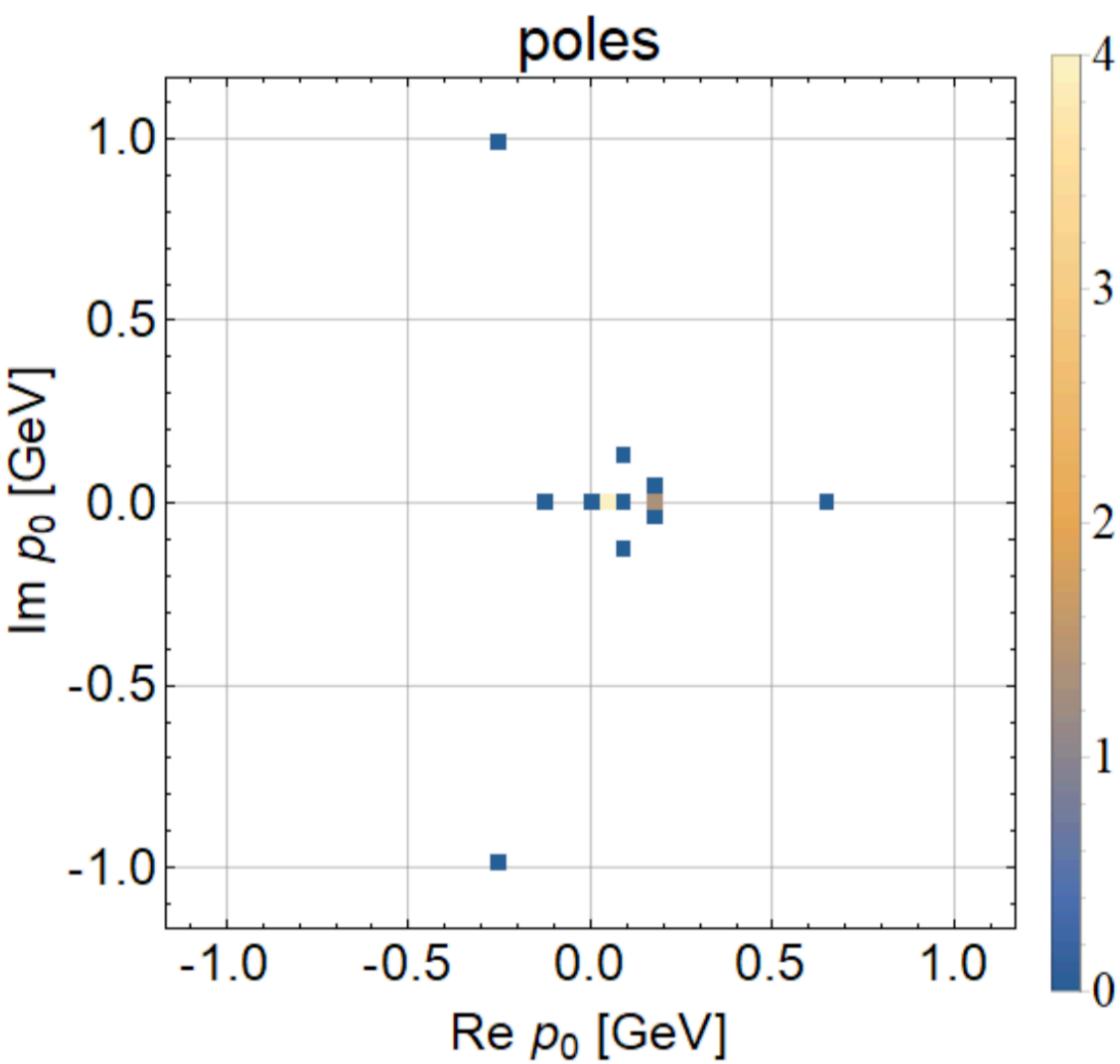


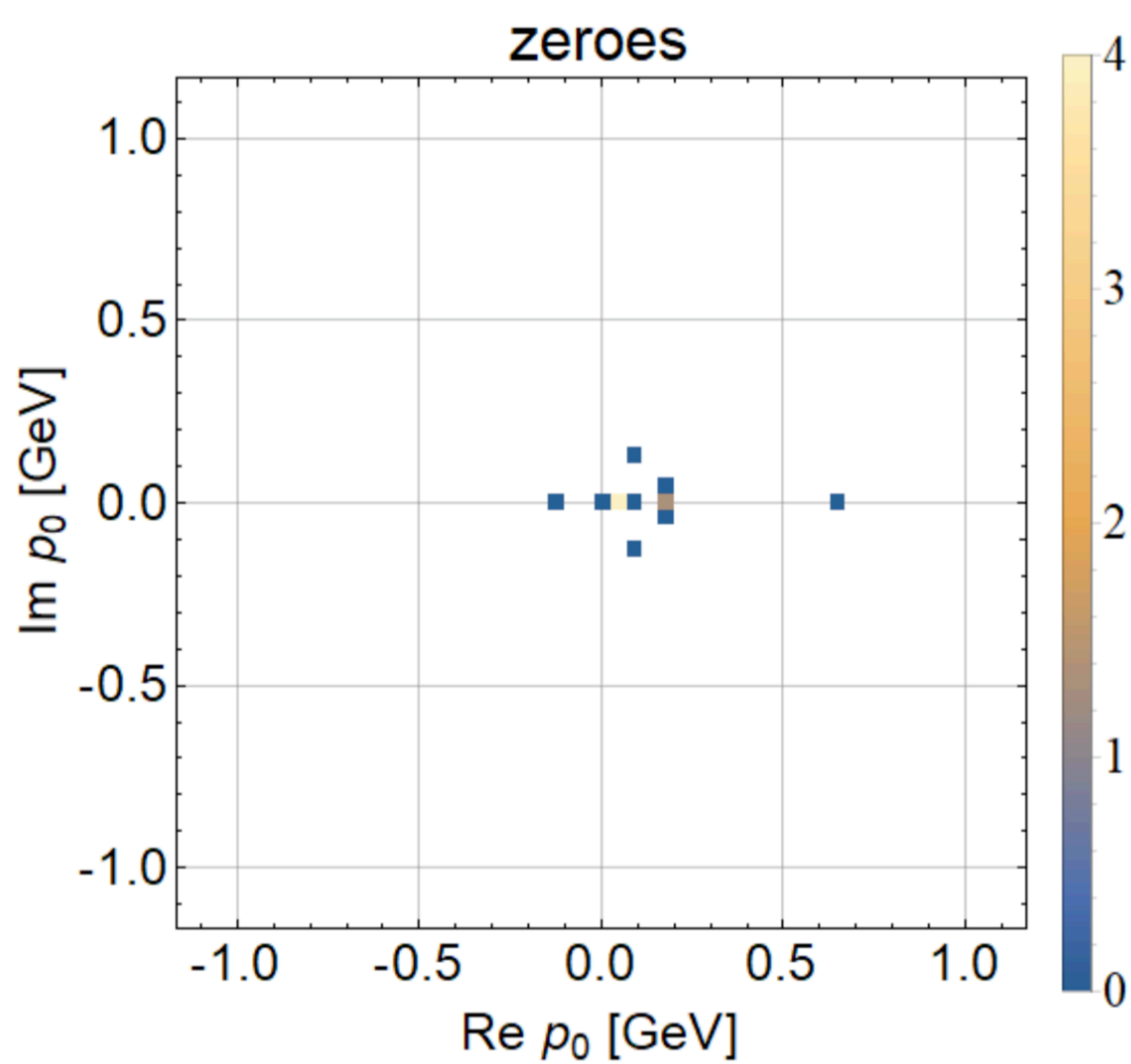
  - **Reconstructed**



BW propagator I: p_0 plane

- **How?**
  Near cancellation between poles and zeros
  - **For $N = 60$**
    there are 30 poles and 29 zeros
  - **Near cancellation**
    leaves poles with very small residue

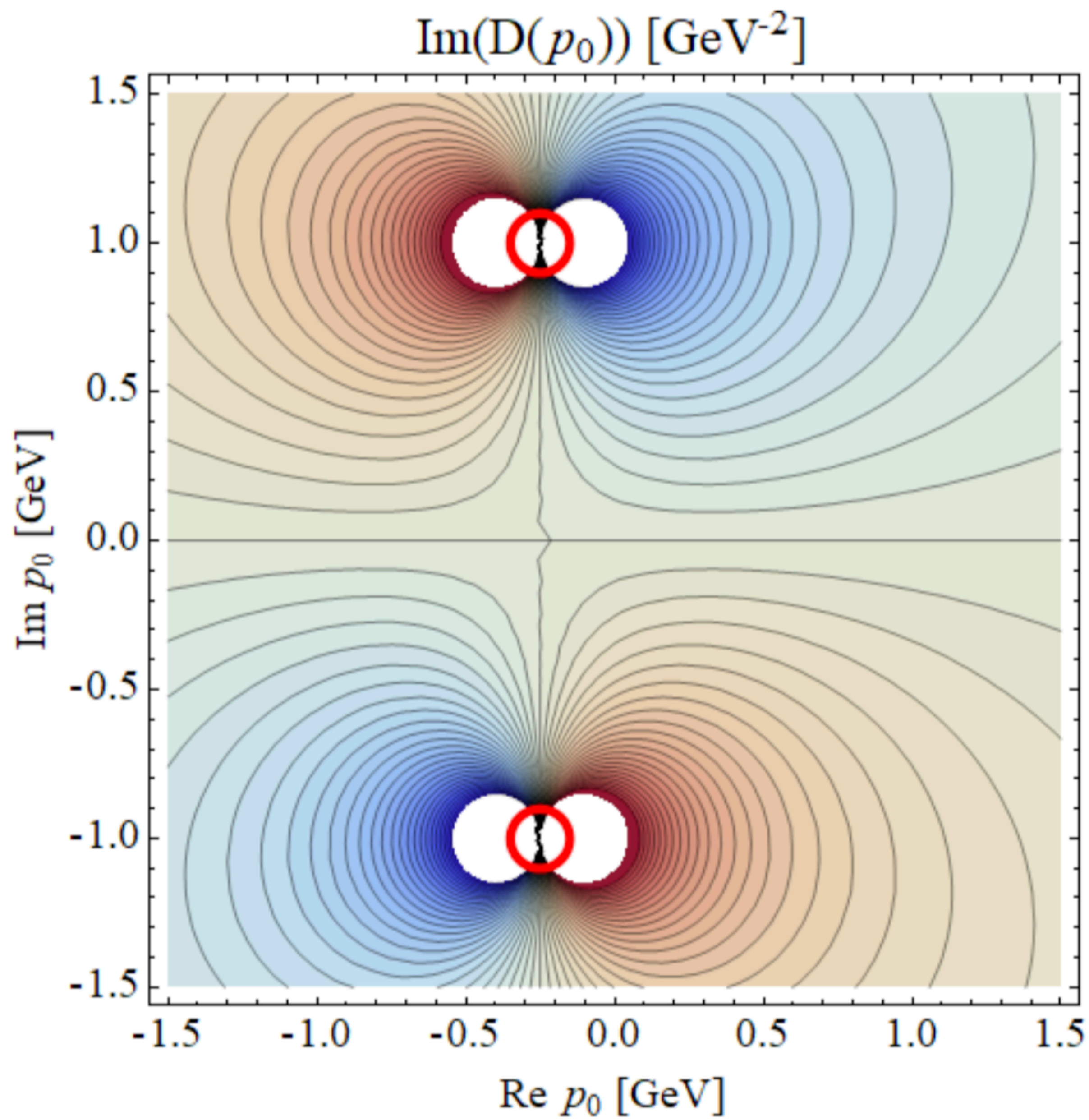




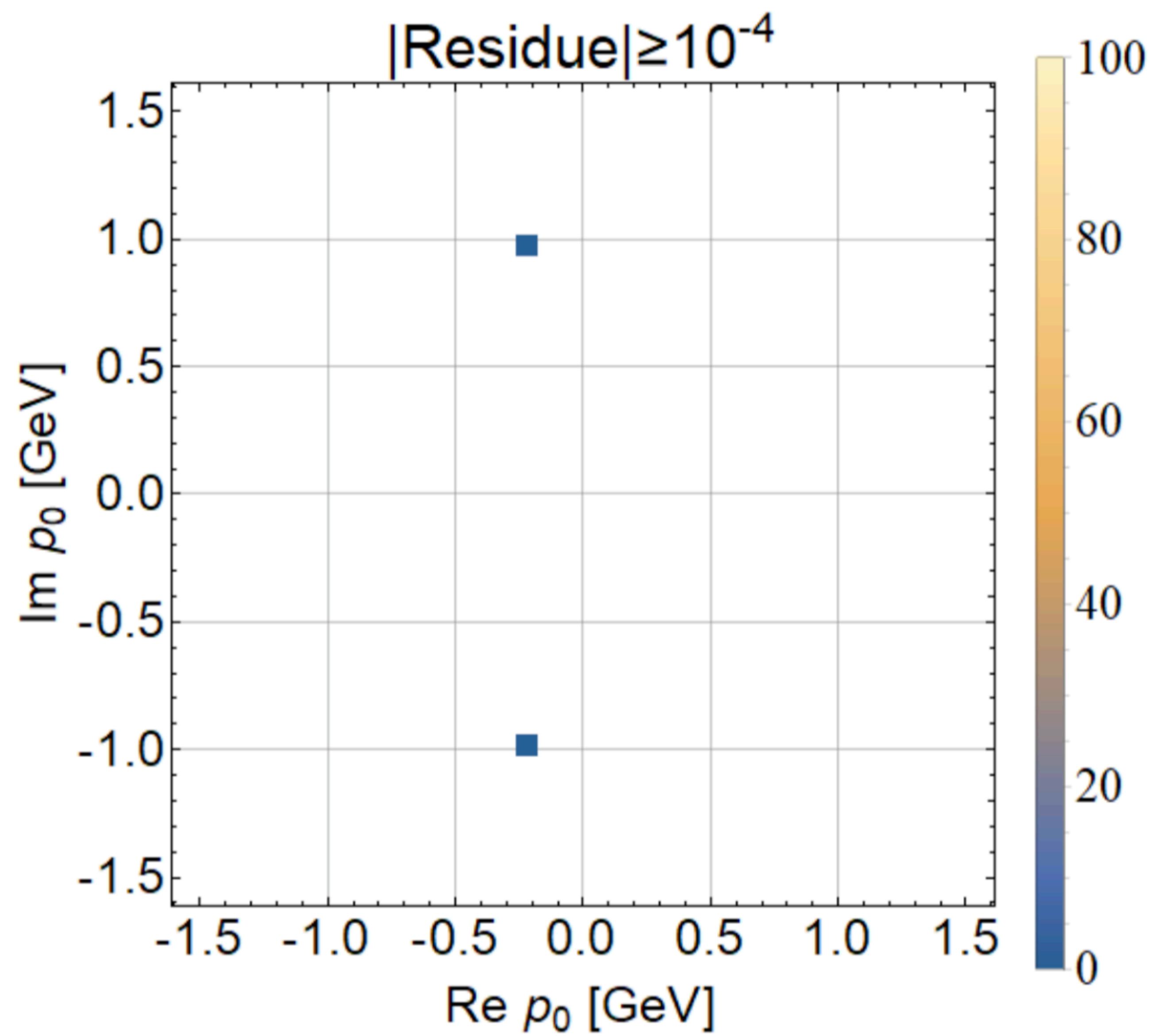
BW propagator I: p_0 plane

- **Filtering**
  physical poles can be identified by using a threshold for the residues

- **Exact**



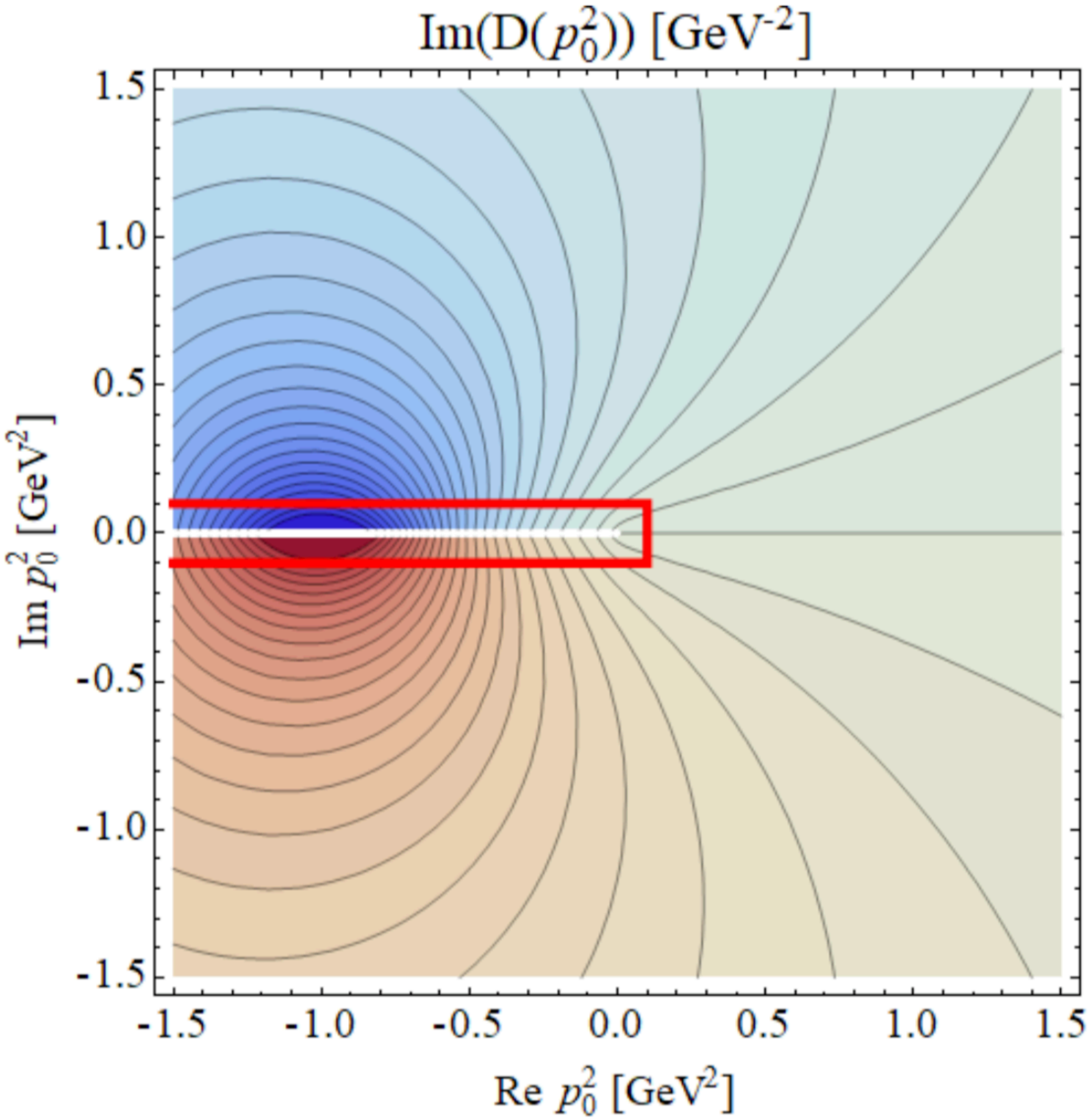
- **Reconstructed**



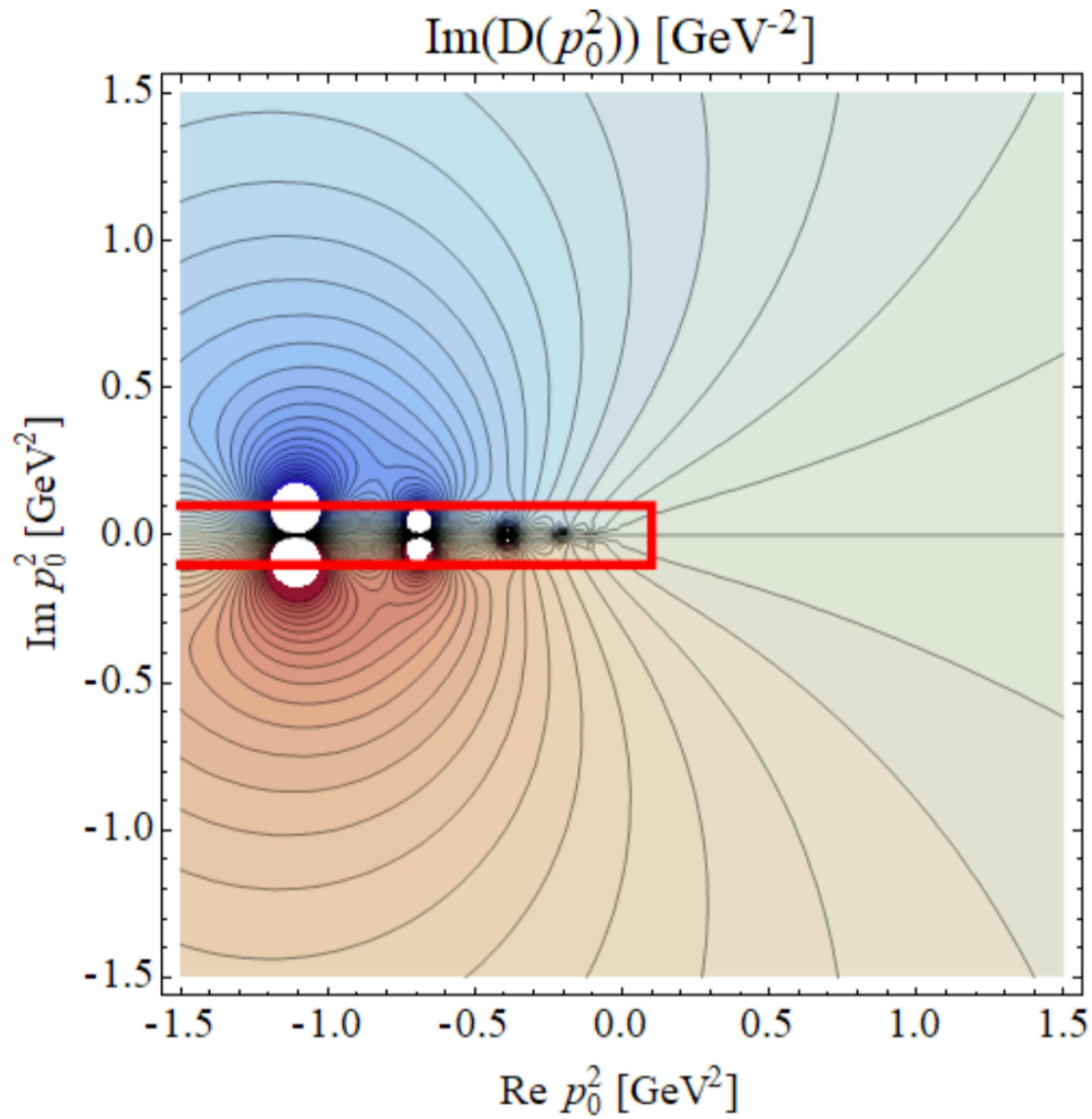
BW propagator II: p_0^2 plane

- **Branch cut:**
  visualized as a series of poles

  - **Exact**

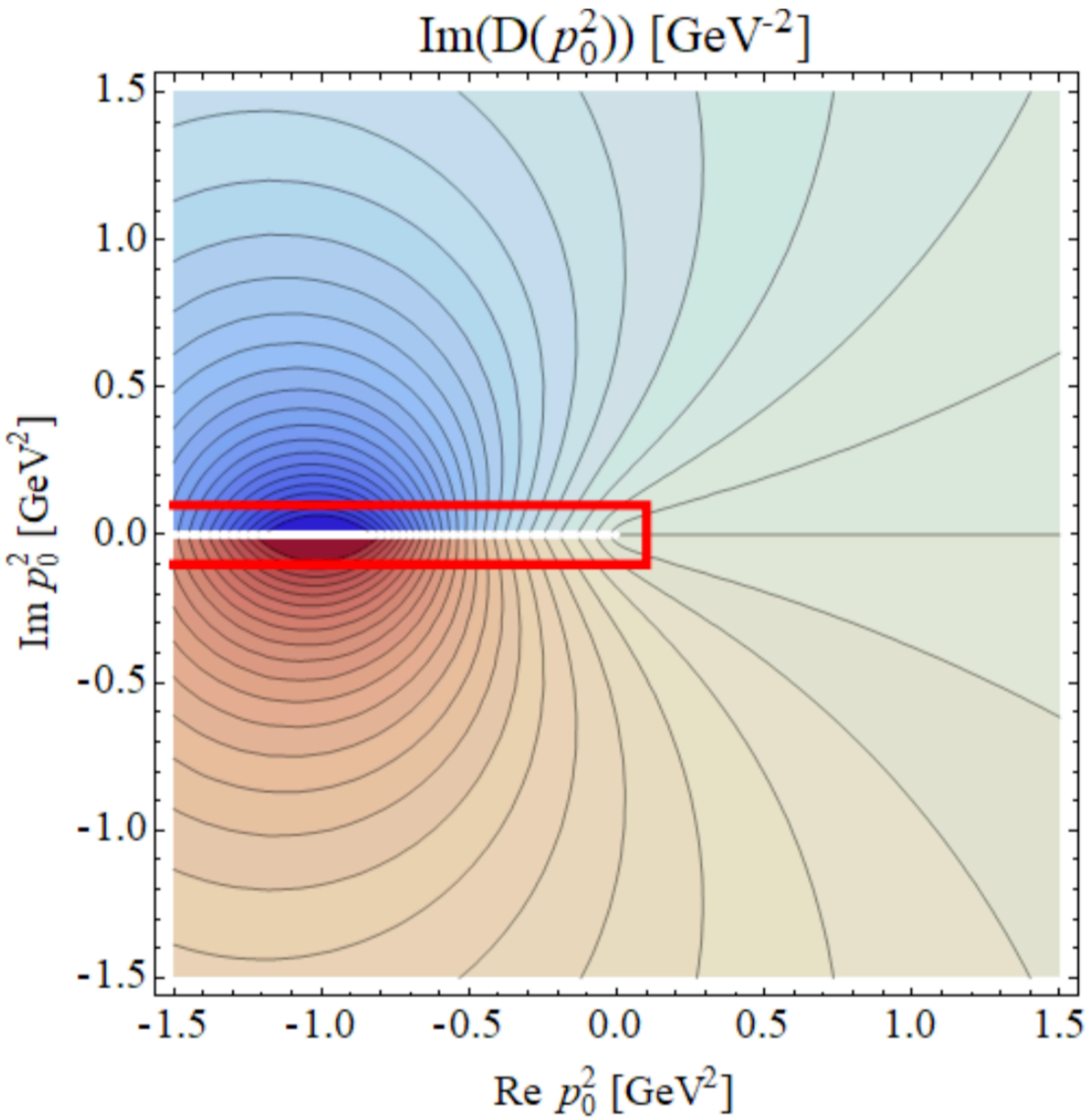


  - **Reconstructed**

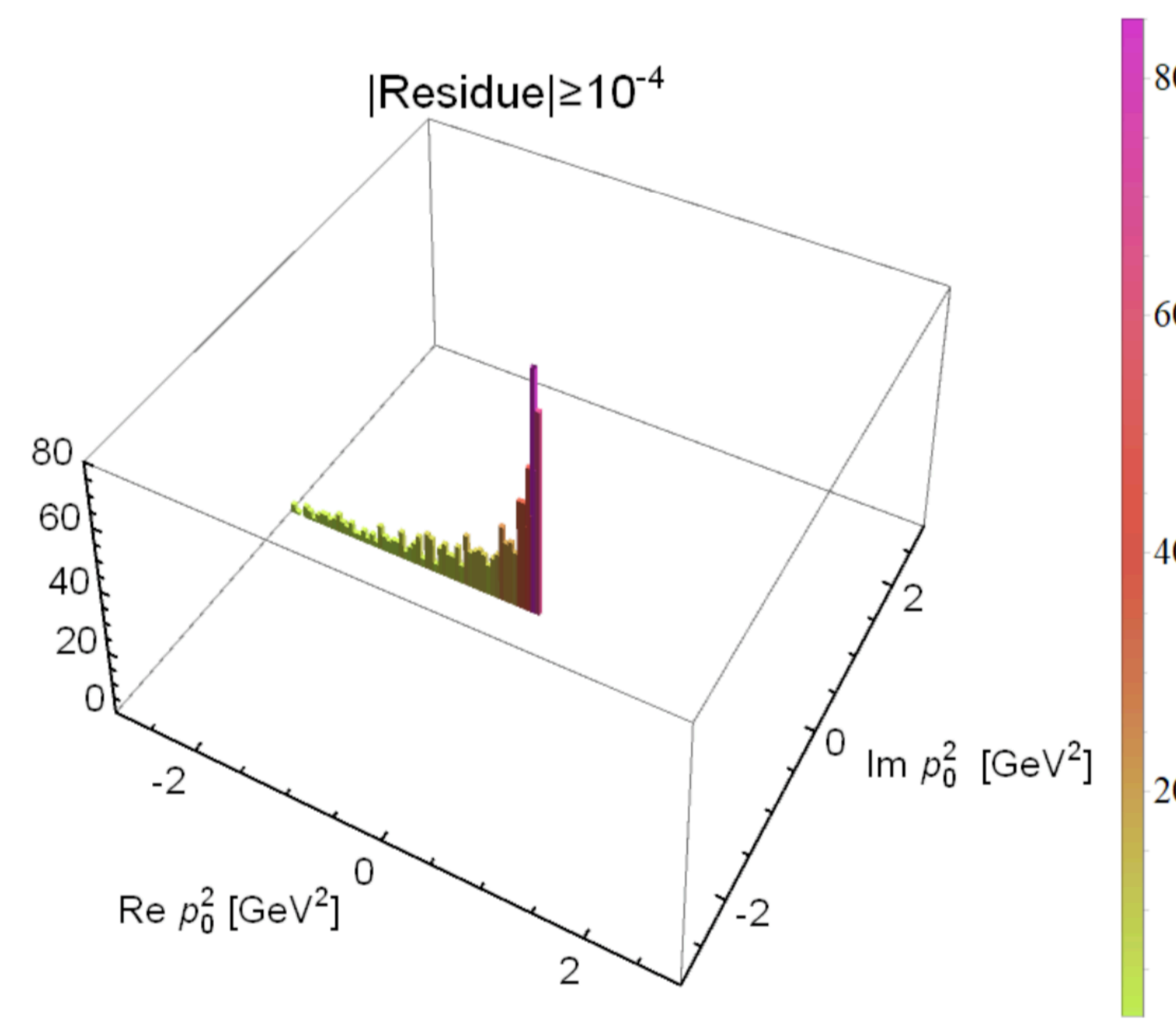


BW propagator II: p_0^2 plane

- **Branch cut:**
  more clearly visible in a histogram, showing the location of the poles for 100 random subsets of the 60 input points
  - **Exact**



  - **Reconstructed**



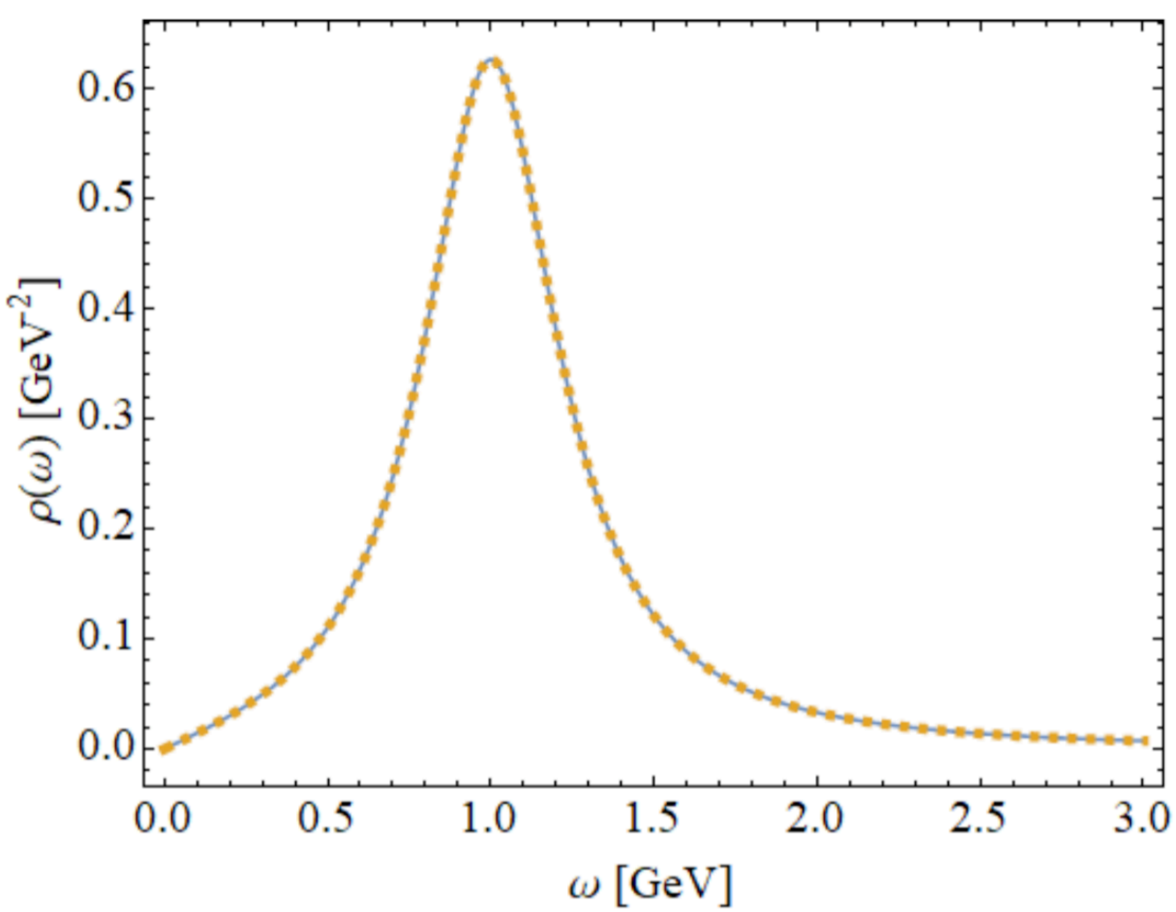
BW spectral function

- **Spectral function**
  Obtained as:

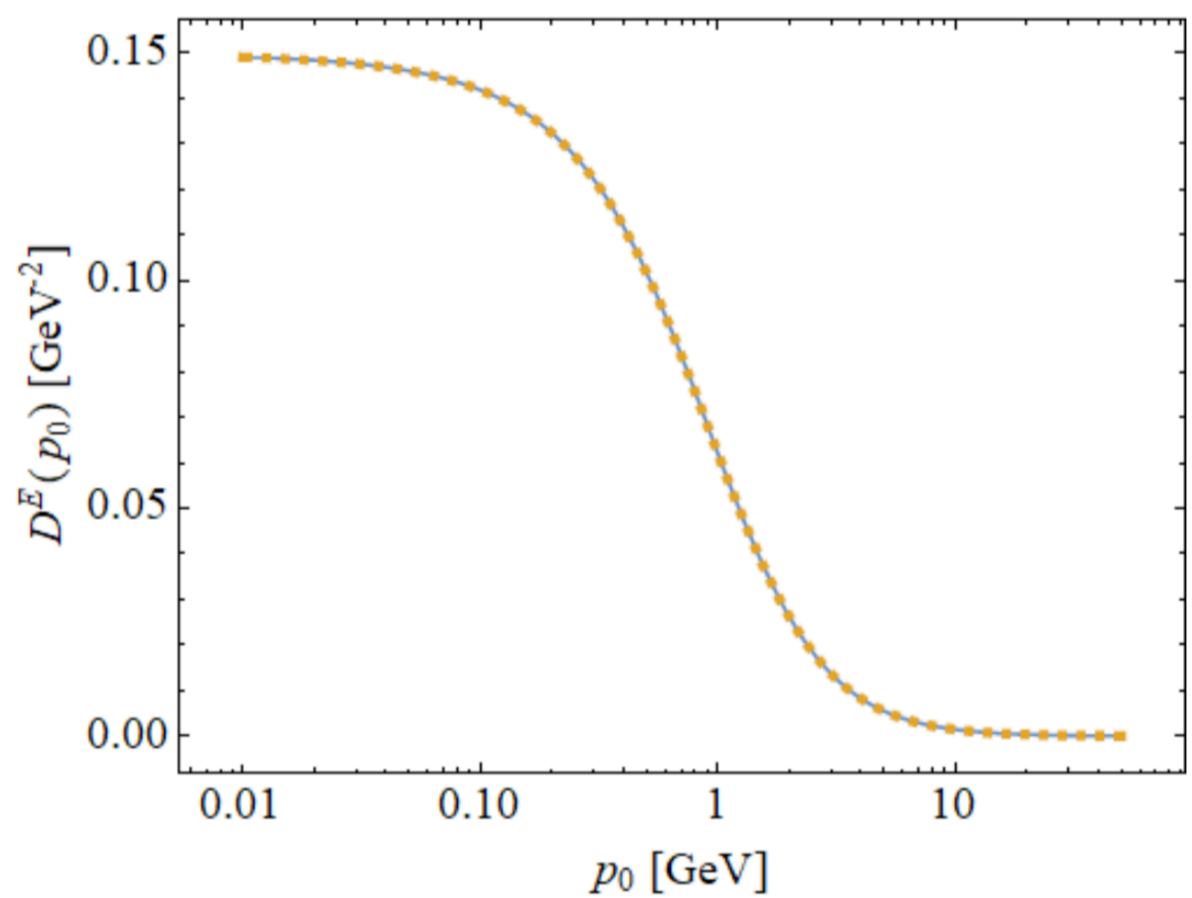
$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+))$$

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2}$$

- **Reconstructed spectral function**



- **Reconstructed propagator**



BW propagator plus poles

- **Add complex conjugated poles**
  to BW propagator

$$D(p_0) = \frac{1}{2\pi}\frac{1}{(p_0+\Gamma)^2+M^2} + \sum_{j=1}^{n}\frac{Z_j}{p_0^2 - z_j}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\omega\rho(\omega)}{\omega^2+p_0^2} + \sum_{j=1}^{n}\frac{Z_j}{p_0^2 - z_j}$$

$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+))$$
$$= \frac{1}{\pi}\frac{2\Gamma\omega}{(\omega^2-\Gamma^2-M^2)^2+4\Gamma^2\omega^2}$$

- **Procedure**
  - **Choose parameters**
    $M = 4\Gamma = 1$ GeV, $Z_1 = Z_2 = 1$, $z_{1,2} = (-1 \pm i)$ GeV^2
  - **Generate Euclidean propagator data**
    100 points between 0.01 and 50 GeV
  - **Apply SPM**
    on your favourite 60 input points
  - **Compare exact/reconstructed analytic structure**
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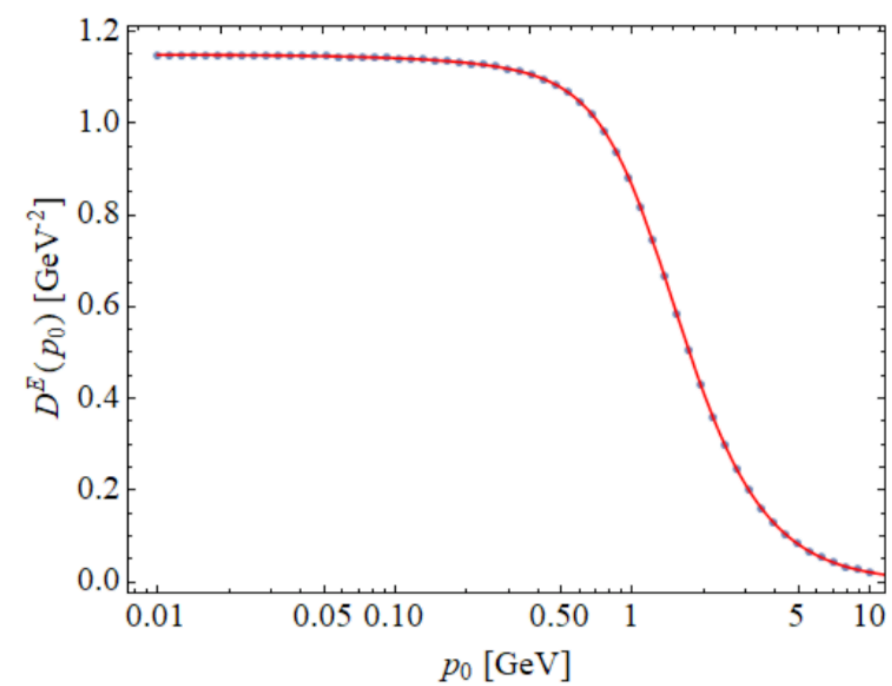
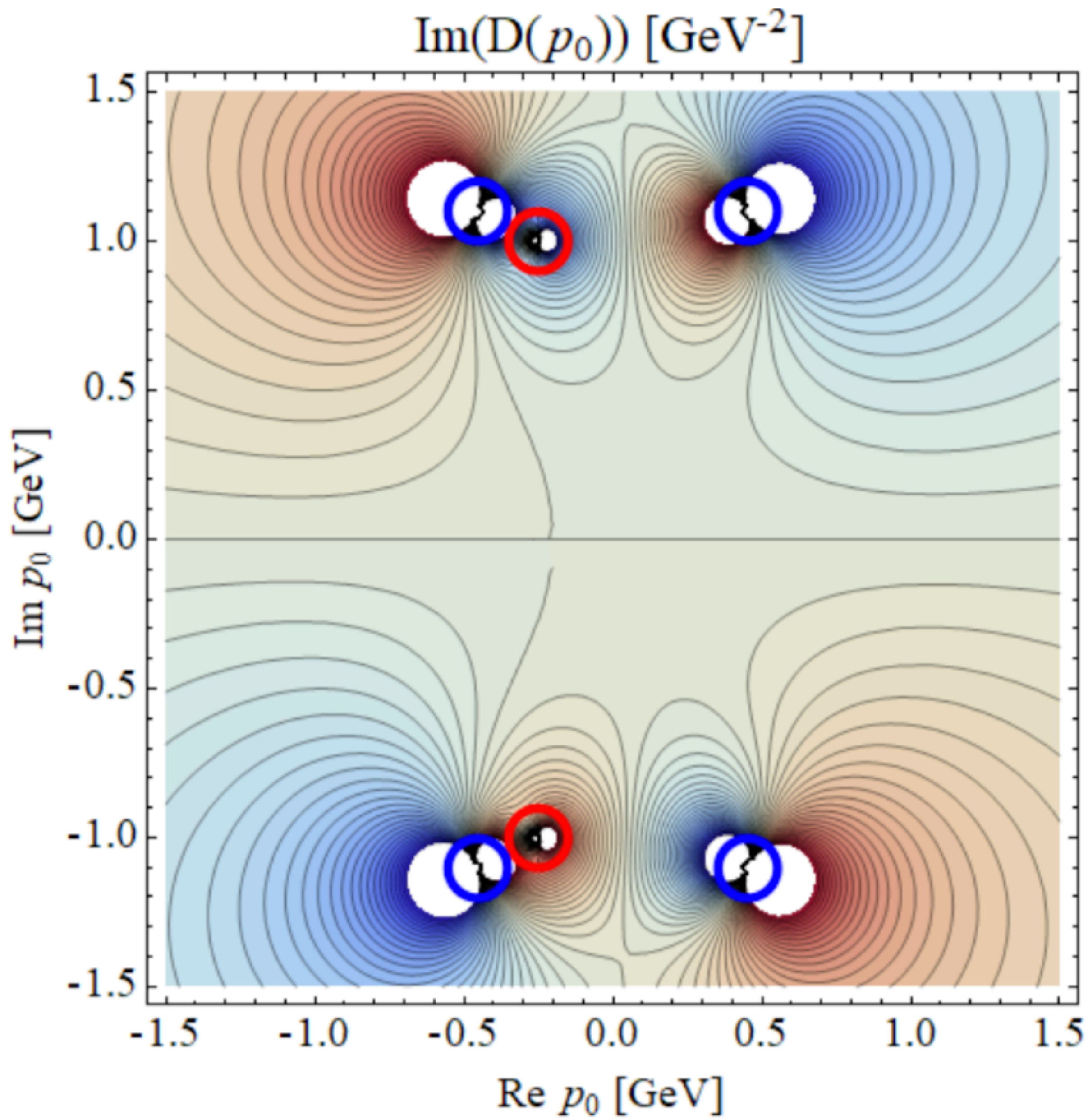


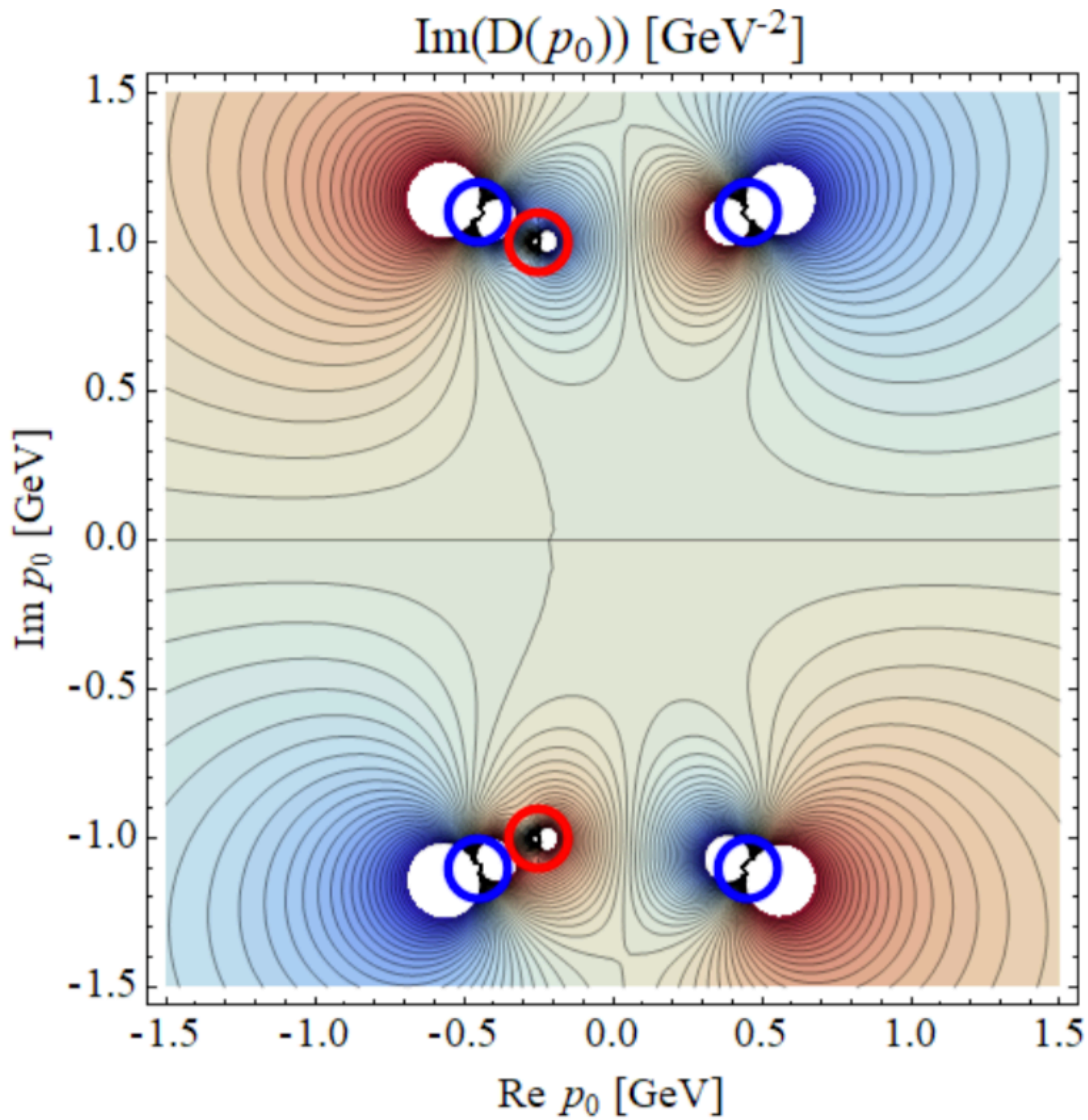
BW propagator plus poles I: p_0 plane

- **Poles:**
  perfectly reconstructed

  - **Exact**

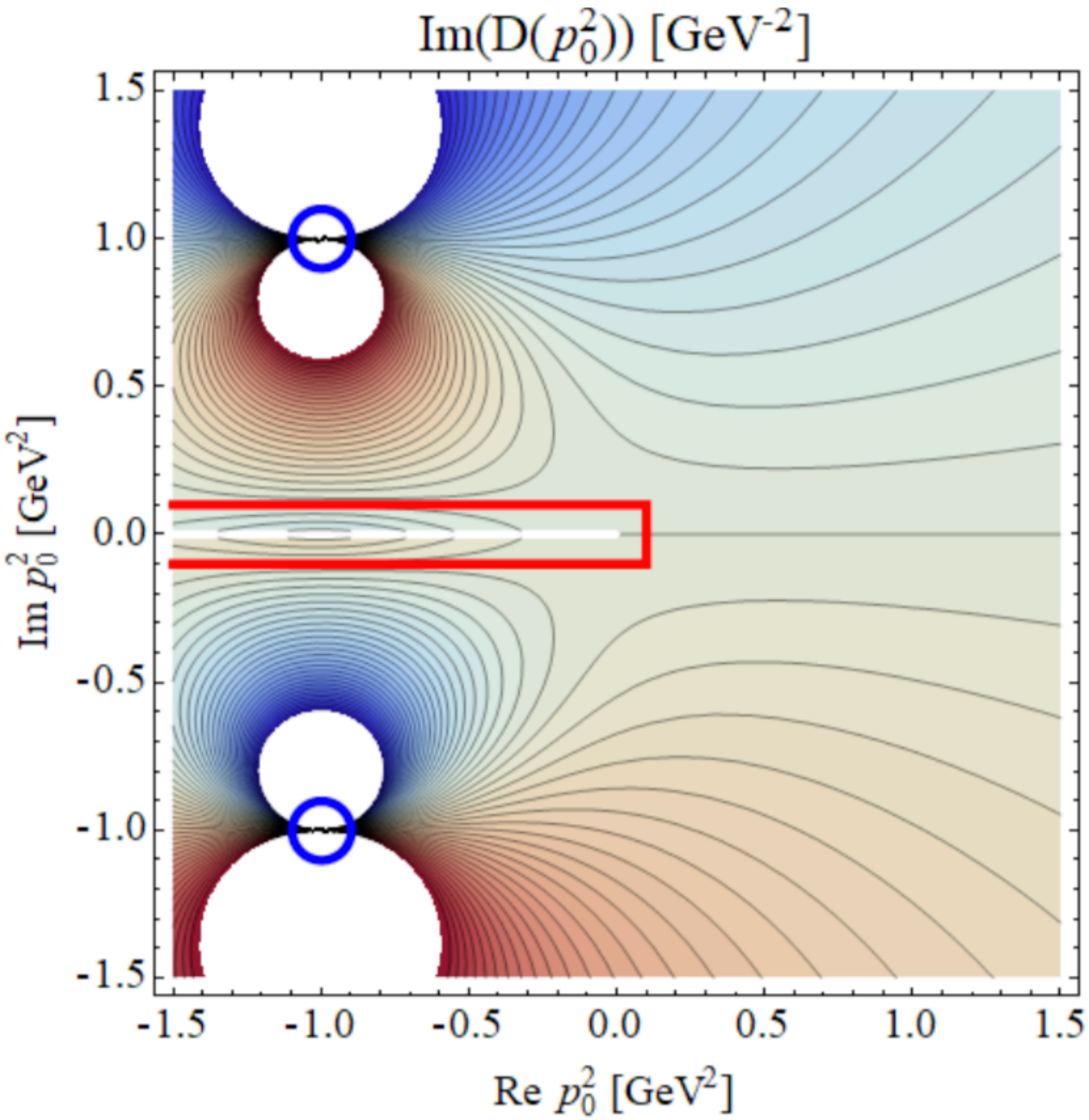


  - **Reconstructed**

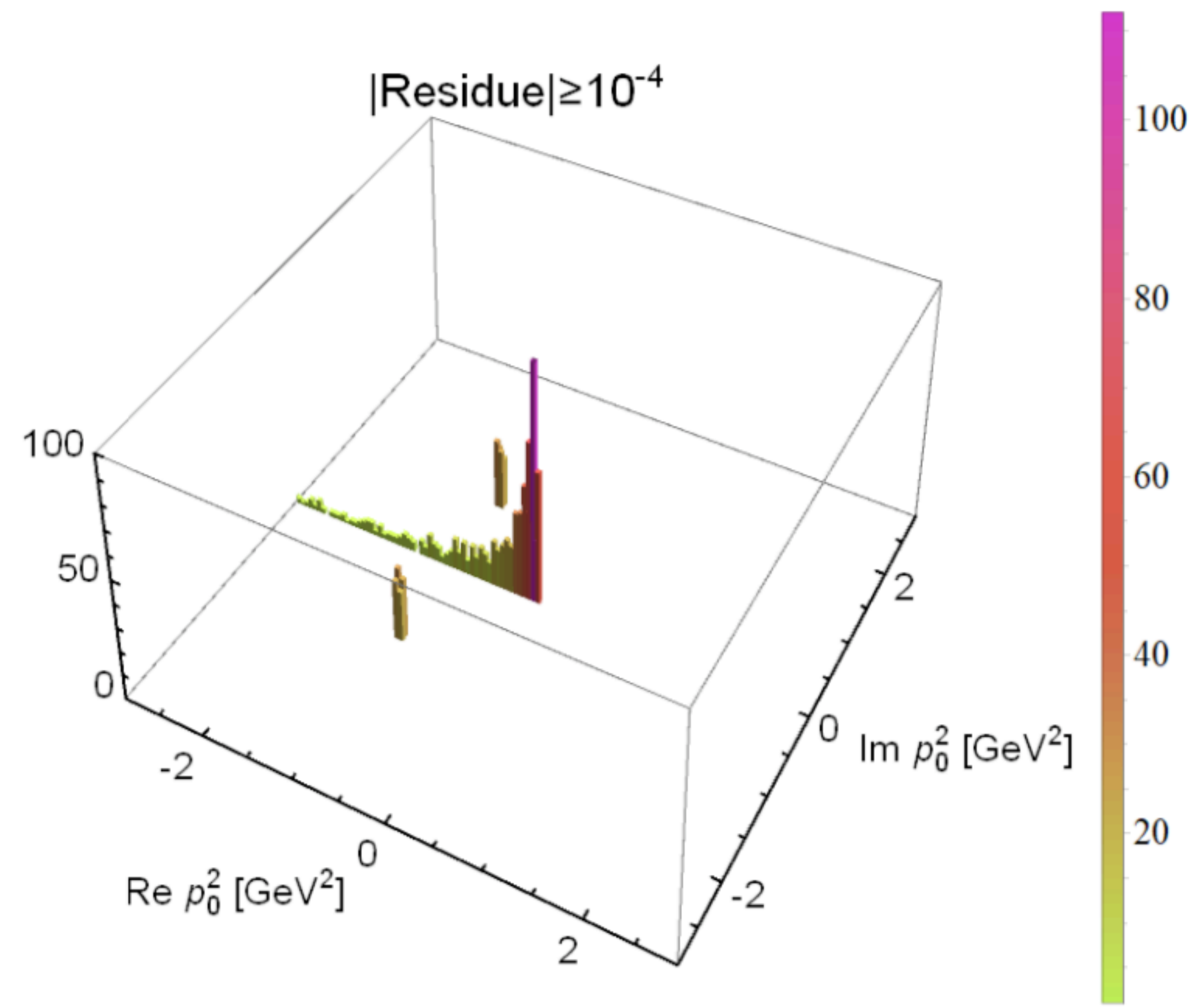


BW propagator plus poles II: p_0^2 plane

- **Branch cut:**
  more clearly visible in a histogram, showing the location of the poles for 100 random subsets of the 60 input points
  - **Exact**



  - **Reconstructed**

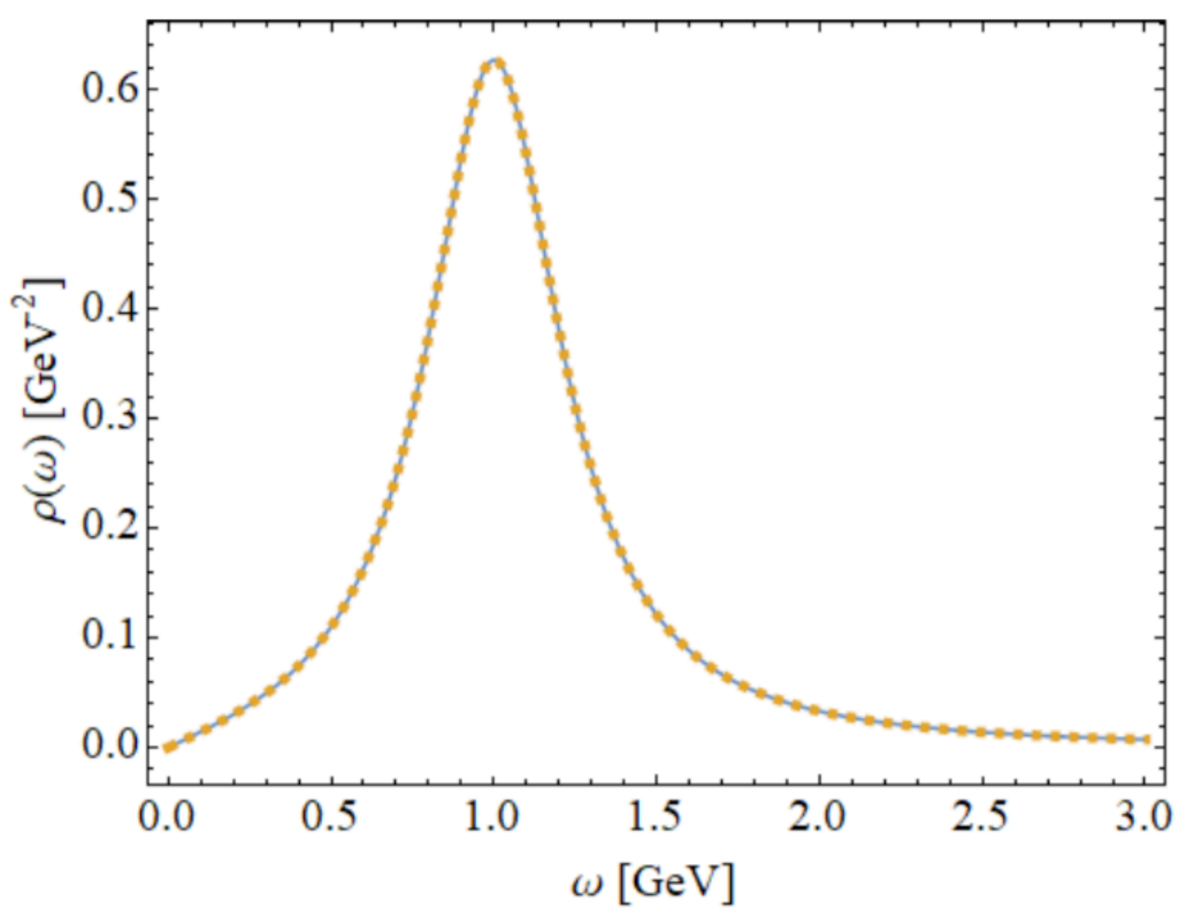


BW plus poles spectral function

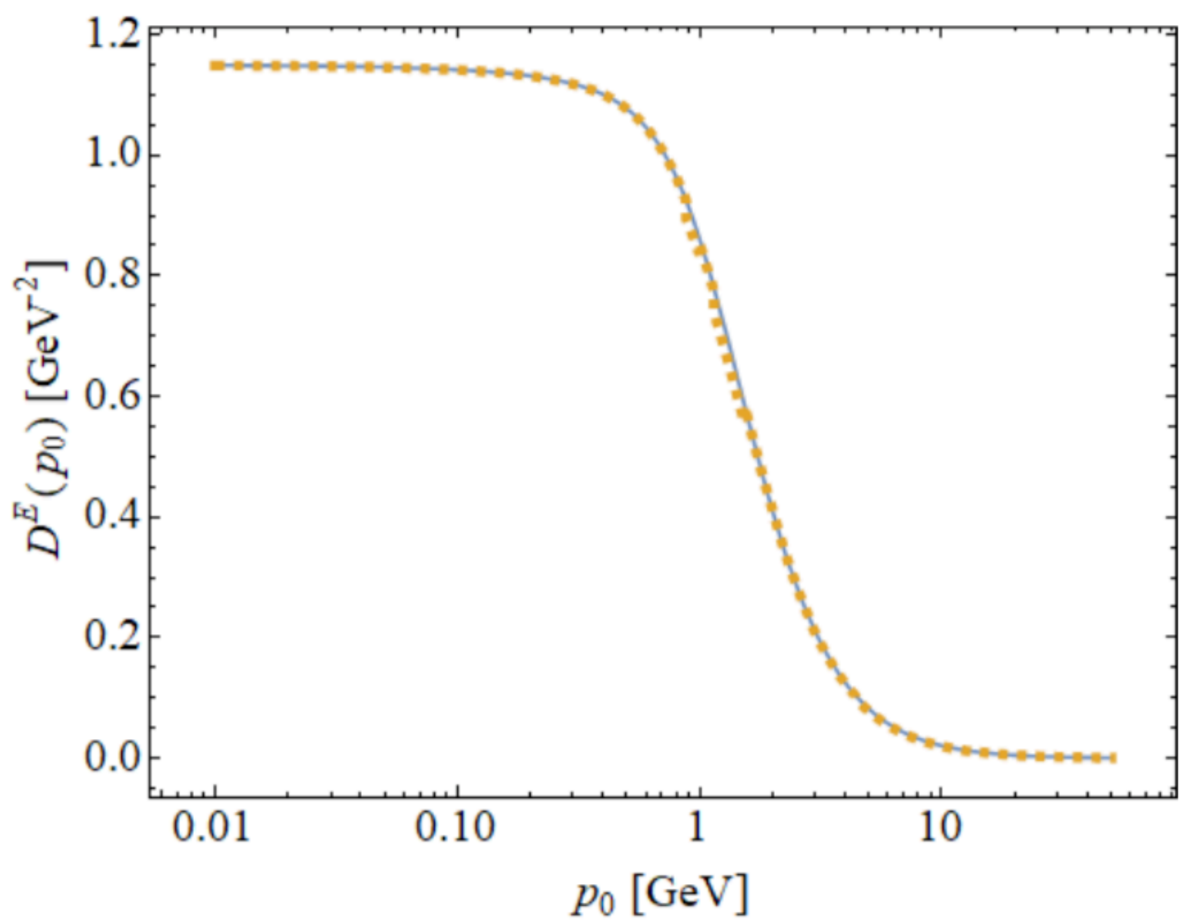
- **Spectral function**
  Obtained as:

$$\rho(\omega) = 2 \ \mathrm{Im} \ D(p_0 \to -i(\omega + i0^+)) \qquad D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2} + \sum_{j=1}^{n} \frac{Z_j}{p_0^2 - z_j} + \mathrm{h.c.}$$

- **Reconstructed spectral function**

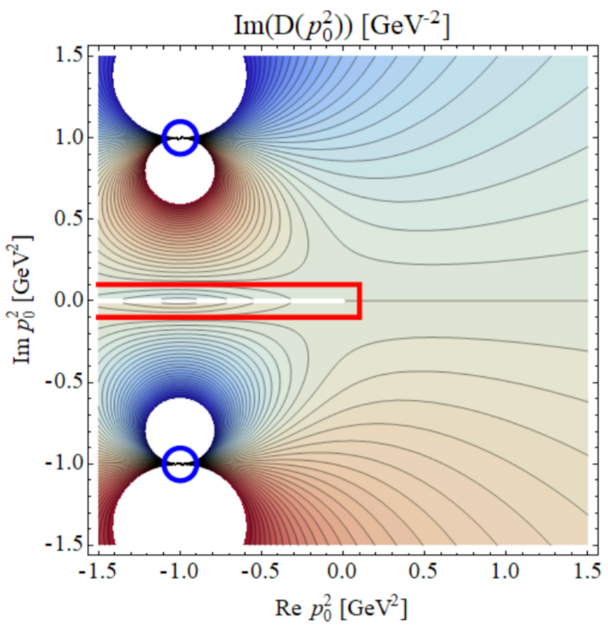


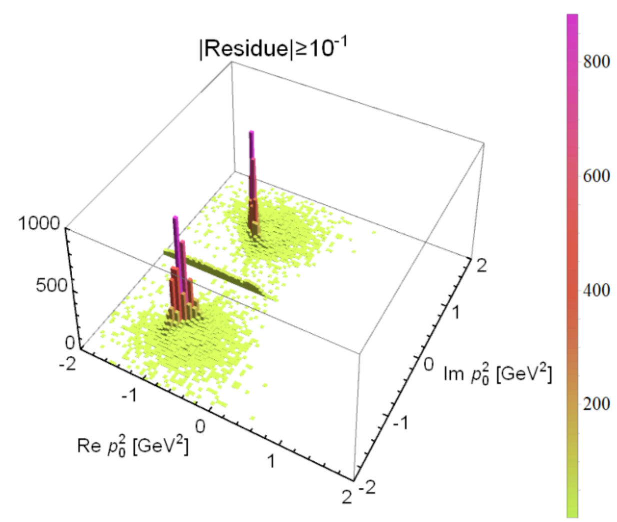
- **Reconstructed propagator**

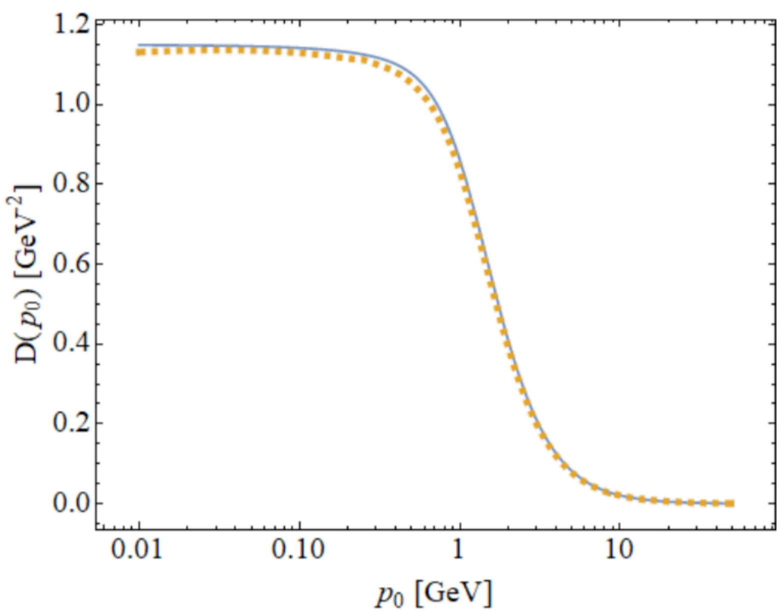


Adding noise

- **Spice up life with some noise!**
  Set $D(p_{0i}) \to D(p_{0i})(1+\varepsilon r_i)$ with $\varepsilon = 10^{-3}$, r_i a random number drawn from a normal distribution with zero mean and unit standard deviation

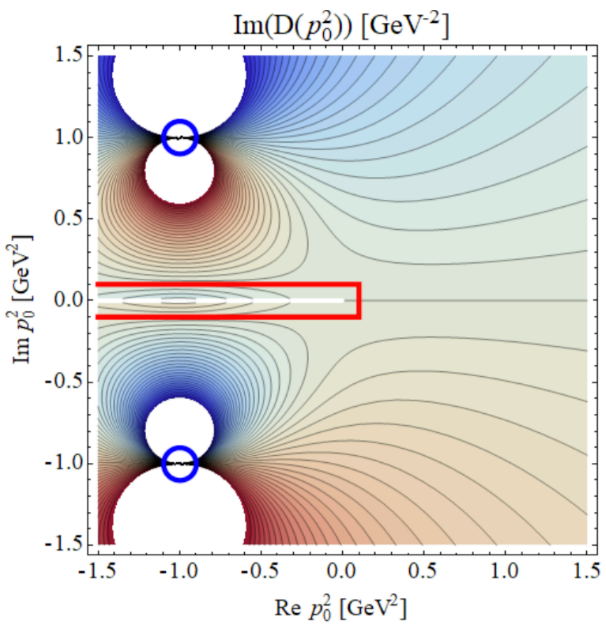


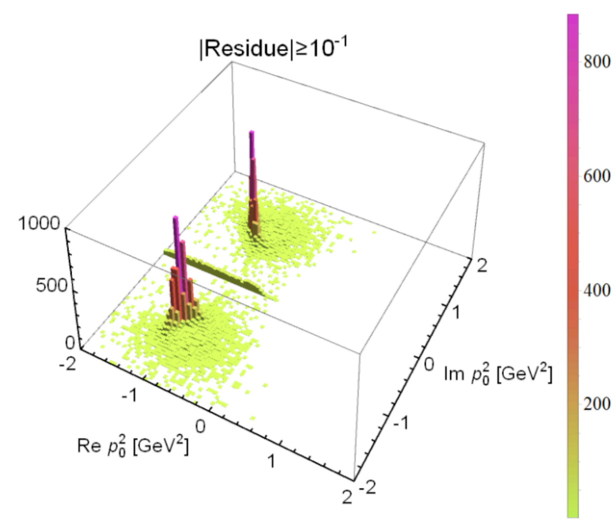


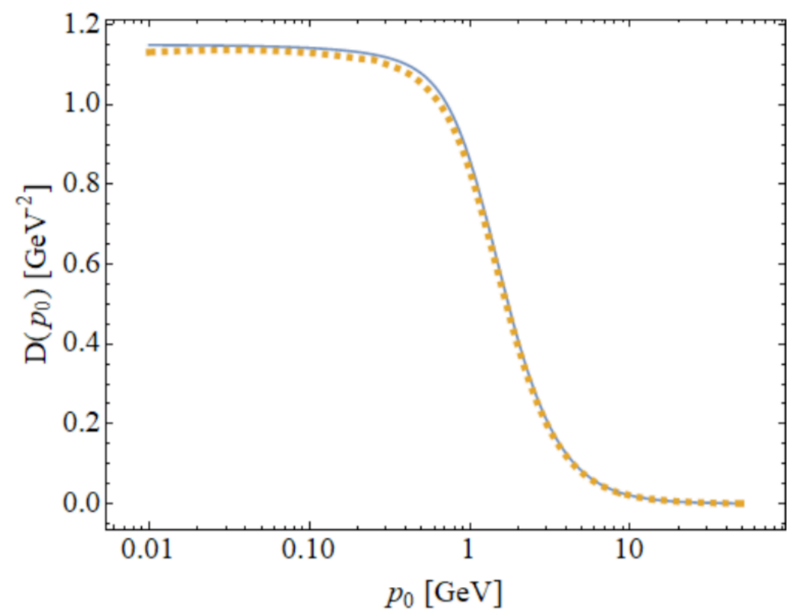


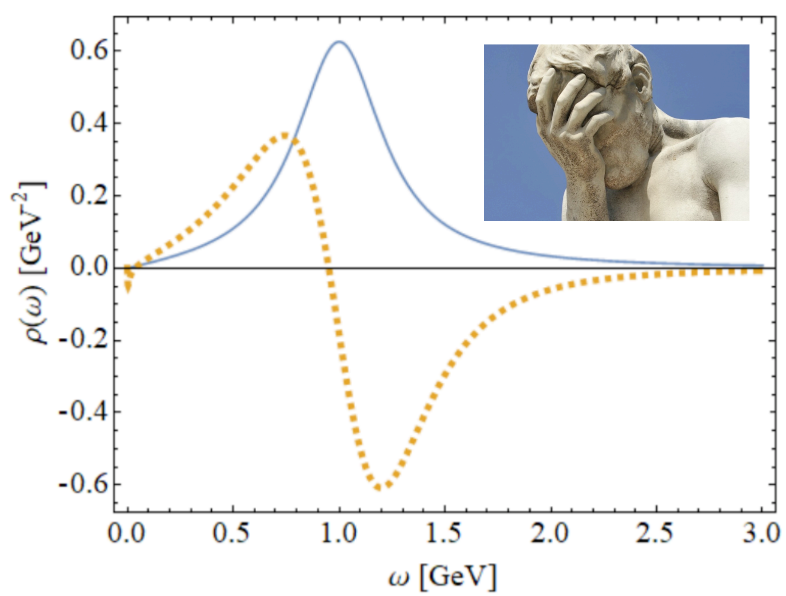
Adding noise

- **Spice up life with some noise!**
  Set $D(p_{0i}) \to D(p_{0i})(1 + \varepsilon r_i)$ with $\varepsilon = 10^{-3}$, r_i a random number drawn from a normal distribution with zero mean and unit standard deviation









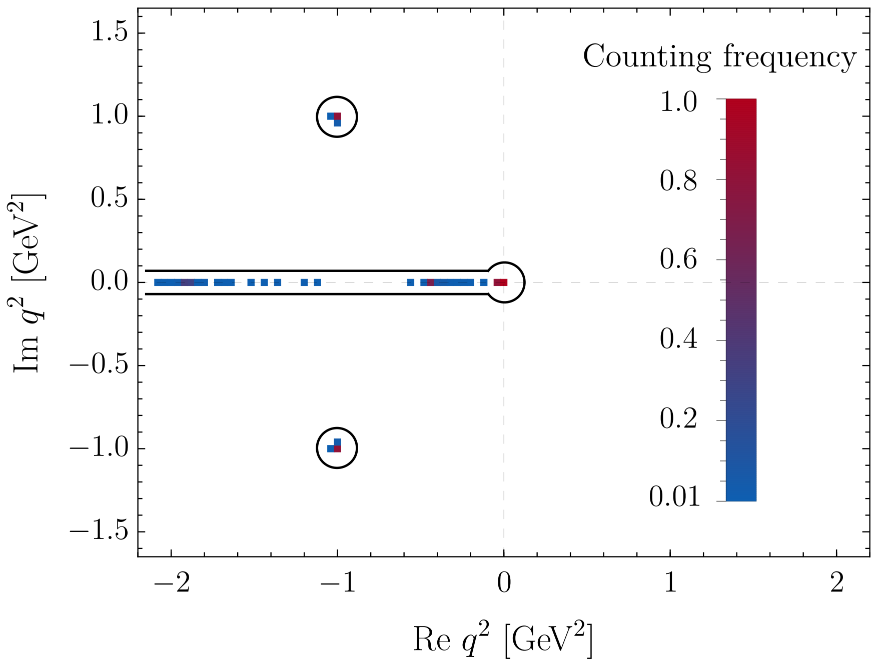
Improving SPM

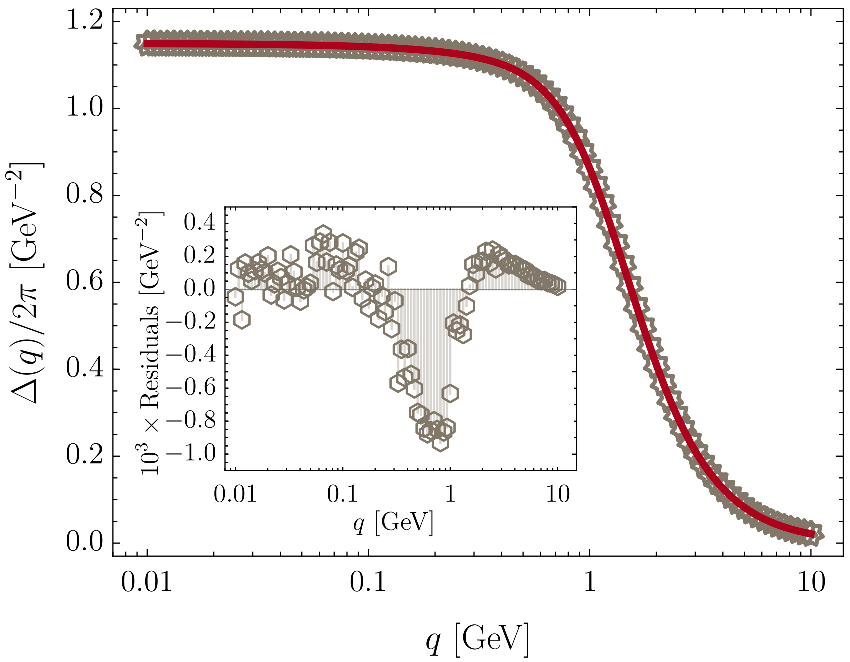
- **Improved SPM algorithm**
  requires the following steps
  DB, Tripolt 1904.08172

1. Select $N = 50$ points randomly from the set of $M > N$ points $(p_{0i}, D(p_{0i}))$
2. Apply the SPM to this subset of points and construct $C_N(p_0)$
3. Obtain the spectral function as $\rho(\omega) = 2\ \mathrm{Im}\ C_N(p_0 \to -i(\omega + i0^+))$
4. Identify the relevant complex poles and compute $D_{\mathrm{rec}}(p_0)$
5. Calculate the χ^2-deviation of the reconstructed propagator, $\chi^2 = \sum_{i=1}^{M} \frac{[D_{\mathrm{rec}}(p_{0i}) - D(p_{0i})]^2}{D(p_{0i})}$
6. Repeat 1-5 $L = 5000$ times and identify the input point $(p_{0j}, D(p_{0j}))$ that appears most often among the $K = 200$ best subsets, i.e. those with the smallest χ^2
7. Repeat 1-6 but always use the points $(p_{0j}, D(p_{0j}))$ among the $N = 50$ points until all optimal input points have been identified

BW propagator plus poles and noise

- **Improved SPM reconstruction**
  of the BW mock data





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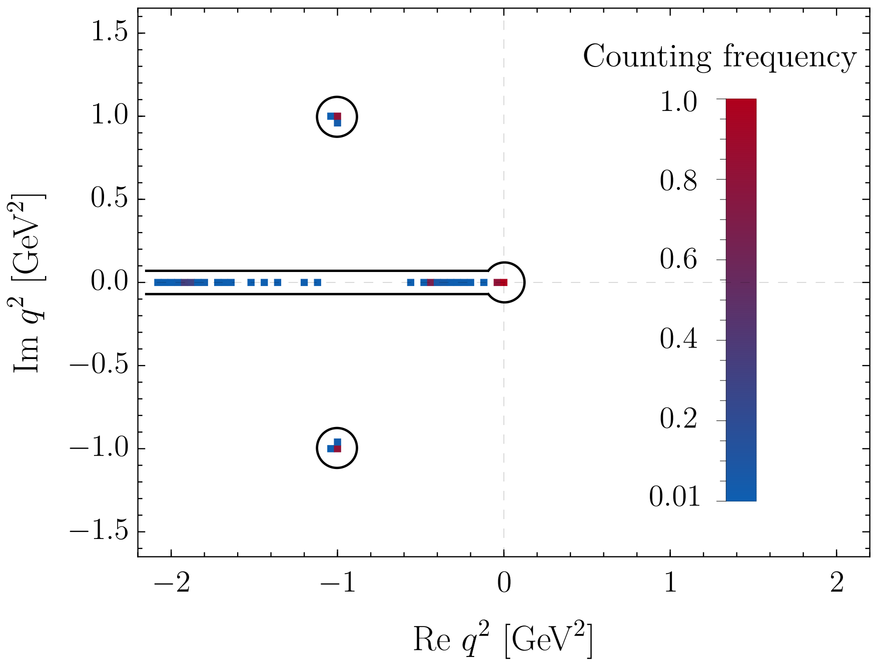

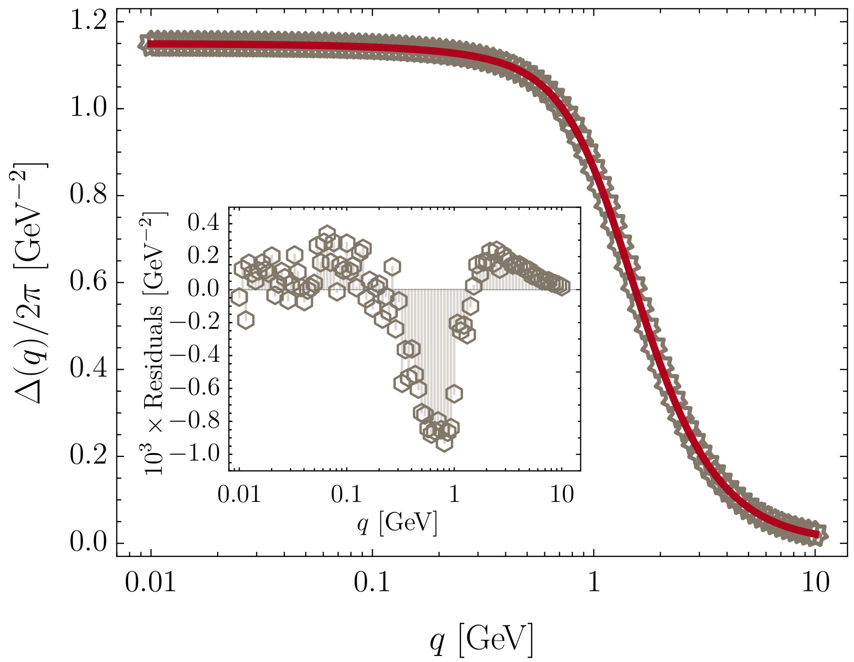

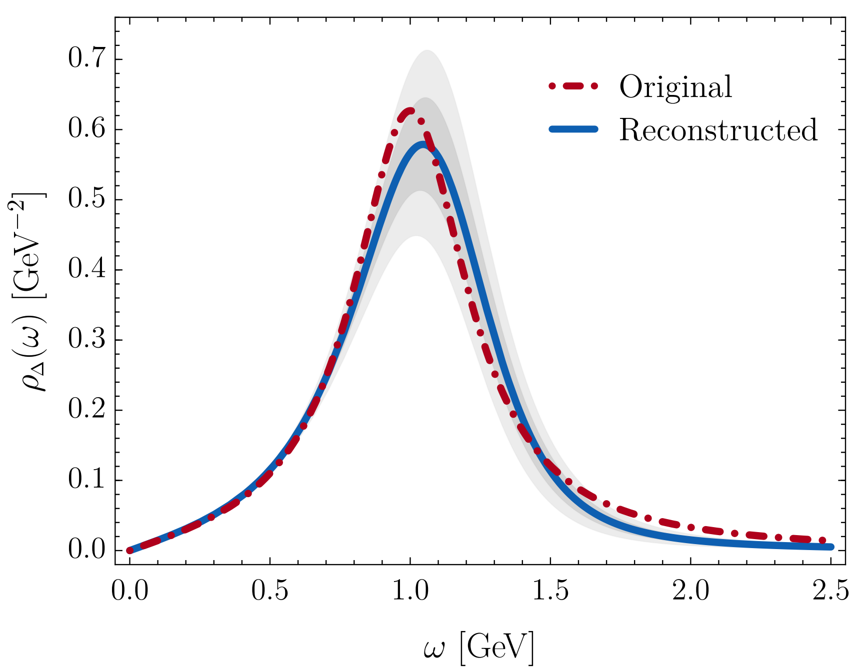

FRG gluon propagator data

- **FRG gluon data**
  have been analyzed by constructing a gluon basis propagator
  A. K. Cyrol, J. M. Pawlowski, A. Rothkopf, N. Wink, SciPost Phys. (2018)

$$\hat{G}^{\mathrm{pole}}_{\mathrm{Ans}}(p_0) = \sum_{k=1}^{N_{\mathrm{ps}}} \prod_{j=1}^{N_{\mathrm{pp}}^{(k)}} \left(\frac{\hat{\mathcal{N}}_k}{(\hat{p}_0 + \hat{\Gamma}_{k,j})^2 + \hat{M}^2_{k,j}} \right)^{\delta_{k,j}}, \quad \hat{G}^{\mathrm{poly}}_{\mathrm{Ans}}(p_0) = \sum_{j=1}^{N_{\mathrm{poly}}} \hat{a}_k \left(\hat{p}_0^2\right)^{\frac{j}{2}}$$

$$\hat{G}^{\mathrm{asy}}_{\mathrm{Ans}}(p_0) = (\hat{p}_0^2)^{-1-2\alpha} \left[\log\left(1 + \frac{\hat{p}_0^2}{\hat{\lambda}^2}\right)\right]^{-1-\beta}, \quad G_{\mathrm{Ans}}(p_0) = \mathcal{K} \hat{G}^{\mathrm{pole}}_{\mathrm{Ans}}(p_0) \hat{G}^{\mathrm{poly}}_{\mathrm{Ans}}(p_0) \hat{G}^{\mathrm{asy}}_{\mathrm{Ans}}(p_0)$$

  - **Optimize the Kallén–Lehmann**
    to estimate the parameters

$\hat{\mathcal{N}}_1$ 1.33678	α -0.428714	β -0.777213	$\hat{\lambda}$ 1.75049		
$\hat{a}_1$ 0.454024	$\hat{a}_2$ 0.241017	$\hat{a}_3$ 3.10257	$\hat{a}_4$ -1.30804	$\hat{a}_5$ 0.63701	
$\hat{\Gamma}_{1,1}$ 0.100169	$\hat{\Gamma}_{1,2}$ 0.100141	$\hat{\Gamma}_{1,3}$ 2.36445	$\hat{\Gamma}_{1,4}$ 1.5564	$\hat{\Gamma}_{1,5}$ 1.22013	$\hat{\Gamma}_{1,6}$ 1.15102
$\hat{M}_{1,1}$ 0.849883	$\hat{M}_{1,2}$ 0.849902	$\hat{M}_{1,3}$ 2.52171	$\hat{M}_{1,4}$ 2.44035	$\hat{M}_{1,5}$ 3.6016	$\hat{M}_{1,6}$ 2.36723
$\delta_{1,1}$ 1.61116	$\delta_{1,2}$ 1.94095	$\delta_{1,3}$ -2.54586	$\delta_{1,4}$ 1.89765	$\delta_{1,5}$ 0.168592	$\delta_{1,6}$ 0.296215

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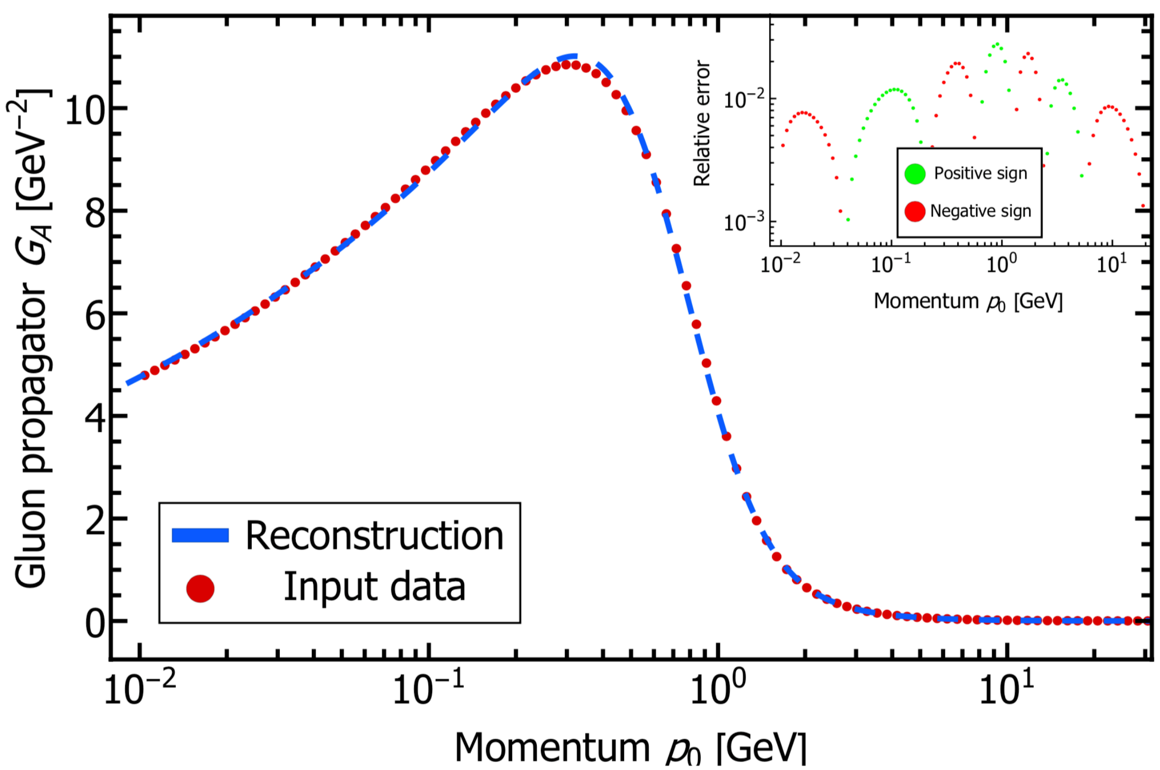

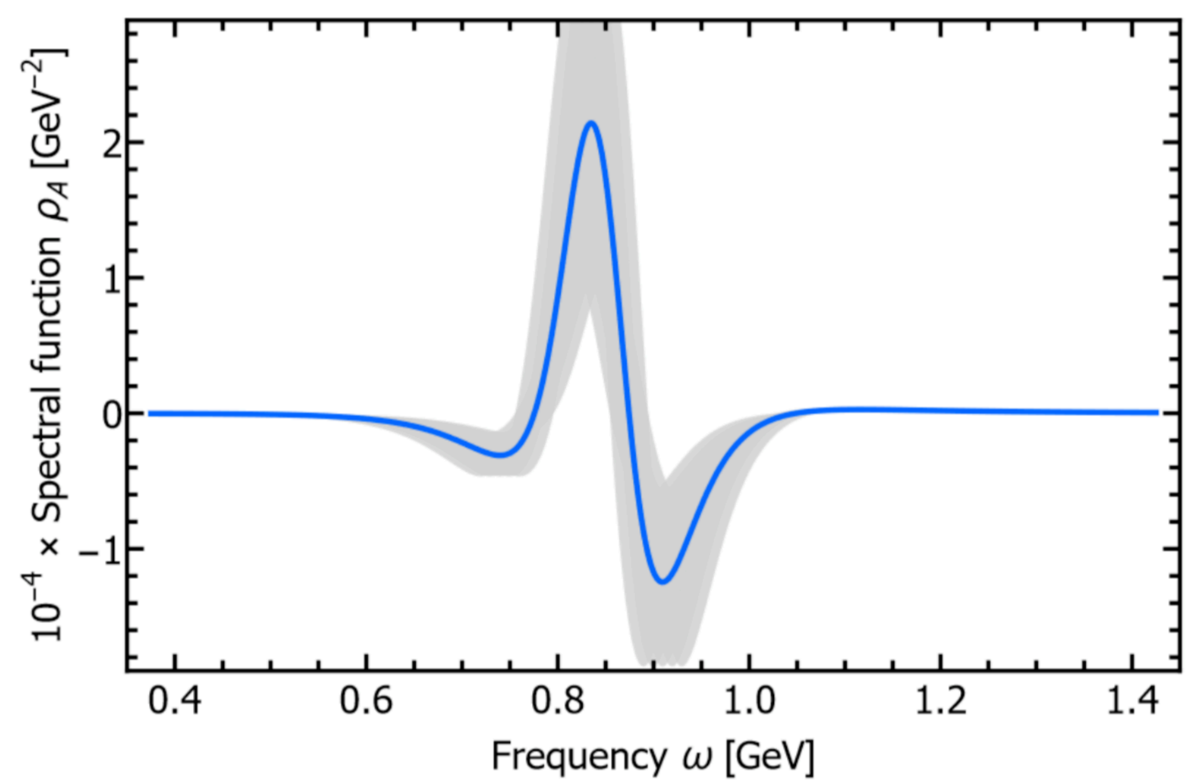

FRG gluon propagator data

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  A. K. Cyrol, J. M. Pawlowski, A. Rothkopf, N. Wink, SciPost Phys. (2018)

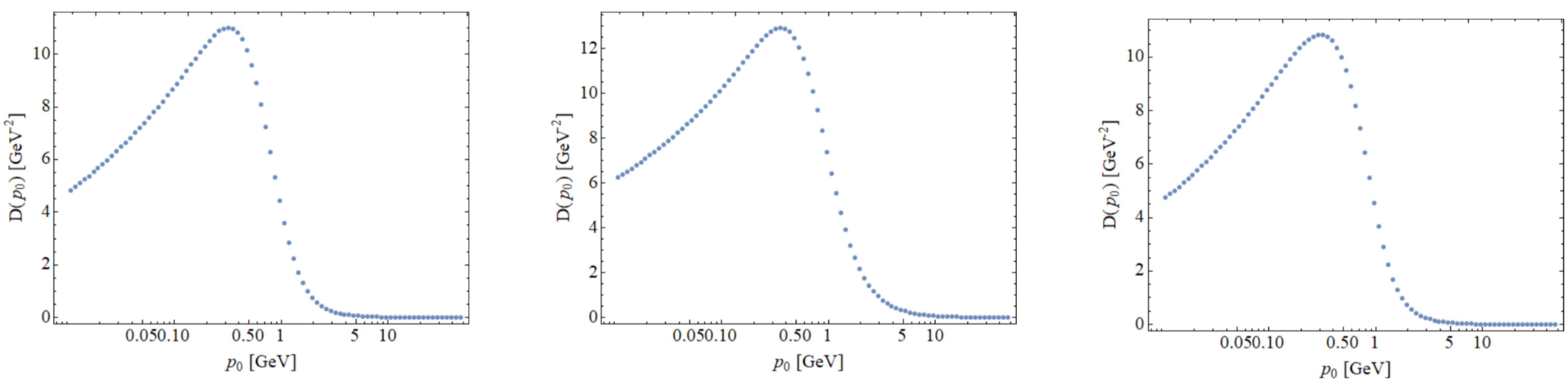
- **Study three data sets**
  with improved SPM

  - **Fit from Cyrol et al.**
    with noise $\varepsilon = 10^{-6}$

  - **Fit from Cyrol et al.**
    with noise and two poles

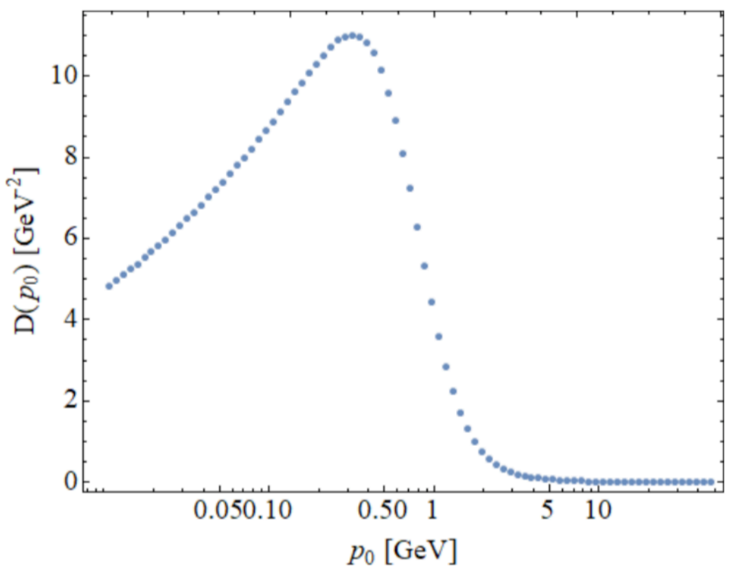
$$G_{\text{Ans}}(p_0) = \mathcal{K}\hat{G}^{\text{pole}}_{\text{Ans}}(p_0)\hat{G}^{\text{poly}}_{\text{Ans}}(p_0)\hat{G}^{\text{asy}}_{\text{Ans}}(p_0) + \frac{3}{p_0^2 - (-0.25+i)} + \frac{3}{p_0^2 - (-0.25-i)}$$

  - **Original data set**

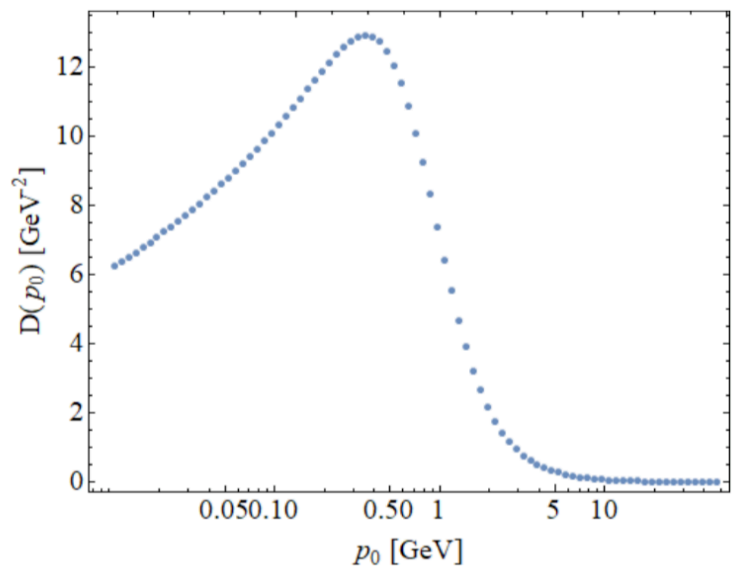


FRG gluon propagator data

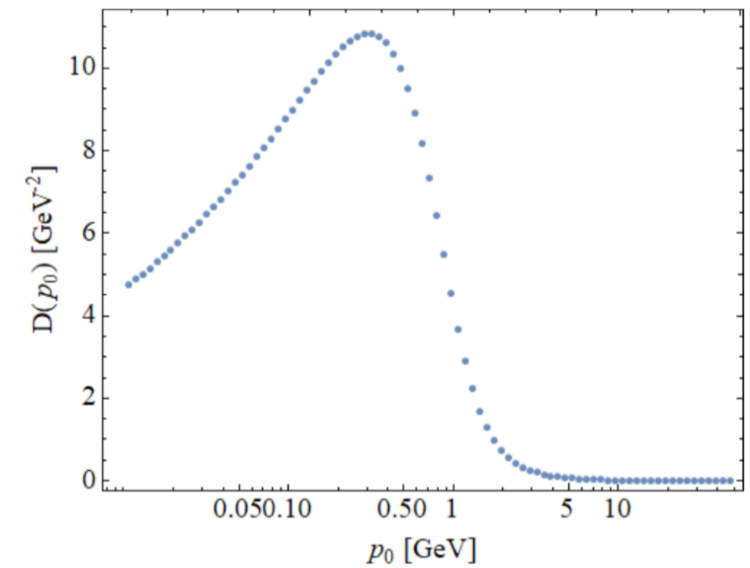
- **Set 1**



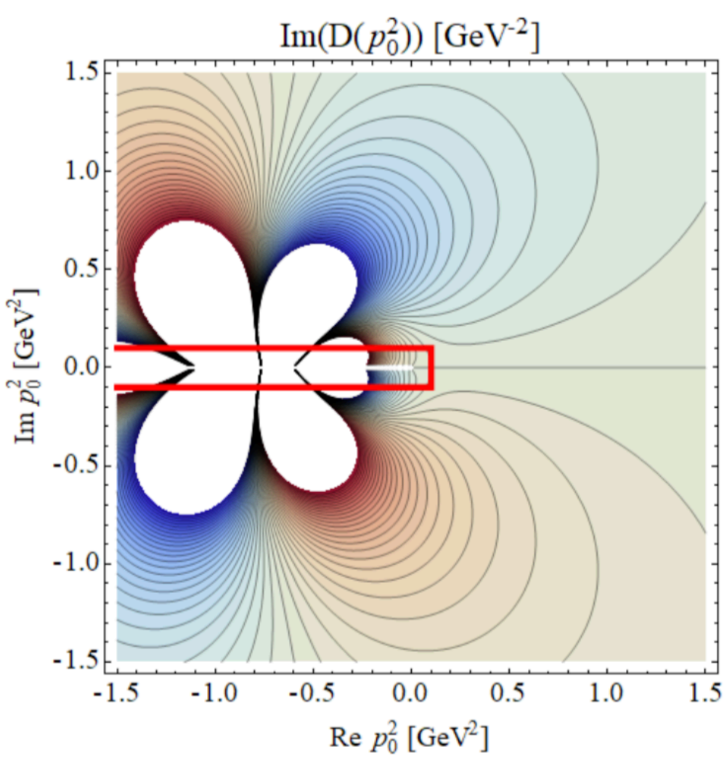
- **Set 2**



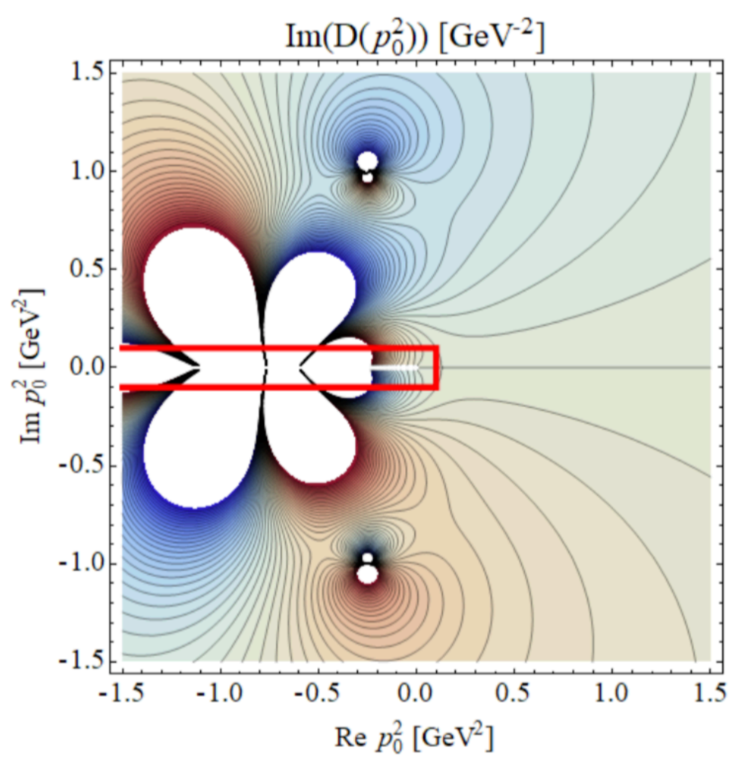
- **Set 3**



- **Exact**



- **Exact**

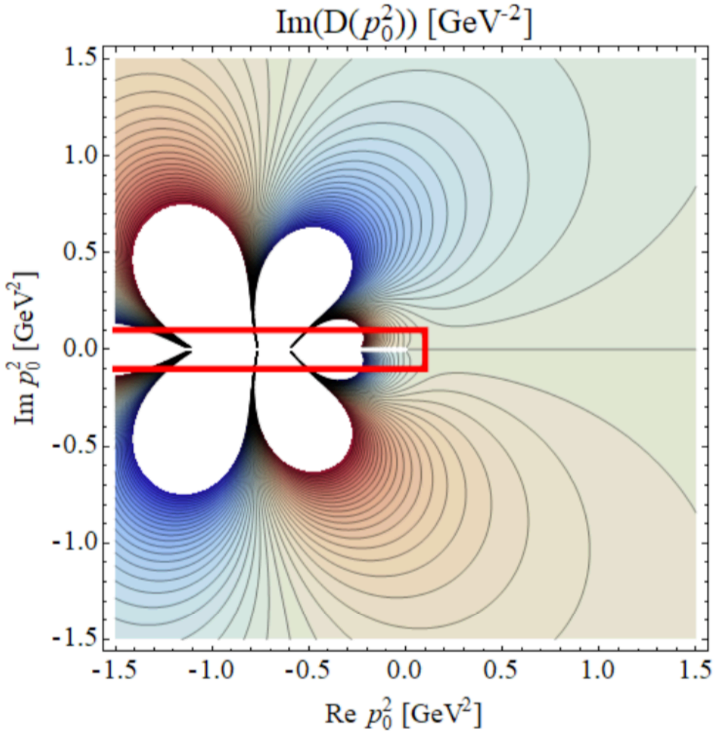


- **Exact**

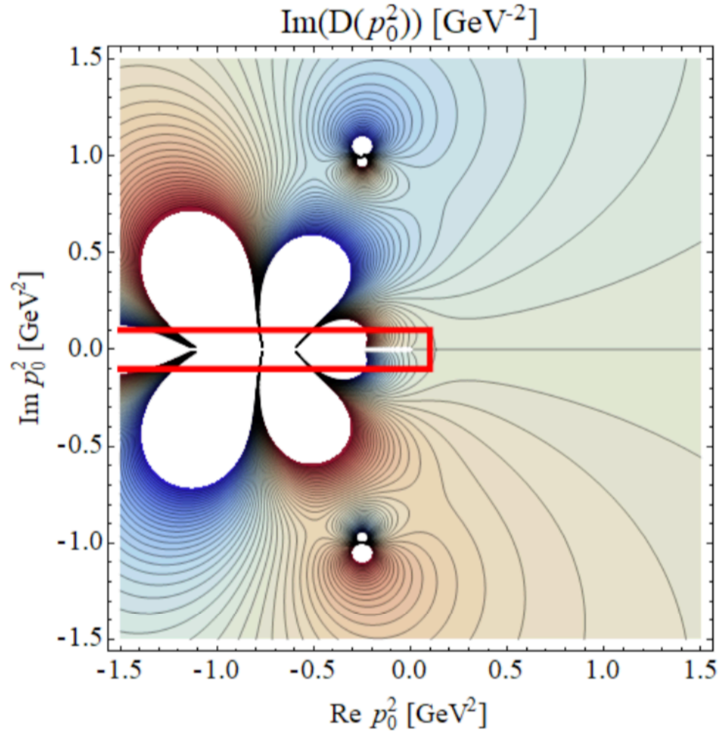
?

FRG gluon propagator data

- **Set 1, exact**



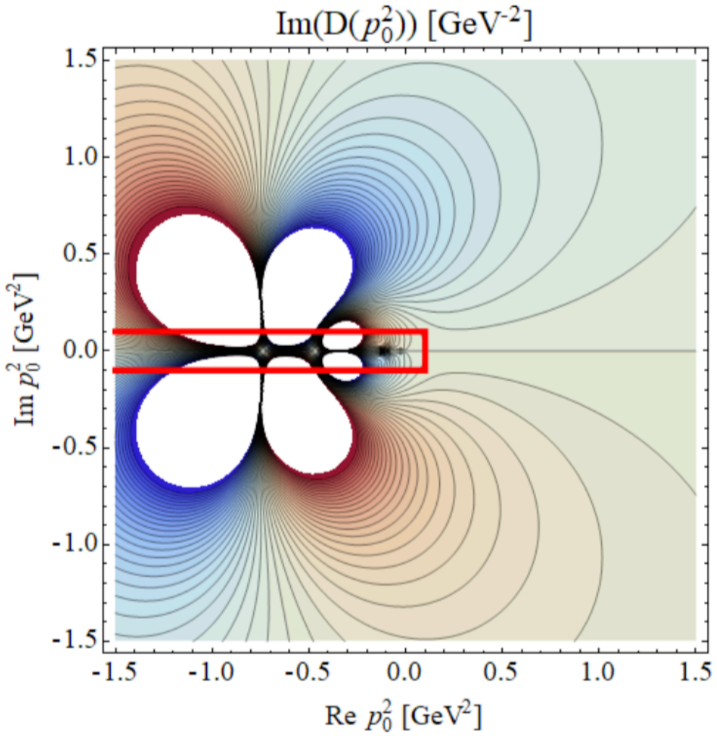
- **Set 2, exact**



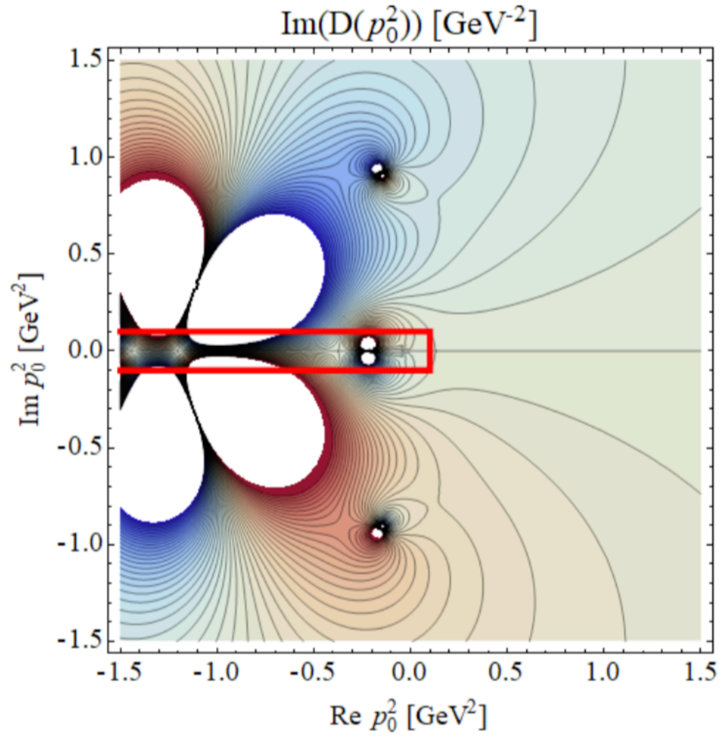
- **Set 3, exact**

?

- **Reconstructed**



- **Reconstructed**



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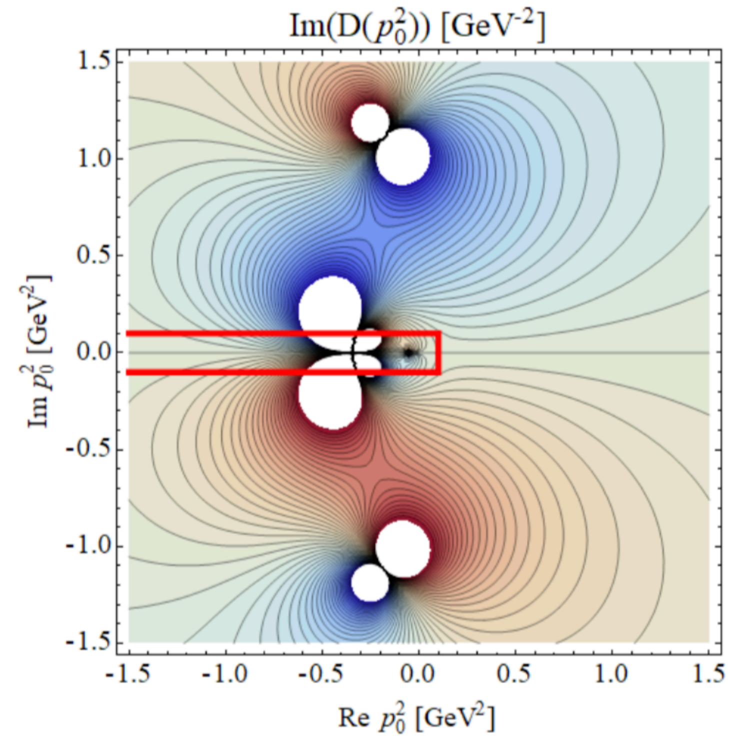

FRG gluon propagator data

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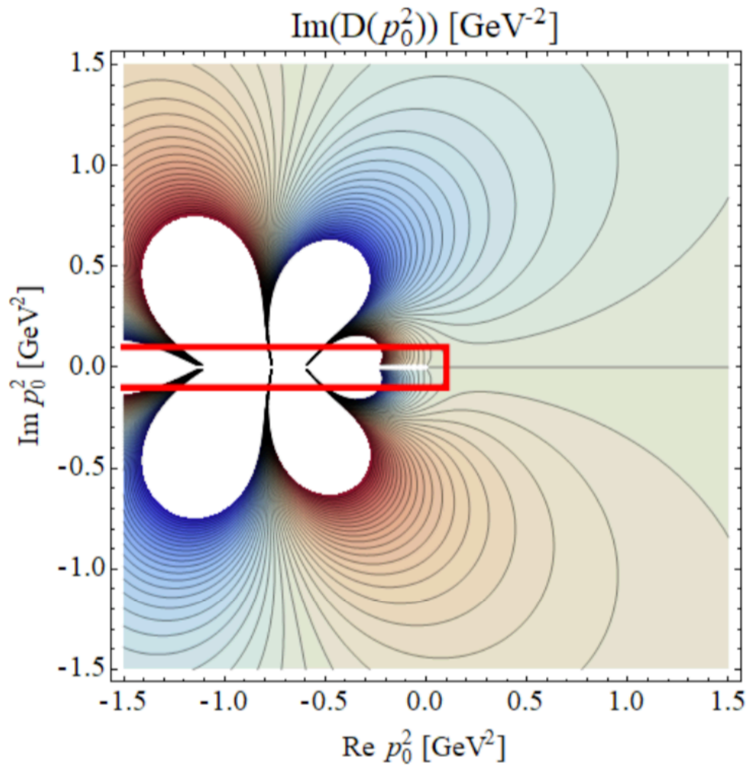
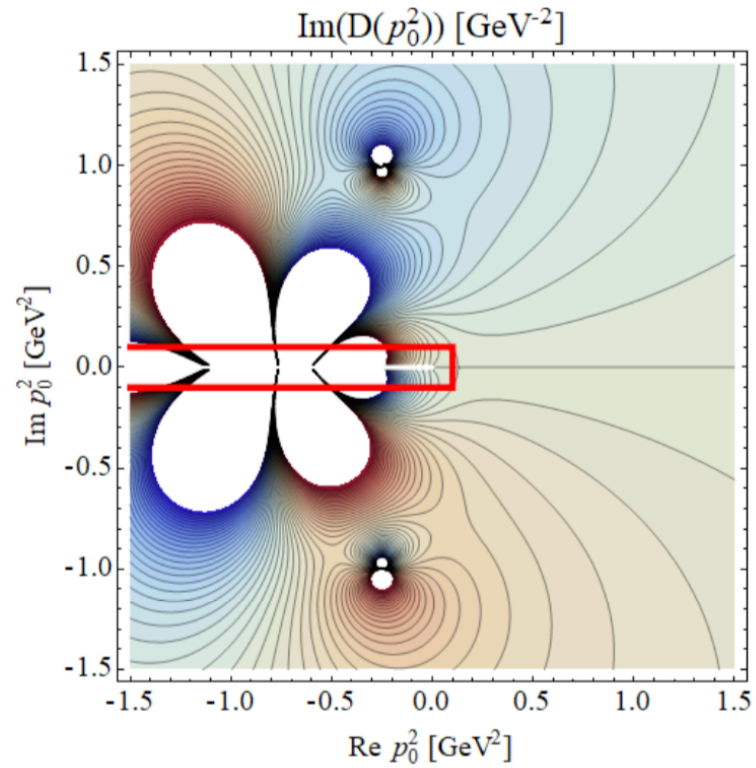


- **Set 2, exact**



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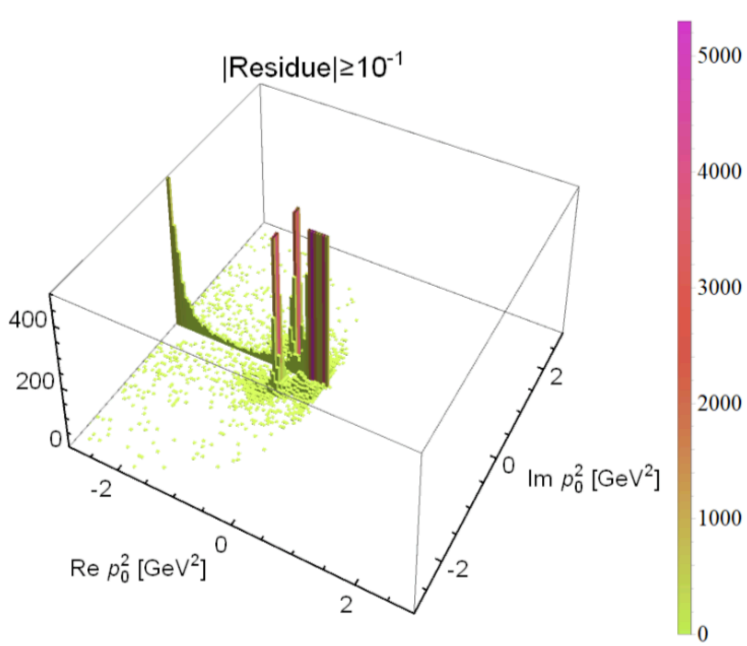
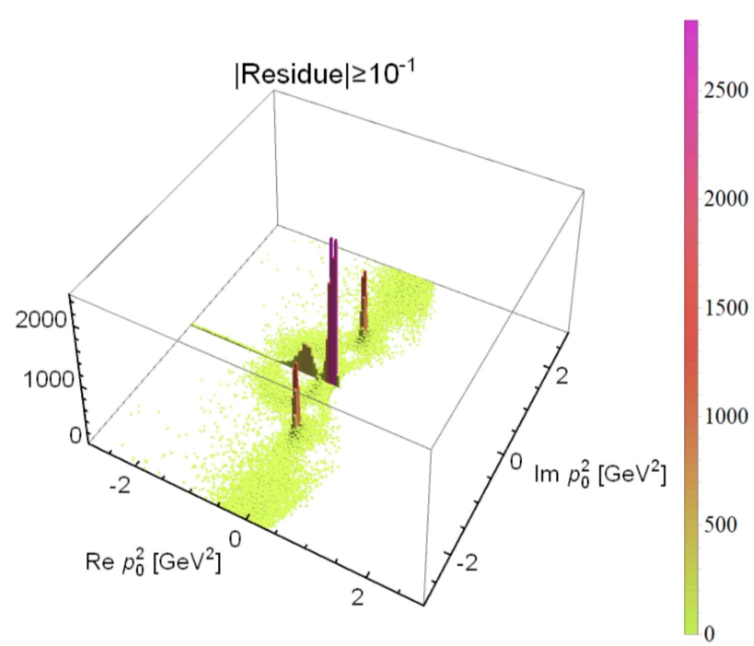


- **Reconstructed**



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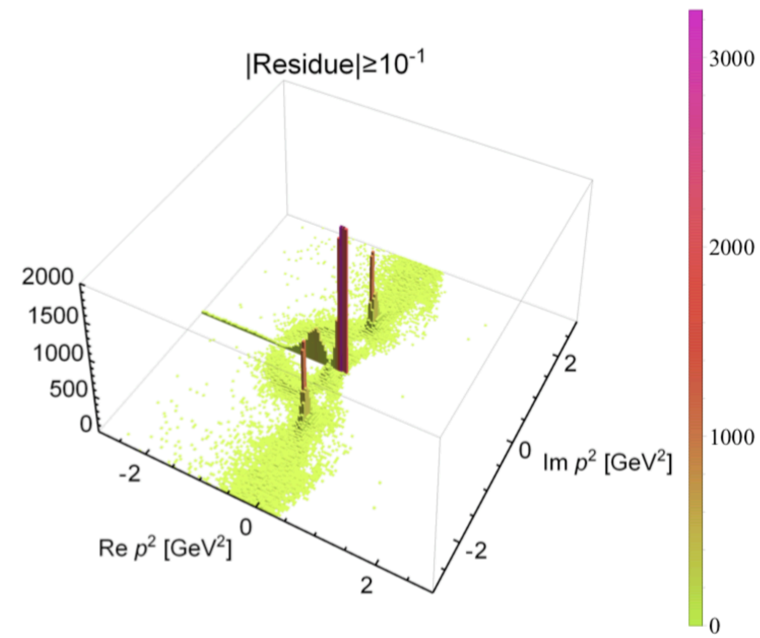
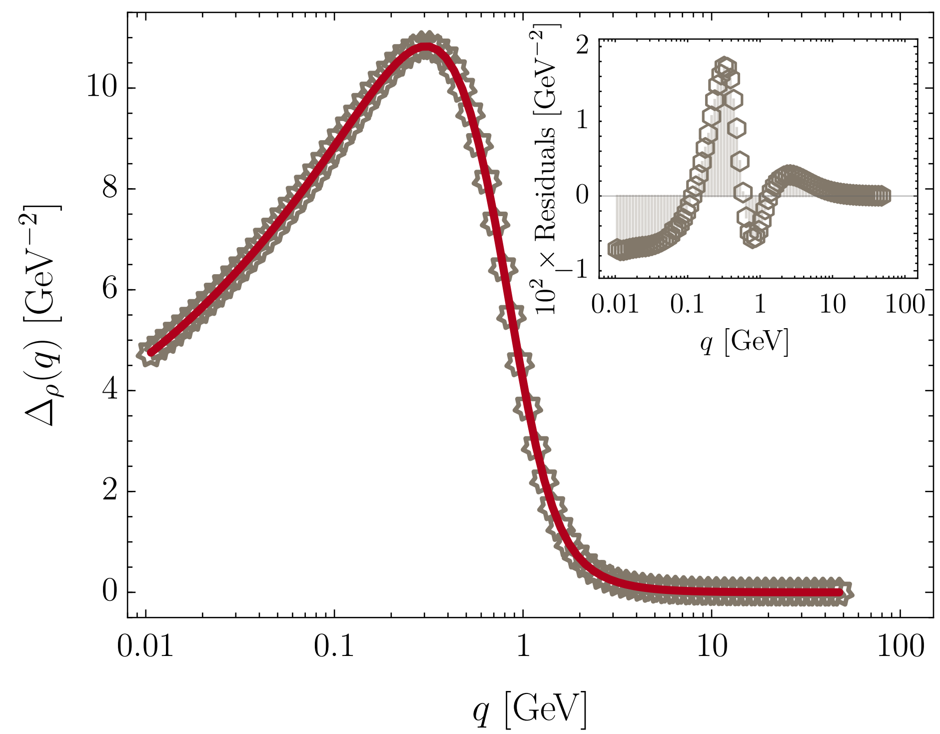
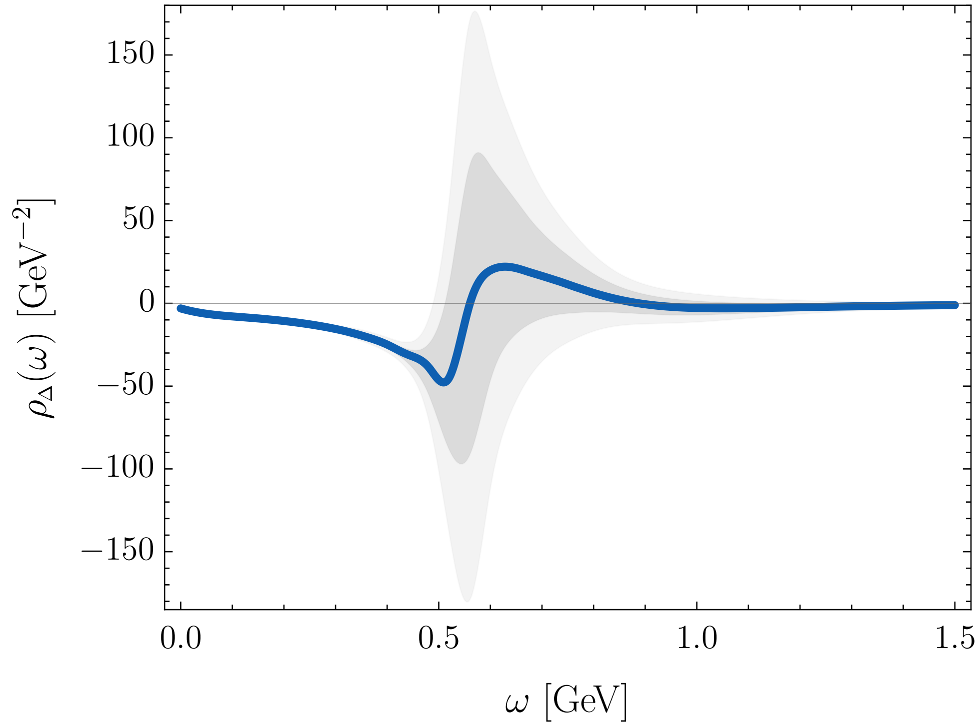
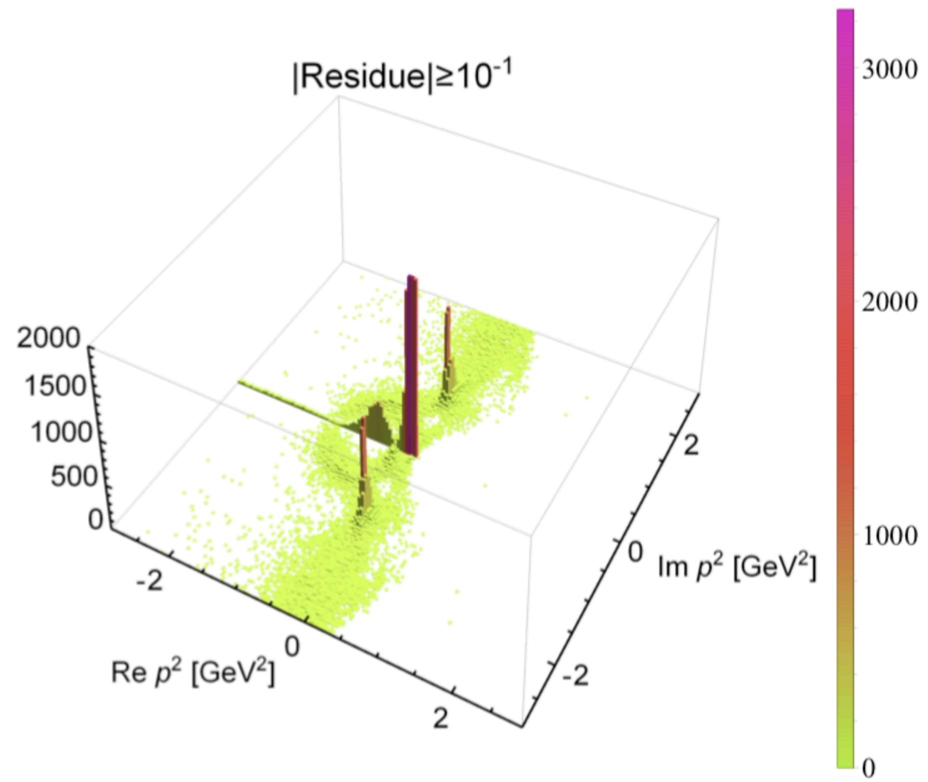


FRG gluon propagator data

- **Improved SPM reconstruction**
  of the FRG data







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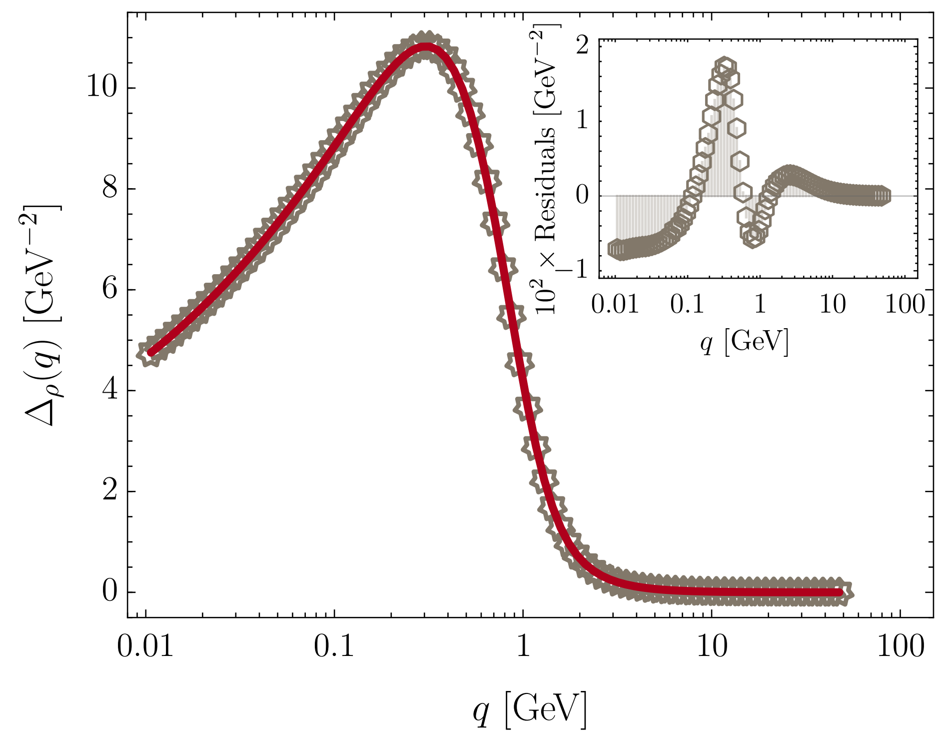
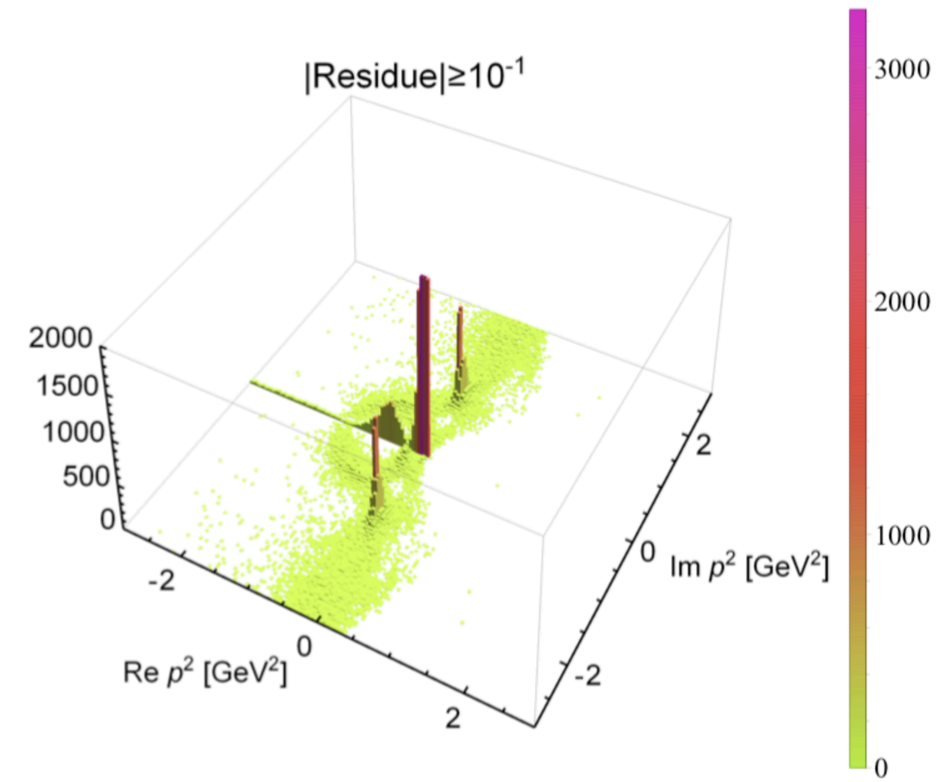
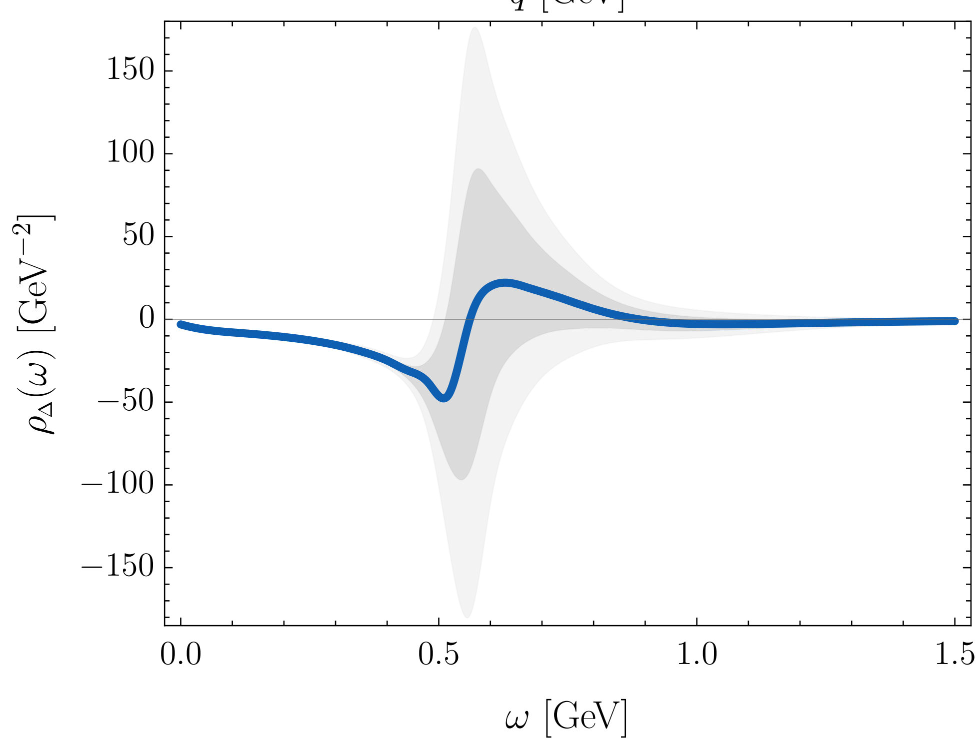
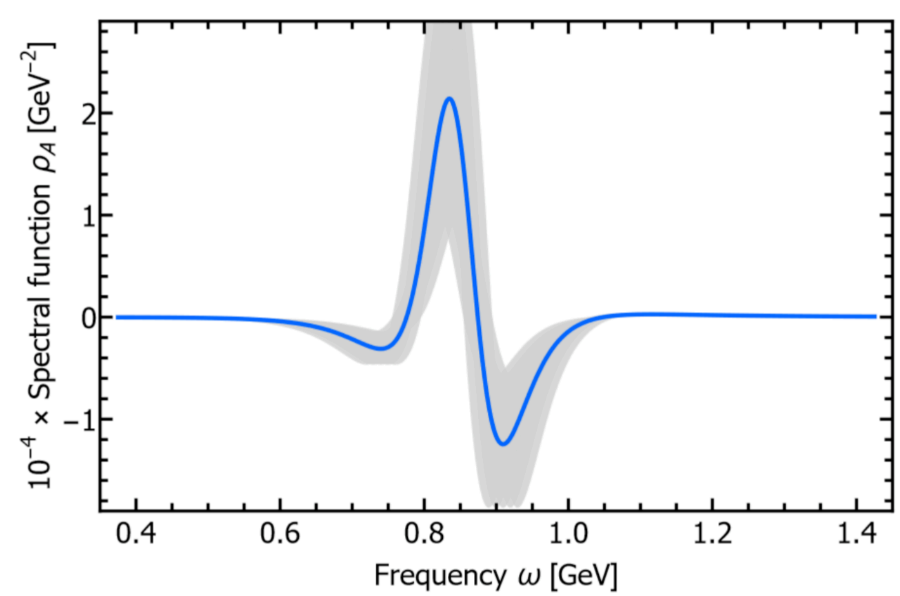


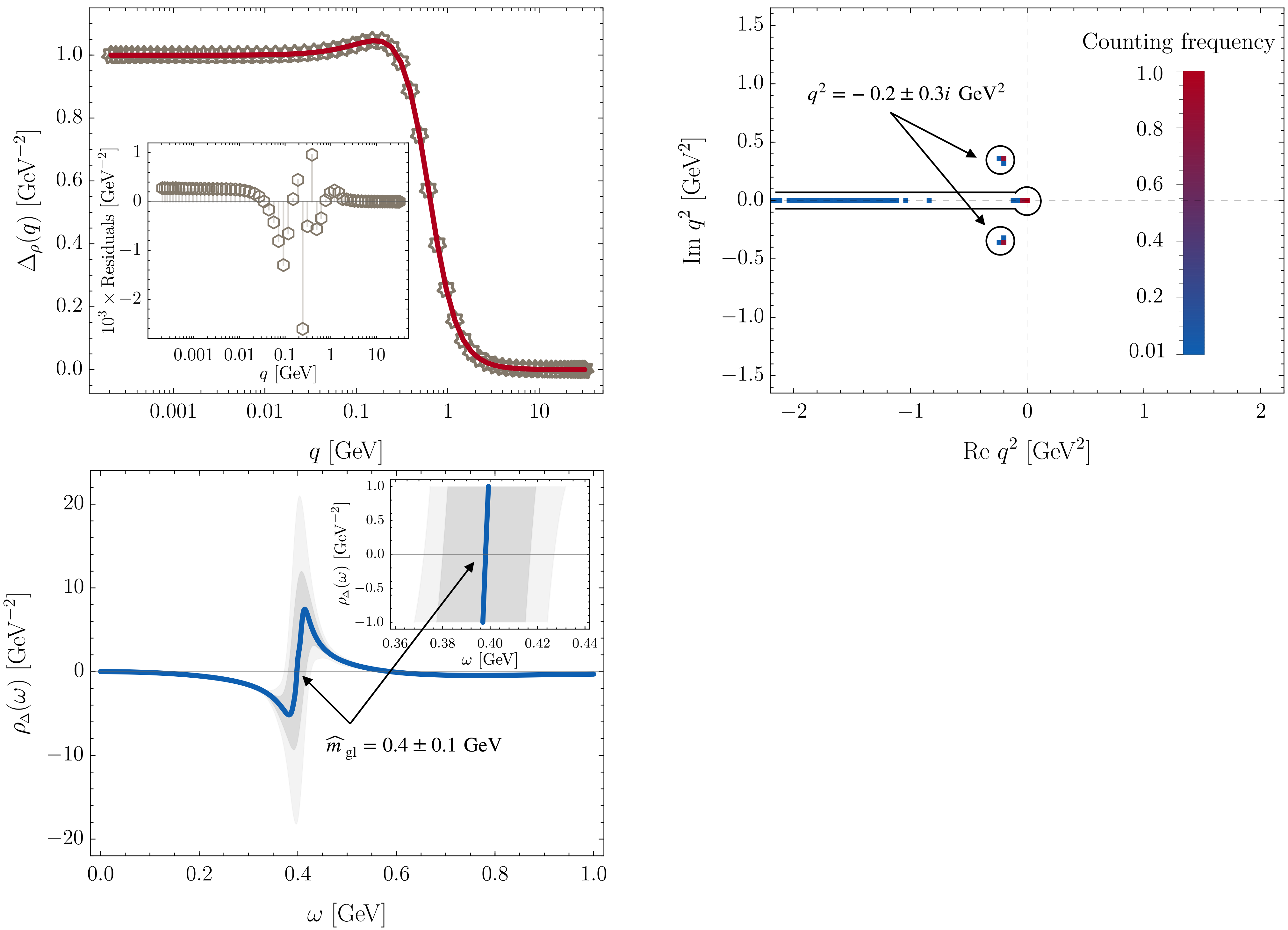






DSE gluon propagator data

- **Improved SPM reconstruction**
  of DSE gluon data
  Strauss, Fischer, Kellermann, PRL (2012)



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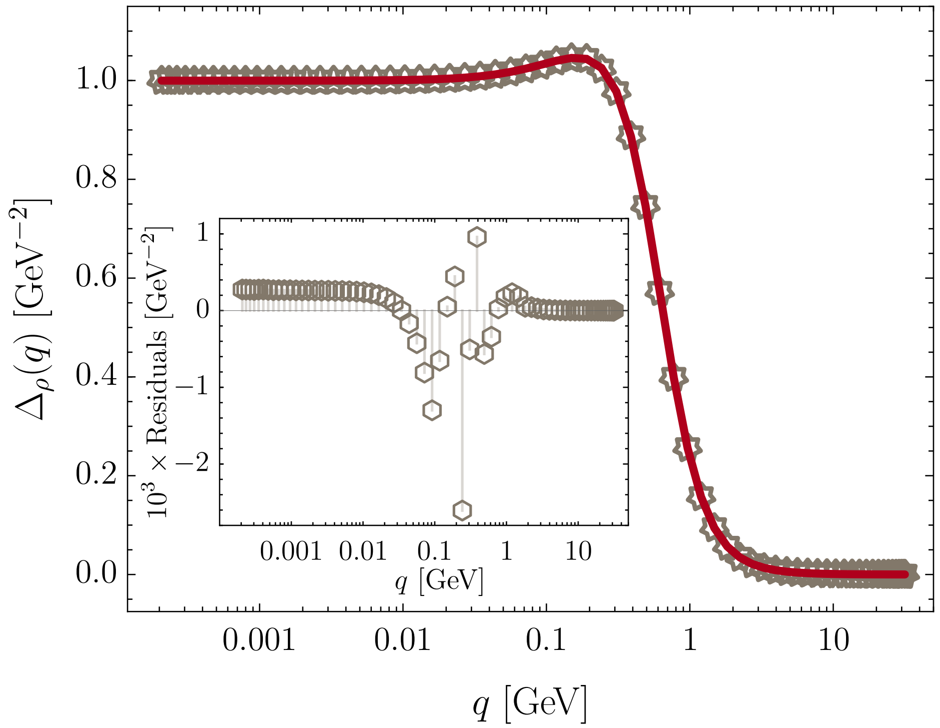

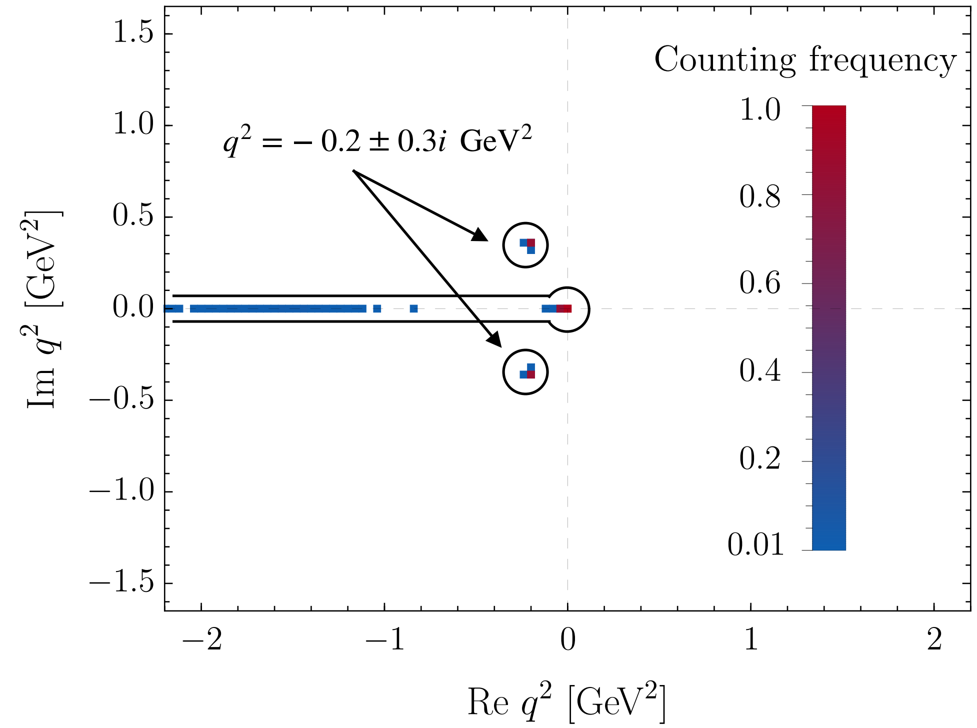

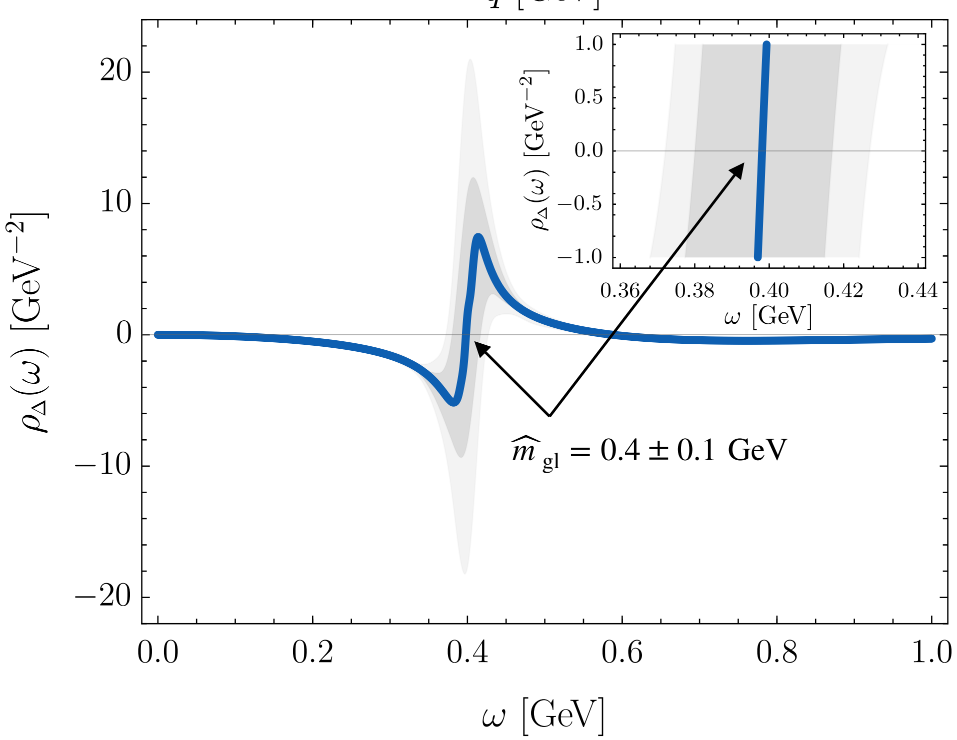

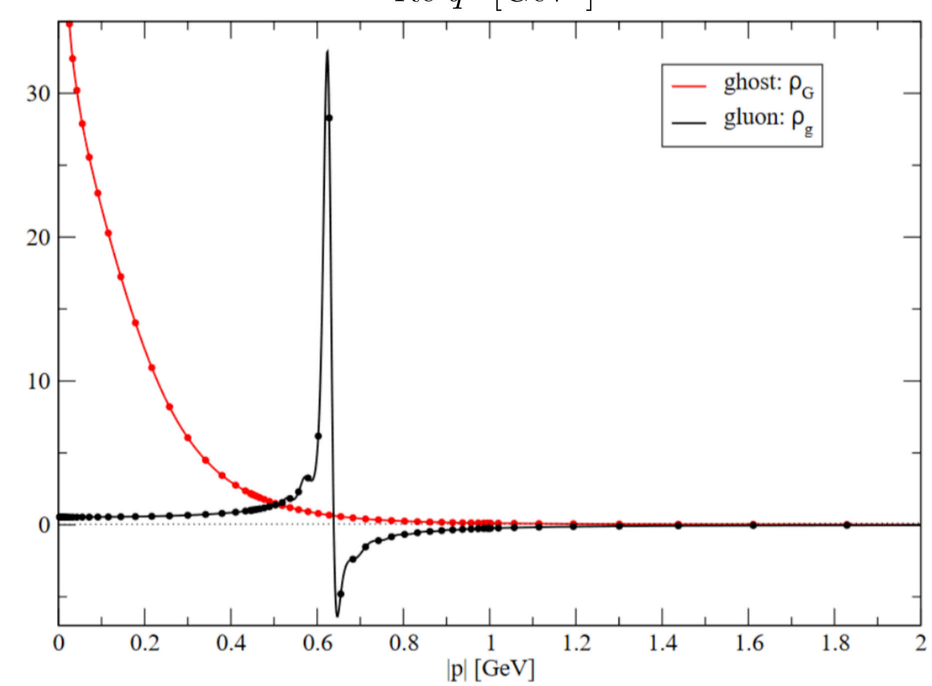

Lattice gluon propagator data

- **Improved SPM reconstruction**
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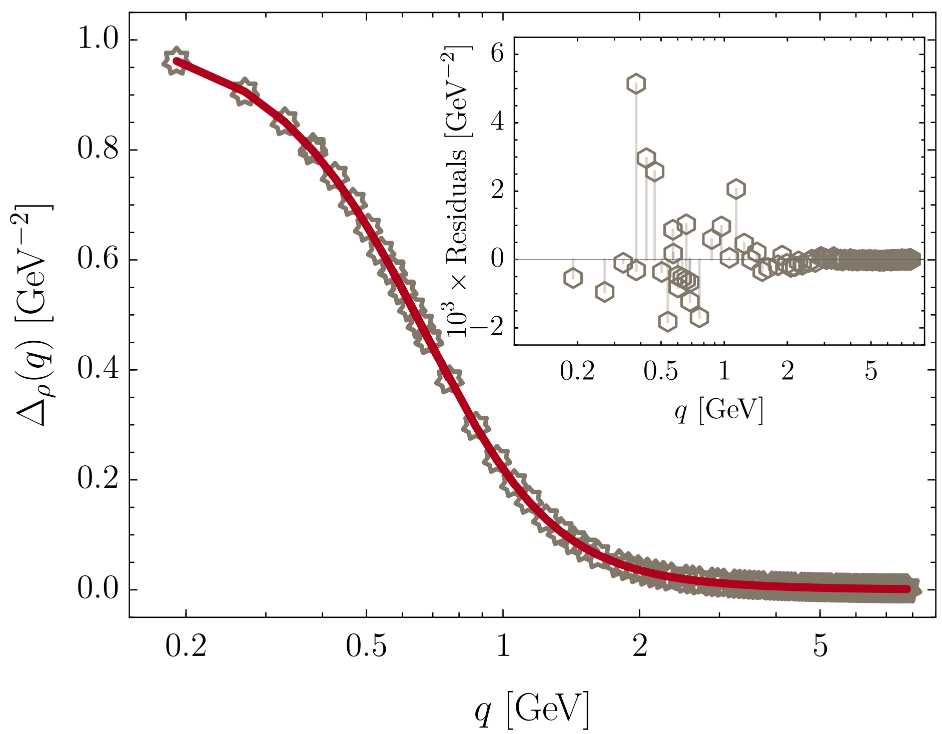
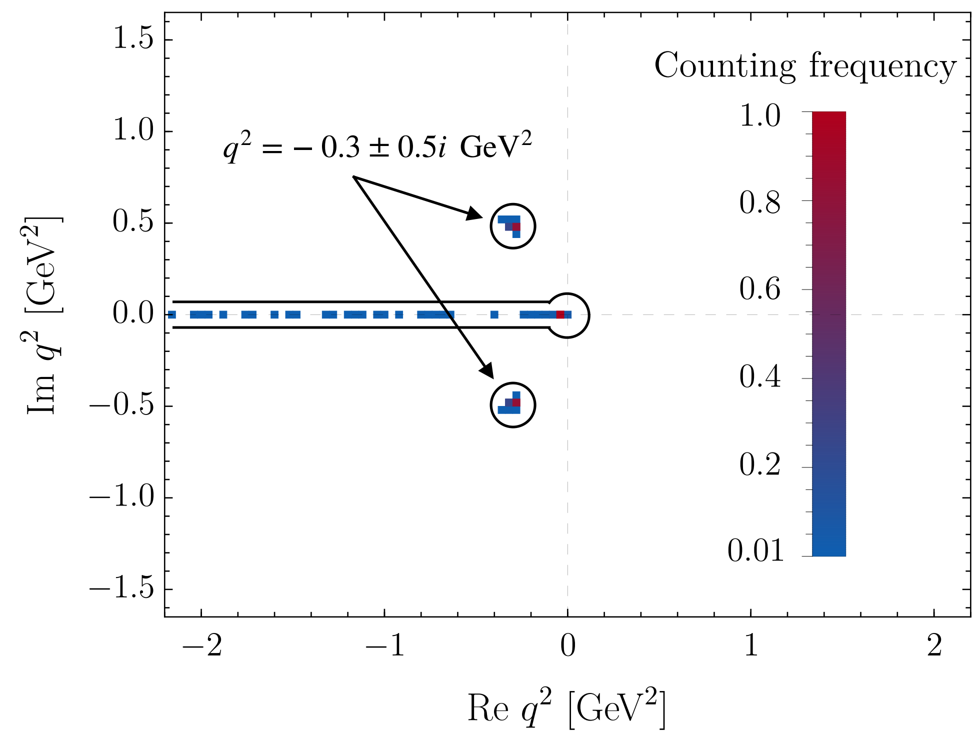
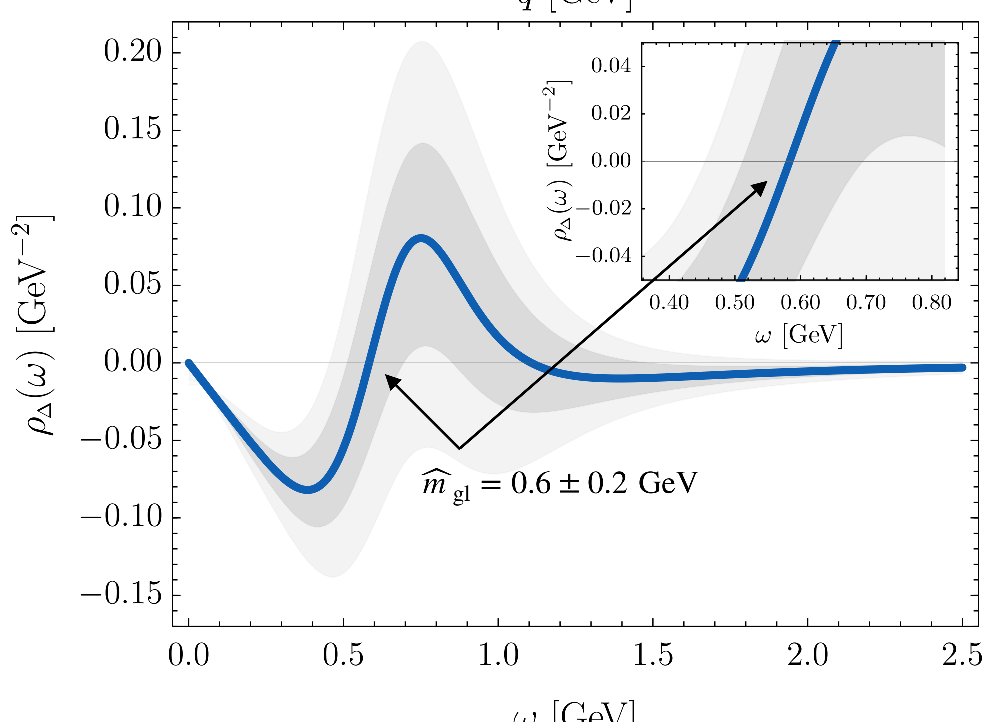
  Duarte, Oliveira, Silva, PRD (2016)







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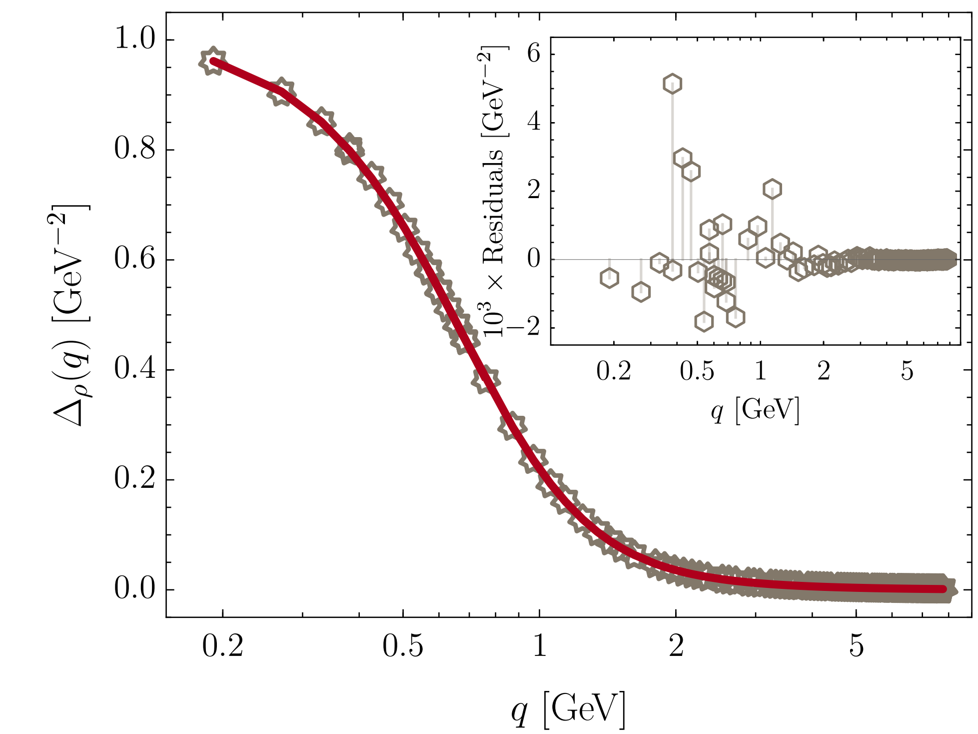
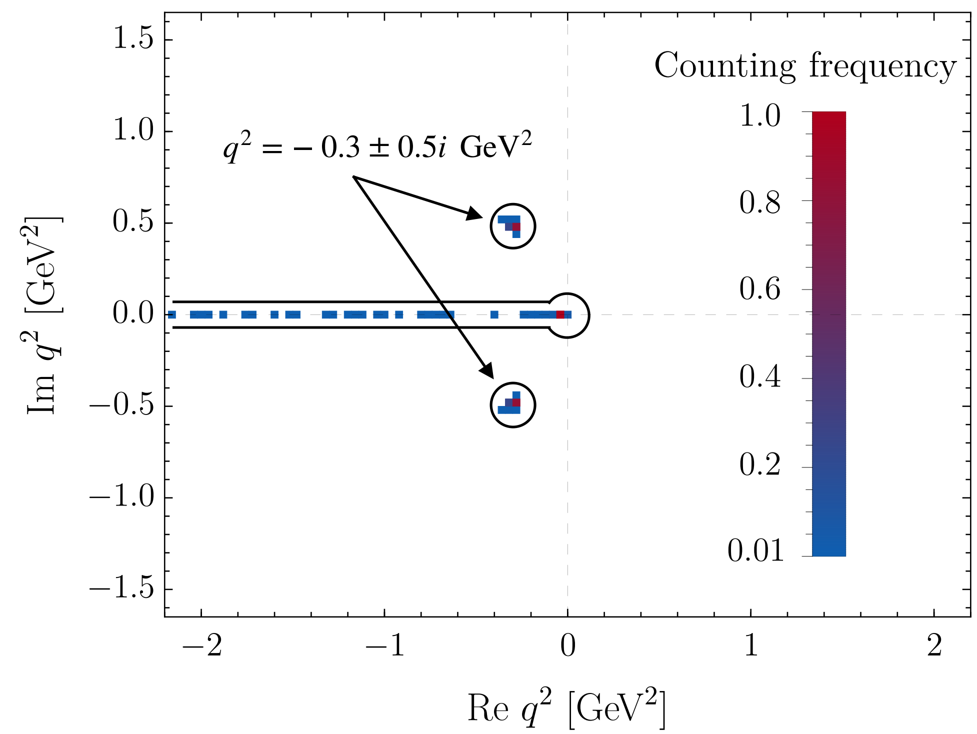
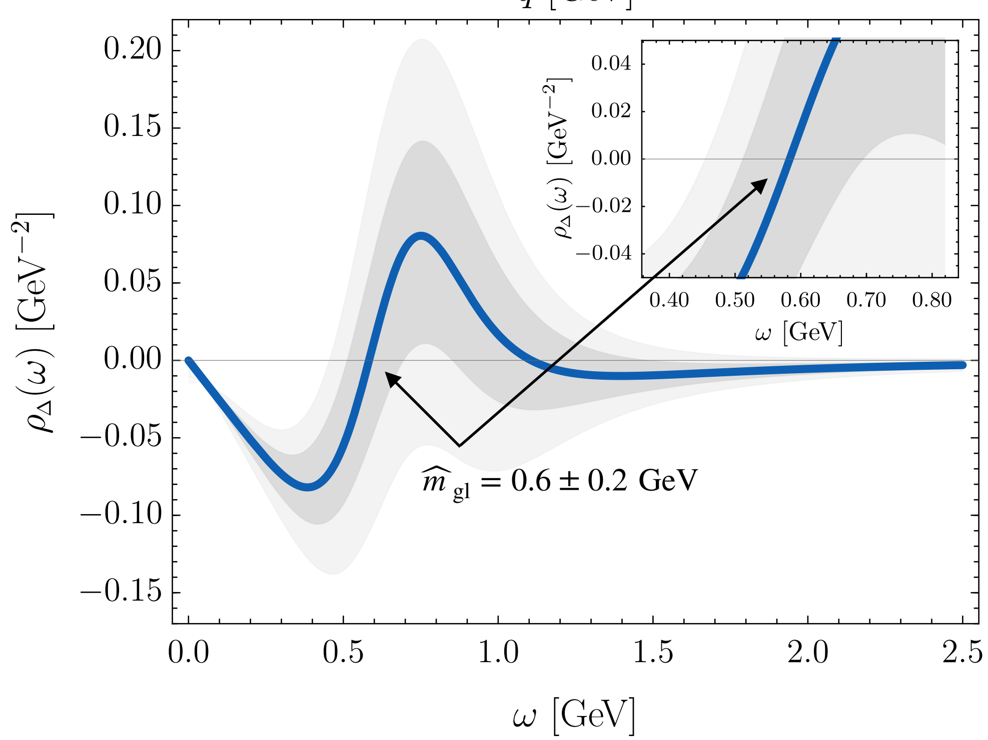
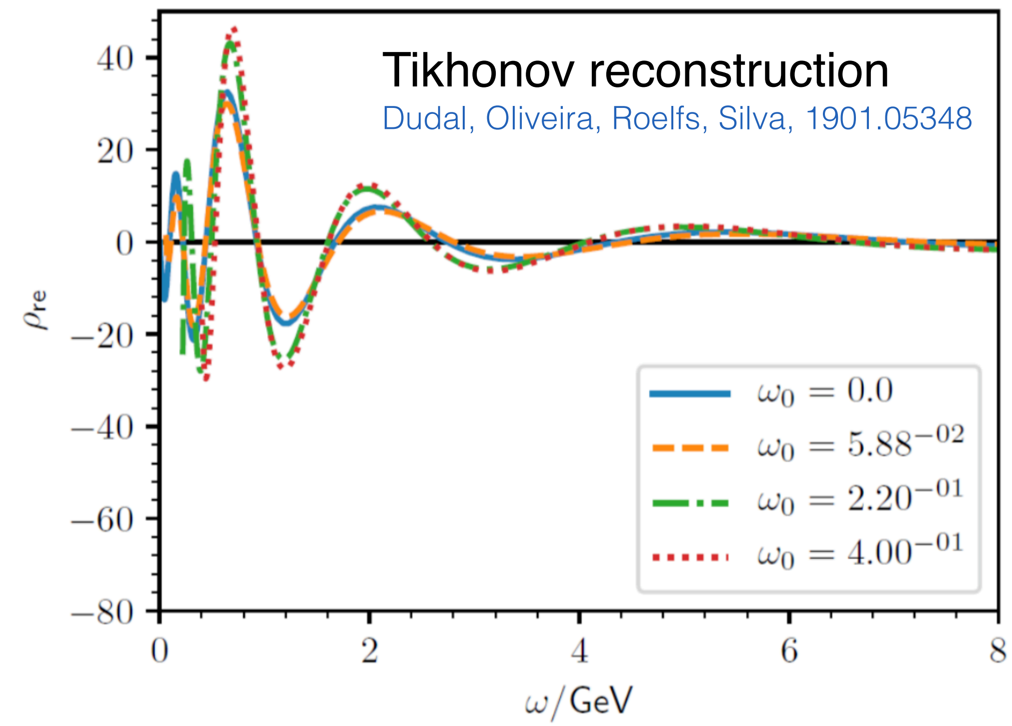






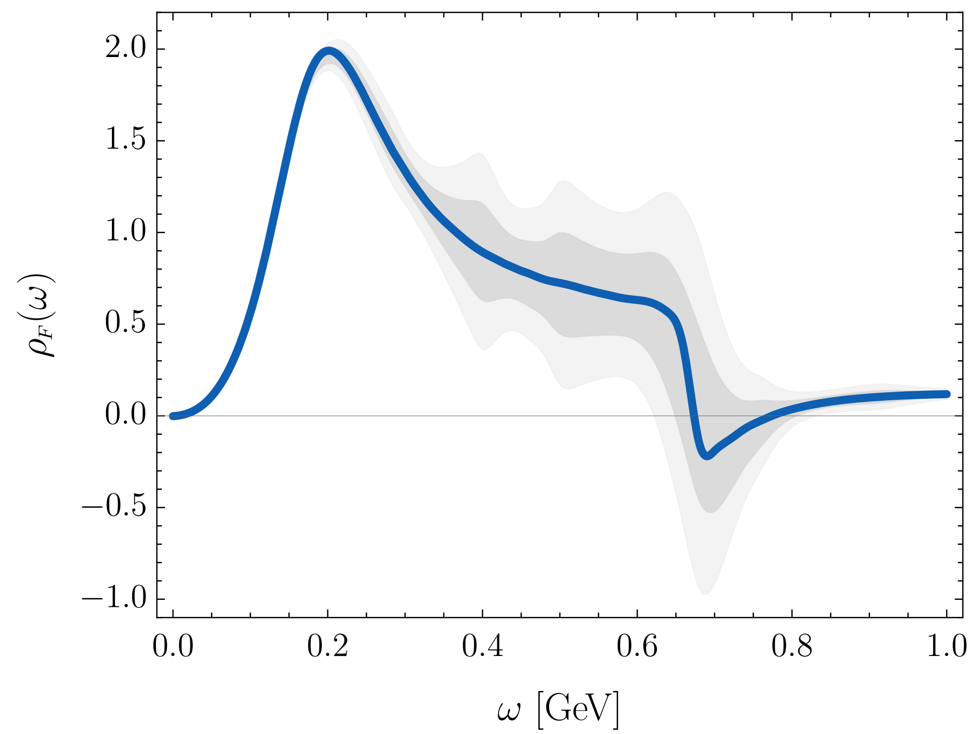


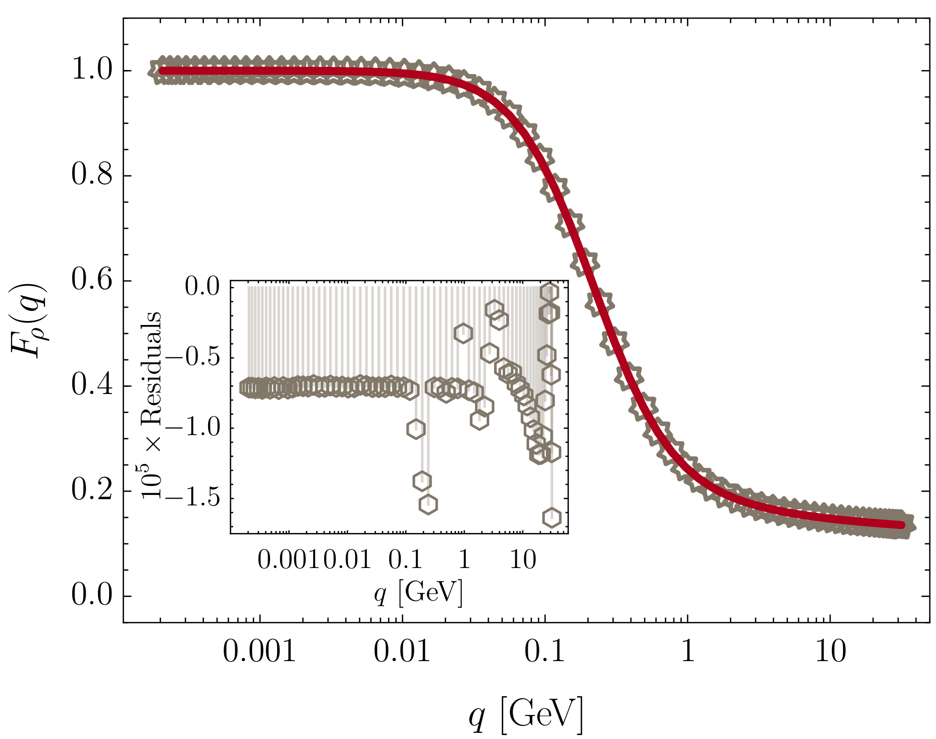
Lattice ghost propagator data

- **Improved SPM reconstruction**
  of DSE ghost data
  Strauss, Fischer, Kellermann, PRL (2012)

- **Need to reconstruct the dressing**
  to avoid massless pole
  $\rho_D(\omega) = F(0)\delta'(\omega) - \rho_F(\omega)/\omega^2$

- **Branch cut only**
  - **Starts at zero**
    gluon has non zero spectral density at arbitrarily low frequencies





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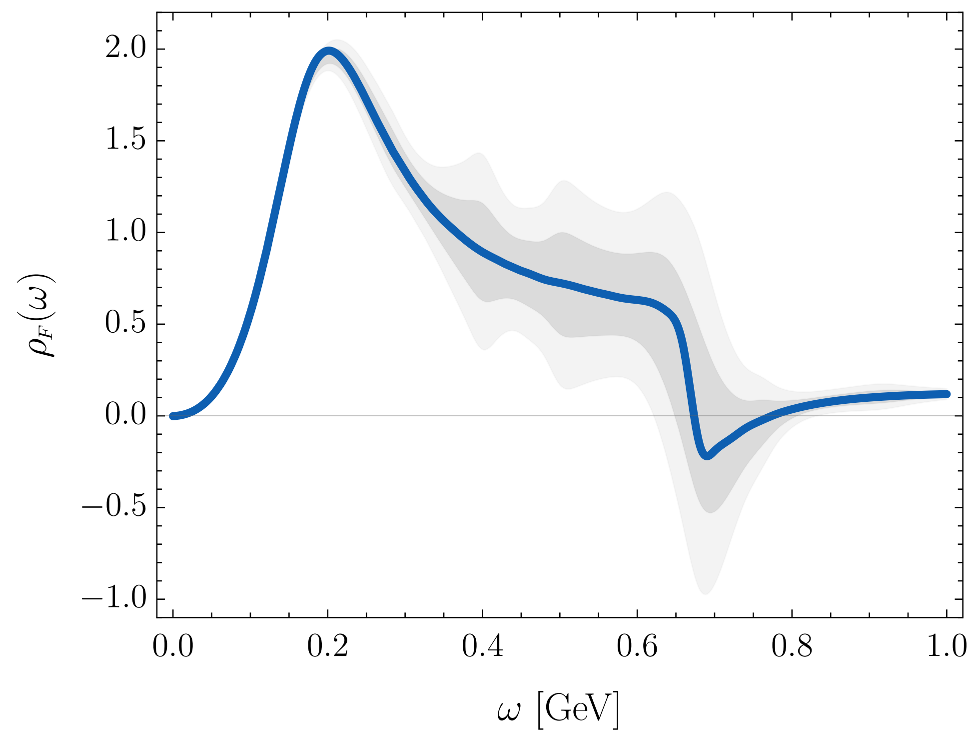

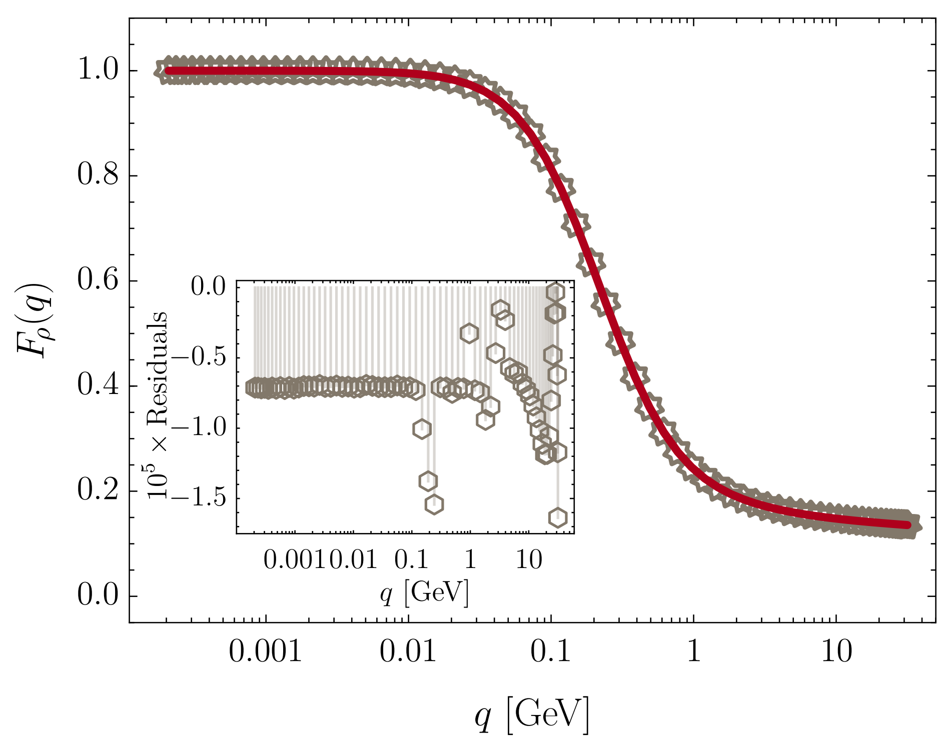

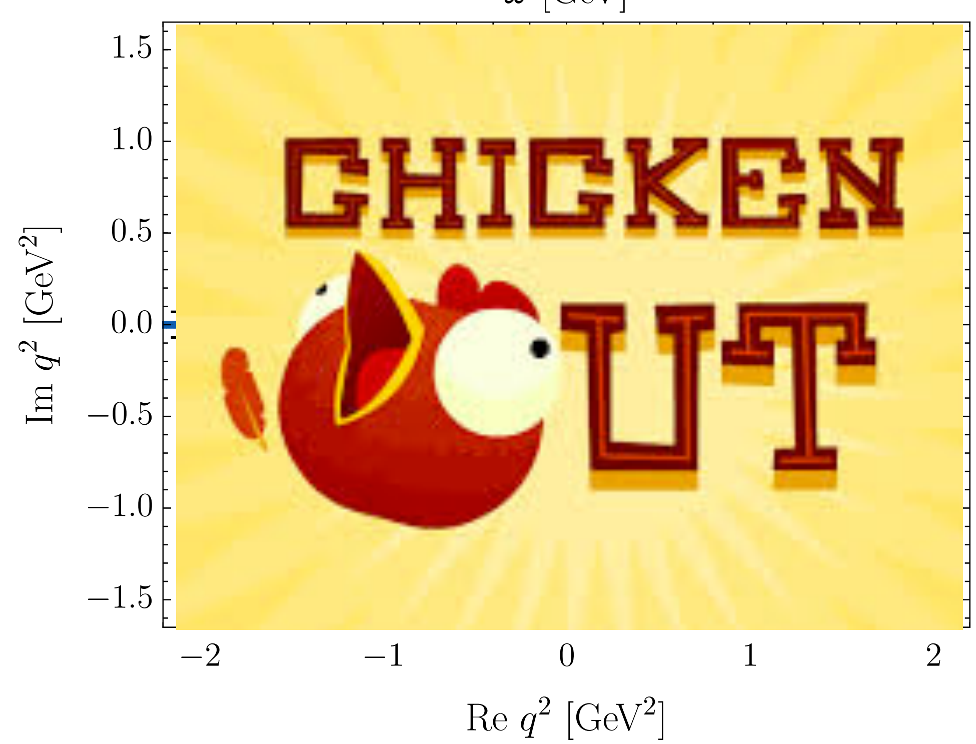

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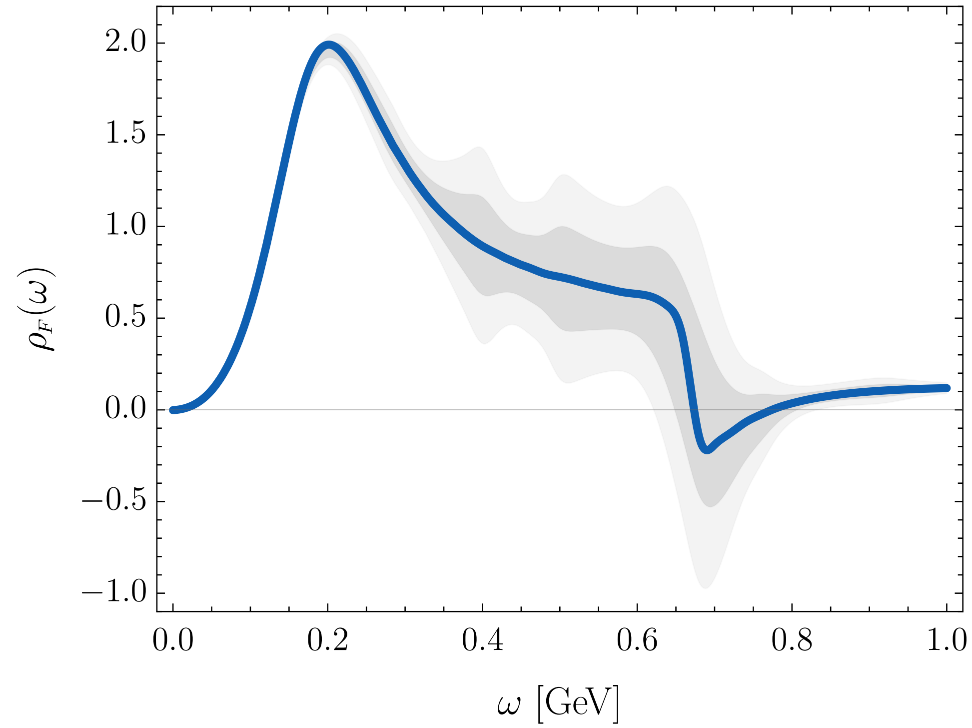
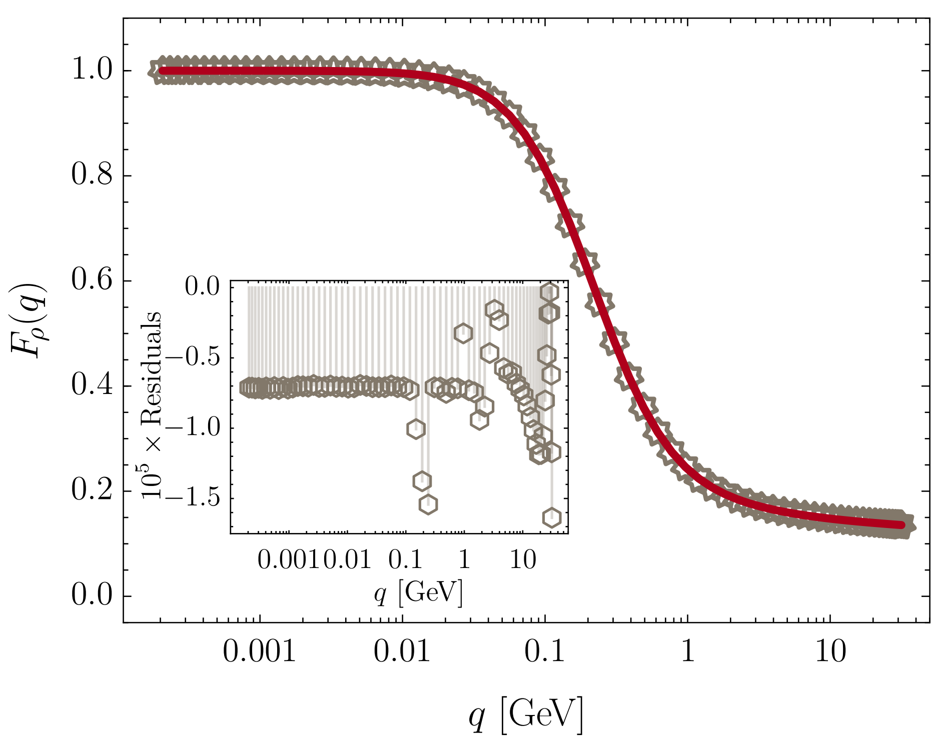
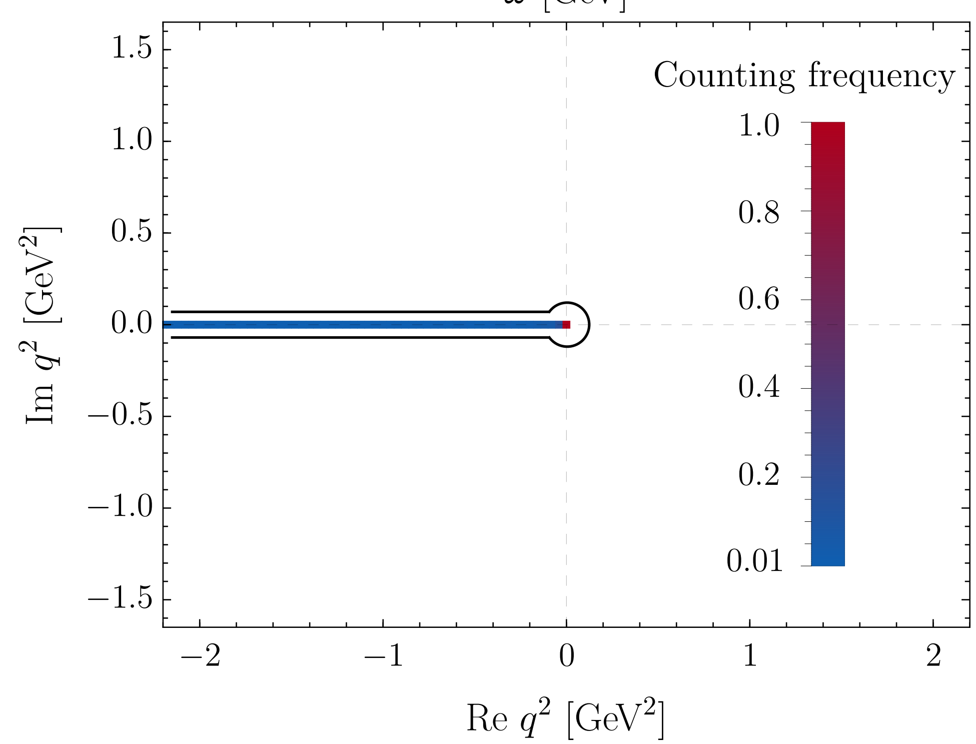


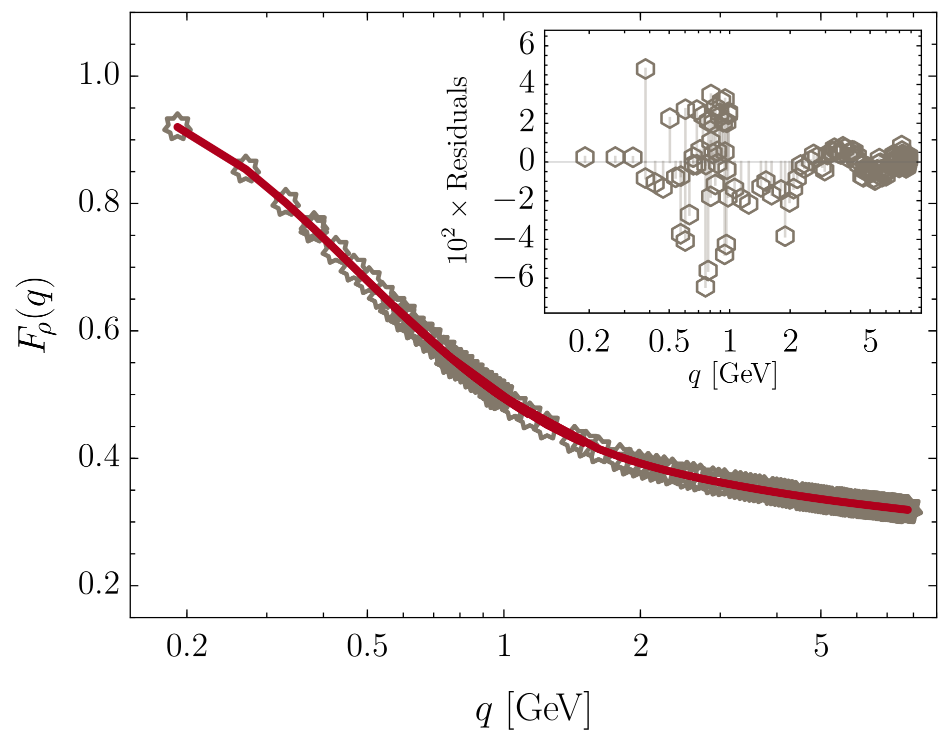


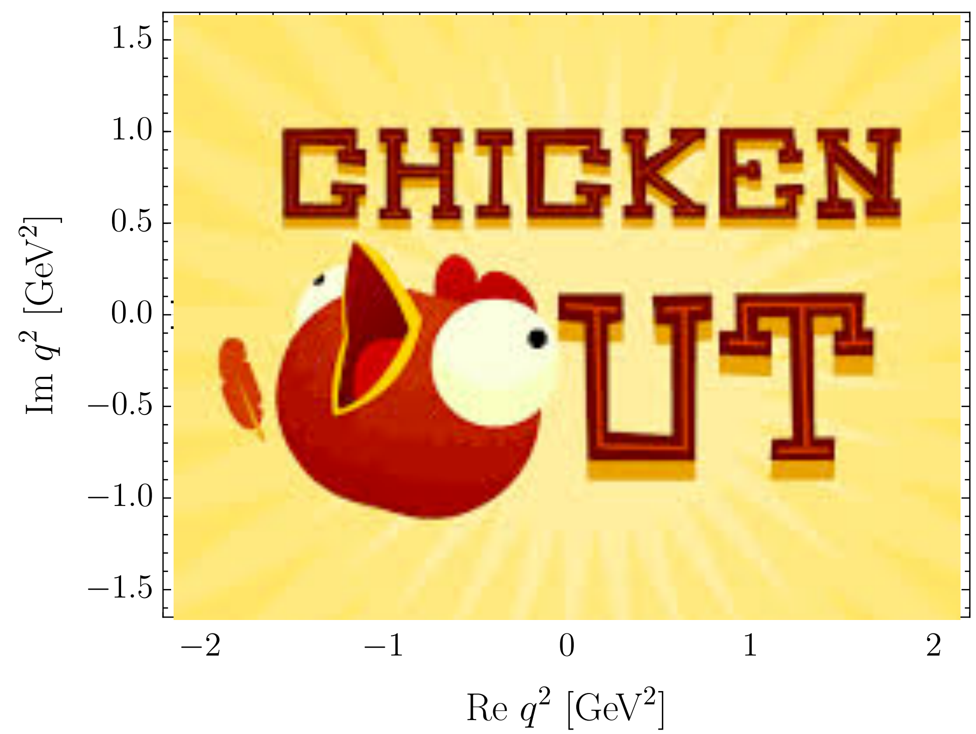


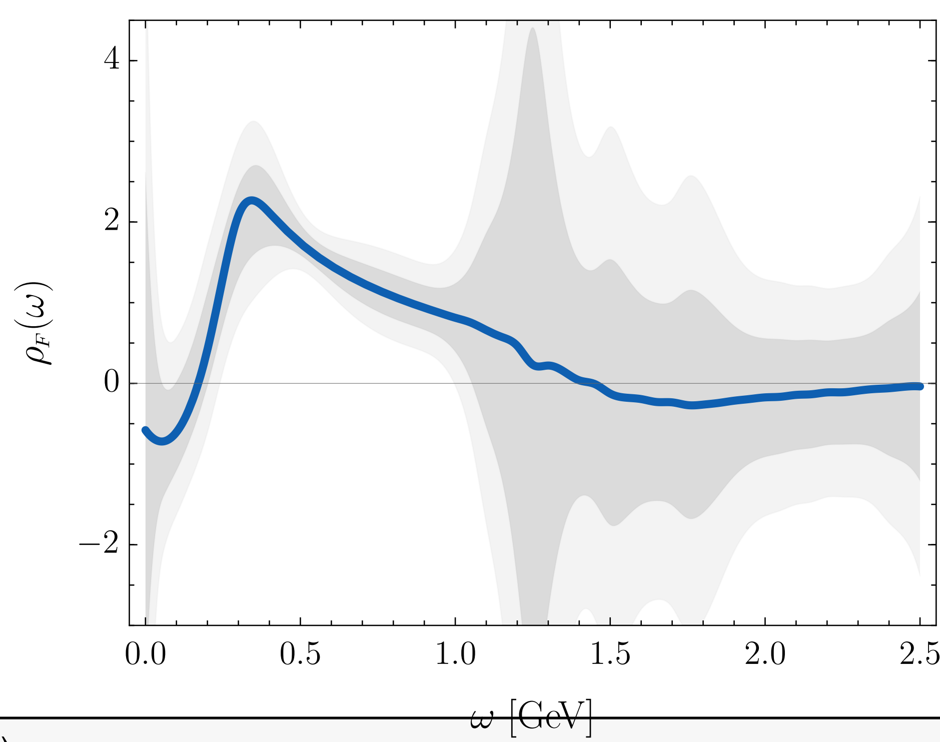
Lattice ghost propagator data

- **Improved SPM reconstruction**
  of SU(3) lattice ghost data at $\beta = 6.0$ and $V = 64^4$
  Duarte, Oliveira, Silva, PRD (2016)

$$\rho_D(\omega) = F(0)\delta'(\omega) - \rho_F(\omega)/\omega^2$$







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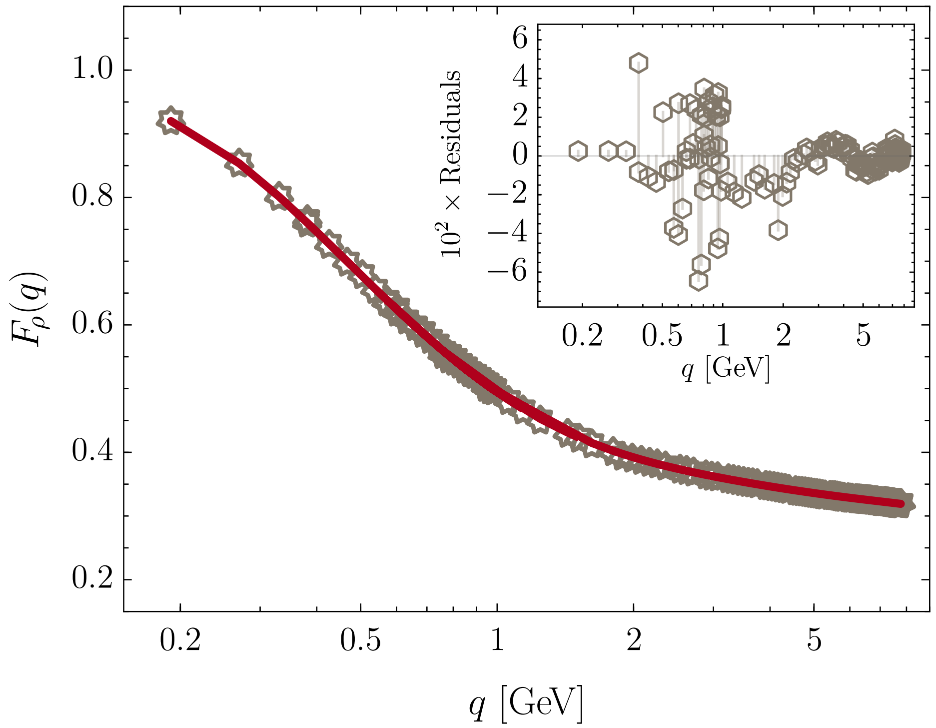

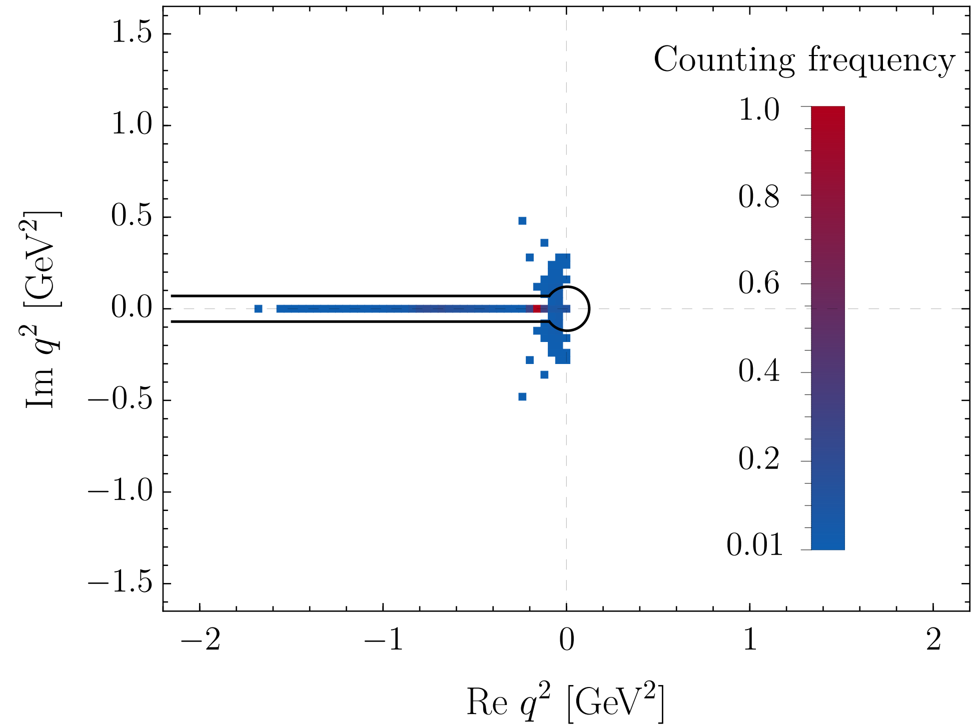

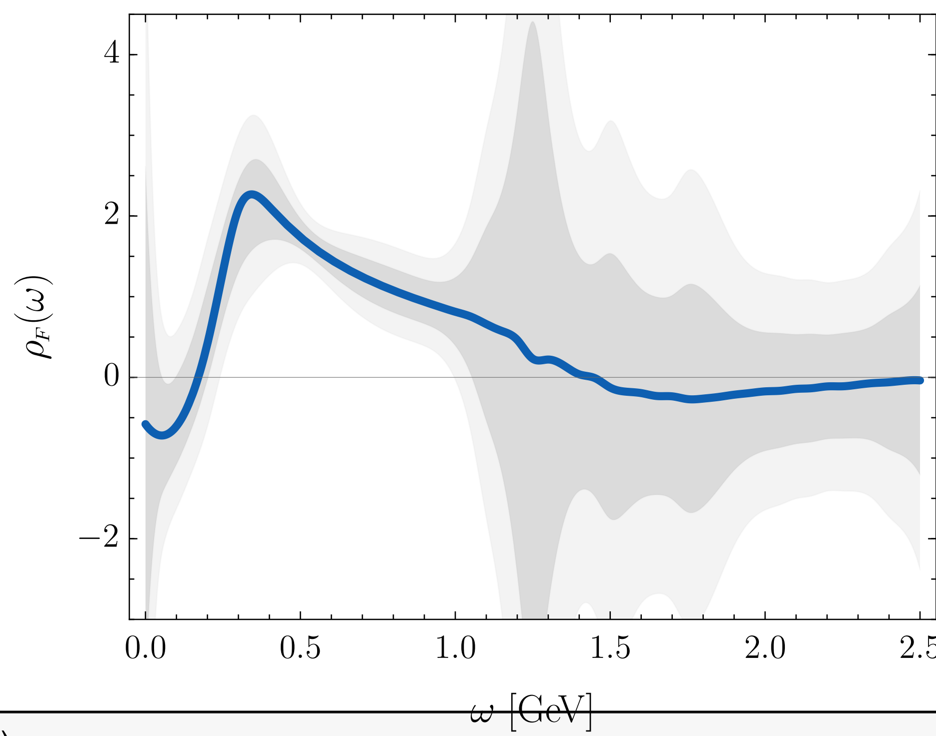

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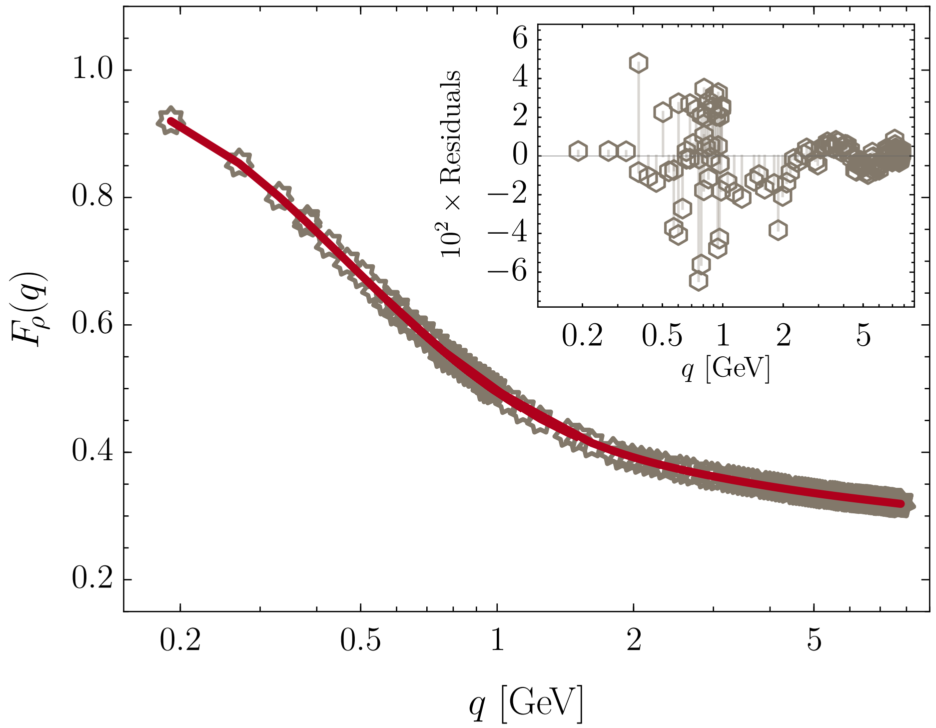

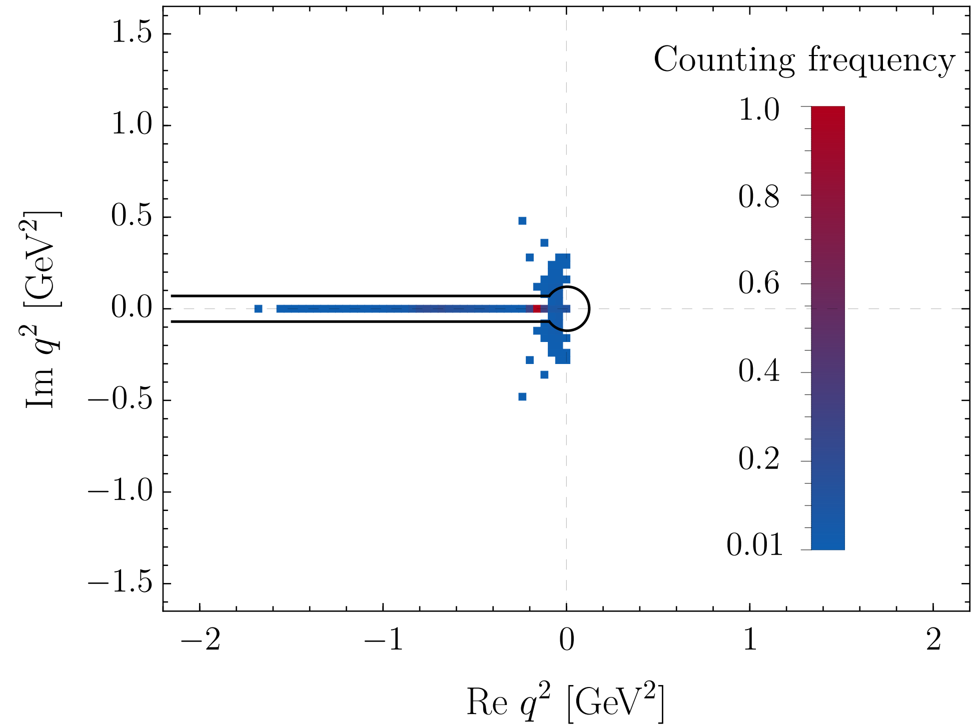

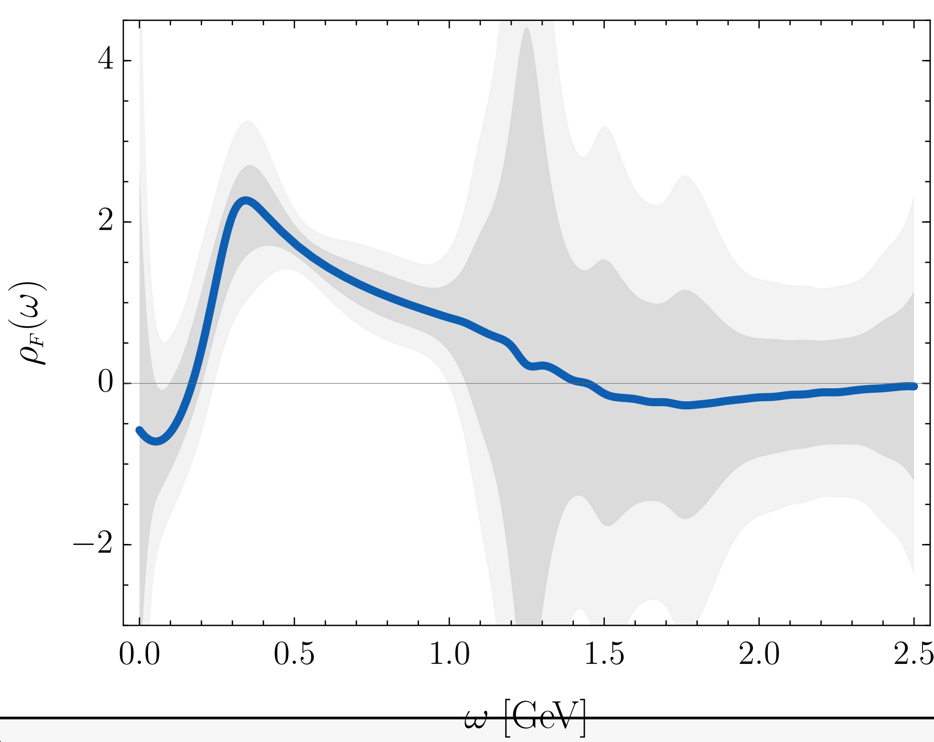

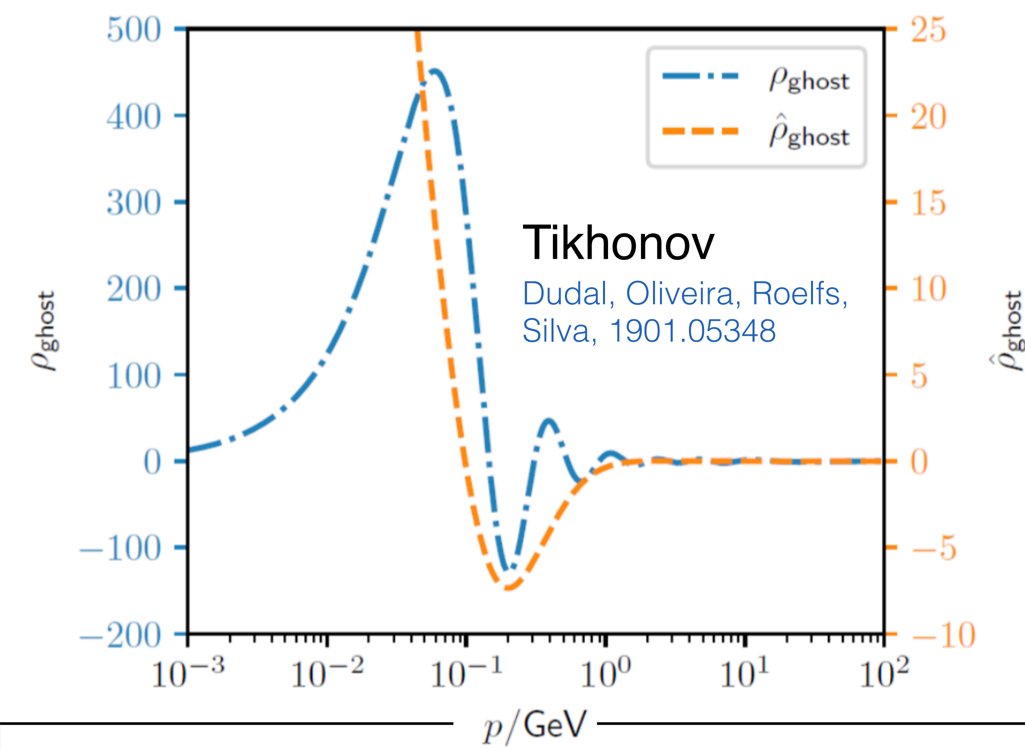

Conclusions

- **SPM interpolating fraction**
  powerful method to capture global features of data sets
- **Multiple uses**
  two discussed today:
  - **Extrapolation**
    form factors, masses, decay constants, moments…
  - **Analytic continuation**
    of two-point functions
- **Most interesting results**