Evgeny Epelbaum, RUB

TALENT School „From Quarks and Gluons to Nuclear Forces and Structure“
ECT*, Trento, July 15 - August 2, 2019

Chiral effective field theory for nuclear forces

Concepts

Effective field theory, chiral perturbation theory, renormalization, predictive power, KSW vs Weinberg, power counting…

Methods

S-matrix matching, method of unitary transformation to derive nuclear forces (and currents), …

Useful literature

- **General:** EE, ***Nuclear forces from chiral EFT: A primer***, arXiv:1001.3229
- **Organization of nuclear chiral EFT (renormalization, power counting, …)**
  - Weinberg, ***Effective chiral Lagrangians for nucleon-pion interactions and nuclear forces,*** Nucl. Phys. B363 (1991) 3
  - Lepage, ***How to renormalize the Schrödinger equation***, nucl/th:9706029
  - Kaplan, Savage, Wise, ***Two nucleon systems from EFT,*** Nucl. Phys. B534 (1998) 617
  - Nogga, Timmermans, van Kolck, ***Renormalization of one-pion exchange and power counting,*** Phys. Rev. C72 (2005) 054006
  - EE, Gegelia, ***Regularization, renormalization and „peratization“ in EFT for two nucleons,*** Eur. Phys. J. A41 (2009) 341
  - Birse, ***The renormalization group and nuclear forces,*** Phil. Trans. Roy. Soc. Lond. A369 (2011) 2662
  - EE, Gegelia, Meißner, ***Wilsonian renormalization group versus subtractive renormalization in EFTs for nucleon-nucleon scattering***, Nucl. Phys. B925 (2017) 161
- **Derivation of nuclear forces and currents using the method of UT**
  - Okubo, ***Diagonalization of Hamiltonian and Tamm-Dancoff equation,*** Prog. Theor. Phys. 12 (1954) 603
  - EE, ***Four-nucleon force using the method of unitary transformation,*** Eur. Phys. J. A34 (2007) 197
  - Krebs, EE, Meißner, ***Nuclear axial current operators to fourth order in chiral EFT,*** Annals Phys. 378 (2017) 317

Useful literature

- **Uncertainty quantification in nuclear chiral EFT**
  - — Ekström et al., ***Statistical uncertainties of a chiral interaction at next-to-next-to-leading order***, J. Phys. G42 (2015) 034003
  - — EE, Krebs, Meißner, ***Improved chiral NN potential up to next-to-next-to-next-to-leading order,*** Eur. Phys. J. A51 (2015) 53
  - — Wesolowski, Klco, Furnstahl, Phillips, ***Bayesian parameter estimation for effective field theories,*** J.Phys. G43 (2016) 074001

- **Review articles on nuclear forces (and beyond)**
  - — EE, ***Few-nucleon forces and systems in chiral EFT,*** Prog. Part. Nucl. Phys. 57 (2006) 654
  - — EE, Hammer, Meißner, ***Modern theory of nuclear forces,*** Rev. Mod. Phys. 81 (2009) 1773
  - — Entem, Machleidt, ***Chiral EFT and nuclear forces,*** Phys. Rept. 53 (2011) 1
  - — EE, Meißner, ***Chiral dynamics of few- and many-nucleon systems,*** Ann. Rev. Nucl. Part. Sci. 62 (2012) 159
  - — Machleidt, Sammarruca, ***Chiral EFT based nuclear forces: Achievements and challenges,*** Phys. Scripta 91 (2016) 083007
  - — Hammer, König, van Kolck, ***Nuclear effective field theory: Status and petrspectives,*** arXiv:1906.12122 [nucl-th]

Lecture Syllabus

Introduction

Part I: Chiral perturbation theory in a nutshell

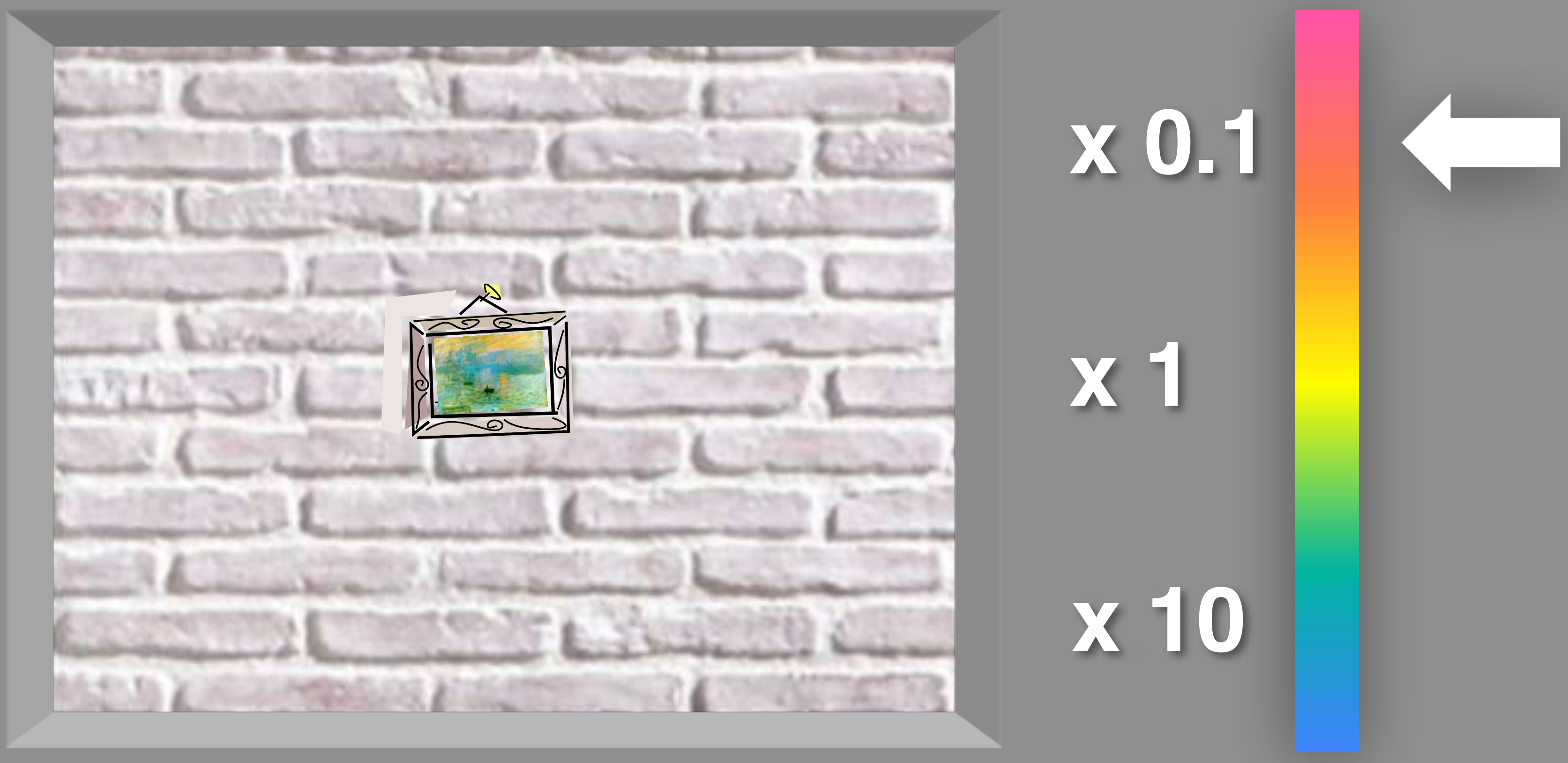
Part II: Two-nucleon scattering: Pionless EFT

Part III: Two-nucleon scattering: Inclusion of pions

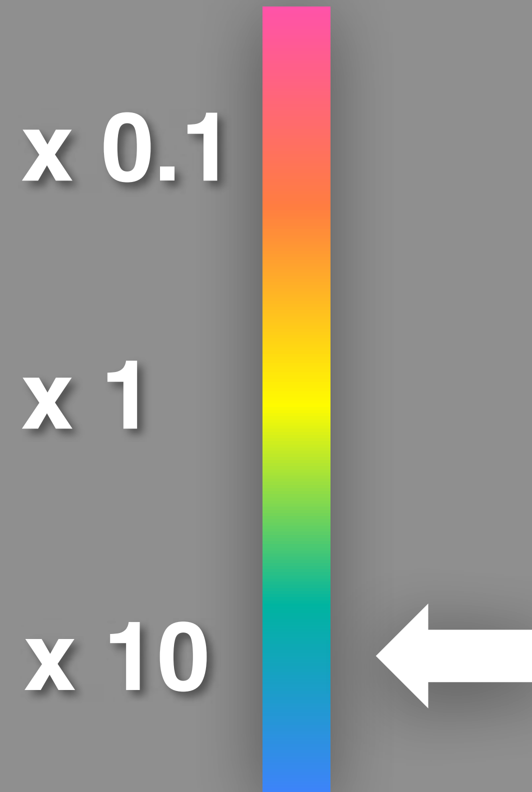
Part IV: Nuclear forces/currents from chiral EFT

Introduction

Effective Theories



Effective Theories



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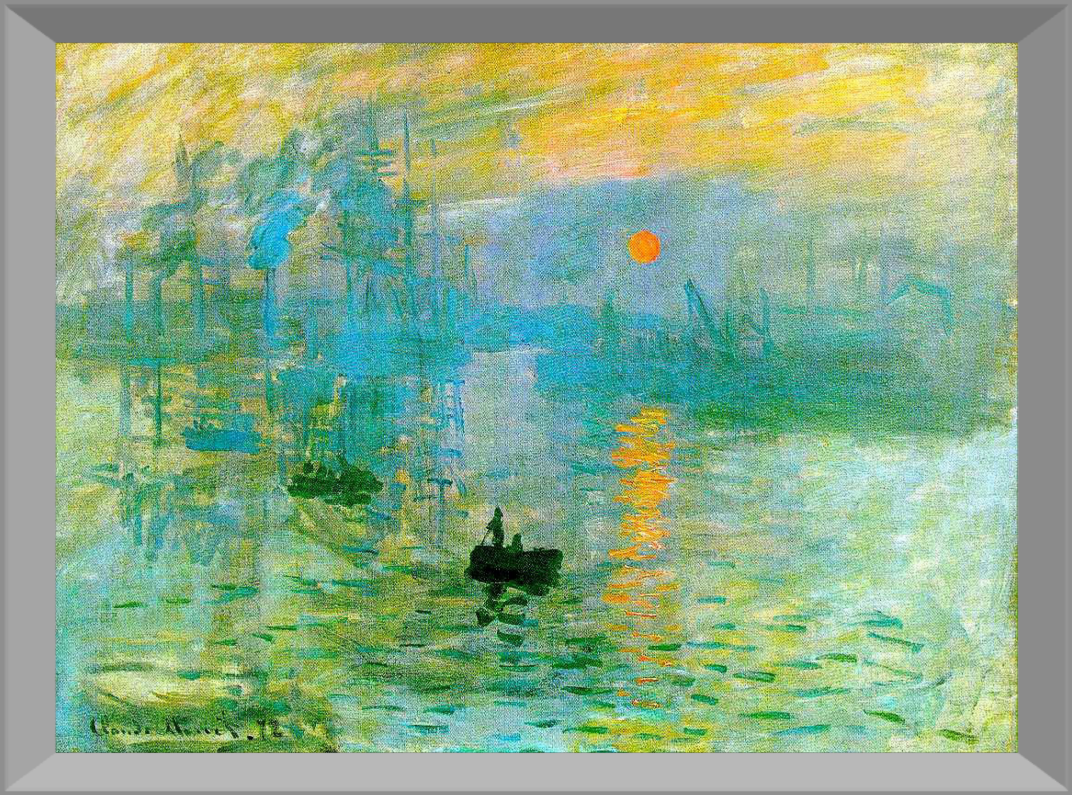
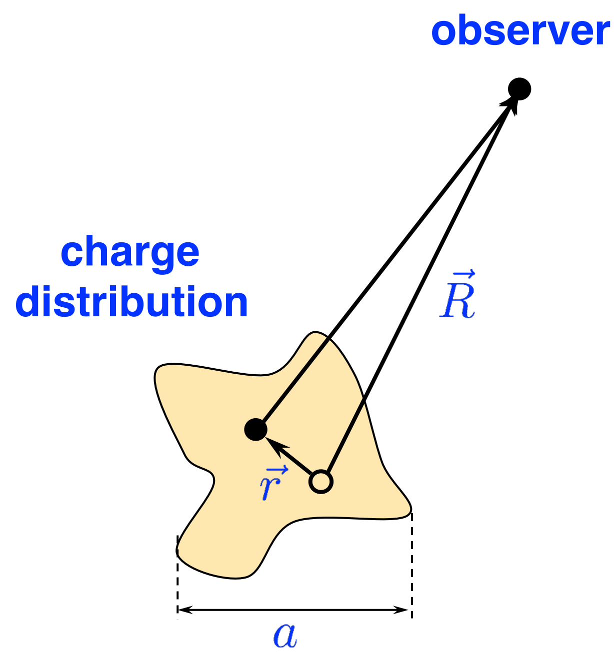


→ **it is crucial to choose a proper resolution !**

What is an effective theory?

Example from electrostatics

The goal: compute electric potential generated by a localized charge distribution $\rho(\vec{r}\,)$



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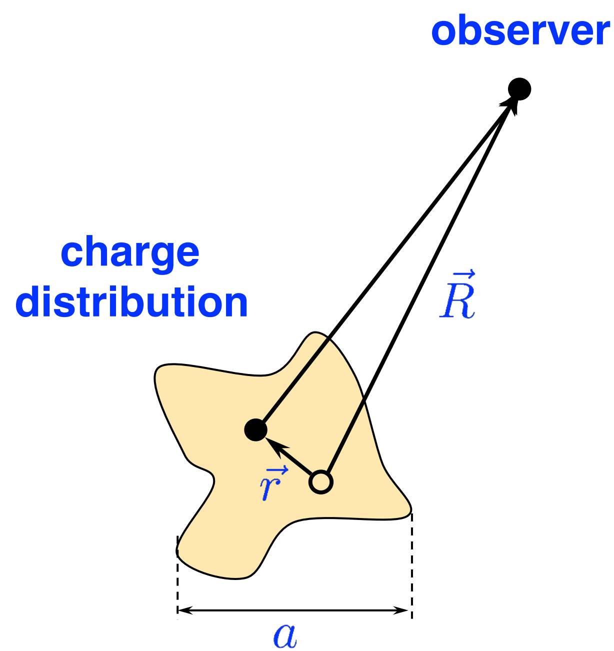
The goal: compute electric potential generated by a localized charge distribution $\rho(\vec{r})$

- The ultimate answer: $V(\vec{R}) \propto \int d^3r \, \frac{\rho(\vec{r})}{|\vec{R}-\vec{r}|}$



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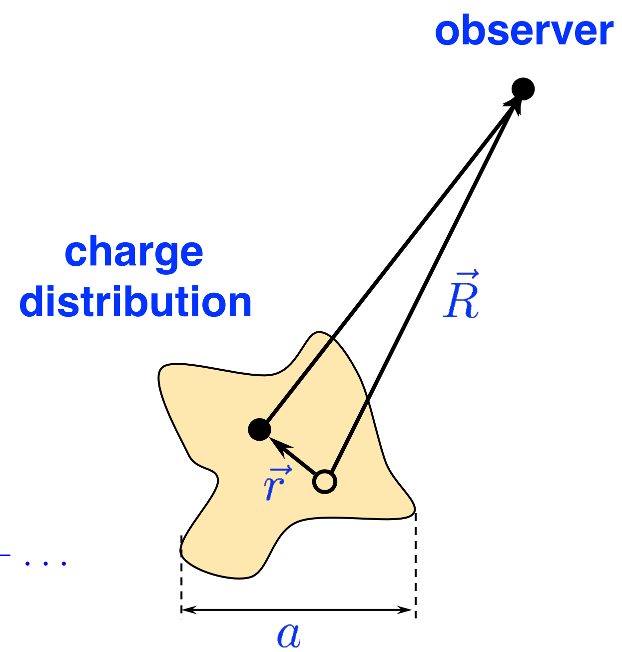
- For $R \gg a$, only moments of $\rho(\vec{r})$ are needed:

$$V(\vec{R}) = \frac{\boldsymbol{q}}{R} + \frac{1}{R^3}\sum_i R_i \boldsymbol{P}_i + \frac{1}{6R^5}\sum_{ij}(3R_iR_j - \delta_{ij}R^2)\boldsymbol{Q}_{ij} + \ldots$$



with multipole moments („low-energy constants“):

$$\boldsymbol{q} = \int d^3r \, \rho(\vec{r}), \qquad \boldsymbol{P}_i = \int d^3r \, \rho(\vec{r})\, r_i, \qquad \boldsymbol{Q}_{ij} = \int d^3r \, \rho(\vec{r})(3r_ir_j - \delta_{ij}r^2)$$

Remember: multipole expansion just follows from:

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R}\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^l P_l(\cos\alpha) = \frac{1}{R}\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^l \frac{4\pi}{2l+1}\sum_m Y_{lm}(\hat{R})Y^{\star}_{lm}(\hat{r})$$

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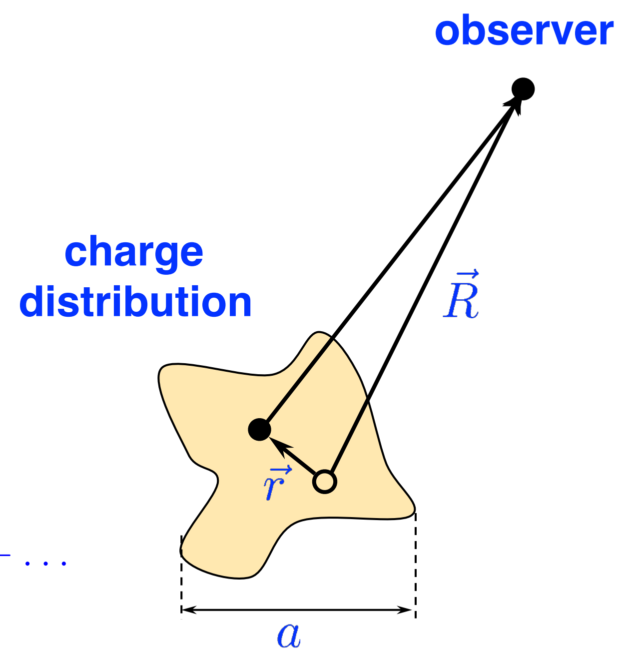

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- Getting the right answer without making calculations (and even without knowing $\rho(\vec{r})$)

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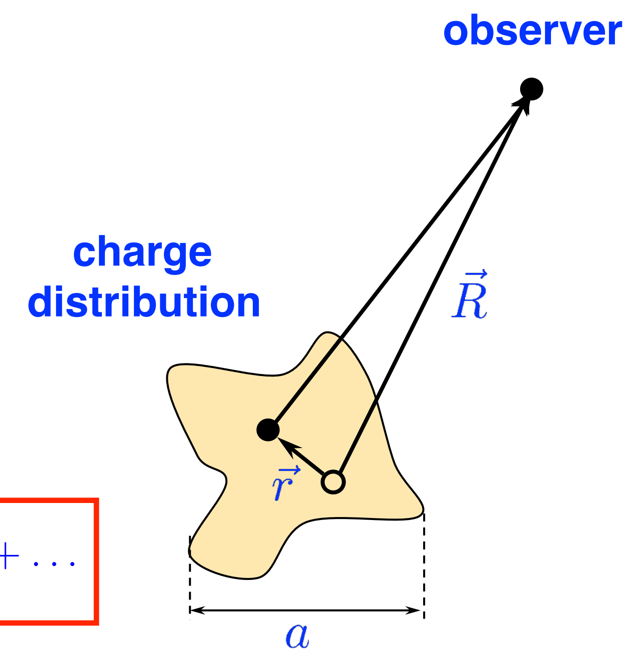

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  - write down the most general rotationally invariant (symmetry!) expression for $V(\vec{R})$

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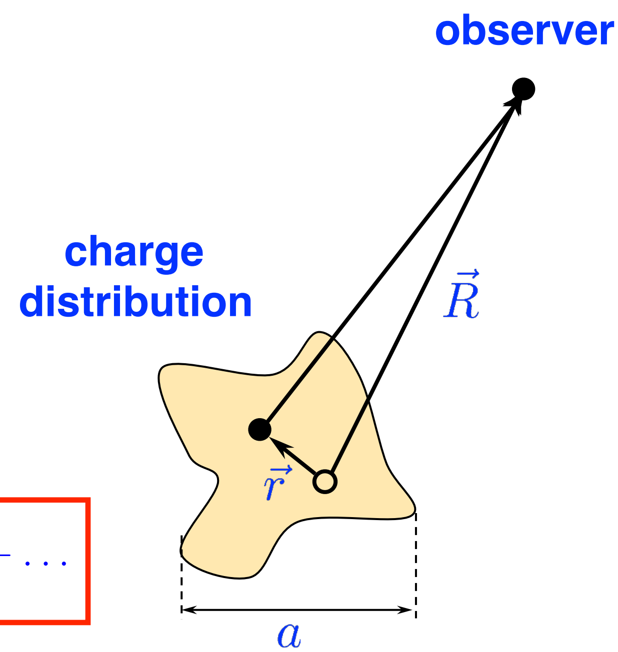

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- Getting the right answer without making calculations (and even without knowing $\rho(\vec{r})$)
  - write down the most general rotationally invariant (symmetry!) expression for $V(\vec{R})$
  - expected natural size of the LECs (dimensional analysis): $\boldsymbol{q} \sim a^0, \;\; \boldsymbol{P}_i \sim a, \;\; \boldsymbol{Q}_{ij} \sim a^2, \;\; \ldots$

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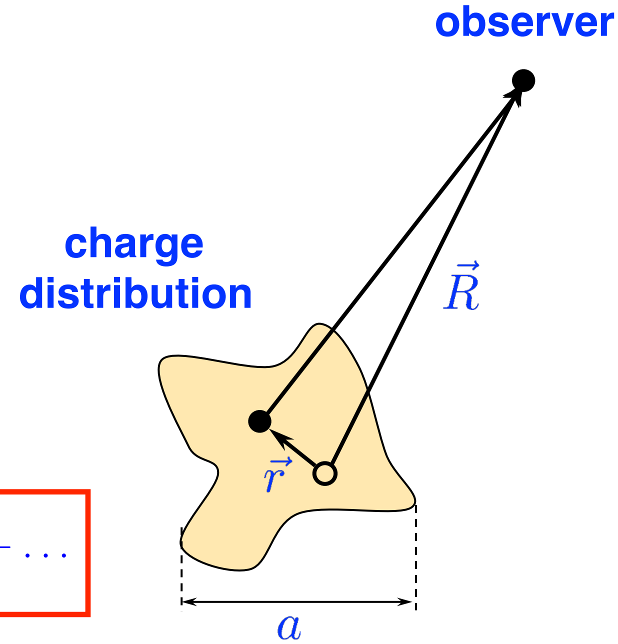

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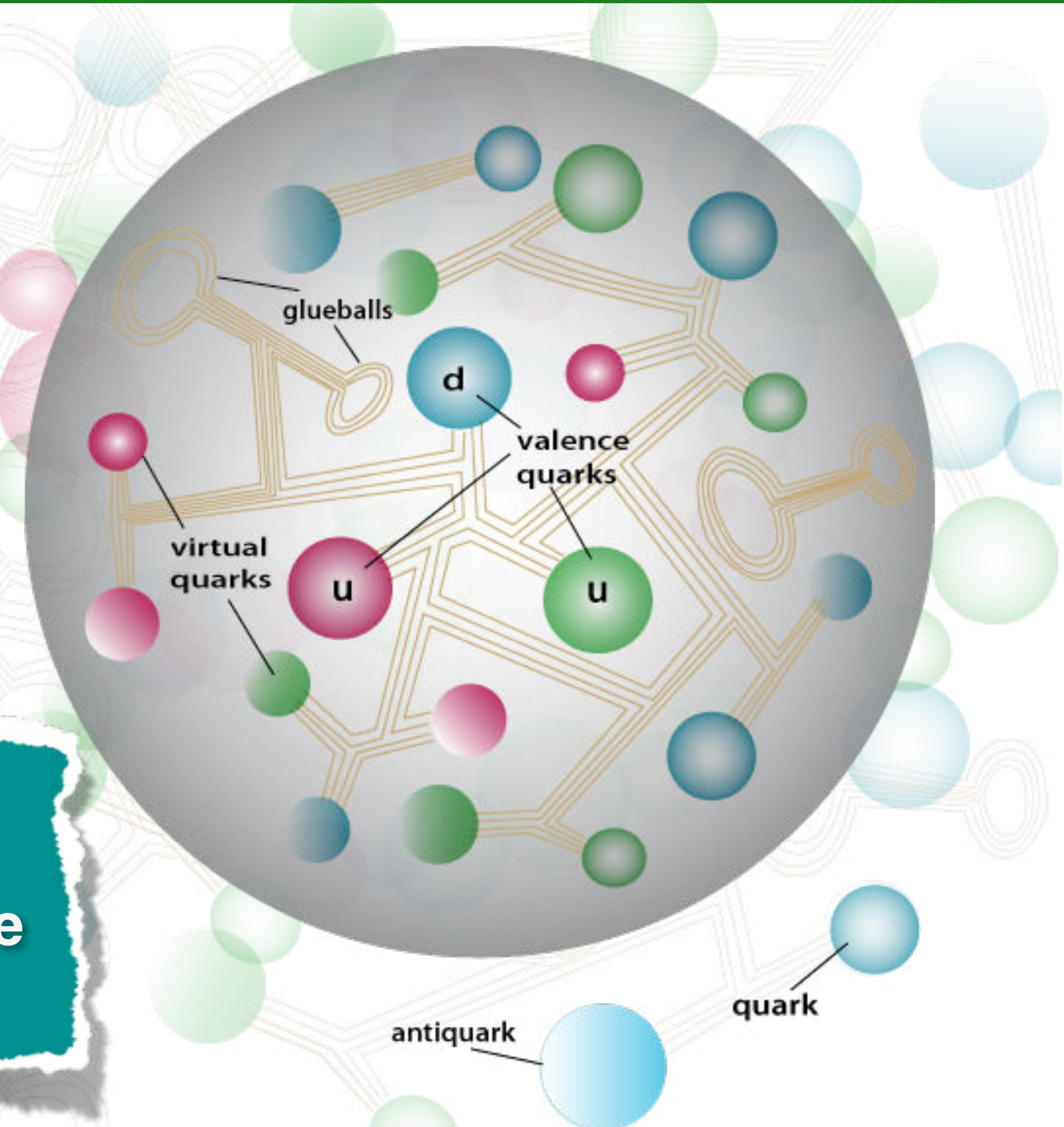
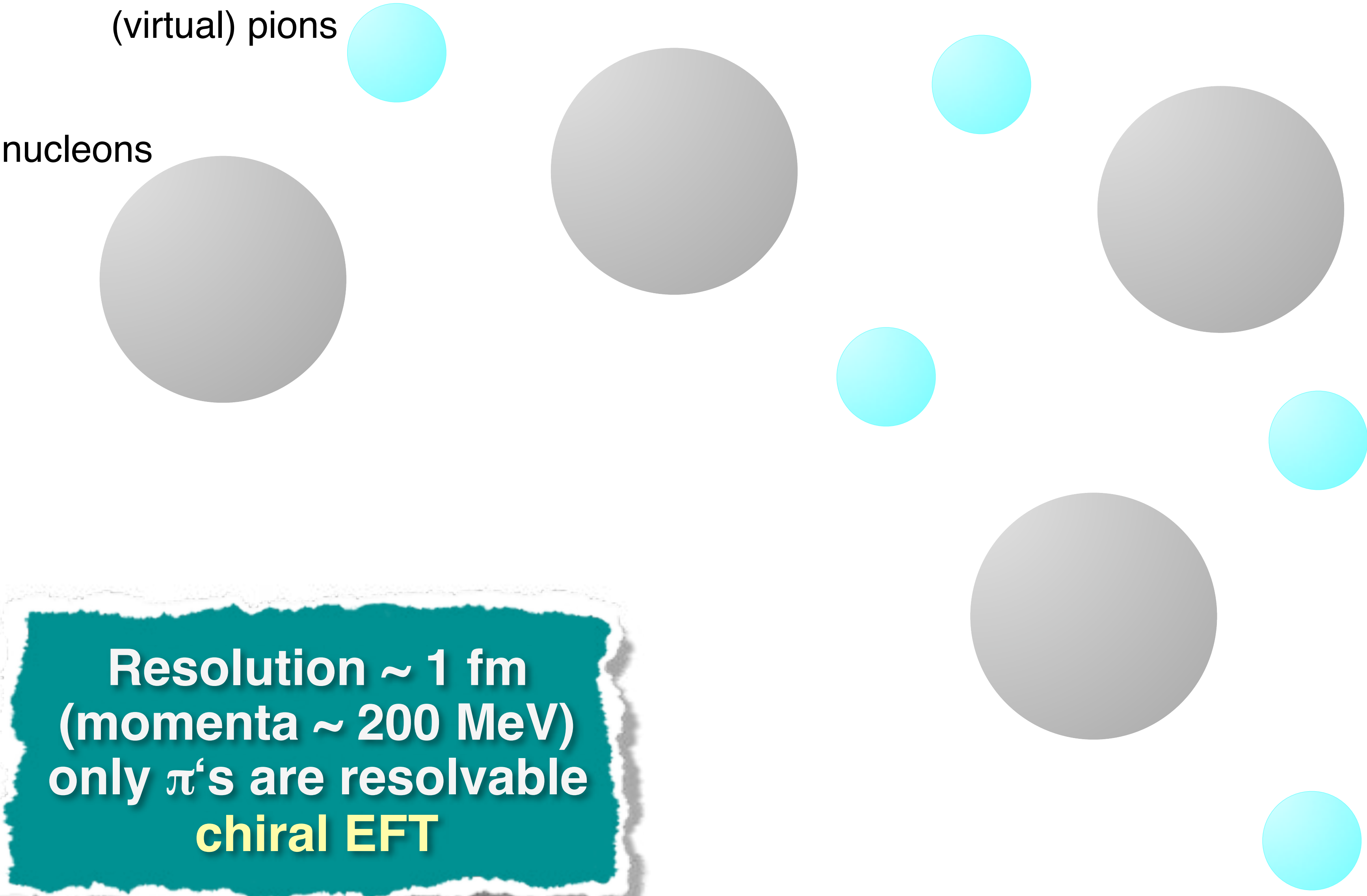
  - measure LECs & compute $V(\vec{R})$ via expansion in $\frac{a}{R}$ (power counting, separation of scales)

NN interaction at different resolutions



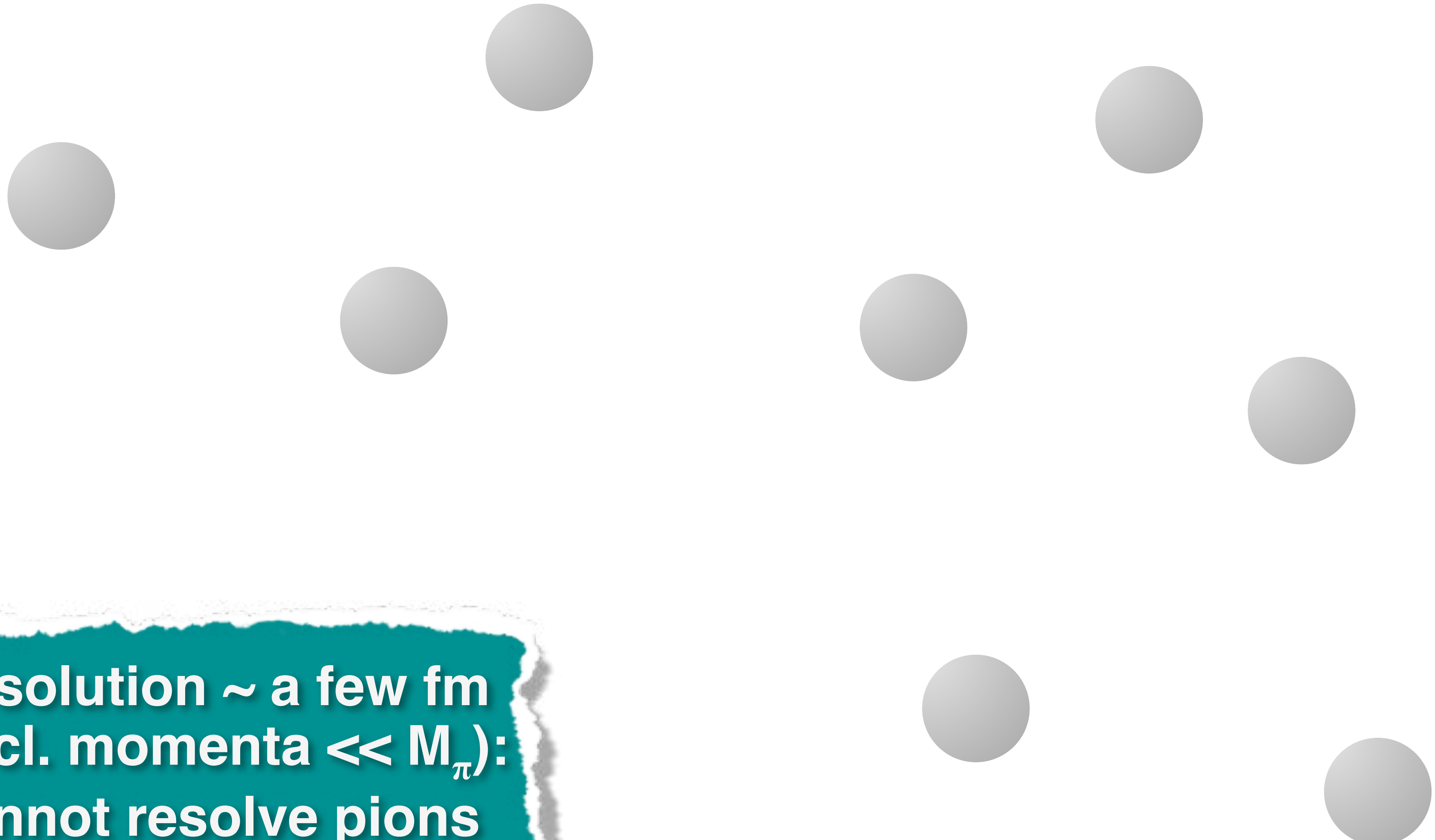
Resolution
scale << 1 fm:
probing the structure
of the nucleons…

NN interaction at different resolutions



Resolution ~ 1 fm
(momenta ~ 200 MeV)
only π's are resolvable
chiral EFT

NN interaction at different resolutions



Resolution ~ a few fm
(nucl. momenta << M_π):
cannot resolve pions
π-less EFT

Part I: Crash course on Chiral Perturbation Theory

Weinberg, Gasser, Leutwyler, Bernard, Kaiser, Meißner, Bijnens, ...

Selected review articles

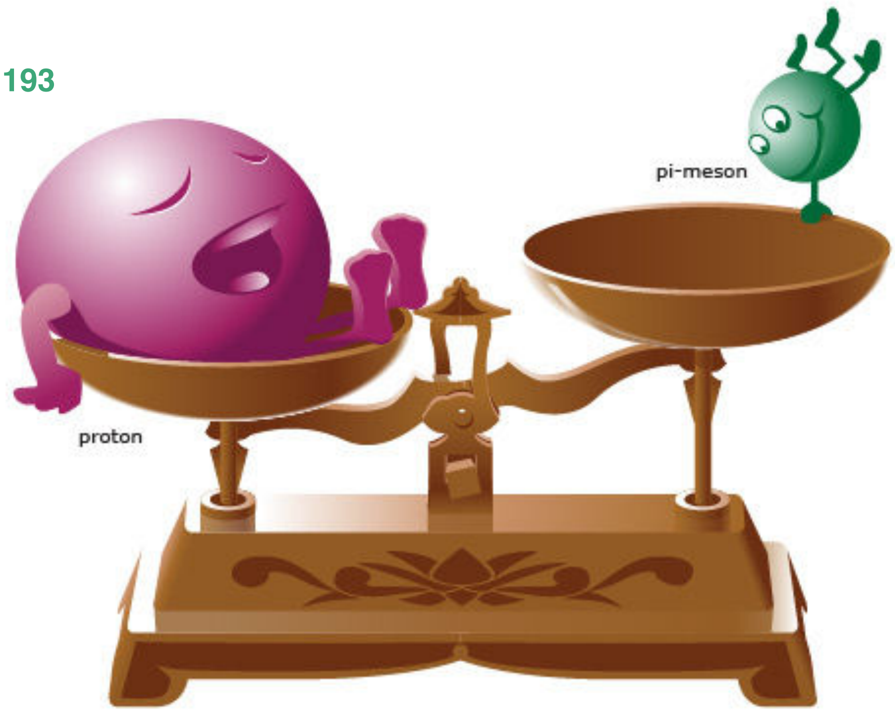
- Bernard, Kaiser, Meißner, Int. J. Mod. Phys. E4 (1995) 193
- Pich, Rep. Prog. Phys. 58 (1995) 563
- Bernard, Prog. Part. Nucl. Phys. 60 (2007) 82
- Scherer, Prog. Part. Nucl. Phys. 64 (2010) 1

Lecture notes

- Scherer, Adv. Nucl. Phys. 27 (2003) 277
- Gasser, Lect. Notes Phys. 629 (2004) 1

Text book

- Scherer, Schindler, A Primer for Chiral Perturbation Theory, Springer, Lecture Notes in Physics, 2012



Part I: Crash course on Chiral Perturbation Theory

Weinberg, Gasser, Leutwyler, Bernard, Kaiser, Meißner, Bijnens, …

1. Effective Lagrangian for pions

2. From effective Lagrangian to S-matrix

3. Inclusion of nucleons

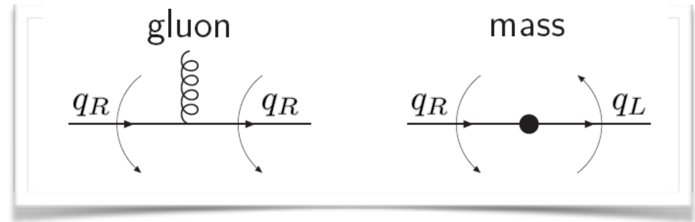
4. Summary of part I

Effective Lagrangian for pions

1. Effective Lagrangian for pions

- **QCD and chiral symmetry**

$$\mathcal{L}_{\rm QCD} = \underbrace{\bar{q}(i\not{D} - \mathcal{M})q}_{\underbrace{\bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R}_{SU(2)_L \times SU(2)_R \text{ invariant}} - \underbrace{\bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M} q_L}_{\text{break chiral symmetry}}} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$



Light quark masses ($\overline{\rm MS},\ \mu = 2\ {\rm GeV}$):

$$\boxed{\begin{array}{rcl} m_u &=& 1.8\ldots 2.8\ {\rm MeV} \\ m_d &=& 4.3\ldots 5.2\ {\rm MeV} \end{array} \ll \Lambda_{QCD}}$$

⟶ $\mathcal{L}_{\rm QCD}$ is approx. $SU(2)_L \times SU(2)_R$ invariant

spontaneous breakdown to $SU(2)_V \subset SU(2)_L \times SU(2)_R$ ⟶ Goldston Bosons (pions)

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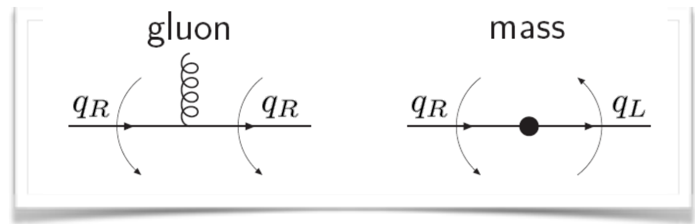

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- **Chiral perturbation theory**
  - Ideal world [$m_u = m_d = 0$], zero-energy limit: non-interacting massless GBs (+ strongly interacting massive hadrons)
  - Real world [$m_u,\ m_d \ll \Lambda_{QCD}$], low energy: weakly interacting light GBs (+ strongly interacting massive hadrons)

⟶ expand about the ideal world (ChPT)

Effective Lagrangian for pions

Pions transform linearly under isospin (isotriplet): $|\pi_1\rangle = \frac{|\pi^+\rangle - |\pi^-\rangle}{\sqrt{2}}\,, \quad |\pi_2\rangle = \frac{|\pi^+\rangle + |\pi^-\rangle}{\sqrt{2}i}\,, \quad |\pi^3\rangle = |\pi^0\rangle$

Pions have to transform nonlinearly under chiral rotations
($SU(2)_L$ x $SU(2)_R$ ~ SO(4) $\longrightarrow$ pion fields as coordinates on a 4-dimentional sphere)

Nonlinear field redefinitions of the kind $\vec{\pi} \to \vec{\pi}' = \vec{\pi}\, F[\vec{\pi}\,],\ F[0] = 1$ do not change physics
$\longrightarrow$ all nonlinear realizations of χ symmetry are equivalent $\longrightarrow$ use most convenient one!
Haag '58; Coleman, Callan, Wess, Zumino '69

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Example of an explicit construction:

Infinitesimal SO(4) rotation of the 4-vector $(\pi_1, \pi_2, \pi_3, \sigma)$:

$$\begin{pmatrix} \boldsymbol{\pi} \\ \sigma \end{pmatrix} \xrightarrow{SO(4)} \begin{pmatrix} \boldsymbol{\pi}' \\ \sigma' \end{pmatrix} = \left[\mathbf{1}_{4\times 4} + \sum_{i=1}^{3} \theta_i^V V_i + \sum_{i=1}^{3} \theta_i^A A_i \right] \begin{pmatrix} \boldsymbol{\pi} \\ \sigma \end{pmatrix}$$

where:

$$\sum_{i=1}^{3} \theta_i^V V_i = \begin{pmatrix} 0 & -\theta_3^V & \theta_2^V & 0 \\ \theta_3^V & 0 & -\theta_1^V & 0 \\ -\theta_2^V & \theta_1^V & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \sum_{i=1}^{3} \theta_i^A A_i = \begin{pmatrix} 0 & 0 & 0 & \theta_1^A \\ 0 & 0 & 0 & \theta_2^A \\ 0 & 0 & 0 & \theta_3^A \\ -\theta_1^A & -\theta_2^A & -\theta_3^A & 0 \end{pmatrix}$$

Switch to nonlinear realization: only 3 out of 4 components of the vector $(\boldsymbol{\pi}, \sigma)$ are independent, i.e. $\boldsymbol{\pi}^2 + \sigma^2 = F^2$

$$\boldsymbol{\pi} \xrightarrow{\theta^V} \boldsymbol{\pi}' = \boldsymbol{\pi} + \boldsymbol{\theta}^V \times \boldsymbol{\pi}\,, \qquad \longleftarrow \quad \textit{linear under } \vec{\theta}^V$$

$$\boldsymbol{\pi} \xrightarrow{\theta^A} \boldsymbol{\pi}' = \boldsymbol{\pi} + \boldsymbol{\theta}^A \sqrt{F^2 - \boldsymbol{\pi}^2} \qquad \longleftarrow \quad \textit{nonlinear under } \vec{\theta}^A$$

Effective Lagrangian for pions

Can be rewritten in terms of a 2 x 2 matrix:

$$U = \frac{1}{F}\left(\sigma\,\mathbf{1}_{2\times2} + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right) \xrightarrow{\textit{nonlinear realization}} U = \frac{1}{F}\left(\sqrt{F^2-\boldsymbol{\pi}^2}\,\mathbf{1}_{2\times2} + i\boldsymbol{\pi}\cdot\boldsymbol{\tau}\right)$$

Chiral rotations: $U \longrightarrow U' = LUR^\dagger$ with $L = \exp[-i(\boldsymbol{\theta}^V - \boldsymbol{\theta}^A)\cdot\boldsymbol{\tau}/2], \quad R = \exp[-i(\boldsymbol{\theta}^V + \boldsymbol{\theta}^A)\cdot\boldsymbol{\tau}/2]$

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Derivative expansion for the effective Lagrangian $\mathcal{L}_{\rm eff} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots$

0 derivatives: $UU^\dagger = U^\dagger U = 1$ - irrelevant $\longleftarrow$ only derivative couplings of GBs

2 derivatives: $\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \xrightarrow{g\in G} \mathrm{Tr}(L\partial_\mu U R^\dagger R\,\partial^\mu U^\dagger L^\dagger) = \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$

$$\longrightarrow \quad \mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)$$

derivatives act only on the next U

4 derivatives: $[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)]^2, \quad \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger)\mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger), \quad \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger)$

(terms with $\partial_\mu\partial_\nu U, \; \partial_\mu\partial_\nu\partial_\rho U, \; \partial_\mu\partial_\nu\partial_\rho\partial_\sigma U$ can be eliminated via EOM/partial integration)

…

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…

Chiral symmetry breaking terms

$\delta\mathcal{L}_{\rm QCD} = -\bar q_L \mathcal{M} q_R - \bar q_R \mathcal{M}^\dagger q_L$ can be made χ-invariant by requiring: $\mathcal{M} \to L\mathcal{M}R^\dagger$

$\longrightarrow$ construct all possible χ-invariant terms involving $\mathcal{M}$ and freeze out $\mathcal{M}$ at the end

LO term: $\delta\mathcal{L}_{\rm SB} = \dfrac{BF^2}{2}\mathrm{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger] = 2BF^2 m_q - Bm_q\vec\pi^2 + \ldots \quad \longrightarrow \quad M_\pi^2 = 2m_q B + \mathcal{O}(m_q^2)$

Effective Lagrangian for pions

The leading and subleading effective Lagrangians for pions

terms involving external fields

$$
\begin{aligned}
\mathcal{L}_\pi^{(2)} &= \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U + \mathcal{M}U^\dagger)\rangle\,, \\
\mathcal{L}_\pi^{(4)} &= \frac{l_1}{4}\langle \partial_\mu U \partial^\mu U^\dagger\rangle^2 + \frac{l_2}{4}\langle \partial_\mu U \partial_\nu U^\dagger\rangle\langle \partial^\mu U \partial^\nu U^\dagger\rangle + \frac{l_3}{16}\langle 2B\mathcal{M}(U + U^\dagger)\rangle^2 + \ldots \\
&- \frac{l_7}{16}\langle 2B\mathcal{M}(U - U^\dagger)\rangle^2
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Gasser, Leutwyler ’84

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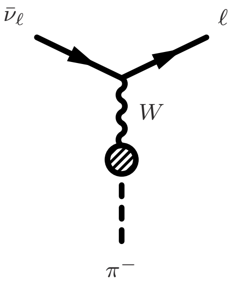
Gasser, Leutwyler '84

Low-energy constants of $\mathcal{L}_\pi^{(2)}$



- F is related to the pion decay constant F_π: $\langle 0 | J^i_{A_\mu}(0) | \pi^j(\vec{p})\rangle = i p_\mu F_\pi \delta^{ij}$

  axial current from $\mathcal{L}_\pi^{(2)}$: $J^i_{A\mu} = i\mathrm{Tr}[\tau^i(U^\dagger \partial_\mu U - U \partial_\mu U^\dagger)] = -F\partial_\mu \pi^i + \ldots$

  $\longrightarrow$ F is F_π in the chiral limit: $F_\pi = F + \mathcal{O}(m_q) \simeq 92.4\ \mathrm{MeV}$
- B is related to the chiral quark condensate

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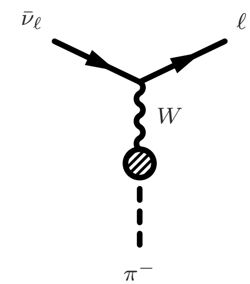
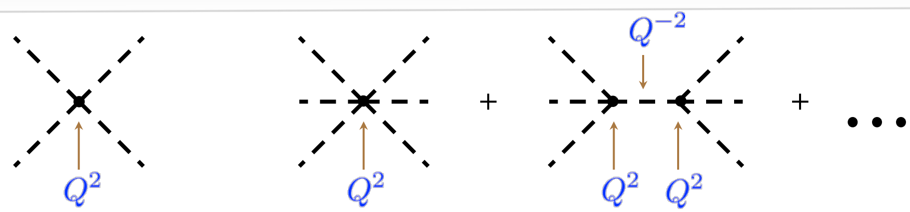


Tree-level multi-pion connected diagrams from $\mathcal{L}_\pi^{(2)}$

$$U(\boldsymbol{\pi}) = \mathbf{1}_{2\times 2} + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F} - \frac{\boldsymbol{\pi}^2}{2F^2} - i\alpha\frac{\boldsymbol{\pi}^2\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{F^3} + \mathcal{O}(\boldsymbol{\pi}^4) \longrightarrow \mathcal{L}_\pi^{(2)} = \frac{\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}}{2} - \frac{M^2\pi^2}{2} + \frac{(\partial_\mu\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2}{2F^2} - \frac{M^2\pi^4}{8F^2} + \ldots$$



- all diagrams scale as Q^2
- insertions from $\mathcal{L}_\pi^{(4)}$, $\mathcal{L}_\pi^{(6)}$, ... suppressed by powers of Q^2
- remarkable predictive power!

From effective Lagrangian to S-matrix

2. From effective Lagrangian to S-matrix

Tree-level diagrams with higher-order vertices are suppressed at low energy.

The argument can be generalized to quantum corrections (loops) $\longrightarrow$ ChPT Weinberg '79

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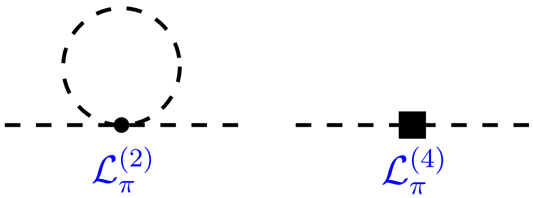
Typical example of a loop integral:



$$I = \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - M^2 + i\epsilon} \xrightarrow{\text{DR}} \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{i}{l^2 - M^2 + i\epsilon}$$

$$= \frac{M^2}{16\pi^2} \ln\left(\frac{M^2}{\mu^2}\right) + 2M^2 L(\mu) + \ldots$$

terms vanishing in d=4

The infinite quantity $L(\mu) = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4} + \text{const}\right)$ can be absorbed into l_i's of $\mathcal{L}_\pi^{(4)}$: $l_i \rightarrow l_i^{\mathrm{r}}(\mu)$

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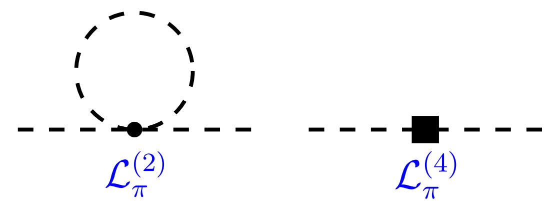

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The bottom line: after renormalization, all momenta flowing through loop graphs are soft, $\sim Q$

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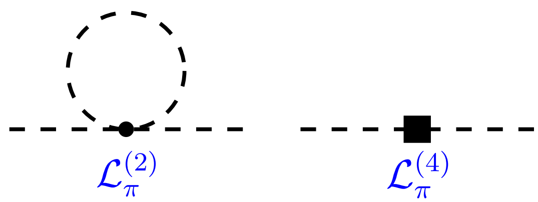

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Power counting (Naive Dimensional Analysis)

- pion propagators: $1/(p^2 - M_\pi^2) \sim 1/Q^2$
- momentum integrations: $d^4 l \sim Q^4$
- delta functions: $\delta^4(p - p') \sim 1/Q^4$
- derivatives: $\partial_\mu \sim Q$

# of loops

# of vertices with d derivatives

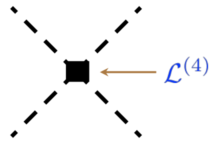
$$D = 2 + 2L + \sum_d N_d (d-2)$$

power of the soft scale Q for a given diagram

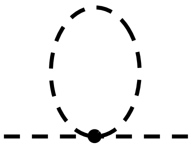
From effective Lagrangian to S-matrix

Examples:

$$\boxed{D = 2 + 2L + \sum_d N_d(d-2)}$$



$D = 2 + 0 + 2 = 4$



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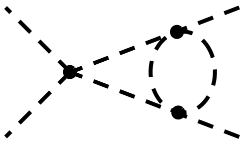

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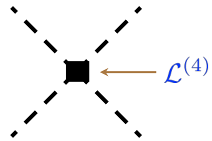

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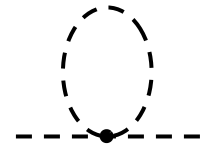

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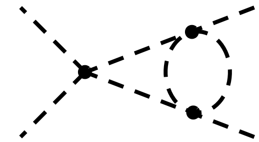

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Scattering amplitude is obtained via an expansion in Q/Λ_χ. What is the value of Λ_χ?

- Chiral expansion breaks down for $E \sim M_\rho \;\rightarrow\; \Lambda_\chi \sim M_\rho = 770 \text{ MeV}$

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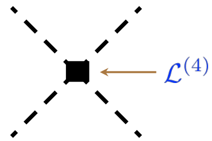

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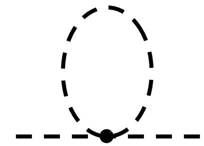

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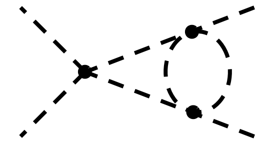

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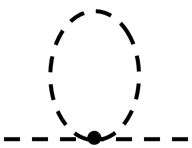

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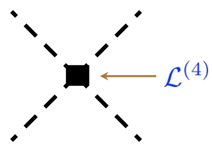

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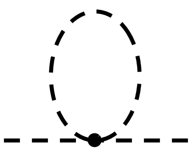

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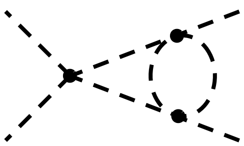

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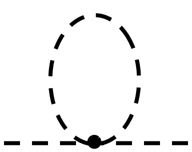

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dimensional arguments

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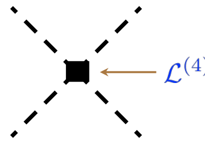

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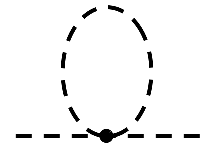

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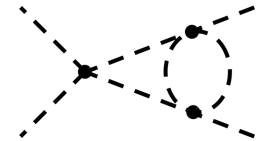

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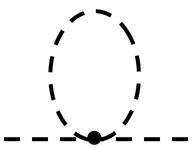

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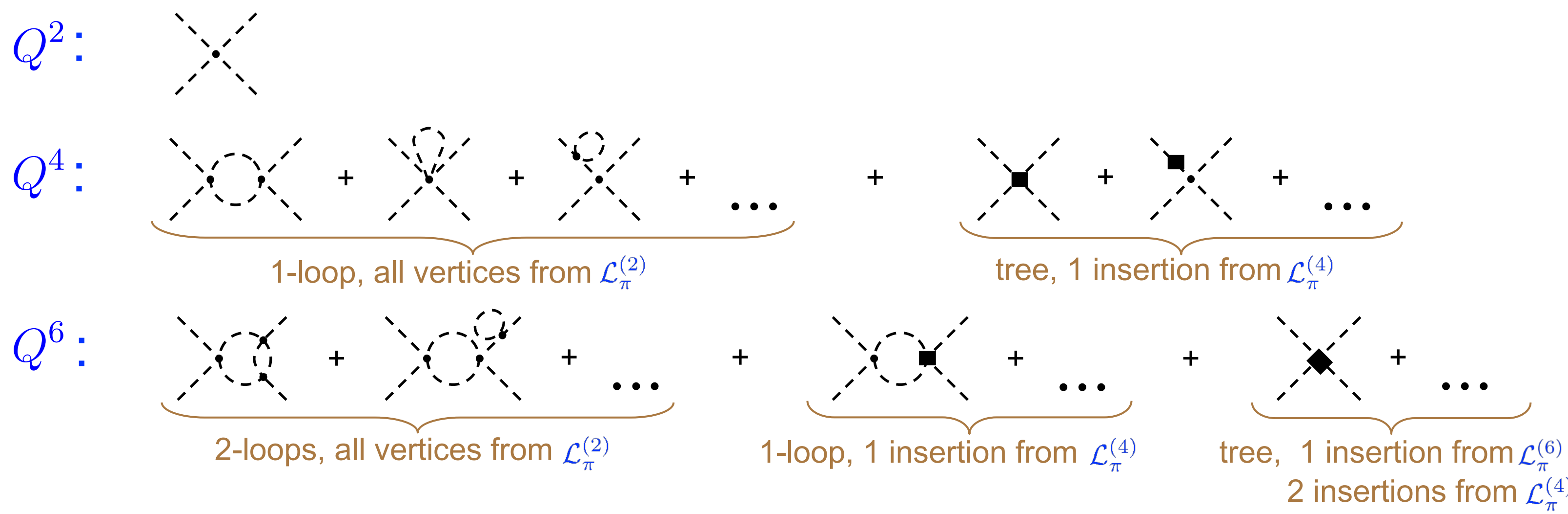
angular integration in 4 dimensions

dimensional arguments

$$\int \frac{d^d l}{(2\pi)^d} = \int \frac{d\Omega_d}{(2\pi)^d}\int l^{d-1} dl = \frac{1}{2^{d-1}\,\pi^{d/2}\,\Gamma(d/2)}\int l^{d-1} dl \xrightarrow{d\to 4} \frac{2}{(4\pi)^2}\int l^3 dl$$

Pion scattering lengths in ChPT



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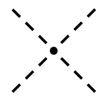
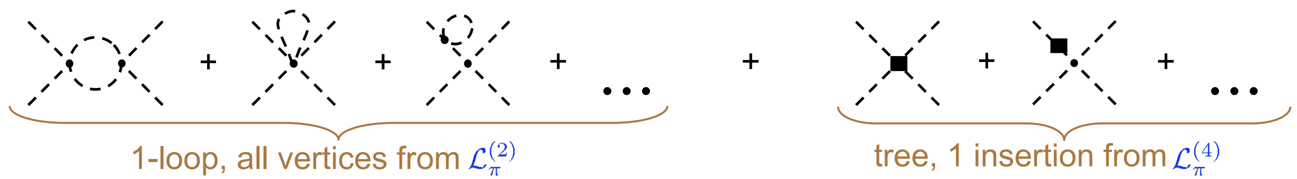
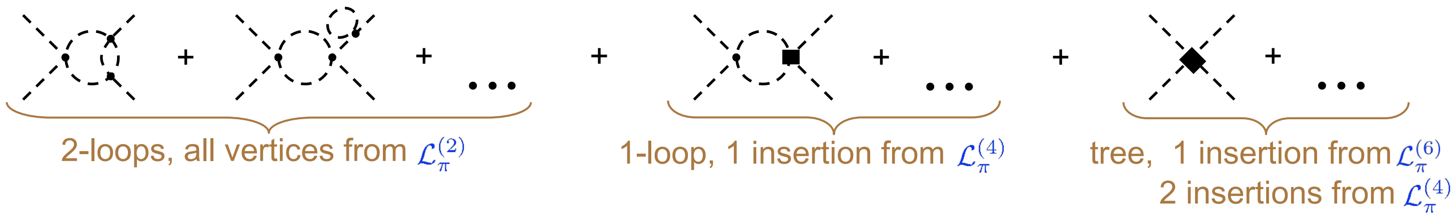
Q^2:



Q^4:



Q^6:



Predictive power?

$$\mathcal{L}_{\rm eff} = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \mathcal{L}_\pi^{(6)} + \ldots$$

# of LECs increasing…

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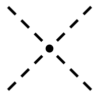

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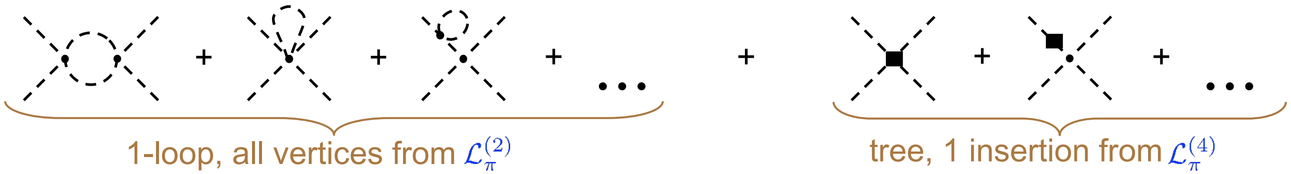

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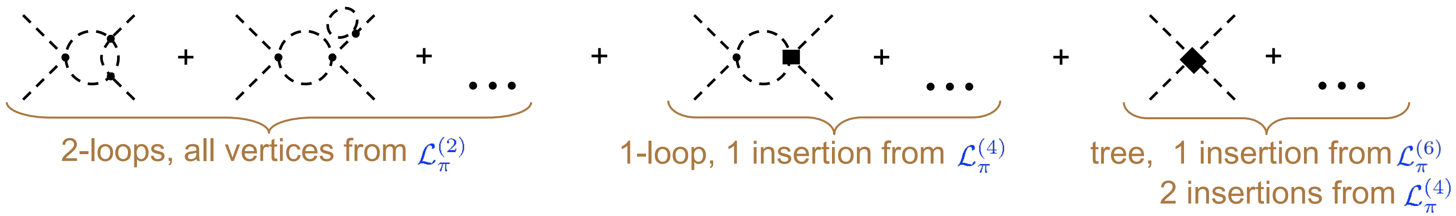

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S-wave $\pi\pi$ scattering length

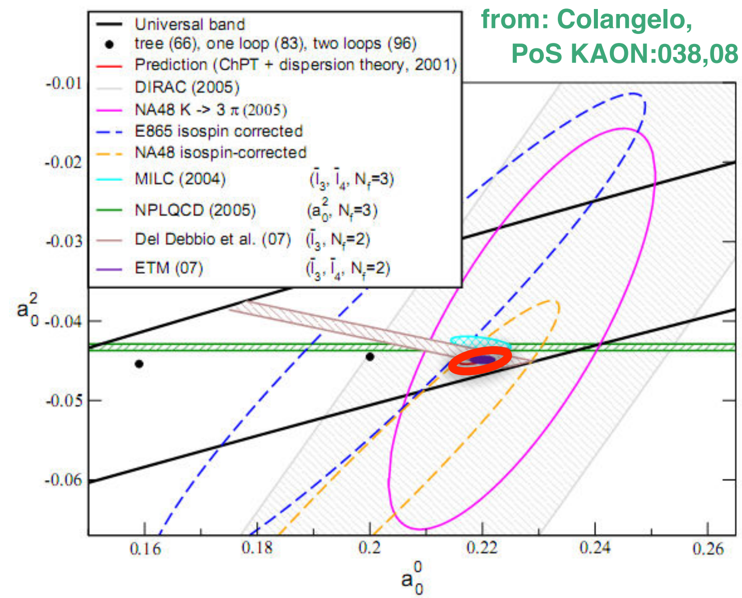
LO: $a_0^0 = 0.16$ **[Weinberg '66]**

NLO: $a_0^0 = 0.20$ **[Gasser, Leutwyler '83]**

NNLO: $a_0^0 = 0.217$ **[Bijnens et al. '95]**

NNLO + disp. relations: **[Colangelo et al.]**

$$a_0^0 = 0.217 \pm 0.008\,(\mathrm{exp}) \pm 0.006\,(\mathrm{th})$$



from: Colangelo, PoS KAON:038,08

ChPT vs Multipole Expansion

Chiral Perturbation Theory

- Most general effective Lagrangian for pions [and matter fields], chiral symmetry!

$$\mathcal{L}_\pi^{(2)} = \frac{\boldsymbol{F}^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U + \mathcal{M}U^\dagger)\rangle\,,$$
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Electric potential

Most general expression for the electric potential (rotational invariance)

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- The size of (ren.) LECs governed by the hard scale $\Lambda_\chi \sim 1$ GeV, LECs can be calculated (lattice-QCD) or fixed from experiment

Electric potential

Most general expression for the electric potential (rotational invariance)

LECs (multipoles) governed by the size a of $\rho(\vec{r})$, they can be calculated or determined from exp.

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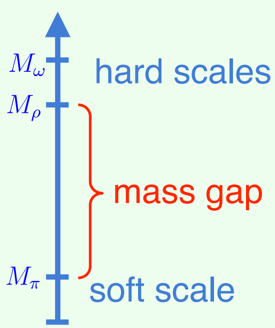
$$\mathcal{L}_\pi^{(4)} = \frac{\boldsymbol{l}_1}{4}\langle \partial_\mu U \partial^\mu U^\dagger\rangle^2 + \frac{\boldsymbol{l}_2}{4}\langle \partial_\mu U \partial_\nu U^\dagger\rangle\langle \partial^\mu U \partial^\nu U^\dagger\rangle + \ldots$$

- The size of (ren.) LECs governed by the hard scale $\Lambda_\chi \sim 1\ \mathrm{GeV}$, LECs can be calculated (lattice-QCD) or fixed from experiment

- Separation of scales: [soft] $Q \sim M_\pi \ll \Lambda_\chi \sim M_\rho$ [hard]



Electric potential

Most general expression for the electric potential (rotational invariance)

LECs (multipoles) governed by the size a of $\rho(\vec{r})$, they can be calculated or determined from exp.

[soft] $1/R \ll 1/a$ [hard]

ChPT vs Multipole Expansion

Chiral Perturbation Theory

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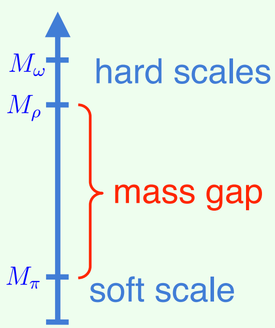
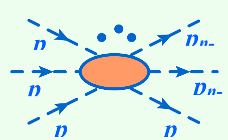


- Chiral expansion of S-matrix elements
  (Feynman graphs, power counting, renorm.)



$$= E^D f\left(\frac{E}{\mu},\, g^r\right)$$

Electric potential

Most general expression for the electric potential (rotational invariance)

LECs (multipoles) governed by the size a of $\rho(\vec{r})$, they can be calculated or determined from exp.

[soft] $1/R \ll 1/a$ [hard]

Multipole expansion for $V(\vec{R})$ in powers of a/R

Summary of Chiral Perturbation Theory

- QCD features approximate $SU(2)_L \times SU(2)_R$ symmetry spontaneously broken down to $SU(2)_V$. Pions are Goldstone Bosons of the broken axial generators.

- ChPT = EFT to describe QCD at low energy using GBs & matter fields as DOF. It provides perturbative (GBs!) expansion of the amplitude in powers of $p \sim M_\pi$ over $\Lambda_\chi \sim M_\rho \sim 1$ GeV.

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- $\mathcal{L}_{\rm eff}$ for GBs: Expansion in ∂_μ and M_π, $M_\pi^2 = 2Bm_q + \mathcal{O}(m_q^2)$; pions transform nonlinearly under SU(2)$_L$ x SU(2)$_R$ [CCWZ]: $U \to LUR^\dagger$.

$$\mathcal{L}_\pi = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots$$

LECs $\qquad = \mathrm{diag}(m_q)$

$$\mathcal{L}_\pi^{(2)} = \frac{F^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U) + \frac{F^2}{2} B\,\mathrm{Tr}(\mathcal{M}U + \mathcal{M}U^\dagger)$$

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- Use DR to calculate Feynman diagrams. Connected diagrams scale as q^ν,

$$\boxed{\nu = 2 + 2L + \sum_i V_i \Delta_i\,, \qquad \Delta_i = d_i - 2}$$

# of loops $\quad$ # of vertices of type i $\quad$ # of derivatives & M_π

[exercises]

- At each order q^ν, a finite number of LECs contribute (exp., lattice, models, …)

Inclusion of nucleons

3. Inclusion of the nucleons

In the baryon sector, it is more convenient to work with u defined via $U =: u^2$. Then:

$$u \longrightarrow u' = \sqrt{RUL^\dagger} =: RuK^{-1} \quad \Rightarrow \quad K = (\sqrt{RUL^\dagger})^{-1} R\sqrt{U}$$

Notice: the transformation property of u can also be written as $u \longrightarrow u' = KuL^\dagger$.

The so-called compensator field K is a complicated SU(2)-valued function of θ^L, θ^R, U (and thus of space-time), $K = K(L,R,U)$, except for isospin (i.e. vector) rotations with $\theta^L = \theta^R = \theta^V$:

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The **Coleman-Callan-Wess-Zumino (CCWZ)** nonlinear realization of the chiral group:

$$\boxed{\begin{pmatrix} U \\ N \end{pmatrix} \xrightarrow{g} \begin{pmatrix} U' \\ N' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L,R,U)N \end{pmatrix}}$$

[proof in the exercises]

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[proof in the exercises]

To construct the effective Lagrangian, one uses building blocks which transform covariantly with respect to $SU(2)_R \times SU(2)_L$

$$N \longrightarrow N' = KN, \quad O_i \longrightarrow O_i' = KO_iK^{-1} = KO_iK^\dagger$$

to write terms like: $\bar{N}O_1 \ldots O_n N \;\; \mathrm{Tr}\,(O_{n+1} \ldots O_m) \;\; \ldots \;\; \mathrm{Tr}\,(O_{m+1} \ldots O_k)$

Inclusion of nucleons

Covariant derivatives of the nucleon and pion fields:

$$D_\mu N := (\partial_\mu + \Gamma_\mu)\, N, \qquad \Gamma_\mu := \frac{1}{2}\left(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger\right)$$

$$u_\mu := i u^\dagger \left(\partial_\mu U\right) u^\dagger$$

[exercises…]

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[exercises…]

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$$g_A = G_A(0): \quad \langle N(p')|A_i^\mu(0)|N(p)\rangle =: \bar{u}(p')\left[\gamma^\mu G_A(t) + \frac{(p'-p)^\mu}{2m} G_p(t)\right]\gamma_5 \frac{\tau_i}{2} u(p)$$

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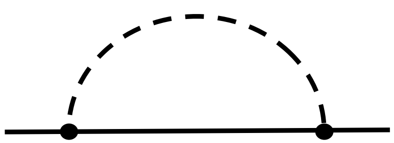
Problem (?): new hard mass scale m $\rightarrow$ power counting ??



$$\longrightarrow \quad \delta m \overset{\mathcal{M}\to 0}{\longrightarrow} -m\frac{3 g_A^2 m^2}{(4\pi F)^2}\left[\log\frac{m}{\mu} + \mu^{d-4}\left(\frac{1}{d-4} + \text{const}\right)\right]$$

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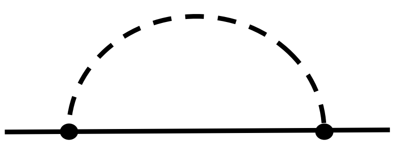

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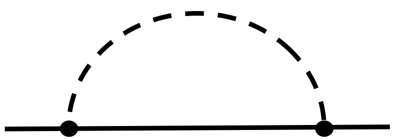

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divergence has to be absorbed by m from the LO Lagrangian…

Inclusion of nucleons

Making power counting manifest: The Heavy-Baryon approach

Jenkins & Manohar '91; Bernard, Keiser, Meißner '92; Mannel, Roberts, Ryzak '92

Write the nucleon momentum as $p^\mu = m v^\mu + l^\mu$ with $v^2 = 1$ and $l_\mu \ll m$

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[exercises]

$\longrightarrow$ the small component h_v behaves as a heavy field and can be integrated out:

$$\boxed{\mathcal{L}_{\pi N}^{(1)} = \bar{N}_v \left(iv\cdot D + g_A S\cdot u\right) N_v \; + \; \mathcal{O}(m^{-1})}$$ where $S_\mu \equiv i\gamma_5 \sigma_{\mu\nu} v^\nu$

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The HB propagator of the nucleon $S(p) = \dfrac{i}{v\cdot p + i\epsilon}$ has an obvious interpretation:

$$S(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p_0 + i\epsilon} e^{-ip\cdot(x-y)} = \theta(x_0 - y_0)\delta^3(\vec{x} - \vec{y})$$

[Exercises: verify the form of the HB propagator by performing 1/m expansion of the standard Dirac propagator for the fermion field.]

Inclusion of nucleons

- In the HB formulation, the nucleon mass does not appear in the nucleon propagator and contribute only through 1/m corrections to vertices

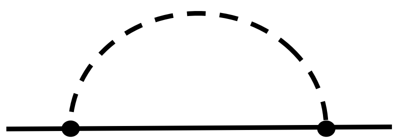
$\longrightarrow$ power counting is manifest!

- Power counting:

# of nucleon fields

$$\boxed{\nu = 1 + 2L + \sum_i V_i \Delta_i \,, \quad \Delta_i = -2 + \frac{n_i}{2} + d_i}$$

[exercises]

- For example: 

$$(\delta m)^{\rm HB} = -\frac{3 g_A^2 M_\pi^3}{32 \pi F_\pi^2}$$

- **Perturbation theory works since GBs interact via derivative couplings…**

Pion-nucleon scattering

Effective chiral Lagrangian:

low-energy constants

$$\mathcal{L}_\pi = \mathcal{L}_\pi^{(2)} + \mathcal{L}_\pi^{(4)} + \ldots$$

$$\mathcal{L}_{\pi N} = \underbrace{\bar{N}\Big(i\gamma^\mu D_\mu[\pi] - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu[\pi]\Big)N}_{\mathcal{L}_{\pi N}^{(1)}} + \underbrace{\sum_i \boldsymbol{c}_i \bar{N}\hat{O}_i^{(2)}[\pi]N}_{\mathcal{L}_{\pi N}^{(2)}} + \underbrace{\sum_i \boldsymbol{d}_i \bar{N}\hat{O}_i^{(3)}[\pi]N}_{\mathcal{L}_{\pi N}^{(3)}} + \underbrace{\sum_i \boldsymbol{e}_i \bar{N}\hat{O}_i^{(4)}[\pi]N}_{\mathcal{L}_{\pi N}^{(4)}} + \ldots$$

Pion-nucleon scattering amplitude for $\pi^a(q_1) + N(p_1) \to \pi^b(q_2) + N(p_2)$:

$$T_{\pi N}^{ba} = \frac{E+m}{2m}\bigg(\delta^{ba}\Big[g^+(\omega,t) + i\vec{\sigma}\cdot\vec{q}_2\times\vec{q}_1\, h^+(\omega,t)\Big] + i\epsilon^{bac}\tau^c\Big[g^-(\omega,t) + i\vec{\sigma}\cdot\vec{q}_2\times\vec{q}_1\, h^-(\omega,t)\Big]\bigg)$$

calculated within the chiral expansion

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calculated within the chiral expansion

Pion-nucleon scattering up to Q^4 in heavy-baryon ChPT

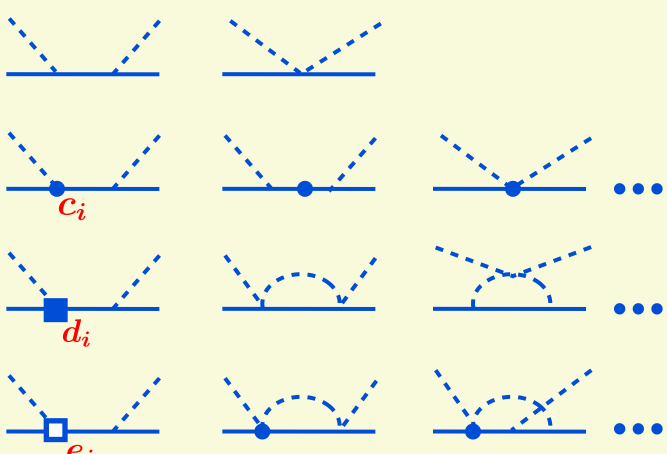
Fettes, Meißner '00; Krebs, Gasparyan, EE '12

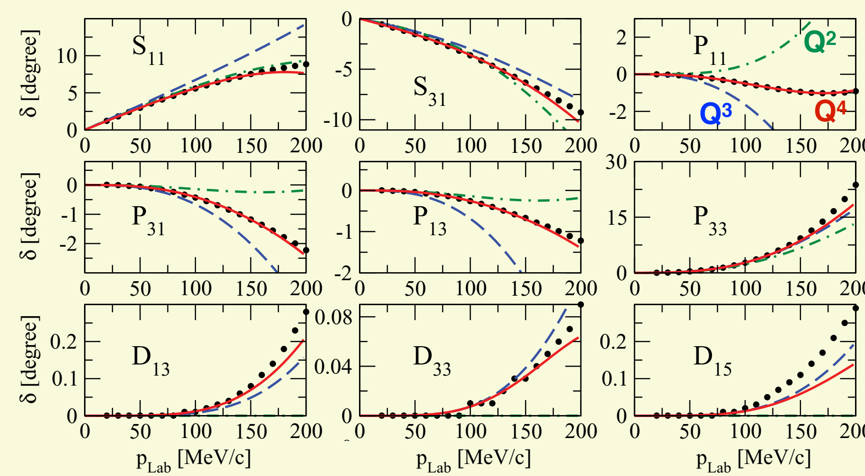
Order Q:

Order Q^2:

Order Q^3:

Order Q^4:





Summary of part I

- QCD is approximately $SU(2)_L \times SU(2)_R$ invariant. The chiral symmetry is spontaneously broken down to $SU(2)_V$ (isospin group). Pions are Goldstone Bosons of the broken axial generators. They would be massless in the chiral limit.

- It is easy to write down the most general chiral invariant effective Lagrangian for pions. The choice of a particular realization of the chiral group is irrelevant (provided proper imbedding of the isospin group). Being Goldstone Bosons, only derivative interactions are allowed → suppression at low energy! Can incorporate explicit breaking due to the quark masses.

- Feynman calculus with DR: every loop is suppressed by Q^2 (power counting) → can calculate quantum corrections!

- It is straightforward to extend the effective Lagrangian to nucleons. Because of the nucleon mass, loops calculated with just DR are not suppressed. Either use additional finite subtractions (EOMS) or perform nonrelativistic expansion of the Lagrangian (the HB approach).

(Some) topics in and beyond ChPT

- **Resummation of leading Log's**

  Weinberg, Bijnens, Colangelo, Bissiger, Fuhrer, Kivel, Polyakov, Vladimirov, …

  Leading logs can be computed for higher loops, all orders possible in certain cases

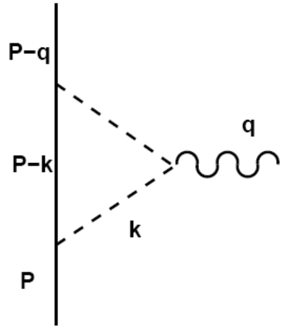
- **Combining ChPT and dispersion theory**

  Colangelo, Gasser, Leutwyler, Bernard, Meißner, Descotes Genon, Knecht, Pelaez, Hoferichter, Kubis, Ruiz de Elvira, …

- **Covariant baryon ChPT**

  Becher, Leutwyler, Bernard, Meißner, Kubis, Gegelia, Scherer, Camalich, Geng, Ren, …

  HB expansion has a very limited convergence range for some types of diagrams $\rightarrow$ better to resum 1/m recoil corrections up to infinite order (IR-ChPT). Alternatively, use manifestly covariant framework + appropriate subtraction (EOMS) to enforce power counting



- **ChPT with explicit spin-3/2 degrees of freedom**

  Hemmert, Bernard, Fettes, Meißner, Pascalutsa, Vanderhaeghen, Kaiser, Gegelia, EE, Gasparyan, Krebs, Siemens, …

  $\Delta(1232)$ has low excitation energy ~ 300 MeV $\rightarrow$ better to include as an explicit DOF…

- **ChPT and/for lattice QCD**

  Colangelo, Beane, Savage, Jiang, Tiburzi, Procura, Weise, Walker Loud, Bernard, Meißner, Rusetsky, Hemmert, …

  Chiral extrapolations, finite volume corrections, quenched ChPT, …

- **Unitarized ChPT and resonance physics**

  Oeller, Meißner, Dobado, Pelaez, Oset, Hanhart, Llanes-Estrada, Kaiser, Weise, ,…

Adding more nucleons…

0N, 1N: Perturbation theory works since GBs interact via derivative couplings…

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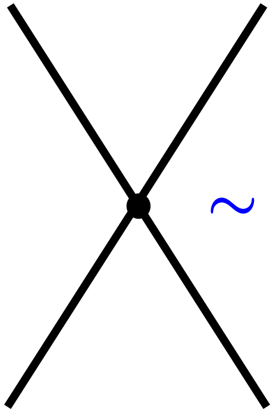
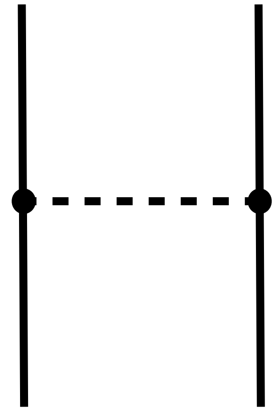
 $\sim\ \text{const}$

 $$\sim \frac{\vec{\sigma}_1 \cdot \vec{q}\; \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^{\,2} + M_\pi^2} = \mathcal{O}(1)$$

Low-energy nucleon-nucleon interactions are <u>NOT</u> suppressed in the chiral limit

No reason to expect perturbation theory to be valid

(indeed, there are shallow bound states…)