Part II: EFTs for two nucleons: Concepts

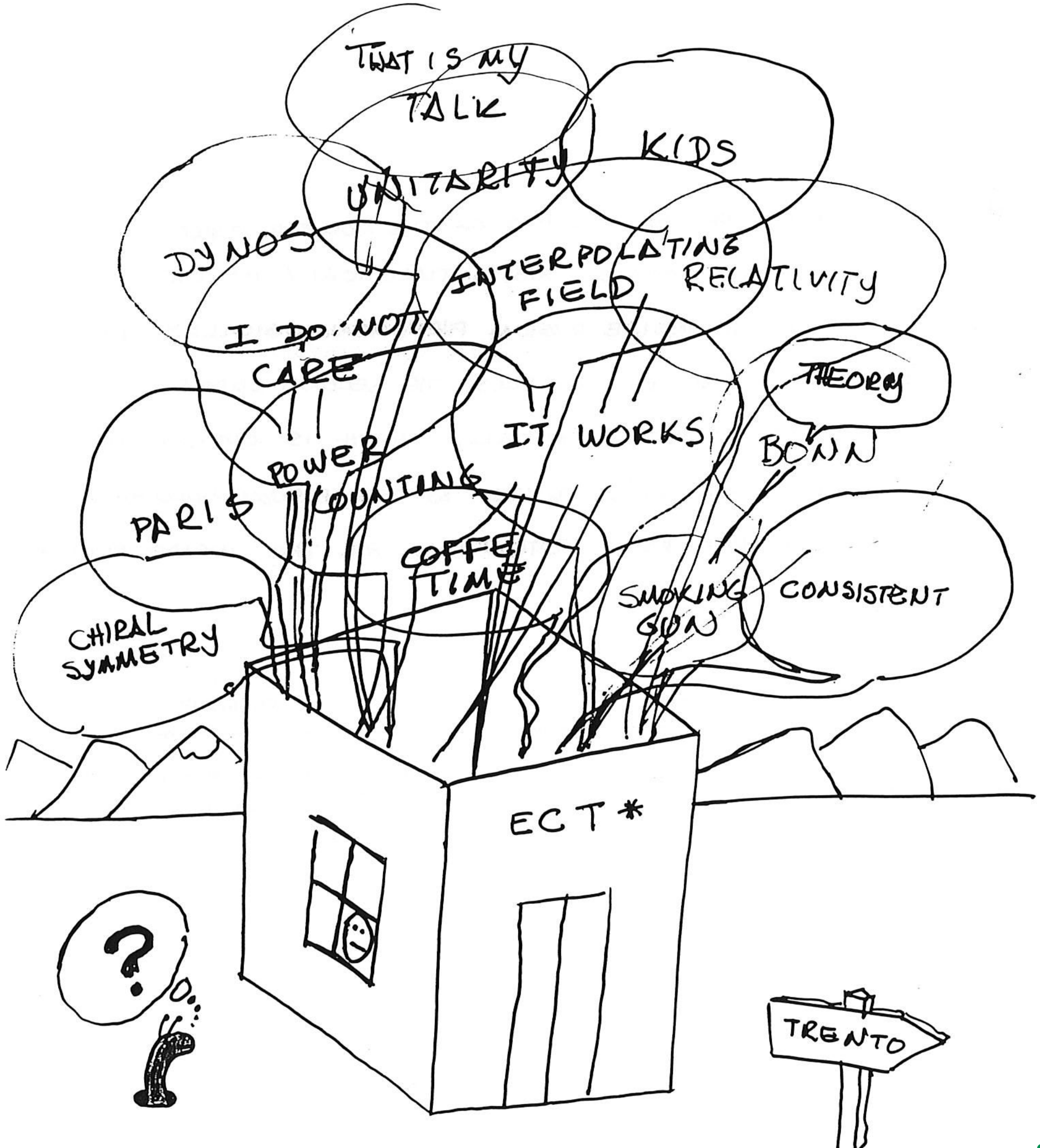
How to design $\not{\pi}$-EFT?

What is the meaning of renormalization/power counting in the context of $\not{\pi}$-EFT?

How to generalize $\not{\pi}$-EFT to chiral EFT (i.e. how to include pions)?

What are the open questions?

Renormalization, power counting and all that…



artwork by Manoel Robilotta
at one of the first nuclear-EFT
ECT* workshops…

Renormalization, power counting and all that…

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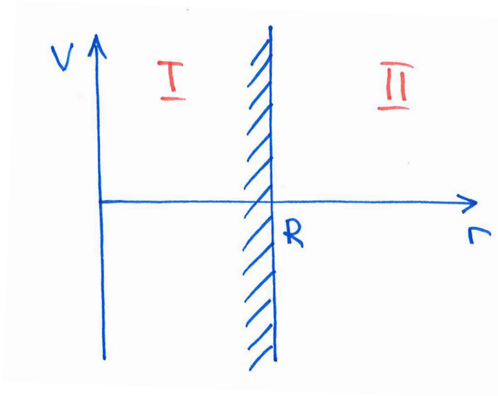
Effective range expansion [Landau, Smorodinsky ’44, Schwinger ’48, Bethe’49]

1. Preliminaries: Effective range expansion



Consider hard-sphere scattering $V(r) = \begin{cases} \infty, & r > R \\ 0, & r < R \end{cases}$

Radial wave function:

$$\begin{cases} R^{I}_{l,p}(r) = 0 \\ R^{II}_{l,p}(r) \propto \alpha j_l(pr) + \beta n_l(pr) \end{cases}$$

$$R^{II}_{l,p}(r) \overset{r\to\infty}{\longrightarrow} \propto \frac{\sin(pr - l\pi/2 + \delta_l)}{pr} \quad \longrightarrow \quad R^{II}_{l,p}(r) = A\Big(\cos\delta_l\, j_l(pr) - \sin\delta_l\, n_l(pr)\Big)$$

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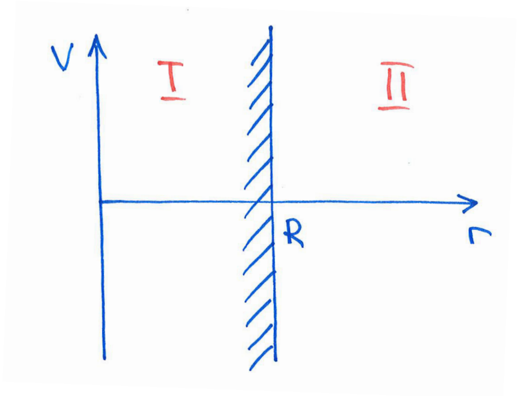

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Matching at the boundary: $R^{II}_{l,p}(R) = 0 \quad \longrightarrow \quad \cot\delta_l(p) = \dfrac{n_l(pR)}{j_l(pR)}$

For s-wave, one then finds: $p\cot\delta_0(p) = -\underbrace{\frac{1}{R}}_{=:\,1/a} + \frac{1}{2}\underbrace{\left(\frac{2R}{3}\right)}_{=:\,r}p^2 + \underbrace{\frac{R^3}{45}}_{=:\,v_2}p^4 + \ldots$

Such an expansion holds for any (regular) short-range potential. The coefficients are specific to the potential. The convergence range is determined by the range of the interaction.

Effective range expansion [Landau, Smorodinsky '44, Schwinger '48, Bethe'49]

The analytic structure of $T_l(E)$ is well known. In addition to the (right-hand) unitarity cut, one observes left-hand cuts specific to V.

$V(r) \propto \frac{e^{-Mr}}{r}$ (or $V(\vec{q}) \propto \frac{1}{\vec{q}^{\,2} + M^2}$), with $\vec{q} = \vec{p}\,' - \vec{p}$ $\longrightarrow$ left-hand cut starts at $p = iM/2$

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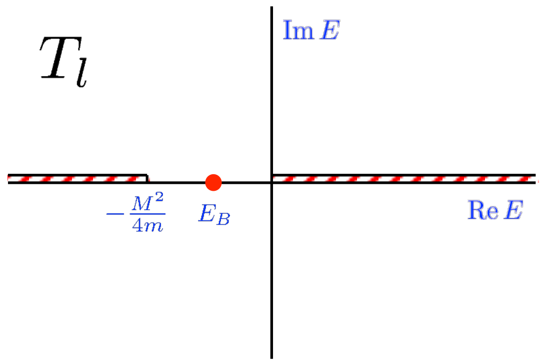

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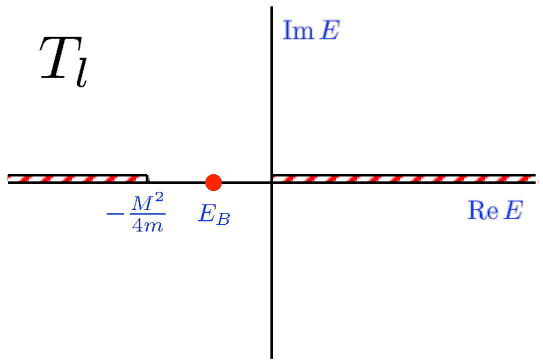
Recall: $S_l(p) = e^{2i\delta_l(p)} = 1 - i\frac{mp}{2\pi} T_l(p)$

$$\rightarrow \; T_l(p) = -\frac{4\pi}{m} \frac{1}{p\cot\delta_l(p) - ip}$$



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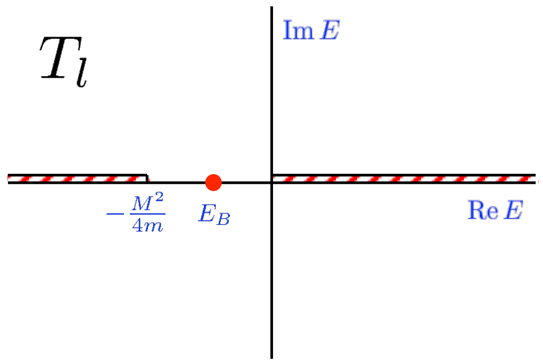

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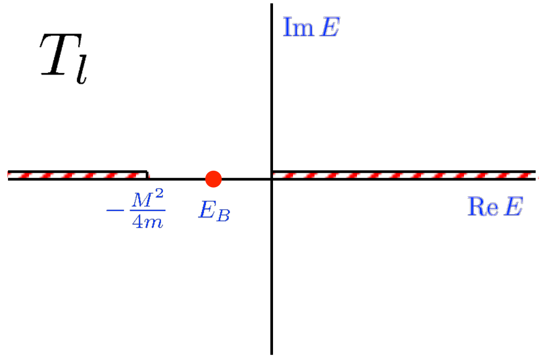
(effective range function)



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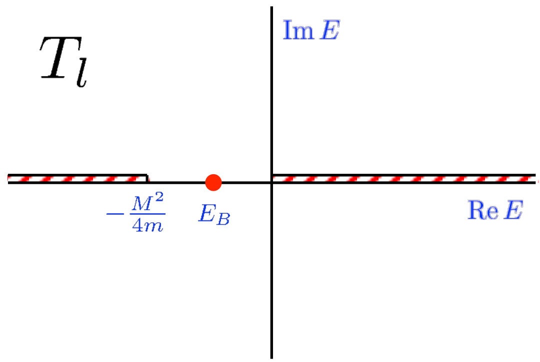
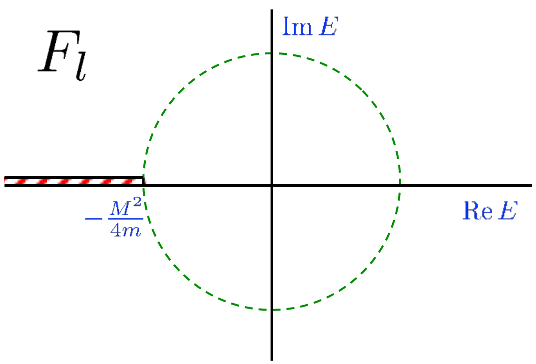


For short-range $V(r)$, $F_l(p)$ is a real meromorphic function of p^2 near the origin

ERE: $$\boxed{F_l(p) = -\frac{1}{a} + \frac{1}{2} r p^2 + v_2 p^4 + v_3 p^6 + \dots}$$



The (maximal) convergence radius is determined by the range M^{-1} of the interaction.

Relevant scales

Weinberg's Folk Theorem:

„if one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition, and the assumed symmetry properties.“

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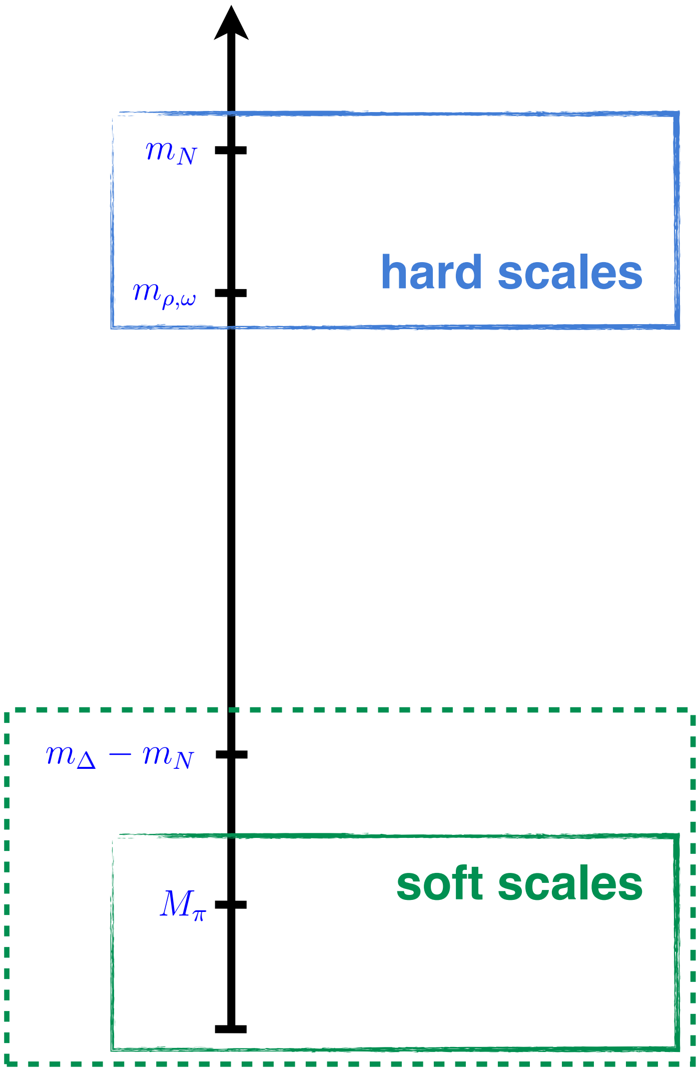
Weinberg's 3rd law of progress in theoretical physics:

„you may use any degrees of freedom you like to describe a physical system, but if you use the wrong ones, you will be sorry.“

in Asymptotic Realms of Physics, MIT Press, Cambridge, 1983

Relevant scales



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$$Q \in \left\{ \frac{M_\pi}{\Lambda}, \frac{|\vec{p}\,|}{\Lambda} \right\}, \quad \Lambda \sim m_\rho \sim 4\pi F_\pi \sim 1 \text{ GeV}$$

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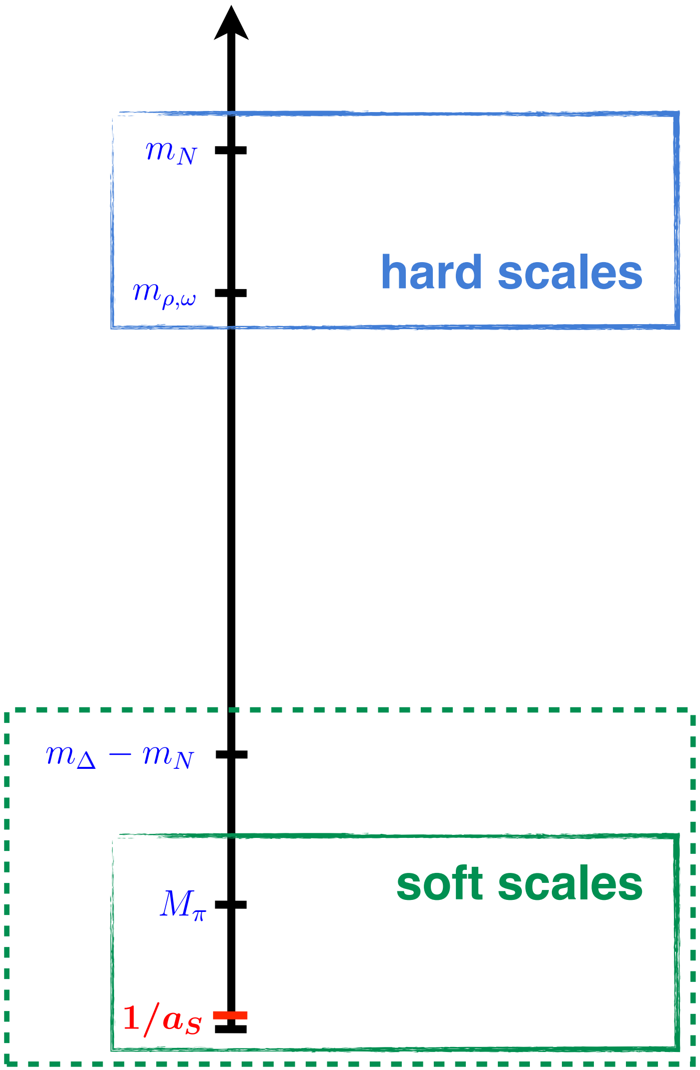

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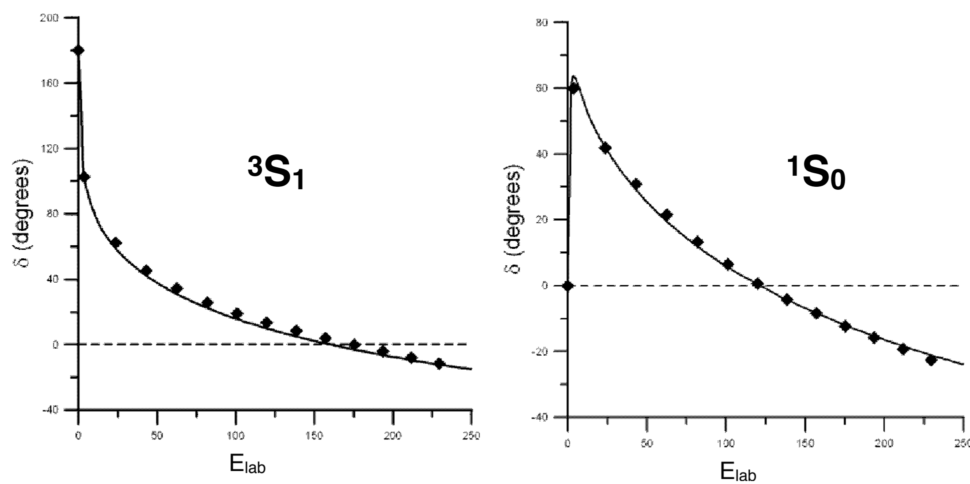

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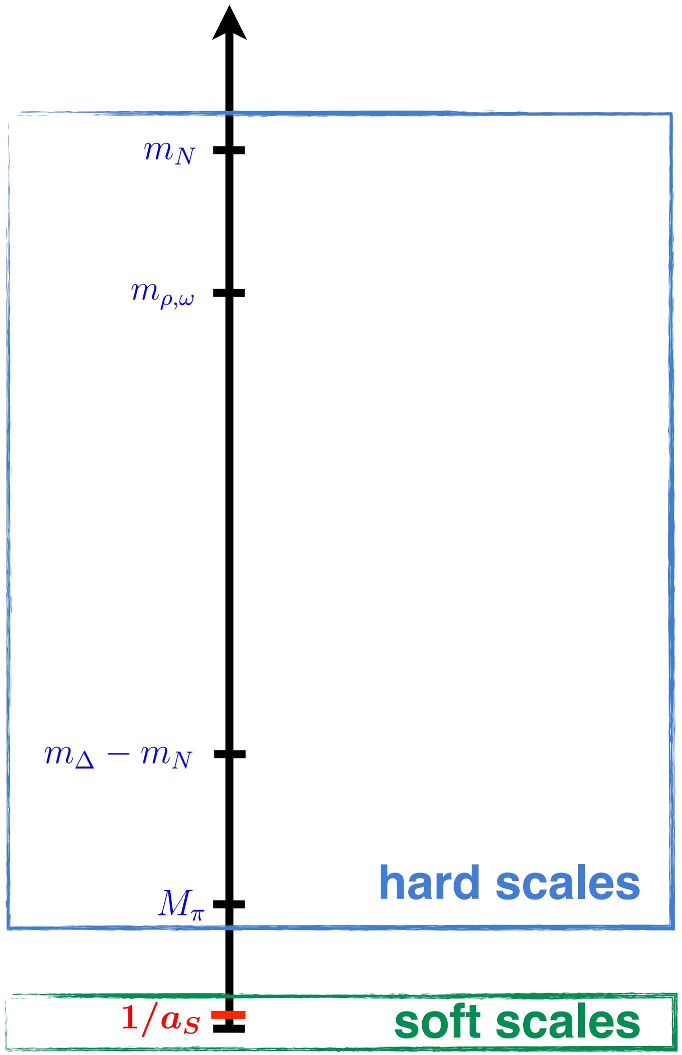

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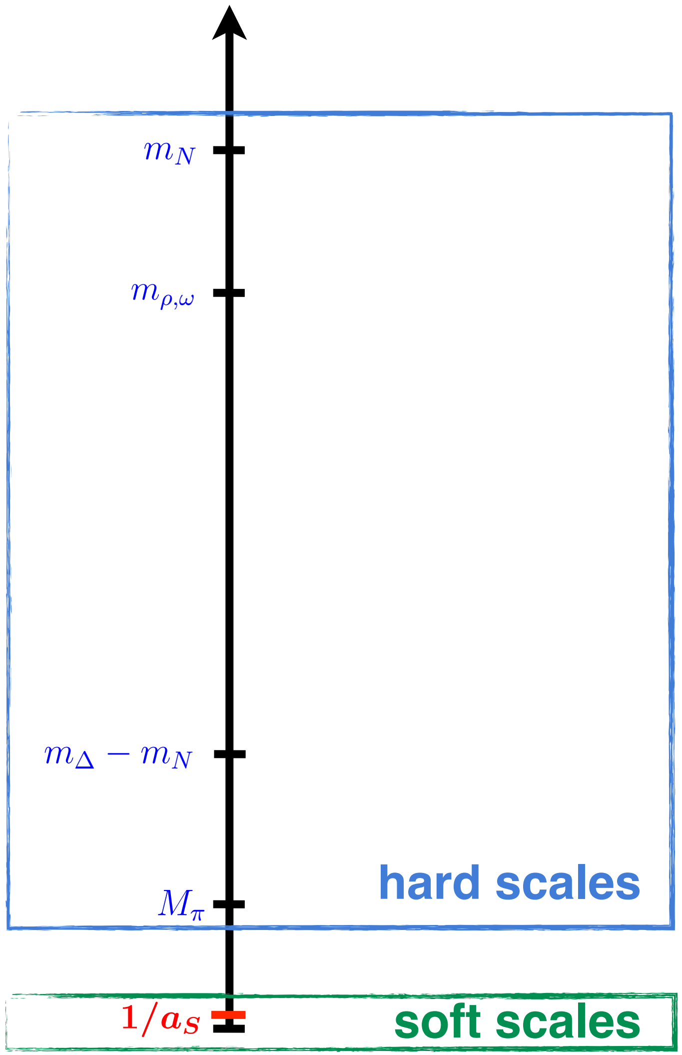

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  Both ERE & $\not{\pi}$-EFT yield an expansion of the amplitude in $|\vec{p}\,|/M_\pi$, have the same validity range and incorporate the same physics ⟶ **ERE ~ $\not{\pi}$-EFT**

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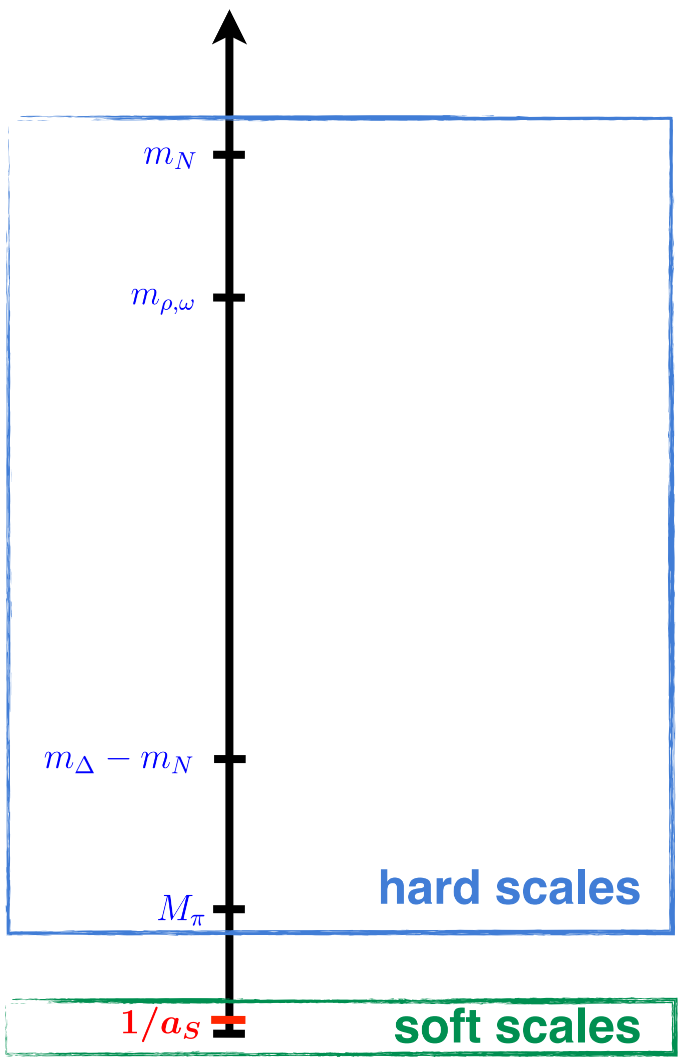

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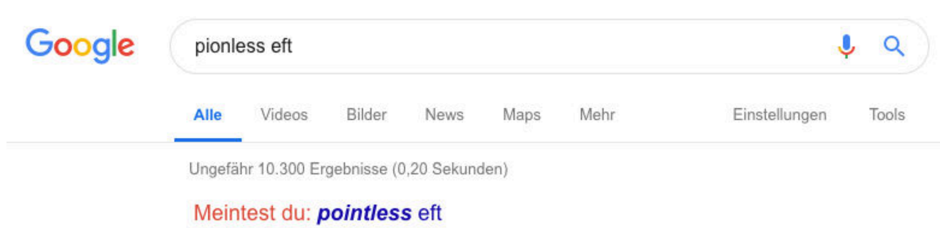

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2. Pionless EFT [Kaplan, Savage, Wise, Nucl. Phys. B534 (1998) 617]

The goal: design an EFT to match ERE (no predictive power beyond ERE)

DOF: nonrelativistic nucleons (use the HB formalism)

Symmetries: rotational invariance, isospin symmetry, usual discrete symmetries…

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$$\mathcal{L} = N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} \right) N - \frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T (N^\dagger \vec{\sigma} N)^2 + \underbrace{\ldots}_{\text{terms with} \geq 2 \text{ derivatives}}$$

Notice: $(N^\dagger \vec{\tau} N)^2$, $(N^\dagger \vec{\tau} \vec{\sigma} N)^2$ are redundant (Pauli principle). Indeed, in there are only 2 independent s-waves ($^1S_0$ and $^3S_1$) in the isospin limit…

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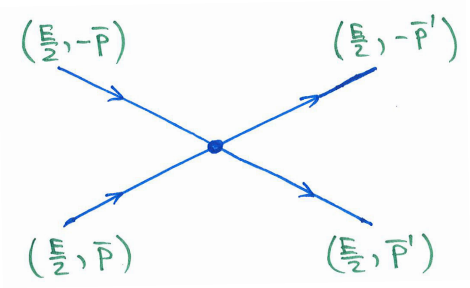
Feynman rule (ignore spin for the moment…):

$$i\mathcal{A}^{\text{tree}} = -i \Big[\underbrace{C_0}_{\substack{\text{linear combination} \\ \text{of } C_S,\, C_T}} + C_2 (\vec{p}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots \Big]$$



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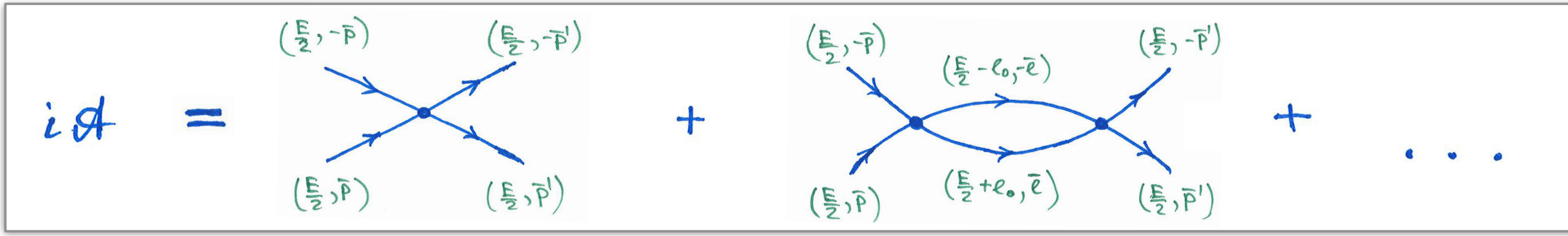

$$
\begin{aligned}
i\mathcal{A}^{1-\mathrm{loop}} &= \int \frac{d^4 l}{(2\pi)^4}(-i)\left[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\right] \frac{i}{\frac{E}{2} + l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon}\, \frac{i}{\frac{E}{2} - l_0 - \frac{\vec{l}^2}{2m_N} + i\epsilon} \\
&\quad \times (-i)\left[C_0 + C_2(\vec{l}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots\right] \\
&= (-i)\int \frac{d^3 l}{(2\pi)^3}\left[C_0 + C_2(\vec{p}^{\,2} + \vec{l}^{\,2}) + \ldots\right] \frac{1}{E - \frac{\vec{l}^2}{m_N} + i\epsilon}\left[C_0 + C_2(\vec{l}^{\,2} + \vec{p}^{\,\prime\,2}) + \ldots\right]
\end{aligned}
$$

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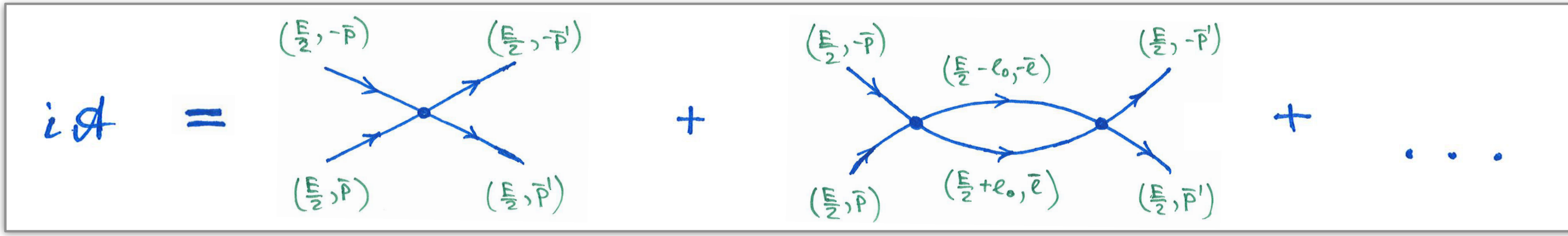

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\end{aligned}
$$

Since loop integrals factorize, the results are trivially generalizable to any number of loops. One finds for $E = p^2/m_N$:

$$
\mathcal{A}(\vec{p}^{\,\prime}, \vec{p}) = V(\vec{p}^{\,\prime}, \vec{p}) - m_N \int \frac{d^3l}{(2\pi)^3} \frac{V(\vec{p}^{\,\prime}, \vec{l}\,)\, \mathcal{A}(\vec{l}, \vec{p}\,)}{\vec{p}^{\,2} - \vec{l}^{\,2} + i\epsilon}
$$

sign convention for V, $\mathcal{A}$

with the potential $V(\vec{p}^{\,\prime}, \vec{p}) = -(C_0 + C_2(\vec{p}^{\,2} + \vec{p}^{\,\prime 2}) + \ldots)$. As expected, the nonrelativistic treatment recovers the quantum mechanical Lippmann-Schwinger equation.

Pionless EFT

In the following, we focus on s-wave scattering. Utilizing the KSW notation,

$$i\mathcal{A}^{\text{tree}} = iV(p', p) = -i(C_0 + \underbrace{C_2(p^2 + p'^2)}_{\substack{\text{terms} \sim pp' \\ \text{contribute to p-waves}}} + \underbrace{\ldots}_{\geq 4 \text{ derivatives}})$$

The LS equation for the half-shell amplitude in the s-waves:

$$\mathcal{A}(p', p) = V(p', p) - m \int \frac{d^3 l}{(2\pi)^3} \frac{V(p', l)\, \mathcal{A}(l, p)}{p^2 - l^2 + i\epsilon}, \quad \text{and} \quad S_0 = 1 + i\frac{mp}{2\pi}\mathcal{A}$$

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- **DR (+PDS):** [KSW '98]

$$J_n(p) = m\int\frac{d^3l}{(2\pi)^3}\frac{l^{2n}}{p^2-l^2+i\epsilon} \xrightarrow{\text{DR}} m\left(\frac{\mu}{2}\right)^{4-d}\int\frac{d^{d-1}l}{(2\pi)^{d-1}}\frac{l^{2n}}{p^2-l^2+i\epsilon}$$

$$= -mp^{2n}(-p^2-i\epsilon)^{\frac{d-3}{2}}\Gamma\left(\frac{3-d}{2}\right)\frac{(\mu/2)^{4-d}}{(4\pi)^{\frac{d-1}{2}}} \xrightarrow{d=4} -mp^{2n}\frac{ip}{4\pi}$$

The DR result is finite for $d=4$ (power-like divergences). Instead of using MS, it is advantageous to subtract poles in $d=3$ (see later). Then: $J_n(p) = -mp^{2n}\frac{\mu+ip}{4\pi}$.

Pionless EFT

- **Cutoff regularization + subtractions:**
  [Gegelia '98]

$$m\int\frac{d^3l}{(2\pi)^3}\frac{l^{2n}}{p^2-l^2+i\epsilon} = \underbrace{-m\int\frac{d^3l}{(2\pi)^3}\,l^{2n-2}}_{=:\,I_{2n+1}} - \ldots - \underbrace{mp^{2n-2}\int\frac{d^3l}{(2\pi)^3}}_{=:\,p^{2n-2}I_3} + \underbrace{mp^{2n}\int\frac{d^3l}{(2\pi)^3}\frac{1}{p^2-l^2+i\epsilon}}_{=:\,p^{2n}I(p)}$$

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Notice: PDS corresponds to the particular choice $\mu\to\mu\pi/2$, $\mu_i = 0$.

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Having defined renormalization scheme, we still need to specify renormalization conditions (i.e. the choice of subtraction scales).

Conventional wisdom suggests: $\mu, \; \mu_i$ ~ soft scale $\sim p \ll M_\pi$. [i.e. loop momenta of the order of the soft scale after renormalization…]

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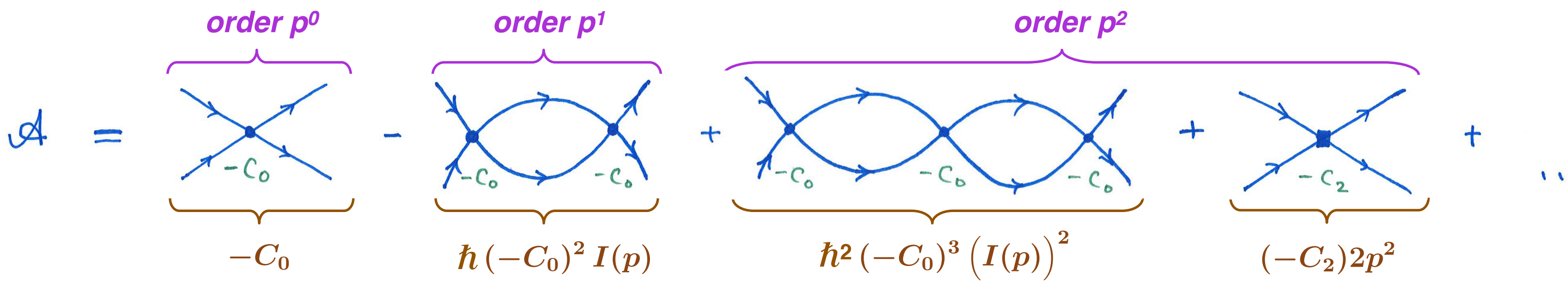

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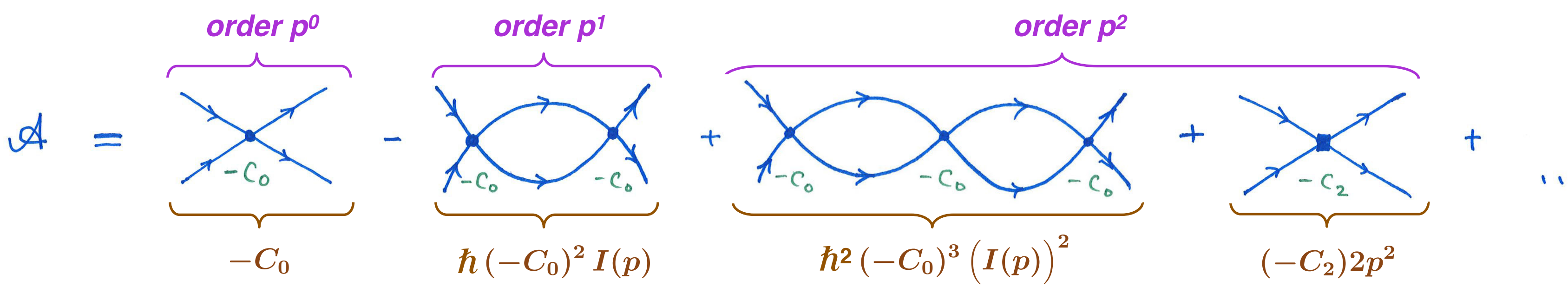

Recall: $$I(p) = \underbrace{\lim_{\Lambda\to\infty}\left(\frac{m}{2\pi^2}(\mu-\Lambda)\right)}_{=\,\Delta(\mu)} \underbrace{-\,\frac{m}{4\pi}\left(ip+\frac{2}{\pi}\mu\right)}_{=\,I^{\rm R}(\mu,p)}$$

Renormalization: $$C_0 = C_0^{\rm R}(\mu) + \hbar\,\underbrace{\delta C_{0,\,1}}_{-\,(C_0^{\rm R})^2\,\Delta} + \hbar^2\,\underbrace{\delta C_{0,\,2}}_{(C_0^{\rm R})^3\,\Delta^2} + \mathcal{O}(\hbar^3)$$

Thus, finally: $$\boxed{\mathcal{A} = -C_0^{\rm R}(\mu) - \left(C_0^{\rm R}(\mu)\right)^2 I^R(\mu,p) - \left(C_0^{\rm R}(\mu)\right)^3\left(I^R(\mu,p)\right)^2 - 2C_2p^2 + \ldots}$$

[If all counter terms are included, renormalization amounts to just replacing $C_i \to C_i^{\rm R}(\mu)$, $I_{[n]} \to I_{[n]}^{\rm R}(\mu)$.]

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To determine LECs, we have to match the amplitude to the ERE:

$$\mathcal{A} = \frac{4\pi}{m}\frac{1}{p\cot\delta - ip} = \frac{4\pi}{m}\frac{1}{\left[-\frac{1}{a}+\frac{1}{2}rp^2+v_2p^4+\ldots\right]-ip}$$

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However, in reality, the scattering lengths are large:

$$a_{^1S_0} = -23.714 \text{ fm} \sim -16.6\, M_\pi^{-1}, \qquad a_{^3S_1} = 5.42 \text{ fm} \sim 3.8\, M_\pi^{-1}$$

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Thus, it seems more appropriate to count $a \sim p^{-1}$. This leads to the expansion:

$$\mathcal{A} = -\frac{4\pi}{m}\left[\underbrace{\frac{1}{a^{-1}+ip}}_{\text{order } p^{-1}} + \underbrace{\frac{rp^2}{2(a^{-1}+ip)^2}}_{\text{order } p^{0}} + \underbrace{\frac{r^2p^4}{4(a^{-1}+ip)^3}}_{\text{order } p} + \ldots\right]$$

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The large scattering length signals non-perturbative physics. In order to accommodate for it, some fine tuning has to be introduced in the EFT.

The resulting power counting depends on the choice of renormalization conditions!

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- **Weinberg:** $\mu \sim \mathcal{O}(1), \quad \mu_i \sim \mathcal{O}(p) \quad \rightarrow \quad x = 0 \quad \rightarrow \quad V^{\mathrm{LO}}_{\mathrm{Weinberg}} \sim \mathcal{O}(1)$
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Scaling of $V^{\rm NLO}$ can be read off from:

$$\hat{\mathcal{A}}^{(0)} = \hat V^{\rm NLO} - \hat V^{\rm NLO} \hat G_0 \hat{\mathcal{A}}^{(-1)} - \hat{\mathcal{A}}^{(-1)} \hat G_0 \hat V^{\rm NLO} + \hat{\mathcal{A}}^{(-1)} \hat G_0 \hat V^{\rm NLO} \hat G_0 \hat{\mathcal{A}}^{(-1)} \longrightarrow \begin{cases} V^{\rm NLO}_{\rm Weinberg} \sim \mathcal{O}(p^2) \\ V^{\rm NLO}_{\rm KSW} \sim \mathcal{O}(1) \end{cases}$$

Pionless EFT

Both choices of the renormalization conditions

— lead to self-consistent approaches,

— are equivalent for pionless EFT (but yield different results in chiral EFT…),

— involve some fine tuning beyond NDA **[see: EE, Gegelia, Meißner, NPB 925 (2017) 161]**

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Both choices of the renormalization conditions

— lead to self-consistent approaches,

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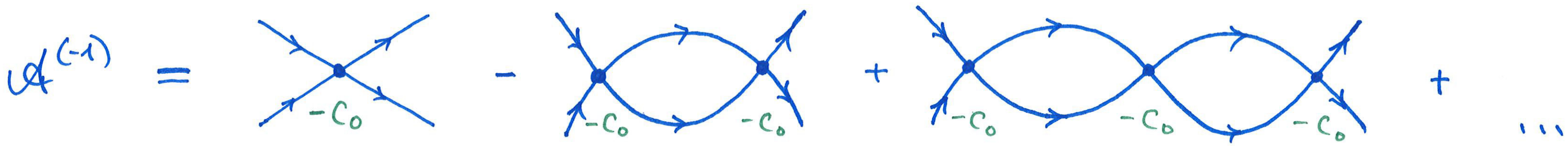
Leading order (p^{-1}):



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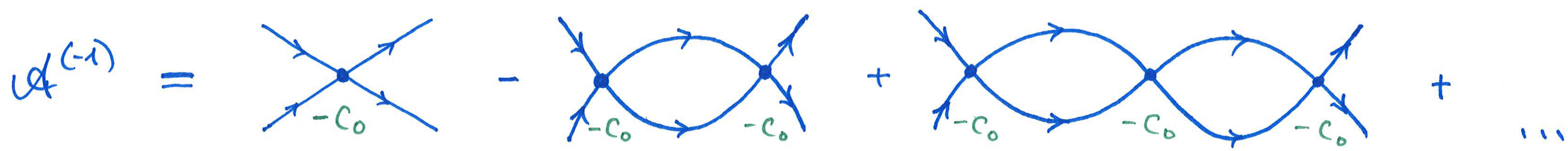

$$\mathcal{A}^{(-1)} = -C_0 - C_0^2 I(p) - C_0^3 (I(p))^2 + \ldots = -\frac{1}{C_0^{-1} - I(p)} = -\frac{1}{(C_0^{\rm R}(\mu))^{-1} - I^{\rm R}(\mu, p)}$$

$$= -\frac{4\pi}{m} \frac{1}{\frac{4\pi}{m}(C_0^{\rm R}(\mu))^{-1} + \frac{2}{\pi}\mu + ip} \stackrel{!}{=} -\frac{4\pi}{m}\frac{1}{a^{-1} + ip} \quad \longrightarrow \quad C_0^{\rm R} = \frac{4\pi}{m}\frac{1}{a^{-1} - \frac{2}{\pi}\mu}$$

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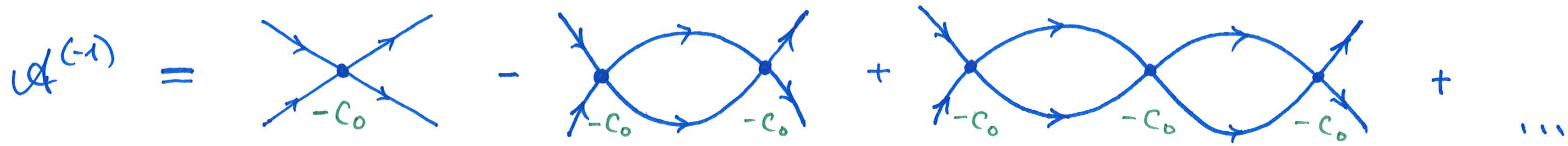

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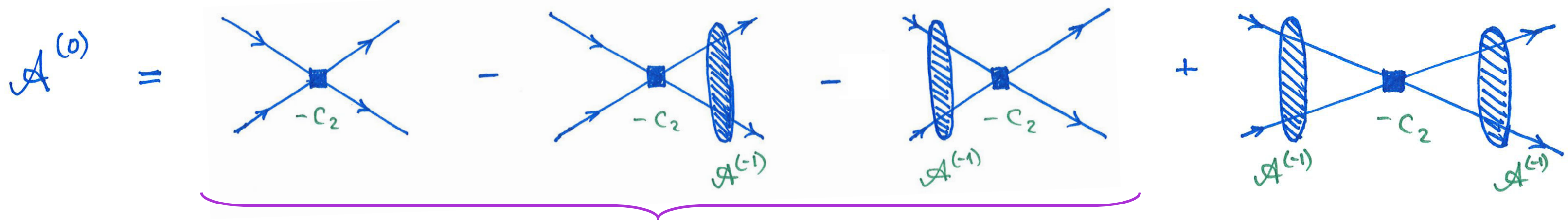
One recovers the Weinberg/KSW scaling of $C_0^{\rm R}$ depending on the choice of μ:

$C_0^{\rm R} \sim \mathcal{O}(1)$ for $\mu \sim \mathcal{O}(1)$; $\quad C_0^{\rm R} \sim \mathcal{O}(p^{-1})$ for $\mu \sim \mathcal{O}(p)$.

Pionless EFT

Subleading order (p^0):



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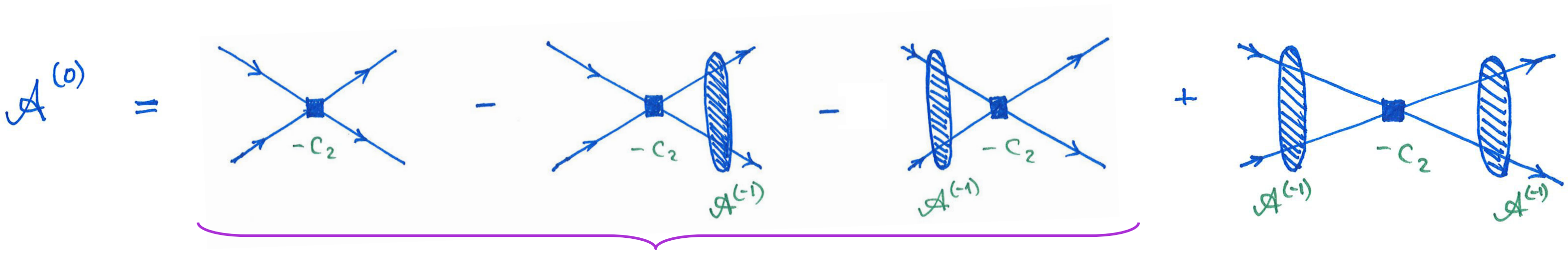

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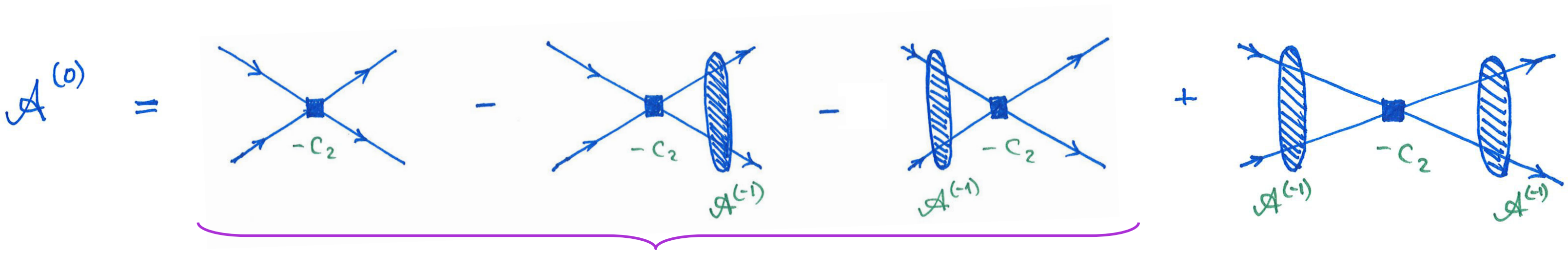

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For the sake of simplicity, choose $\mu_3 = 0$ (so that $I_3^{\rm R} = 0$).

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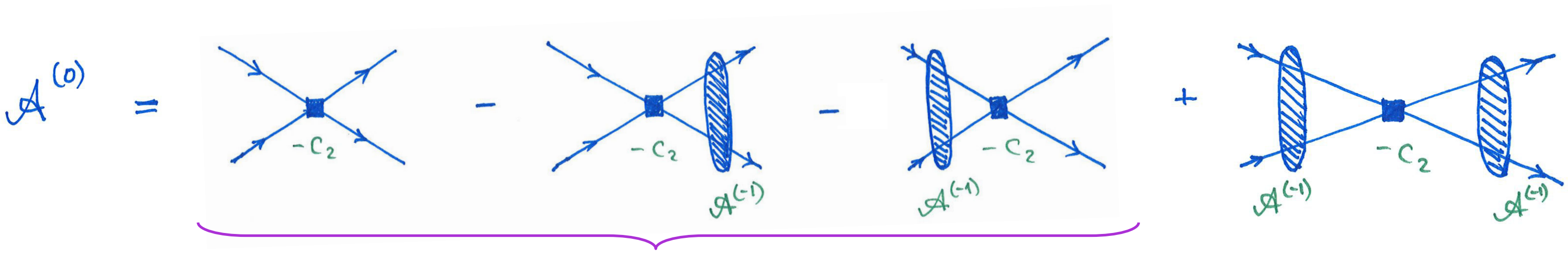

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Again, we recover: $C_2^{\rm R} \sim \mathcal{O}(1)$ for $\mu \sim \mathcal{O}(1)$; $\quad C_2^{\rm R} \sim \mathcal{O}(p^{-2})$ for $\mu \sim \mathcal{O}(p)$.

Pionless EFT

3. Wilsonian RG analysis

Birse et al.; Harada et al.

Lippmann-Schwinger equation for the off-shell K-matrix

$$K(k', k, p) = V(k', k, p, \Lambda) + 2M\,\mathcal{P}\int \frac{d^3l\,\theta(\Lambda - l)}{(2\pi)^3}\,\frac{V(k', l, p, \Lambda)\,K(l, k, p)}{p^2 - l^2}$$

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{\pi^2}\,V(k', \Lambda, p, \Lambda)\,\frac{\Lambda^2}{\Lambda^2 - p^2}\,V(\Lambda, k, p, \Lambda)$$

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Express all low-energy scales in units of Λ, $\hat V(\hat k',\hat k,\hat p,\Lambda) := \frac{M\,\Lambda}{\pi^2}\,V(\Lambda\hat k',\Lambda\hat k,\Lambda\hat p,\Lambda)$,

to obtain the RG equation for the rescaled potential:

$$\Lambda\,\frac{\partial \hat V}{\partial \Lambda} = \hat k'\,\frac{\partial \hat V}{\partial \hat k'} + \hat k\,\frac{\partial \hat V}{\partial \hat k} + \hat p\,\frac{\partial \hat V}{\partial \hat p} + \hat V(\hat k',\hat k,\hat p,\Lambda) + \hat V(\hat k',1,\hat p,\Lambda)\,\frac{1}{1-\hat p^2}\,\hat V(1,\hat k,\hat p,\Lambda)$$

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For Λ → 0, the system approaches a fixed point (scale-invariant solutions):

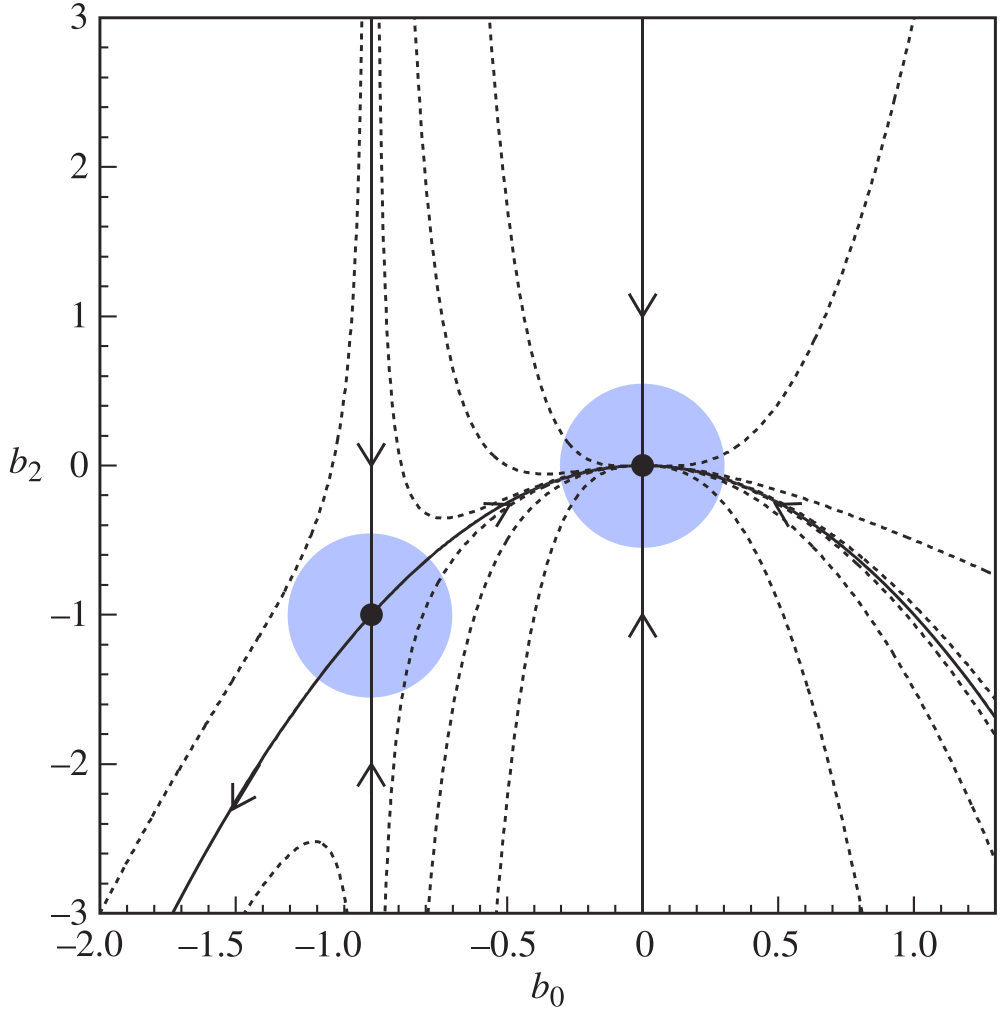
— trivial: $\hat{V} = 0$ (stable, no scattering)

— non-trivial, p-dependent: $\frac{1}{\hat{V}} = -1 + \frac{\hat{p}}{2}\ln\frac{1+\hat{p}}{1-\hat{p}}$ (unstable, unitary limit)

Notice: there exist infinitely many other fixed points [Birse, EE, Gegelia ’16]

Pionless EFT

Power counting can be identified by considering perturbations around the fixed points, e.g. of the form ~ p^{2n}.



RG flow of the potential

$$\hat{V}(\hat{p}, \Lambda) = b_0(\Lambda) + b_2(\Lambda)\hat{p}^2 + \ldots$$

(from: Birse, Phil. Trans. R. Soc. A (2011) 369, 2662-2678)

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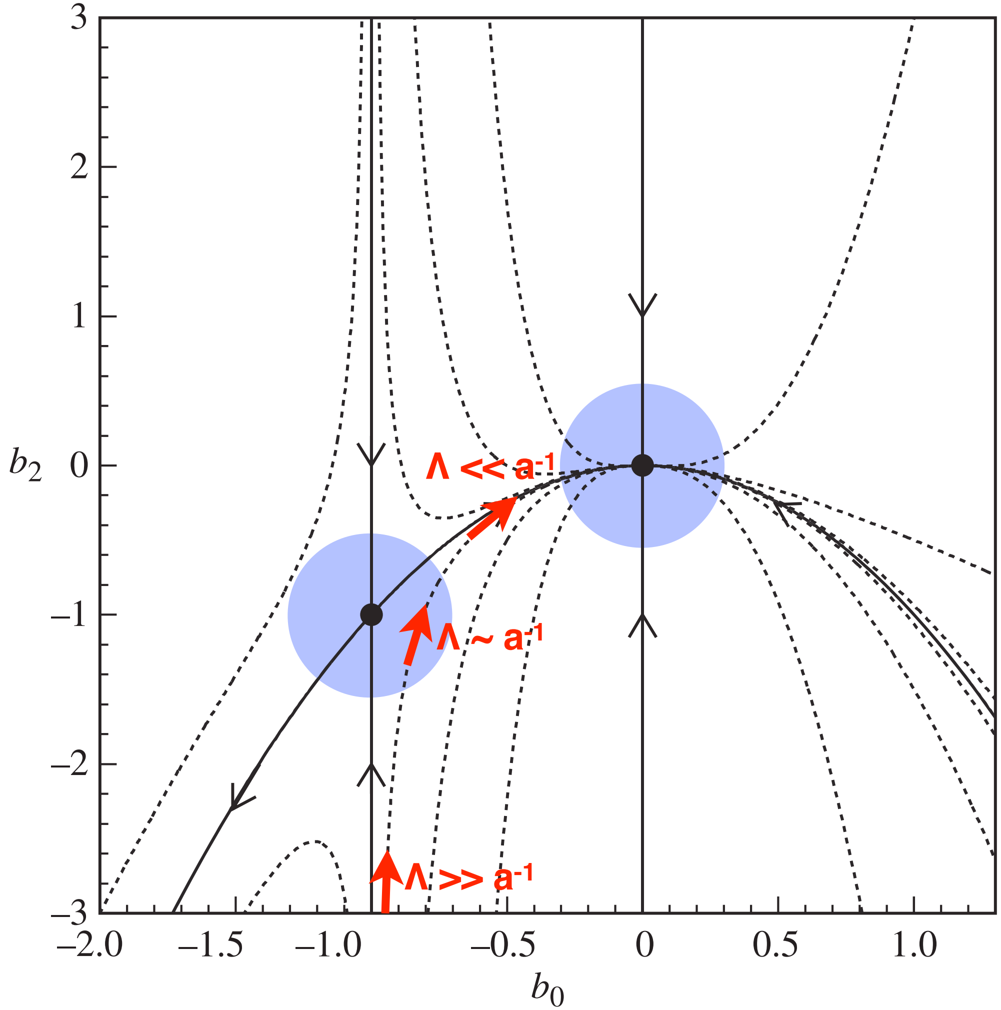

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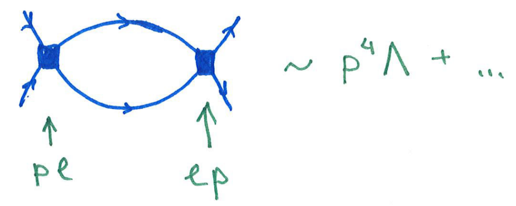
The power counting near the nontrivial fixed point is found to agree with the KSW one (same choice of renorm. conditions…)

4. Finite-cutoff EFT for pionless theory

Suppose, one needs to include the range term nonperturbatively (e.g. one wants describe p-wave resonance).

Problem: need infinite number of counter terms…



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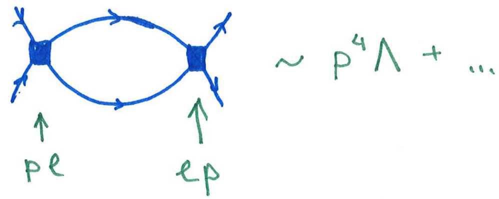


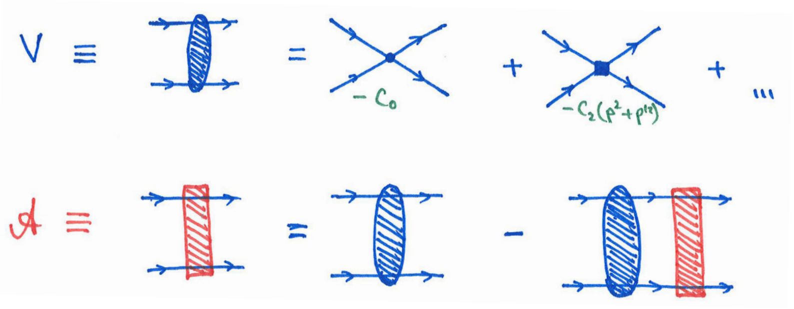
Fortunately, that's not the end of EFT!

„The theory is fully specified by the values of the bare constants once a suitable regularization procedure is chosen. In principle, the renormalization program is straightforward: one calculates quantities of physical interest in terms of the bare parameters at given, large value of (ultraviolet cutoff) Λ. Once a sufficient number of physical quantities have been determined as functions of the bare parameters one inverts the result and expresses the bare parameters in terms of physical quantities, always working at some given, large value of Λ. Finally, one uses these expressions to eliminate the bare parameters in all other quantities of physical interest."

Gasser, Leutwyler, Phys. Rep. **87** (1982) 77

Pionless EFT

Define the contact potential; introduce a UV cutoff $\Lambda \sim M_\pi$; solve the LS equation; tune **bare** LECs $C_0(\Lambda)$, $C_2(\Lambda)$ to a, r.



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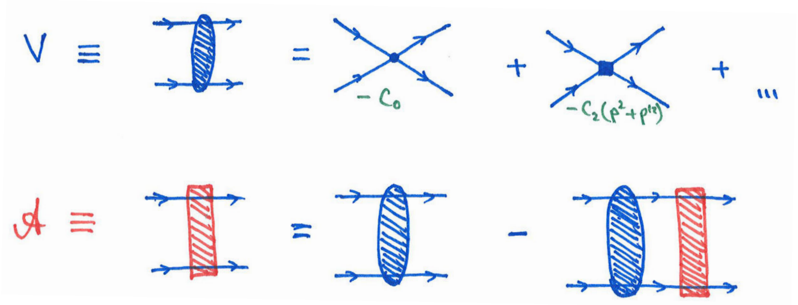

For a sharp cutoff, one finds at NLO:

$$mC_0 = \frac{6\pi^2\left(\beta - 6\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2}\right)}{5\alpha\Lambda}\,, \qquad mC_2 = \frac{6\pi^2\left(\sqrt{3}\sqrt{\alpha(\pi - 2a\Lambda)^2} - \alpha\right)}{\alpha\Lambda^3}$$

where I have introduced:

$$\alpha \equiv 16a^2\Lambda^2 - \pi a\Lambda\left(a\Lambda^2 r + 12\right) + 3\pi^2\,, \qquad \beta \equiv 64a^2\Lambda^2 - \pi a\Lambda\left(3a\Lambda^2 r + 62\right) + 18\pi^2$$

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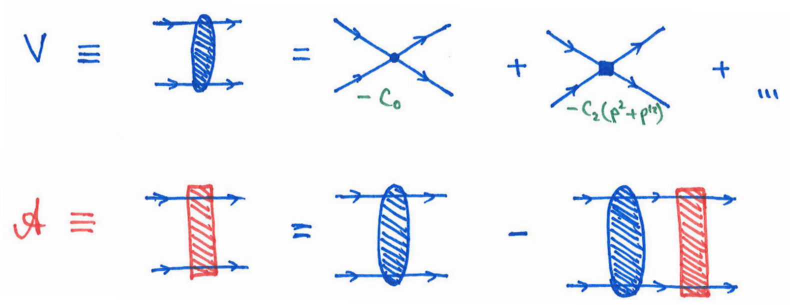

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Implicitly renormalized expression for the inverse amplitude:

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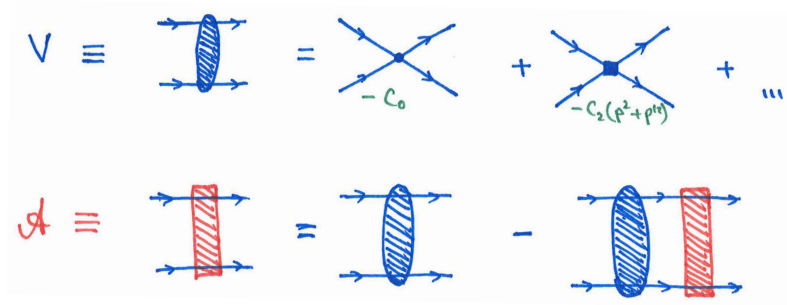

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⟶ well-defined & correct (up to higher-order terms) result for $\Lambda \sim r^{-1} \sim M_\pi$;
things may (and generally will!) go wrong for $\Lambda \gg r^{-1}$
(complex C_i, Wigner bound, peratization…)

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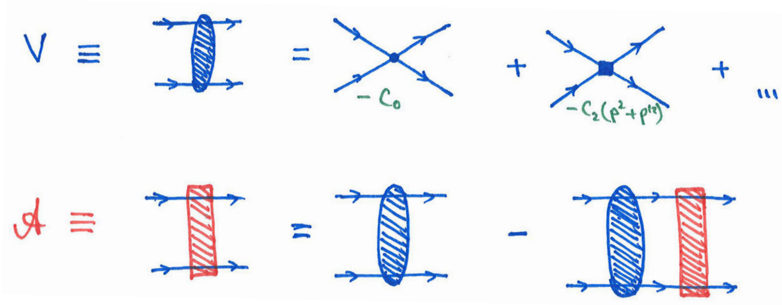

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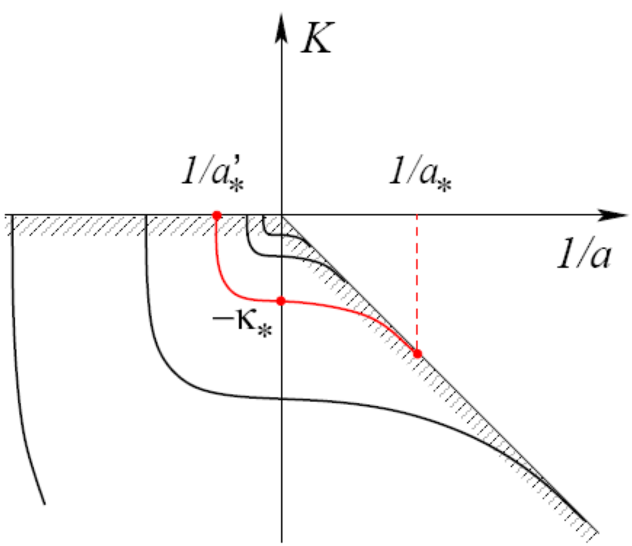
Notice:

- Contrary to the previous cases, not all c.t. needed to remove UV divergences are included ⟶ it is not legitimate to take the limit $\Lambda \to \infty$.
- Higher-order terms are indeed small provided $\Lambda \sim$ hard scale (NDA…).
- Implicit renormalization (i.e. no explicit splitting of C_i into $C_i^{\mathrm{R}}(\mu)$ and $\Delta(\mu)$).
- The bare LECs $C_i(\Lambda)$ must be re-fitted at every order.

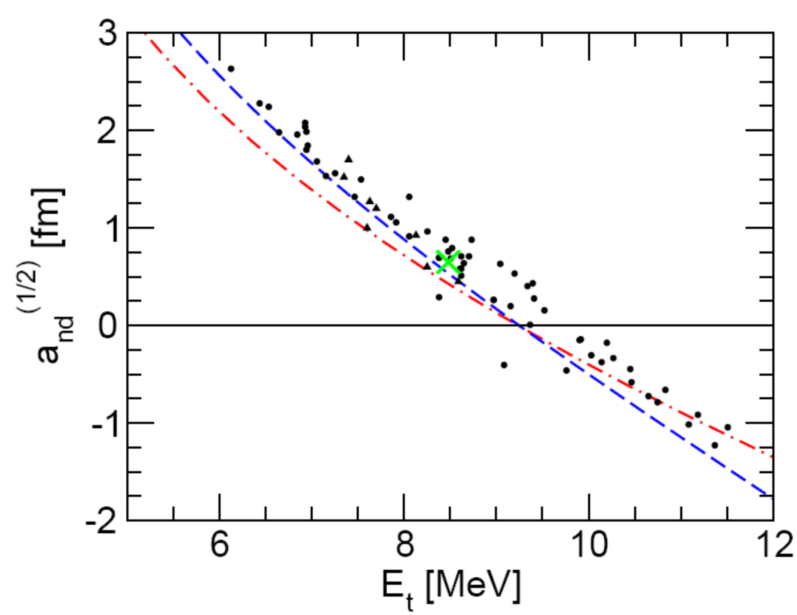
Pionless EFT: (some) applications

- Astrophysical reactions Butler, Chen, Kong, Ravndal, Rupak, Savage, …
- Efimov physics and universality in few-body systems with large 2-body scatt. length (e.g. Phillips/Tjon „lines“) Braaten, Hammer, Meißner, Platter, von Stecher, Schmidt, Moroz, …
- Halo-nuclei Bedaque, Bertulani, Hammer, Higa, van Kolck, Phillips, …
- Parity violation Schindler, Springer, Vanasse, … any many other topics…

Efimov effect (3-body spectrum)



Phillips line



Braaten, Hammer, Phys. Rept 428 (06) 259