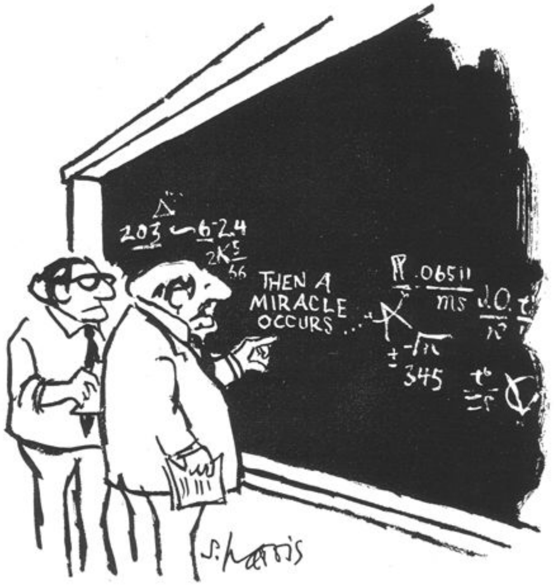
Inclusion of pions

Goal: EFT for NN scattering at typical momenta Q ~ M_π

Are pions perturbative?

How to test whether pion dynamics is being treated properly?

Low-Energy Theorems as a tool to test predictive power beyond ERE



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Modified Effective Range Expansion (MERE)

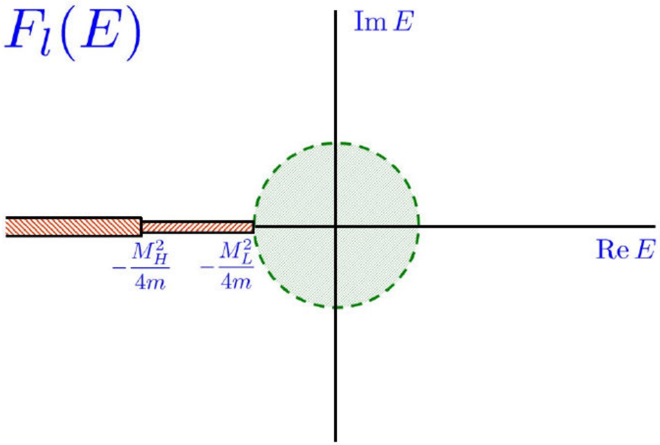
5. LETs and the MERE

What are the low-energy theorems?

Two-range potential: $V(r) = V_L(r) + V_S(r)$

with $M_L^{-1} \gg M_H^{-1}$

- $F_l(k^2)$ is meromorphic in $|k| < M_L/2$



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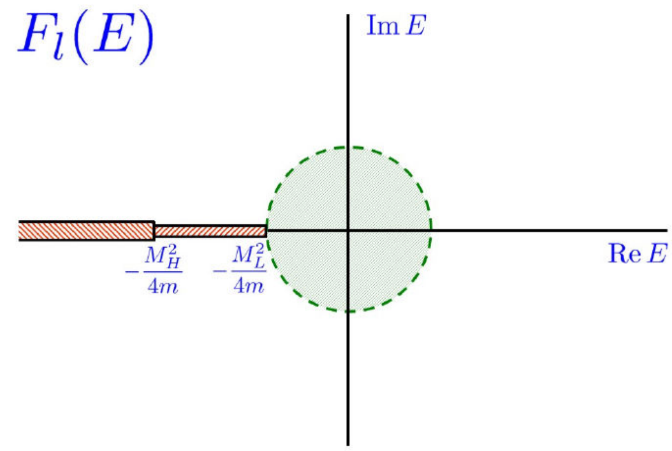


- $F_l(k^2)$ is meromorphic in $|k| < M_L/2$

- $$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

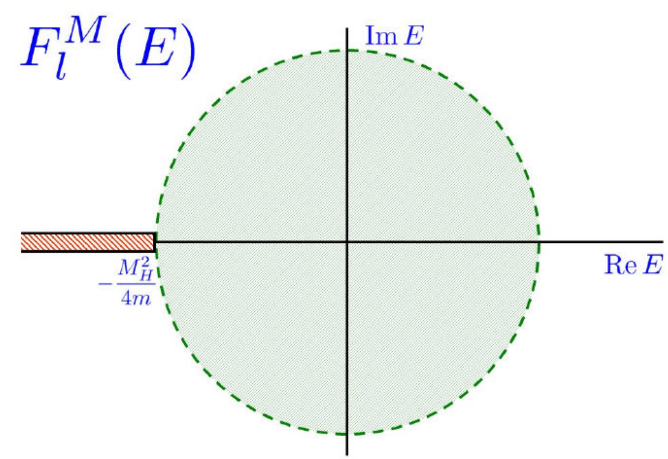
← modified effective range function
van Haeringen, Kok '82

$$f_l^L(k) = \lim_{r \to 0} \left(\frac{l!}{(2l)!} (-2ikr)^l f_l^L(k, r) \right)$$

Jost function for $V_L(r)$ — Jost solution for $V_L(r)$

$$M_l^L(k) = Re\left[\frac{(-ik/2)^l}{l!} \lim_{r \to 0} \left(\frac{d^{2l+1}}{dr^{2l+1}} \frac{r^l f_l^L(k, r)}{f_l^L(k)} \right) \right]$$

Per construction, F_l^M reduces to F_l for $V_L = 0$ and is meromorphic in $|k| < M_H/2$



MERE and low-energy theorems

Example: proton-proton scattering

$$F_C(k^2) = C_0^2(\eta)\, k\, \cot[\delta(k) - \delta^C(k)] + 2k\,\eta\, h(\eta) = -\frac{1}{a^M} + \frac{1}{2} r^M k^2 + v_2^M k^4 + \ldots$$

where $\underbrace{\delta^C \equiv \arg\Gamma(1+i\eta)}_{\text{Coulomb phase shift}}$, $\eta = \frac{m}{2k}\alpha$, $\underbrace{C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta}-1}}_{\text{Sommerfeld factor}}$, $h(\eta) = \mathrm{Re}\Big[\underbrace{\Psi(i\eta)}_{\text{Digamma function}}\Big] - \ln(\eta)$

$\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

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Coulomb phase shift | Sommerfeld factor | Digamma function $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$

MERE and low-energy theorems

Long-range forces impose correlations between the ER coefficients (low-energy theorems)

[Cohen, Hansen '99; Steele, Furnstahl '00; Baru, EE, Filin, Gegelia '15,'16]

The emergence of the LETs can be understood in the framework of MERE:

$$F_l^M(k^2) \equiv M_l^L(k) + \frac{k^{2l+1}}{|f_l^L(k)|^2} \cot\left[\delta_l(k) - \delta_l^L(k)\right]$$

$F_l^M(k^2)$: meromorphic for $k^2 < (M_H/2)^2$

$M_l^L(k)$, $f_l^L(k)$, $\delta_l^L(k)$: can be computed if the long-range force is known

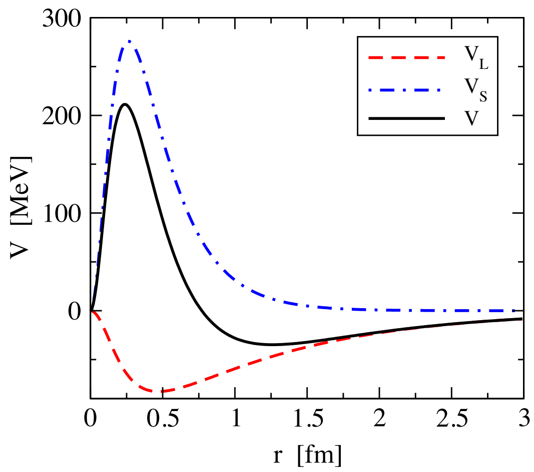
- approximate $F_l^M(k^2)$ by first 1,2,3,… terms in the Taylor expansion in k^2
- calculate all "soft" quantities
- reconstruct $\delta_l^L(k)$ and predict all coefficients in the ERE

Toy model: Low-energy theorems

$$V(r) = \underbrace{v_L e^{-M_L r} f(r)}_{V_L} + \underbrace{v_H e^{-M_H r} f(r)}_{V_H}$$

where $f(r) = \frac{(M_H r)^2}{1 + (M_H r)^2}$

and $M_L = 1.0\,,\ v_L = -0.875\,,\ M_H = 3.75\,,\ v_H = 7.5$ (all in fm^{-1})



ERE and MERE

	a	r	v_2	v_3	v_4
F_0 [fm^n]	5.458	2.432	0.113	0.515	-0.993
F_0^M [M_S^{-n}]	1.710	-1.063	-0.434	-0.680	2.624

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Low-Energy Theorems

	LO	NLO	NNLO	"Exp"
r				2.432197161
v_2				0.112815751
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for an analytic example, see EE, Gegelia, EPJ A41 (2009) 341

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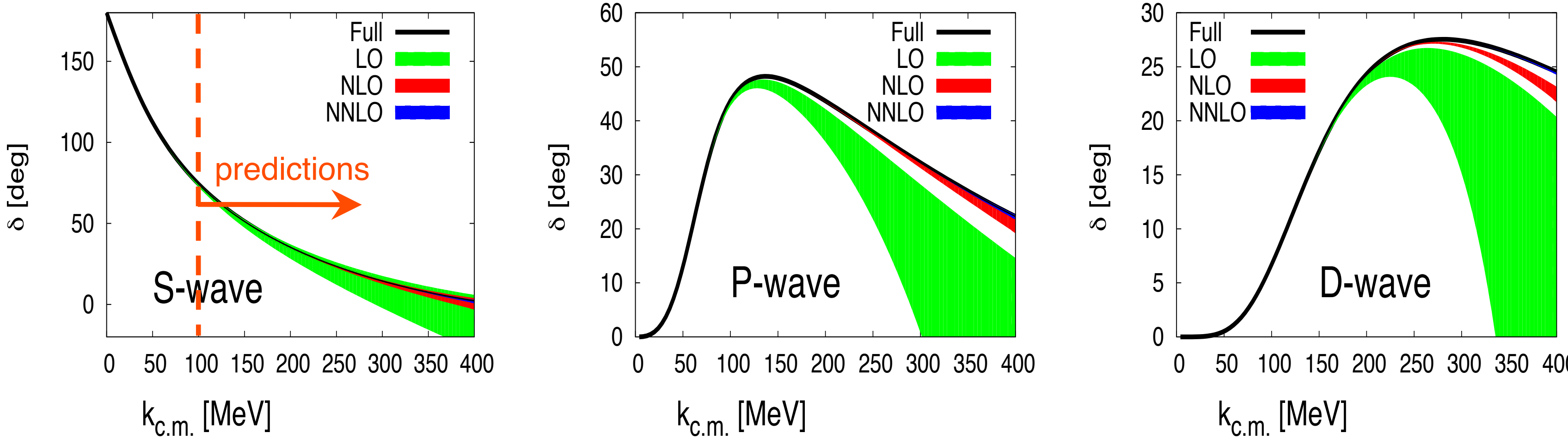
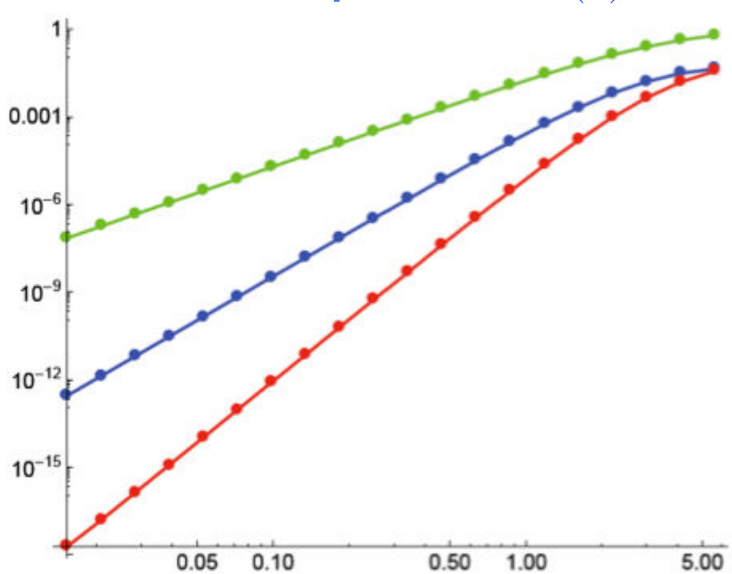
Toy model: phase shifts & error plots



Error plots for $\delta^M(k)$



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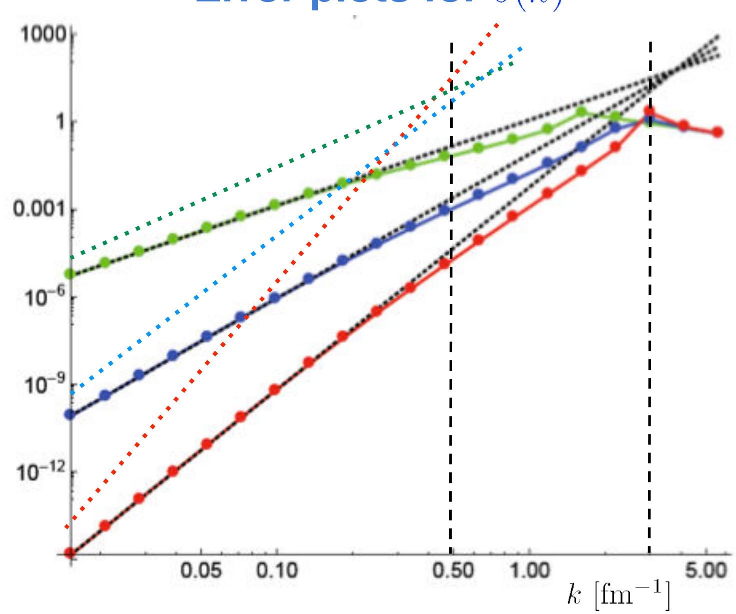

Chiral EFT for NN scattering

6. KSW with perturbative pions

Recall the differences between the W and KSW counting schemes:

- **Weinberg:** $\mu \sim \mathcal{O}(1), \;\; \mu_i \sim \mathcal{O}(p)$ → $V^{\rm LO}_{\rm Weinberg} \sim \mathcal{O}(1), \;\; V^{\rm NLO}_{\rm Weinberg} \sim \mathcal{O}(p^2)$

  [i.e. scaling of C_{2n} according to NDA (~ O(1))]

- **KSW:** $\mu, \mu_i \sim \mathcal{O}(p)$ → $V^{\rm LO}_{\rm KSW} \sim \mathcal{O}(p^{-1}), \;\; V^{\rm NLO}_{\rm KSW} \sim \mathcal{O}(1)$

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  [i.e. scaling of C_{2n} as $C_{2n} \sim O(p^{-1-n})$]

While the two schemes are equivalent for pionless theory, they suggest different scenarios for pionful (chiral) EFT:

$$V_{1\pi} = -\left(\frac{g_A}{2F_\pi}\right)^2 \frac{\vec{\sigma}_1 \cdot \vec{q}\, \vec{\sigma}_2 \cdot \vec{q}}{q^2 + M_\pi^2} \, \vec{\tau}_1 \cdot \vec{\tau}_2 \sim \mathcal{O}(1)$$

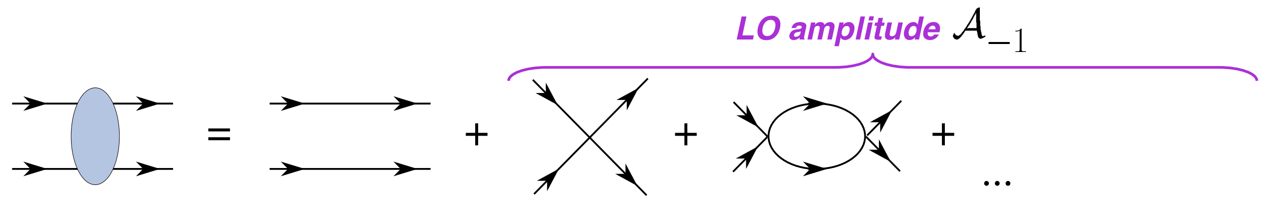
OPE is expected to be:

— LO contribution (nonperturbative) in the Weinberg scheme,

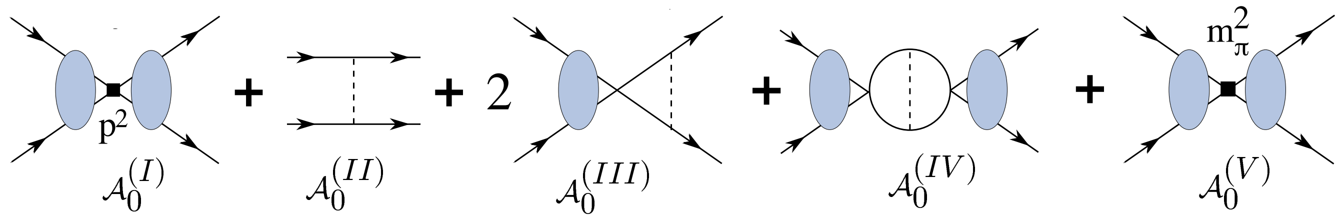
— NLO contribution (perturbative) in the KSW scheme.

Chiral EFT for NN scattering

- **Leading order:**



- **NLO:** $\mathcal{A}_0$ =



$$\mathcal{A}_0^{(I)} = -C_2^{(^1S_0)} p^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}}\right]^2, \qquad \mathcal{A}_0^{(II)} = \left(\frac{g_A^2}{2f^2}\right)\left(-1 + \frac{m_\pi^2}{4p^2}\ln\left(1+\frac{4p^2}{m_\pi^2}\right)\right)$$

$\sqrt{2}F_\pi$

$$\mathcal{A}_0^{(III)} = \frac{g_A^2}{f^2}\left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi}\right)\left(-\frac{(\mu+ip)}{m_\pi} + \frac{m_\pi}{2p}\left[\tan^{-1}\left(\frac{2p}{m_\pi}\right) + \frac{i}{2}\ln\left(1+\frac{4p^2}{m_\pi^2}\right)\right]\right)$$

$$\mathcal{A}_0^{(IV)} = \frac{g_A^2}{2f^2}\left(\frac{m_\pi M \mathcal{A}_{-1}}{4\pi}\right)^2\left(-\left(\frac{\mu+ip}{m_\pi}\right)^2 + \left[i\tan^{-1}\left(\frac{2p}{m_\pi}\right) - \frac{1}{2}\ln\left(\frac{m_\pi^2+4p^2}{\mu^2}\right) + 1\right]\right)$$

$$\mathcal{A}_0^{(V)} = -D_2^{(^1S_0)} m_\pi^2 \left[\frac{\mathcal{A}_{-1}}{C_0^{(^1S_0)}}\right]^2$$

For more details see:
Kaplan, Savage, Wise, Nucl. Phys. B534 (1998) 329.

LETs for NN S-waves

Use these results to test the LETs for S-waves: [Cohen, Hansen, PRC 59 (1999) 13]

$$p \cot \delta_0(p) \;=\; \frac{4\pi}{m} \left[\frac{1}{\mathcal{A}_{-1}} - \frac{\mathcal{A}_0}{(\mathcal{A}_{-1})^2} + \ldots \right] + ip \;\stackrel{!}{=}\; -\frac{1}{a} + \frac{1}{2} r p^2 + v_2 p^4 + v_3 p^6 + v_4 p^8 + \ldots$$

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Express the LECs C_0, C_2, in terms of a and r to predict the shape parameters, e.g.:

$$v_2 \; = \; \frac{g_A^2 m}{16\pi F_\pi^2} \left(-\frac{16}{3a^2 M_\pi^4} + \frac{32}{5a M_\pi^3} - \frac{2}{M_\pi^2} \right), \qquad v_3 \; = \; \frac{g_A^2 m}{16\pi F_\pi^2} \left(\frac{16}{a^2 M_\pi^6} - \frac{128}{7a M_\pi^5} + \frac{16}{3 M_\pi^4} \right), \; \ldots$$

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$^1S_0$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-3.3	18	-108
Nijmegen PWA	-23.7	2.67	-0.5	4.0	-20

$^3S_1$ partial wave	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
NLO KSW Cohen, Hansen '99	fit	fit	-0.95	4.6	-25
Nijmegen PWA	5.42	1.75	0.04	0.67	-4.0

→ large deviations suggest that pions should be treated nonperturbatively…

[Even stronger evidence at N^{2}LO, see: Fleming, Mehen, Stewart, NPA 677 (2000) 313. See however: Kaplan, arXiv:1905.07485.]

Nonperturbative inclusion of pions

7. Nonperturbative inclusion of pions

LO scattering amplitude:

$$T(\vec{p}\,', \vec{p}) \;=\; \left[V_{\rm cont}(\vec{p}\,', \vec{p}) + V_{1\pi}(\vec{p}\,', \vec{p})\right] + m \int \frac{d^3 l}{(2\pi)^3} \frac{\left[V_{\rm cont}(\vec{p}\,', \vec{l}) + V_{1\pi}(\vec{p}\,', \vec{l})\right] T(\vec{l}, \vec{p})}{p^2 - l^2 + i\epsilon}$$

Complications (as compared to pionless theory):

— $V_{1\pi}$ is not separable, no analytic results beyond 2 loops are available,

— $1/r^3$ singularity of $V_{1\pi}$

Static OPEP in coordinate space:

tensor operator: $S_{12} = 3\,\vec{\sigma}_1 \cdot \hat{r}\,\vec{\sigma}_2 \cdot \hat{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2$

$$V_{1\pi}(\vec{r}) = \left(\frac{g_A}{2F_\pi}\right)^2 \tau_1 \cdot \tau_2 \left[M_\pi^2 \frac{e^{-M_\pi r}}{12\pi r} \left(S_{12}(\hat{r}) \left(1 + \frac{3}{M_\pi r} + \underbrace{\frac{3}{(M_\pi r)^2}}\right) + \vec{\sigma}_1 \cdot \vec{\sigma}_2\right) - \frac{1}{3}\,\vec{\sigma}_1 \cdot \vec{\sigma}_2\,\delta^3(r)\right]$$

singular potential in all S=1 channels
(solutions to the Schröd./LS equation still exist in repulsive cases)

→ Need counter terms in all spin-triplet waves! In fact, infinitely many c.t.'s are needed in every spin-triplet channel to remove UV divergences from iterations…

Nonperturbative inclusion of pions

Consider iterations of the LO potential $V_{\rm LO} = V_{1\pi} + C_S + C_T \vec{\sigma}_1 \cdot \vec{\sigma}_2$ in the LS equation

$$T = V + \int V G_0 V + \int\int V G_0 V G_0 V + \ldots \quad \text{where} \quad G_0 = \frac{m}{\vec{p}^{\,2} - \vec{l}^{\,2} + i\epsilon}$$

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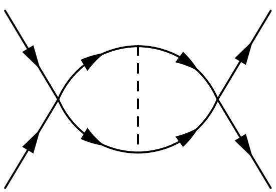
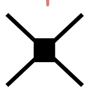
The 2n iteration will generally produce (among other) overall Log-divergences $\times (Q m_N)^{2n}$, where $Q \in \left\{ |\vec{p}|,\ M_\pi \right\}$ (in s=0 channels no powers of $|\vec{p}|$ can appear):



$$\propto \frac{1}{d-4} \frac{g_A^2 C^2}{256\pi^2 F^2} m_N^2 M_\pi^2$$

$\longrightarrow$ need to include $\underbrace{D_0 M_\pi^2}$ $= \left[\delta D_0 + \underbrace{D(\mu_0) + \frac{g_A^2 C^2}{256\pi^2 F^2} m_N^2 \ln\left(\frac{\mu}{\mu_0}\right)}_{D_0^r(\mu)} \right] M_\pi^2$



[However, numerical estimations show no enhancement of renormalized higher-order counter terms, **Gegelia, Scherer, Int. J. Mod. Phys. A21 (2006) 1079**]

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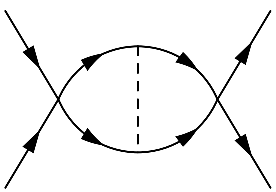

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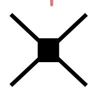
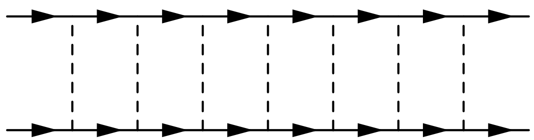


$D_0^r(\mu)$

[However, numerical estimations show no enhancement of renormalized higher-order counter terms, **Gegelia, Scherer, Int. J. Mod. Phys. A21 (2006) 1079**]

Another example:



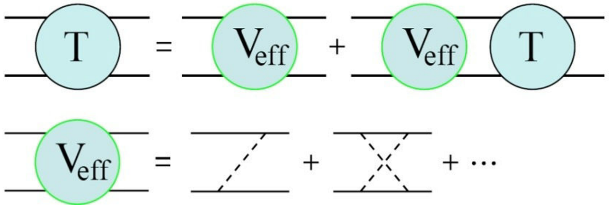
$$\propto \frac{1}{d-4} \vec p^{\,6} m_N^6$$ (spin-triplet)

Nonperturbative inclusion of pions

Nuclear chiral EFT

Weinberg, van Kolck, Kaiser, EE, Glöckle, Meißner, Entem, Machleidt, Krebs, ...

$$\left[\left(\sum_{i=1}^{A}\frac{-\vec{\nabla}_i^2}{2m_N}+\mathcal{O}(m_N^{-3})\right)+\underbrace{V_{2N}+V_{3N}+V_{4N}+\ldots}_{\text{derived in ChPT}}\right]|\Psi\rangle=E|\Psi\rangle$$



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⟶ infinitely many CTs are needed to absorb all UV divergences from iterations!

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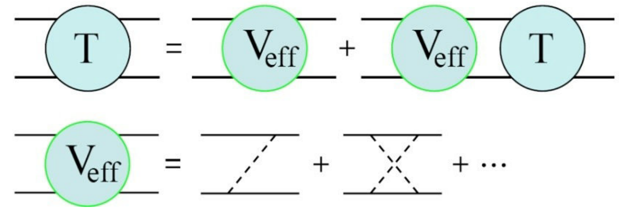

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See: Lepage, „How to renormalize the Schrödinger equation“, nucl-th/9607029 and talk@INT in 2000

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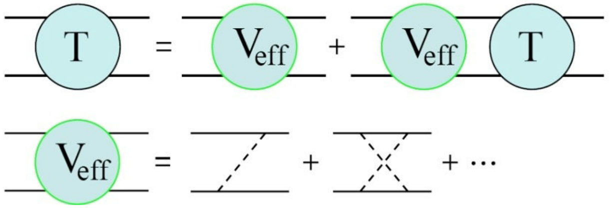

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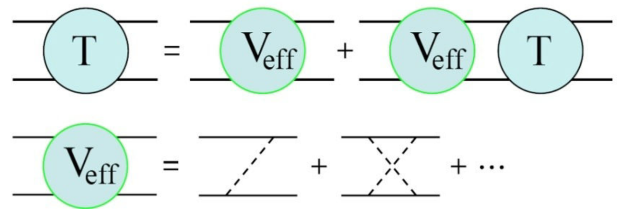

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Nonperturbative inclusion of pions

LETs for neutron-proton scattering: nonperturbative vs perturbative OPEP

	a [fm]	r [fm]	v_2 [fm^3]	v_3 [fm^5]	v_4 [fm^7]
$^1S_0$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.50	−1.9	8.6(8)	−37(10)
NLO EE et al., EPJA51 (15) 71	fit	fit	−0.61 . . . − 0.55	5.1 . . . 5.5	−30.8 . . . − 29.6
NLO KSW Cohen, Hansen '98	fit	fit	−3.3	18	−108
Empirical values	−23.7	2.67	−0.5	4.0	−20

$^3S_1$ partial wave					
LO EE, Gegelia, PLB617 (12) 338	fit	1.60	−0.05	0.82	−5.0
NLO Baru et al., PRC94 (16) 014001	fit	fit	0.06	0.70	−4.0
NLO KSW Cohen, Hansen '98	fit	fit	−0.95	4.6	−25
Empirical values	5.42	1.75	0.04	0.67	−4.0

- perturbative inclusion of pions (KSW approach) fails
- $^1S_0$ channel: limited predictive power of the LETs due to the weakness of the OPEP; taking into account the range correction (NLO) leads to improvement
- $^3S_1$ channel: LETs work as advertised (strong tensor part of the OPEP)

Renormalization vs. peratization

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  However:

  — in the EFT context, the issue is irrelevant as the singularity of the OPEP is beyond the region one can trust the theory **[EE, Meißner, Few Body Syst. 54 (13) 2175]**

  — no reason why „peratized" results should be correct **[EE, Gegelia, EPJA 41 (09) 341]**

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

An analytical example showing the failure of „peratization“

A toy model with separable interactions: $V(p,\, p') = v_l\, F_l(p)\, F_l(p') + v_s\, F_s(p)\, F_s(p')$

with the form-factors: $F_l(p) \equiv \frac{\sqrt{p^2 + m_s^2}}{p^2 + m_l^2}\,, \quad F_s(p) \equiv \frac{1}{\sqrt{p^2 + m_s^2}}$

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„Chiral expansion“ of the ERE coefficients:

$$
\begin{aligned}
a &= \frac{1}{m_l}\left(\alpha_a^{(0)} + \alpha_a^{(1)}\frac{m_l}{m_s} + \alpha_a^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right) \\
r &= \frac{1}{m_l}\left(\alpha_r^{(0)} + \alpha_r^{(1)}\frac{m_l}{m_s} + \alpha_r^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right) \\
v_i &= \frac{1}{m_l^{2i-1}}\left(\alpha_{v_i}^{(0)} + \alpha_{v_i}^{(1)}\frac{m_l}{m_s} + \alpha_{v_i}^{(2)}\frac{m_l^2}{m_s^2} + \ldots\right)
\end{aligned}
$$

The dimensionless coefficients $\alpha_a^{(n)}$, $\alpha_r^{(n)}$ and $\alpha_{v_i}^{(n)}$ are determined by the form of the interaction and expressible in terms of $\alpha_{l,s}$.

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

For example, the scattering length:

$$\alpha_a^{(0)} = \alpha_l \,, \qquad \alpha_a^{(1)} = (\alpha_l - 1)^2 \alpha_s \,, \qquad \alpha_a^{(2)} = (\alpha_l - 1)^2 \alpha_l \alpha_s^2 \,, \qquad \ldots$$

Similarly, for the effective range: $\alpha_r^{(0)} = \dfrac{3\alpha_l - 4}{\alpha_l} \,, \qquad \alpha_r^{(1)} = \dfrac{2\,(\alpha_l - 1)\,(3\alpha_l - 4)\,\alpha_s}{\alpha_l^2} \,,$

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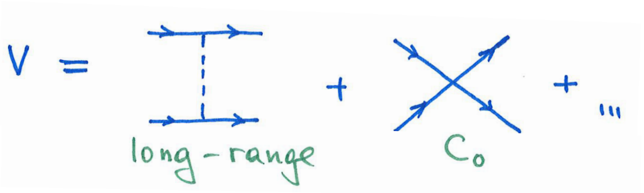

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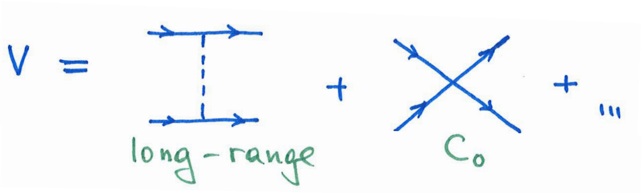

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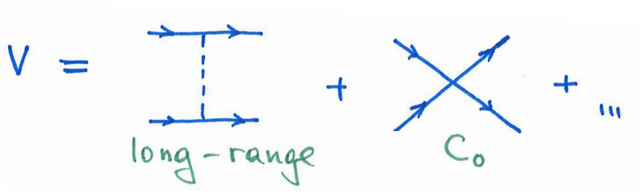

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NLO: C_0 is insufficient to absorb all UV divergences $\longrightarrow$ do a finite-Λ theory:

— calculate the amplitude for a fixed Λ,

— renormalize by tuning $C_0(\Lambda)$ to the scattering length (viewed as „datum“)

Renormalization vs. peratization

EE, Gegelia, EPJA 41 (09) 341

According to the LETs, expect to reproduce $\alpha_r^{(1)}$, $\alpha_{v_i}^{(1)}$. E.g. the effective range:

$$r_\Lambda = \frac{1}{m_l}\Big[\overbrace{\frac{3\alpha_l - 4}{\alpha_l}}^{\text{LO LET}} + \overbrace{\frac{2(\alpha_l-1)(3\alpha_l-4)\alpha_s}{\alpha_l^2 m_s}}^{\text{NLO LET}} m_l + \left(\frac{4(\alpha_l-2)\alpha_s}{\pi\alpha_l m_s^2}\left(\ln\frac{m_s}{2\Lambda}+1\right) + \frac{(\alpha_l-1)(3\alpha_l-4)(5\alpha_l-3)\alpha_s^2 + (2-\alpha_l)\alpha_l^2}{\alpha_l^3 m_s^2}\right) m_l^2 + \mathcal{O}\left(m_l^3\right)\Big]$$

Works as advertised. Λ-dependence appears in terms beyond the accuracy of the calculation. For Λ ~ m_s, their contributions are suppressed (NDA).

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Infinite-Λ limit (peratization)

Take the limit $T_\infty := \lim\limits_{\Lambda\to\infty} T_\Lambda(p,p)$. Fixing again $C_0(\infty)$ from the scattering length we get Λ-independent predictions for the effective range (and shape parameters):

$$r_\infty = \frac{1}{m_l}\Bigg[\frac{3\alpha_l-4}{\alpha_l} + \frac{4\left(\alpha_l-1\right)^2\alpha_s}{\alpha_l^2 m_s} m_l + \frac{\alpha_l^3\left(8\alpha_s^2-1\right)+\alpha_l^2\left(2-20\alpha_s^2\right)+16\alpha_l\alpha_s^2-4\alpha_s^2}{\alpha_l^3 m_s^2} m_l^2 + \ldots\Bigg]$$

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LO LET — NLO LET

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← while Λ-independent, the results violate the LETs which is unacceptable from the EFT point of view

Summary of part II

- NN interaction is strong, some diagrams (ladder) need to be resummed, large scattering lengths require some fine tuning…
- The simplest EFT for NN is pionless EFT. While equivalent to the ERE (for 2N), it may serve as a simple playground to test various concepts.
- Power counting depends on the choice of renormalization conditions. The KSW and W counting schemes are equivalent for pionless EFT (but differ when pions are included…). Alternatively, can do finite-cutoff EFT with implicit renormalization (without actually specifying power counting).
- A proper inclusion of pions must fulfill LETs (MERE). The KSW approach (perturbative pions) strongly violates the LECs, W's approach works fine.
- A nonperturbative inclusion of the 1π-exchange in the nonrelativistic framework (Lippmann-Schwinger) requires infinitely many counterterms to absorb all divergences. It is, therefore, not legitimate to take the infinite-cutoff limit. This may lead to results incompatible with the LECs (peratization rather than renormalization).

Discussion

Open questions:

What expansion of the amplitude does the W. approach correspond to?

— π-less theory: ERE (regardless of the size of the scattering length)
— theory with known long-range forces: MERE
— chiral EFT (long-range force from ChPT): no rigorous answer known…

Are there alternative approaches?

Yes! In particular, the RG analysis by Birse, studies by Pavon-Valderrama and Yang/Long suggest different specific pattern for contact operators…

Can these scenarios be tested/discriminated?

Yes, possibly by looking at the convergence pattern (requires high orders + uncertainty estimation…)