Pinning Down the Inner Radiative Correction in Beta Decays

Chien-Yeah Seng

Helmholtz-Institut für Strahlen- und Kernphysik
and
Bethe Center for Theoretical Physics,
Universität Bonn

“Precise beta decay calculations for searches for new physics” workshop,
ECT*, Trento, Italy

9 April, 2019

Outline

1. The Inner Radiative Correction
2. Dispersive Approach
3. First-Principle Calculation
4. Summary

1. The Inner Radiative Correction

The Inner Radiative Correction

- Extraction of V_{ud} from beta decays:
  - (1) Superallowed beta decay

ft values corrected by nuclear structure effects: see Misha's talk

$$|V_{ud}|^2 = \frac{2984.432(3)\,\mathrm{s}}{\mathcal{F}t(1+\Delta_R^V)}$$

"nucleus-independent" correction

  - (2) Neutron beta decay

$$|V_{ud}|^2 = \frac{5099.34s}{\tau_n(1+3\lambda^2)(1+\Delta_R)}$$

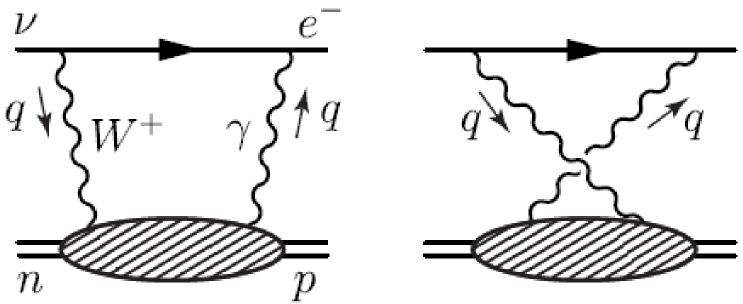
$$\Delta_R = (\alpha/2\pi)\bar{g}(E_m) + \Delta_R^V.$$

"outer" correction: sensitive to electron spectrum: see Leendert's talk

- "Inner radiative correction": the part of radiative correction (RC) which is insensitive to the electron spectrum

The Inner Radiative Correction

- Main source of uncertainty in inner RC: **γW-box** diagram



Sensitive to loop momentum q at ALL scales!

The "model-dependent" piece involves the axial component of the charged weak current:

$$T^{\mu\nu}_{\gamma W} = \frac{1}{2}\int d^4x e^{iq\cdot x} \langle p(p)| T[J^\mu_{em}(x) J^\nu_W(0)] |n(p)\rangle$$

$$= \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] T_1 + \frac{\hat{p}^\mu \hat{p}^\nu}{p\cdot q} T_2 + \frac{i\varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{2p\cdot q} T_3$$

$$(\Delta^V_R)^{VA}_{\gamma W} = 8\pi\alpha \mathrm{Re} \int \frac{d^4q}{(2\pi)^4} \frac{M_W^2}{M_W^2 + Q^2} \frac{Q^2+\nu^2}{Q^4} \frac{T_3(\nu, Q^2)}{m_N \nu}$$

$$\nu = p\cdot q / m_N$$

The Inner Radiative Correction

- Previous best determination: Marciano and Sirlin (M&S)

Marciano and Sirlin, Phys.Rev.Lett. 96 (2006) 032002

- Write the RC as a single-variable integral over Q^2, and identify the dominant physics as a function of Q^2.

$$(\Delta_R^V)_{\gamma W}^{VA} = \frac{\alpha}{4\pi}\int_0^\infty dQ^2 \frac{M_W^2}{M_W^2+Q^2}F(Q^2)$$

1. Short distance: leading OPE + perturbative QCD
2. Intermediate distance: VMD-inspired interpolating function + 100% uncertainty
3. Long distance: Elastic contribution

Combined: $\Delta_R^V(\text{M}\,\&\,\text{S}) = 0.02361(38)$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9994(5)$$

With largest, non-improvable uncertainty

2. Dispersive Approach

Dispersive Approach

- T_3 depends **on virtual intermediate states**: theoretical modeling is less transparent
- **Dispersive treatments** to box diagrams are developed since the last ten years, relating the former to matrix elements of **on-shell intermediate states**

$$T^{\mu\nu}_{\gamma W} = \frac{1}{2}\int d^4x e^{iq\cdot x} \langle p(p)| T[J^\mu_{em}(x) J^\nu_W(0)] |n(p)\rangle$$

$$= \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] T_1 + \frac{\hat{p}^\mu \hat{p}^\nu}{p\cdot q} T_2 + \frac{i\varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{2p\cdot q} T_3$$

Hadronic tensor in inclusive scattering:

$$W^{\mu\nu}_{\gamma W} = \frac{1}{8\pi}\int d^4x e^{iq\cdot x} \langle p(p)| [J^\mu_{em}(x), J^\nu_W(0)] |n(p)\rangle$$

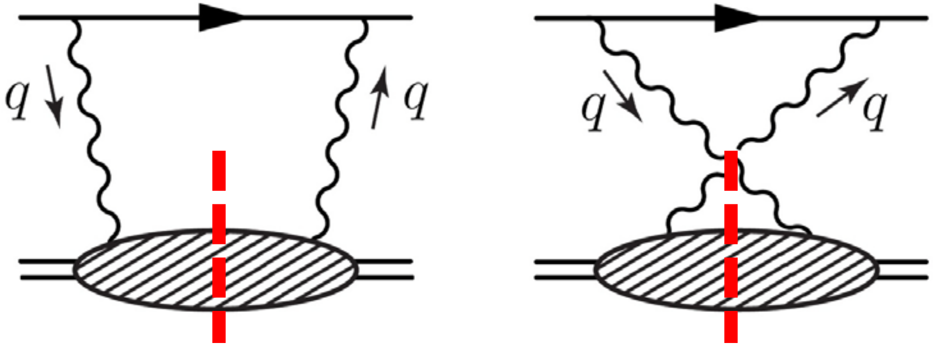
$$= \left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] F_1 + \frac{\hat{p}^\mu \hat{p}^\nu}{p\cdot q} F_2 + \frac{i\varepsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta}{2p\cdot q} F_3$$

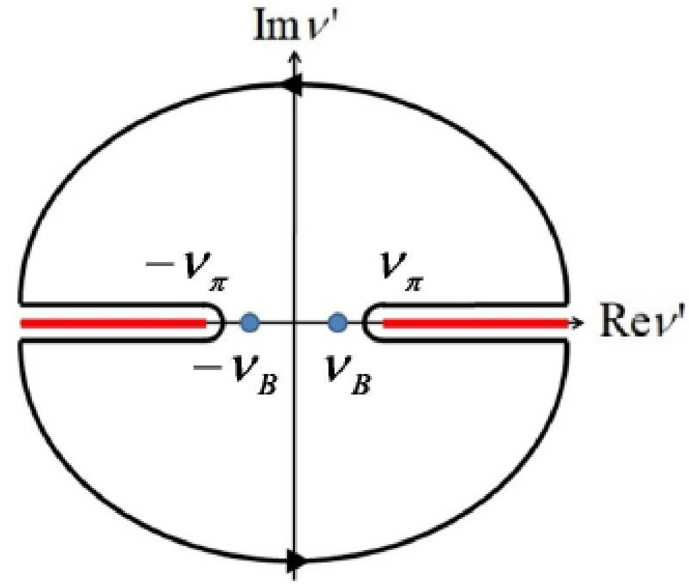
- We need only the contribution from the **isoscalar EM current (0)**

$$J^\mu_{em} = J^{(I=0),\mu}_{em} + J^{(I=1),\mu}_{em}$$

Dispersive Approach

- **Dispersion relation:**





$$T_3^{(0)}(\omega, Q^2) = -4i\omega \int_0^1 dx \frac{F_3^{(0)}(x, Q^2)}{1 - \omega^2 x^2}$$

$$\omega = 1/x_B = 2p{\cdot}q/Q^2$$

- Box diagrams are expressed in terms of the "**First Nachtmann moment**" of $F_3^{(0)}$:

$$(\Delta_R^V)_{\gamma W}^{VA} = \int_0^\infty \frac{dQ^2}{Q^2} \frac{3\alpha}{\pi} \frac{M_W^2}{M_W^2 + Q^2} M_1[F_3^{(0)}]$$

Central result!!!

$$M_1[F_3^{(0)}] = \int_0^1 dx \Pi(x, Q^2) F_3^N(x, Q^2) \qquad \Pi(x, Q^2) = \frac{4}{3} \frac{1 + 2\sqrt{1 + 4m_N^2 x^2/Q^2}}{(1 + \sqrt{1 + 4m_N^2 x^2/Q^2})^2}$$

Dispersive Approach

- Isospin symmetry:

$$F_3^{(0)} = -\frac{1}{4}\left(F_3^p - F_3^n\right)$$

where the **flavor-diagonal structure functions F_3^N** are defined through:

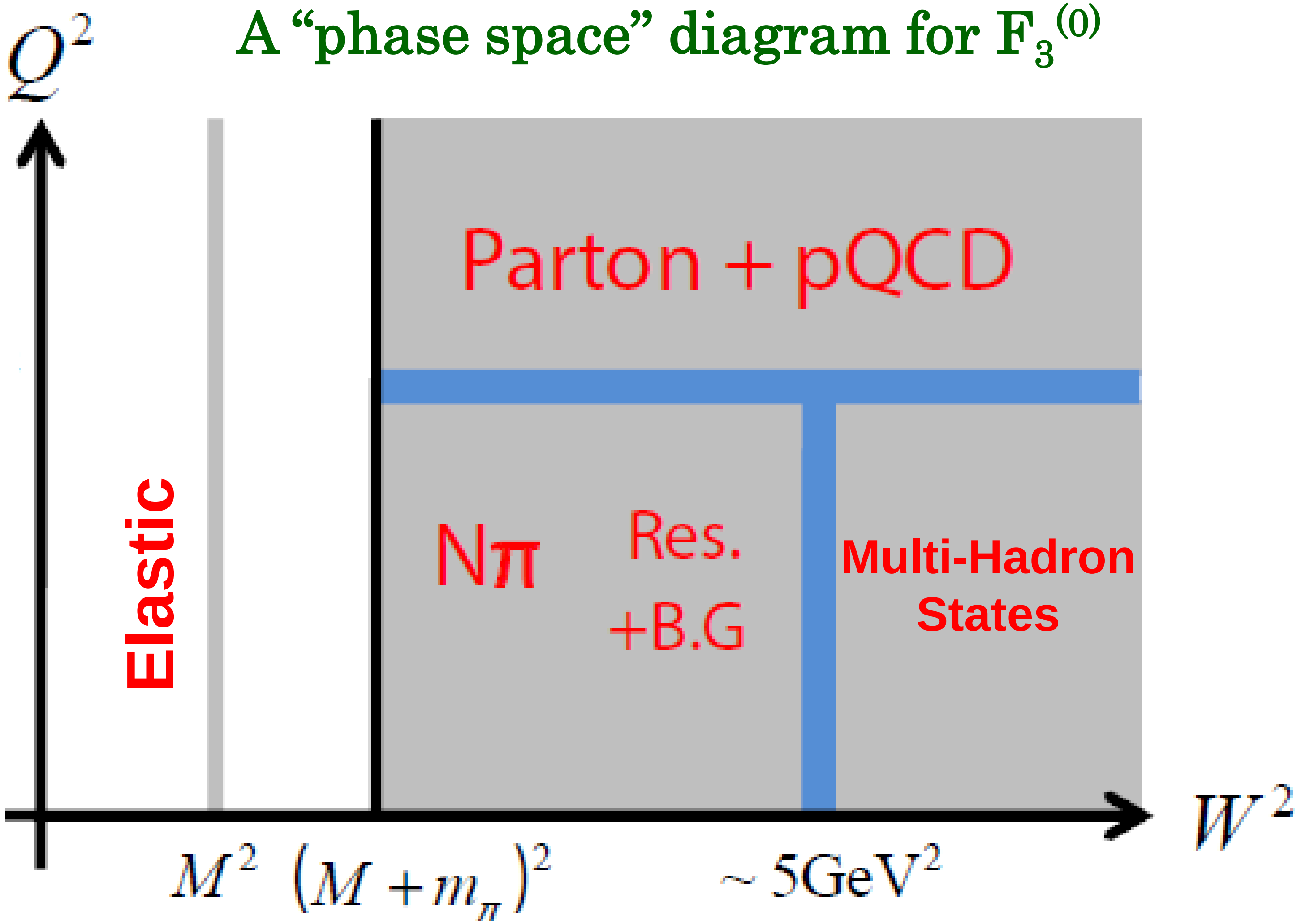
$$\begin{aligned} W_N^{\mu\nu} &= \frac{1}{4\pi}\int d^4x e^{iq\cdot x}\left\langle N(p)\right|\left[J_{em}^{\mu}(x), J_A^{\nu}(0)\right]\left|N(p)\right\rangle \\ &= \frac{i\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta}{2p\cdot q}F_3^N \end{aligned}$$

involving the interference between the **FULL electromagnetic current** and the **ISOVECTOR axial current**:

$$J_A^\mu = \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$$

Dispersive Approach

A "phase space" diagram for $F_3^{(0)}$



Dispersive Approach

$$F_3^{(0)} = F_{3,\mathrm{el}}^{(0)} + F_{3,\mathrm{inel}}^{(0)}$$

$$F_{3,\mathrm{inel}}^{(0)} = \begin{cases} F_{3,\mathrm{DIS}}^{(0)} & Q^2 > 2\mathrm{GeV}^2 \\ F_{3,N\pi}^{(0)} + F_{3,\mathrm{res}}^{(0)} + F_{3,\mathbb{R}}^{(0)} & Q^2 < 2\mathrm{GeV}^2 \end{cases}$$

Elastic: (isoscalar) magnatic Sach FF and axial FF

$$F_{3,\mathrm{el}}^{(0)} = -\frac{1}{4} G_A(Q^2) G_M^S(Q^2) \delta(1-x)$$

Z.Ye, J.Arrington, R.J.Hill and G.Lee, Phys.Lett.B777,8 (2018)

B.Bhattacharya, R.J.Hill and G.Paz,Phys.Rev.D84,073006 (2011)

$$0.0019(2) \to 0.0021(1)$$

DIS: polarized Bjorken sum rule +pQCD correction

$$M_1[F_{3,\mathrm{DIS}}^{(0)}] = \frac{1}{12}\left[1 - \tilde{C}_1\left(\frac{\alpha_S}{\pi}\right) - \tilde{C}_2\left(\frac{\alpha_S}{\pi}\right)^2 - \tilde{C}_3\left(\frac{\alpha_S}{\pi}\right)^3 + \ldots\right]$$

$$0.00427 \to 0.00434$$ (mere change of integration limit)

Nπ+ Resonance: Negligible

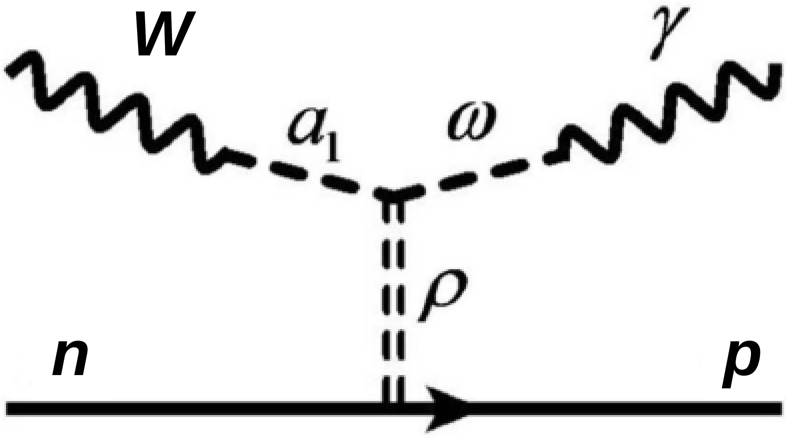
(Only I=1/2 intermediate states contributes)

Dispersive Approach

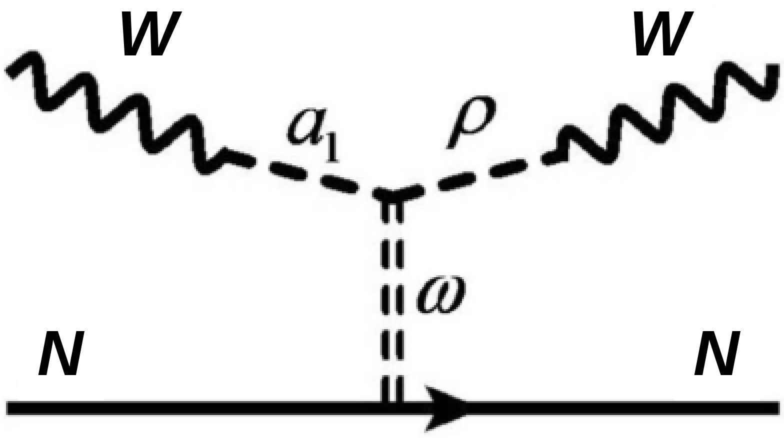
$$F_3^{(0)} = F_{3,\mathrm{el}}^{(0)} + F_{3,\mathrm{inel}}^{(0)}$$

$$F_{3,\mathrm{inel}}^{(0)} = \begin{cases} F_{3,\mathrm{DIS}}^{(0)} & Q^2 > 2\mathrm{GeV}^2 \\ F_{3,N\pi}^{(0)} + F_{3,\mathrm{res}}^{(0)} + F_{3,\mathbb{R}}^{(0)} & Q^2 < 2\mathrm{GeV}^2 \end{cases}$$

Multi-hadron states: Regge model + VDM

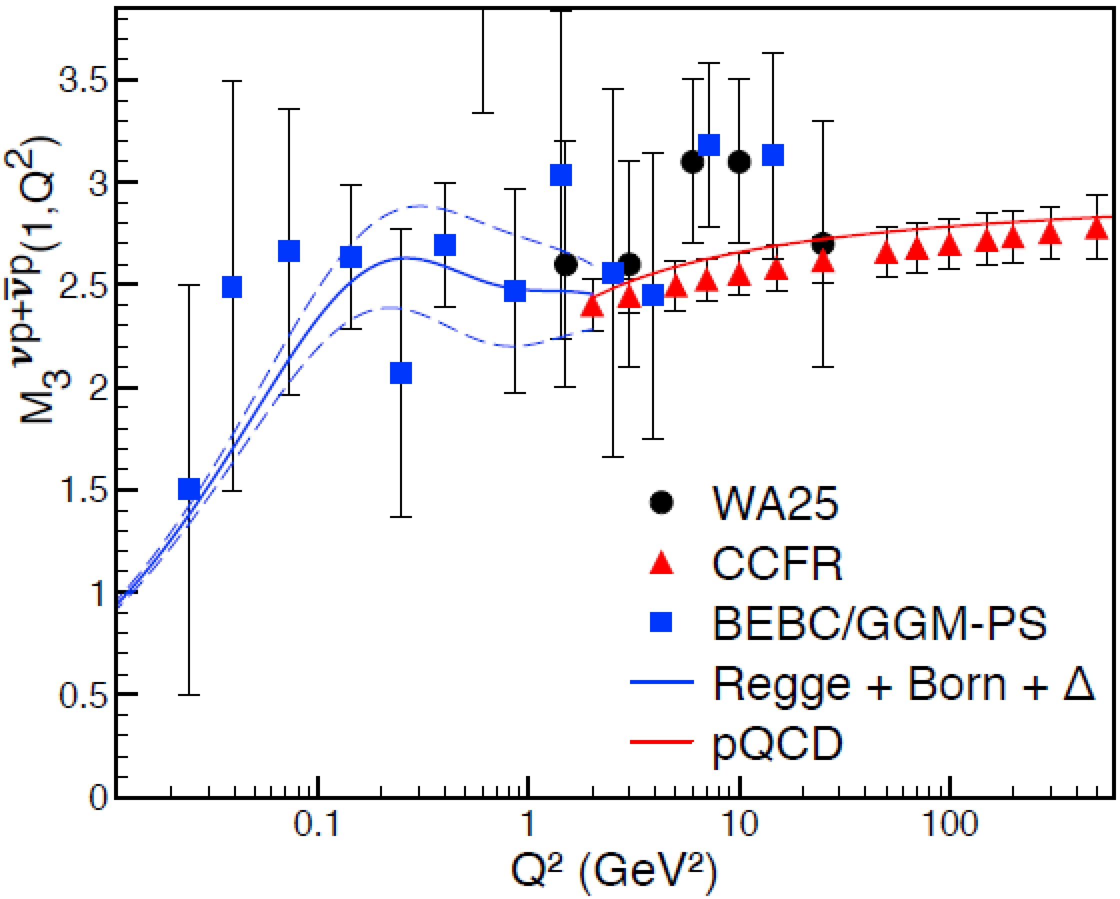


(I=1)*(I=0)



(I=1)*(I=1)

Dispersive Approach

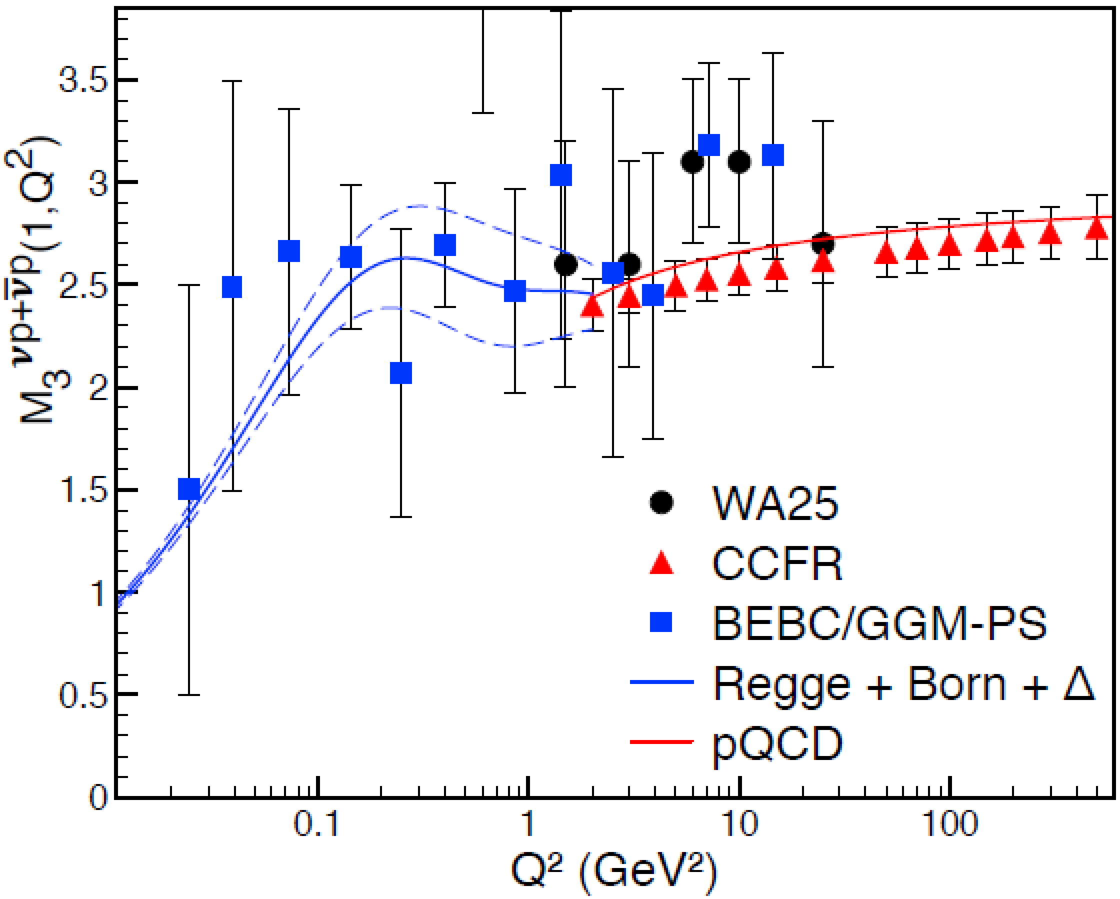


Matching the 1st Nachtmann moment of the **(I=1)*(I=1)** piece **to ν p/νbar p scattering data**

$$\frac{d^2\sigma^{\nu(\bar{\nu})}}{dx\,dy} = \frac{G_F^2 m_N E}{\pi(1+Q^2/m_W^2)^2}\left[xy^2 F_1^{\nu(\bar{\nu})} + \left(1 - y - \frac{m_N x y}{2E}\right) F_2^{\nu(\bar{\nu})} \pm x\left(y - \frac{y^2}{2}\right) F_3^{\nu(\bar{\nu})}\right]$$

(I=1)*(I=0) piece is then deduced using **Regge model+VDM**

Dispersive Approach



Significant increase in the **multi-hadron contribution** compare to M&S result, with **reduced uncertainty**:

$$0.0003(3) \to 0.0011(2)$$

Dispersive Approach

- Reduced hadronic uncertainty in the **determination of V_{ud}**:

CYS, M.Gorchtein, H.H.Patel and M.J.Ramsey-Musolf, Phys.Rev.Lett. 121 (2018) no. 24, 241804

CYS, M.Gorchtein and M.J.Ramsey-Musolf, arXiv:1812.03352

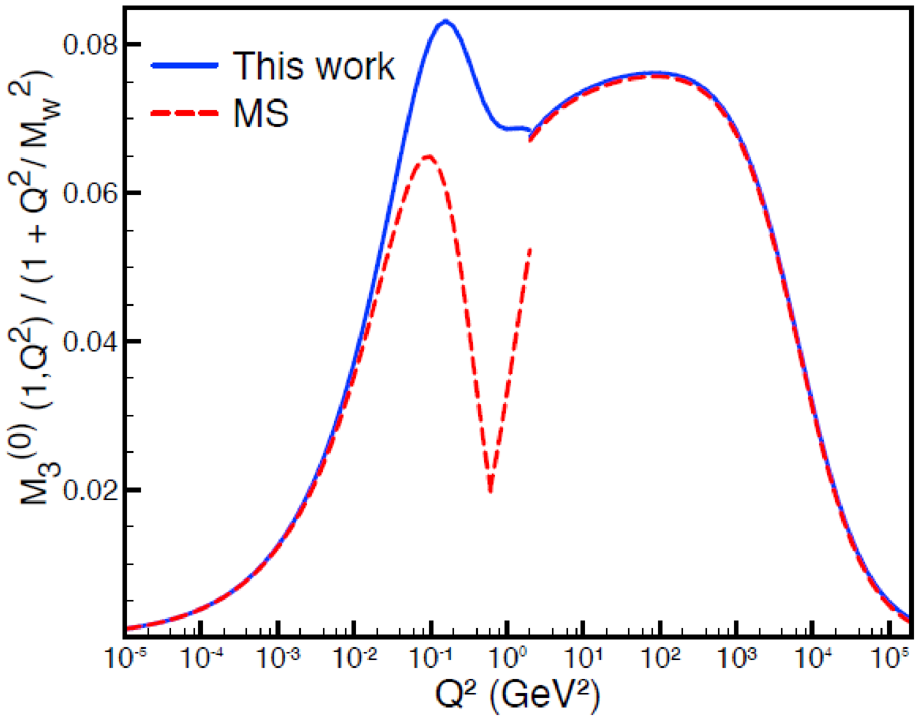
$$\Delta_R^V: \quad 0.02361(38) \to 0.02467(22)$$

$$|V_{ud}|: \quad 0.97420(21) \to 0.97370(14)$$

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2:$$

$$0.9994(5) \to 0.9984(4)$$

(assume nothing else changes)



- **Possible issues**:
  - Quality of the neutrino data?
  - Residual model-dependence?

which leads to the discussions below.

3. First-Principle Calculation

First-Principle Calculation

$$(\Delta_R^V)_{\gamma W}^{VA} = -\frac{3\alpha}{4\pi}\int_0^\infty \frac{dQ^2}{Q^2}\frac{M_W^2}{M_W^2+Q^2}\{M_1[F_3^p]-M_1[F_3^n]\}$$

- Recall the that we are interested in $M_1[F_3^N]$ as a function of Q^2 . Neutrino data helps identifying **dominant contributors at different Q^2**:

$$M_1[F_3^N] = \begin{cases} \approx \text{elastic} + \Delta & Q^2 < 0.1\text{GeV}^2 \\ \text{multi}-\text{hadron states} & 0.1\text{GeV}^2 < Q^2 < 2\text{GeV}^2 \\ \text{DIS} & Q^2 > 2\text{GeV}^2 \end{cases}$$

- Therefore, to remove the hadronic uncertainties in the box diagrams, we need to have a good handle of the **first Nachtmann moment of F_3 at moderate Q^2**.
- Question: is there a way to calculate $M_1[F_3^N]$ from **FIRST-PRINCIPLE**?

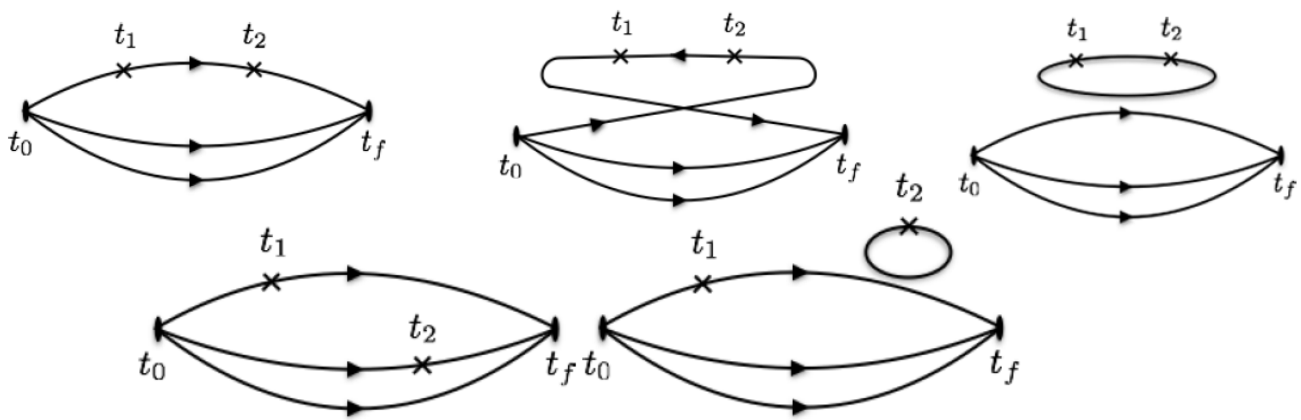
First-Principle Calculation

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X (2\pi)^4 \delta^4(p+q-p_X)\,\langle N(p)|\,J^{\mu}_{em}\,|X\rangle\,\langle X|\,J^{\nu}_{A}\,|N(p)\rangle$$

- Difficult because it involves **a sum of all on-shell intermediate states.**

- Recently-developed techniques in lattice calculation of PDFs (quasi-PDF, pseudo-PDF, lattice cross-section etc) do not apply because they rely on OPE that holds only at large Q^2.

- We wish to **avoid direct calculations of four-point functions** (noisy contractions, complicated finite-volume effect…)



J. Liang, K-F. Liu and Y-B. Yang, EPJ Web Conf. 175 (2018) 14014

First-Principle Calculation

- A more promising approach is through the **Feynman-Hellmann theorem (FHT)**:

$$\frac{dE_{n,\lambda}}{d\lambda} = \left\langle n_\lambda \middle| \frac{\partial H_\lambda}{\partial \lambda} \middle| n_\lambda \right\rangle$$

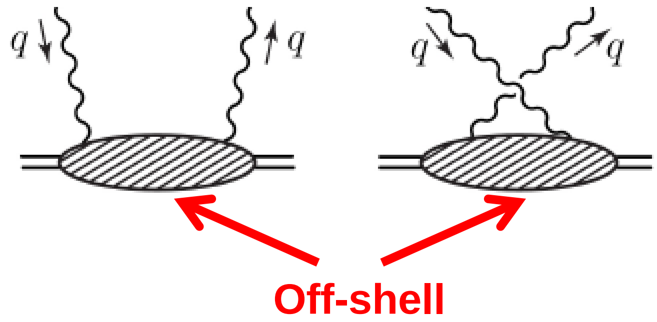
- **Shift in energy level→ matrix element**. Extraction of energy levels on lattice are more straightforward, avoid complicated contraction diagrams.
- **Momentum transfer** could be introduced through **periodic external potential**.
- Shows great potential in studies of:
  - Nucleon axial charge and sigma term
  - EM form factors
  - Compton amplitude
  - P-even structure functions
  - Hadron resonances
  - ……

First-Principle Calculation

Some warm-up:

Kinematics: $q^\mu = (0, \vec{q}) \implies \omega = -\frac{2\vec{p}\cdot\vec{q}}{Q^2}$

"Off-shell condition": $|\omega| < 1 \implies E(\vec{p} \pm \vec{q}) > E(\vec{p})$



- Consider a **periodic potential**: $V(\vec{x}) = V_0 \cos(\vec{q}\cdot\vec{x}) = \frac{1}{2}V_0(e^{i\vec{q}\cdot\vec{x}} + e^{-i\vec{q}\cdot\vec{x}})$

$$V(x)\psi_{\vec{p}}(\vec{x}) \sim \psi_{\vec{p}+\vec{q}}(\vec{x}) + \psi_{\vec{p}-\vec{q}}(\vec{x})$$

- The off-shell condition prohibits mixing of degenerate states through perturbation. Thus, non-degenerate perturbation theory at 1^{st}-order gives:

$$\langle\vec{p}|\, V\, |\vec{p}\rangle \sim \langle\vec{p}|\, \vec{p} \pm \vec{q}\rangle = 0$$

No first-order energy shift!

First-Principle Calculation

Our Strategy:

- Introduce **TWO** periodic source terms, and study the **SECOND ORDER ENRGY SHIFT**:

$$H_\lambda = H_0 + 2\lambda_1 \int d^3x \cos(\vec{q}\cdot\vec{x}) J^2_{em}(\vec{x}) - 2\lambda_2 \int d^3x \sin(\vec{q}\cdot\vec{x}) J^3_A(\vec{x})$$

$$\left(\frac{\partial^2 E_{N,\lambda}(\vec{p})}{\partial\lambda_1 \partial\lambda_2}\right)_{\lambda=0} = \frac{iq_x}{Q^2\omega} T_3^N(\omega, Q^2).$$

CYS and U.G-Meissner, hep-ph/1903.07969

- Plugging it into the dispersion relation of T_3^N:

$$\left(\frac{\partial^2 E_{N,\lambda}(\vec{p})}{\partial\lambda_1 \partial\lambda_2}\right)_{\lambda=0} = \frac{4q_x}{Q^2}\int_0^1 dx \frac{F_3^N(x, Q^2)}{1-\omega^2 x^2} \,,$$

Central result!!!

2nd order Energy shift → (FHT) → Generalized Forward Compton tensor → (DR) → Structure Function

First-Principle Calculation

- Lattice momenta are **discrete**:

$$\vec{p} = \frac{2\pi}{L}(n_{px}, n_{py}, n_{pz}), \qquad \vec{q} = \frac{2\pi}{L}(n_{qx}, n_{qy}, n_{qz})$$

- Requiring Q^2 at the hadronic scale and the off-shell condition imply:

$$\frac{4\pi^2}{L^2}(n_{qx}^2 + n_{qy}^2 + n_{qz}^2) \lesssim 1\,\mathrm{GeV}^2$$

$$\frac{2|n_{px}n_{qx} + n_{py}n_{qy} + n_{pz}n_{qz}|}{n_{qx}^2 + n_{qy}^2 + n_{qz}^2} < 1.$$

- **A concrete example**: $L \approx 2.8\mathrm{fm}$ $\quad \vec{q} = \frac{2\pi}{L}(2,1,0)$ impose the restriction:

$Q^2 \approx 1\mathrm{GeV}^2$

Allowed values for ω: $|\omega| = 0, \ \frac{2}{5}, \ \frac{4}{5}$

First-Principle Calculation

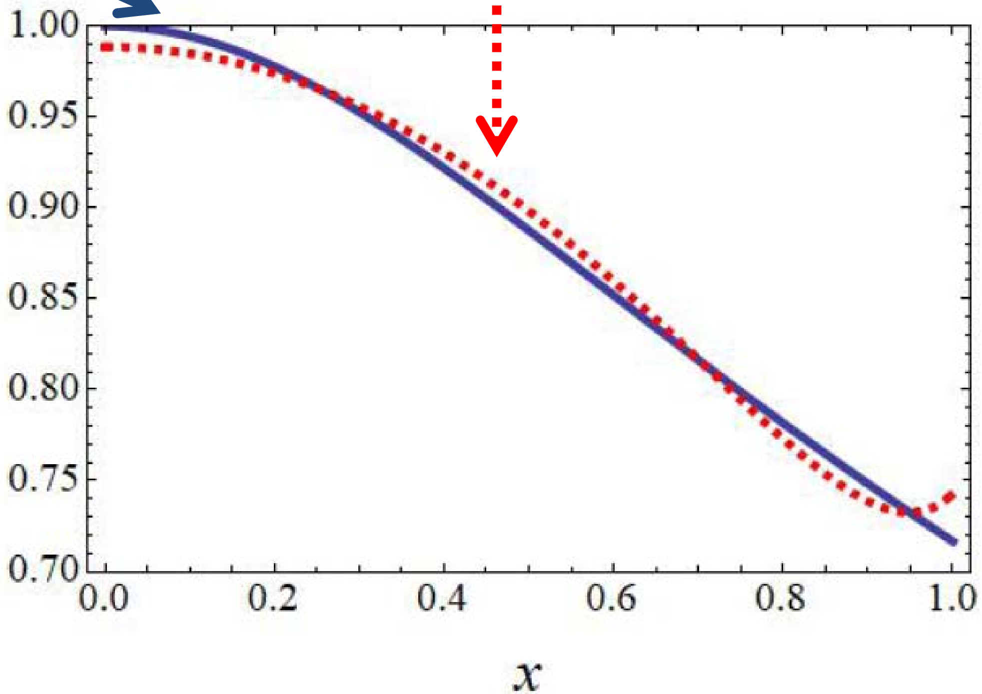
Reconstructing the first Nachtmann moment from energy shifts

$$M_1[F_3^N] = \int_0^1 dx \Pi(x, Q^2) F_3^N(x, Q^2),$$

$$\frac{Q^2}{4q_x}\left(\frac{\partial^2 E_{N,\lambda}}{\partial\lambda_1 \partial\lambda_2}\right)_{\lambda=0} = \int_0^1 dx \Lambda(x,\omega) F_3^N(x, Q^2)$$

$$\Lambda(x,\omega) = (1-\omega^2 x^2)^{-1}$$

$$\Pi(x, Q^2) \approx a\Lambda(x,0) + b\Lambda(x,2/5) + c\Lambda(x,4/5)$$



$$\begin{aligned} a &= 3.81529 \\ b &= -3.01135 \\ c &= 0.184491 \end{aligned}$$

A very good reconstruction!

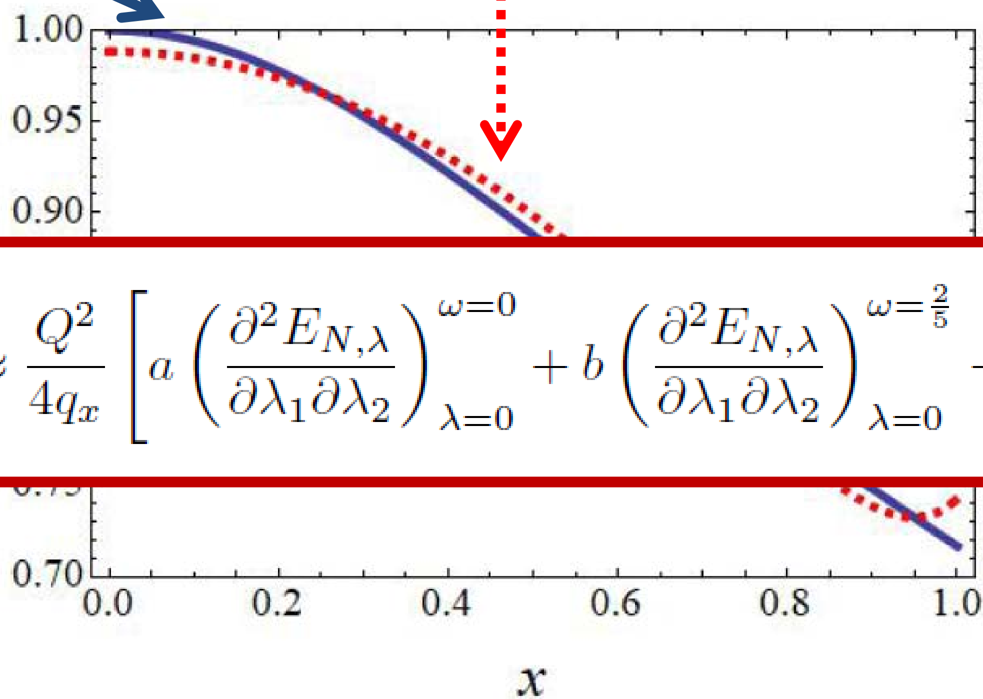
First-Principle Calculation

Reconstructing the first Nachtmann moment from energy shifts

$$M_1[F_3^N] = \int_0^1 dx\,\Pi(x,Q^2)F_3^N(x,Q^2), \qquad \frac{Q^2}{4q_x}\left(\frac{\partial^2 E_{N,\lambda}}{\partial\lambda_1\partial\lambda_2}\right)_{\lambda=0} = \int_0^1 dx\,\Lambda(x,\omega)F_3^N(x,Q^2)$$

$$\Lambda(x,\omega) = (1-\omega^2x^2)^{-1}$$

$$\Pi(x,Q^2) \approx a\Lambda(x,0) + b\Lambda(x,2/5) + c\Lambda(x,4/5)$$



$$a = 3.81529$$
$$b = -3.01135$$
$$c = 0.184491$$

$$M_1[F_3^N] \approx \frac{Q^2}{4q_x}\left[a\left(\frac{\partial^2 E_{N,\lambda}}{\partial\lambda_1\partial\lambda_2}\right)_{\lambda=0}^{\omega=0} + b\left(\frac{\partial^2 E_{N,\lambda}}{\partial\lambda_1\partial\lambda_2}\right)_{\lambda=0}^{\omega=\frac{2}{5}} + c\left(\frac{\partial^2 E_{N,\lambda}}{\partial\lambda_1\partial\lambda_2}\right)_{\lambda=0}^{\omega=\frac{4}{5}}\right]$$

Summary

1. The axial γW box diagram is one of the main sources of theoretical uncertainty in the extraction of V_{ud} through neutron and superallowed beta decay.
2. The application of dispersion relation utilizing ν p/ νbar p scattering data reduces the uncertainty in the γW box diagram by a half, but at the same time raises tension with the first row CKM unitarity.
3. A recent proposal that hybridizes the dispersion relation and computations of shifted energy levels on lattice may, for the first time, lead to a first-principle theoretical calculation of the γW box.