# O(a) improvement


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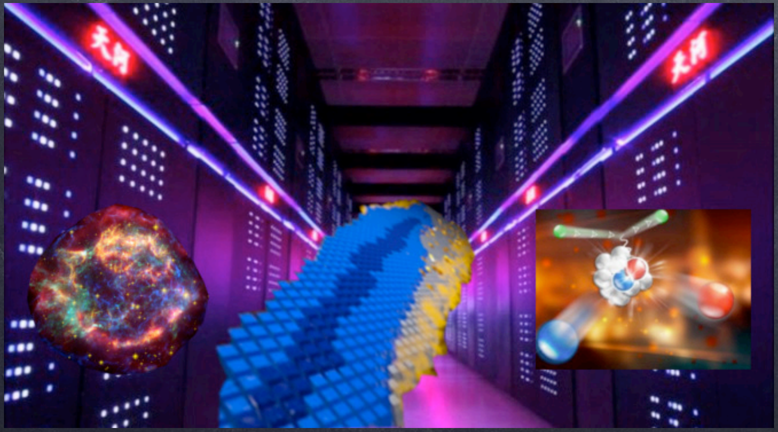
TALENT School - From quarks and gluons to nuclear forces and structures



ECT* - Trento
07.24.2019

# Discretization effects

$$S_\phi = \frac{1}{2}a^4 \sum_x \phi(x)\left[-\partial_\mu^*\partial_\mu + m^2\right]\phi(x) \to S_\phi^{(\mathrm{cont})} + O(a^2)$$

$$S_\phi^{(\mathrm{cont})} = \frac{1}{2}\int d^4x\ \phi(x)\left[-\partial_\mu^2 + m^2\right]\phi(x)$$

$$S_G[U] = \frac{\beta}{3}\sum_x \sum_{\mu<\nu} \mathrm{Retr}[1 - U(x,x;\Box)] \quad \to S_G^{(\mathrm{cont})} + O(a^2)$$

$$S_F = a^4 \sum_x \overline{\psi}(x)\left[D_W + m\right]\psi(x) \quad \to S_F^{(\mathrm{cont})} + O(a)$$

$$D_W = \frac{1}{2}\left[\gamma_\mu\left(\nabla_\mu + \nabla_\mu^*\right) - a\nabla_\mu^*\nabla_\mu\right]$$

Continuum limit a->0 (for QCD $g_0(a) \to 0$ )

Before $N,\ N_T \to \infty$ or large enough

# Discretization effects

To improve discretization effects we can add irrelevant operators changing discretization effects but not the continuum limit

$$S_\phi = \frac{1}{2} a^4 \sum_x \phi(x) \left[ -\partial_\mu^* \partial_\mu + c a^2 \partial_\mu^* \partial_\mu \partial_\mu^* \partial_\mu + m^2 \right] \phi(x)$$

$c = \frac{1}{12}$ tree-level $\qquad$ $c = c(g_0)$ in general

$\to S_\phi^{(\mathrm{cont})} + O(a^4)$ $\qquad$ $S_\phi \to S_\phi^{(\mathrm{cont})} + a^4 \sum_x a^2 c(g_0) \mathcal{L}_6$

$$\mathcal{L}_6 = \phi(x) \left[ \partial_\mu^* \partial_\mu \partial_\mu^* \partial_\mu \right] \phi(x)$$

Systematic? → YES! Symanzik improvement

# Symanzik improvement

$$\langle O_1(x_1)\ldots O_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U]\int \mathcal{D}[\psi,\overline{\psi}] \times O_1(x_1)\ldots O_n(x_n)\exp\left\{-S_F[\psi,\overline{\psi},U]-S_G[U]\right\}$$

→ Energy spectrum

→ Matrix elements

Close to the continuum limit

$$S_{\text{eff}} = S_0 + aS_1 + a^2 S_2 + \cdots \qquad S_0 = S_{QCD}$$

# Symanzik improvement

$$S_{\rm eff} = S_0 + aS_1 + a^2 S_2 + \cdots \qquad S_0 = S_{QCD}$$

$$S_1 = \int d^4x \, \sum_k c_k(g_0)\mathcal{L}_k^{(5)}(x)$$

→ d=5

$$S_2 = \int d^4x \, \sum_k d_k(g_0)\mathcal{L}_k^{(6)}(x)$$

→ d=6

$\mathcal{L}_k^{(5)}, \mathcal{L}_k^{(6)}$ → contain the fundamental fields, derivatives, quark masses

Classify and reduce

# Symanzik improvement

Reduce ⟶

- Integration by parts $\int d^4x$
- Gauge symmetry
- Vector symmetry $SU(N_f)_V \times U(1)_V$
- <u>All lattice symmetries</u>
- Charge conjugation

# Symanzik improvement

$$\mathcal{L}_1^{(5)} = \overline{\psi}(x)\sigma_{\mu\nu}G_{\mu\nu}(x)\psi(x)$$

$$\mathcal{L}_2^{(5)} = \overline{\psi}(x)\overrightarrow{D_\mu}\overrightarrow{D_\mu}\psi(x) + \overline{\psi}(x)\overleftarrow{D_\mu}\overleftarrow{D_\mu}\psi(x)$$

$$\mathcal{L}_3^{(5)} = m\mathrm{Tr}\left[G_{\mu\nu}(x)G_{\mu\nu}(x)\right]$$

$$\mathcal{L}_4^{(5)} = m\left[\overline{\psi}(x)\gamma_\mu\overrightarrow{D_\mu}\psi(x) + \overline{\psi}(x)\gamma_\mu\overleftarrow{D_\mu}\psi(x)\right]$$

$$\mathcal{L}_5^{(5)} = m^2\overline{\psi}(x)\psi(x)$$

E.o.m. reduce the list

$$\left(\gamma_\mu D_\mu + m\right)\psi(x) = 0$$

$$\left.\begin{aligned} &\mathcal{L}_1^{(5)} - \mathcal{L}_2^{(5)} + 2\mathcal{L}_5^{(5)} = 0 \\ &\mathcal{L}_4^{(5)} + 2\mathcal{L}_5^{(5)} = 0 \end{aligned}\right\}$$

We can eliminate

$$\mathcal{L}_2^{(5)}, \mathcal{L}_4^{(5)}$$

# Symanzik improvement

$$\mathcal{L}_3^{(5)} = m\mathrm{Tr}\left[G_{\mu\nu}(x)G_{\mu\nu}(x)\right] = m\mathcal{L}_G$$

$$\mathcal{L}_5^{(5)} = m^2\overline{\psi}(x)\psi(x) = m\mathcal{L}_F^{(\mathrm{mass})}$$

Can be reabsorbed in the definition of the coupling and the mass

$$\begin{cases} g_0^2 \to g_0^2\left(1 + a\ b_g(g_0)\ m\right) \\ m \to m\left(1 + a\ b_m(g_0)\ m\right) \end{cases}$$

$$S_{\mathrm{eff}} = S_{QCD} + a\int d^4x\ c_1(g_0)\mathcal{L}_1^{(5)} = S_{QCD} + a\int d^4x\ c_1(g_0)\overline{\psi}(x)\sigma_{\mu\nu}G_{\mu\nu}\psi(x)$$

At finite lattice spacing use a discretization of $\mathcal{L}_1^{(5)}$

$$S_F = a^4\sum_x \overline{\psi}(x)\left[D_W + m + a\frac{i}{4}c_{\mathrm{sw}}(g_0)\widehat{G}_{\mu\nu}(x)\right]\psi(x)$$

Clover fermions

# Symanzik improvement

$$\langle O(x)O(y)\rangle = \frac{1}{Z}\int \mathcal{D}\left[U,\overline{\psi},\psi\right] \mathrm{e}^{-S[U,\overline{\psi},\psi]} O(x)O(y)$$

$$O_{\mathrm{eff}} = O_0 + a\sum_k c_k(g_0)O_k^{(1)} + O(a^2)$$

$$O_k^{(1)} \longrightarrow d_k = d_0 + 1$$

Transform under the lattice symmetries as O(x)

$$\langle O(x)O(y)\rangle_{\mathrm{latt}} \xrightarrow{\text{effective}}$$

$$\langle O_0(x)O_0(y)\rangle_{\mathrm{cont}} - a\,\langle O_0(x)O_0(y)S_1\rangle_{\mathrm{cont}} + \sum_k c_k(g_0)\,\langle O_k(x)O_0(y)\rangle_{\mathrm{cont}} \sum_k c_k(g_0)\,\langle O_0(x)O_k(y)\rangle_{\mathrm{cont}}$$

# Symanzik improvement

$$A_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\frac{\sigma^a}{2}\psi(x)$$

$$\left(O_1^{(1)}\right)_\mu^a = \overline{\psi}(x)\gamma_5\frac{\sigma^a}{2}\sigma_{\mu\nu}\overrightarrow{D}_\nu\psi(x) - \overline{\psi}(x)\overleftarrow{D}_\nu\gamma_5\frac{\sigma^a}{2}\sigma_{\mu\nu}\psi(x)$$

$$\left(O_2^{(1)}\right)_\mu^a = \partial_\mu\left[\overline{\psi}(x)\gamma_5\frac{\sigma^a}{2}\psi(x)\right]$$

$$\left(O_3^{(1)}\right)_\mu^a = m\overline{\psi}(x)\gamma_\mu\gamma_5\frac{\sigma^a}{2}\psi(x)$$

redundant

Axial current normalization

$$[A_\mu^a(x)] = A_\mu^a(x) + c_A(g_0)a\frac{1}{2}\left(\partial_\mu + \partial_\mu^*\right)\overline{\psi}(x)\gamma_5\frac{\sigma^a}{2}\psi(x)$$

# Continuum limit

K. Wilson: 1974

Wilson fermions $$P_{\text{latt}} = P_{\text{QCD}} + aP_1 + \ldots$$

J. Kogut, L. Susskind: 1975
K. Symanzik: 1982-1983
B. Sheikholeslami, R. Wohlert:1985
D. Kaplan: 1992
Y. Shamir: 1993
Narayanan, Neuberger: 1993
M. Lüscher, S. Sint, R. Sommer, P. Weisz:1995
R. Frezzotti, P. Grassi, S. Sint., P. Weisz: 2001
A. S.: Phys.Rept. 461 (2008) 37-110

Non-perturbative clover-fermions
Wilson twisted mass fermions
Domain-wall fermions
Staggered fermions
Overlap fermions

$$P_{\text{latt}} = P_{\text{QCD}} + a^2 P_2 + \ldots$$

# Signal-to-noise

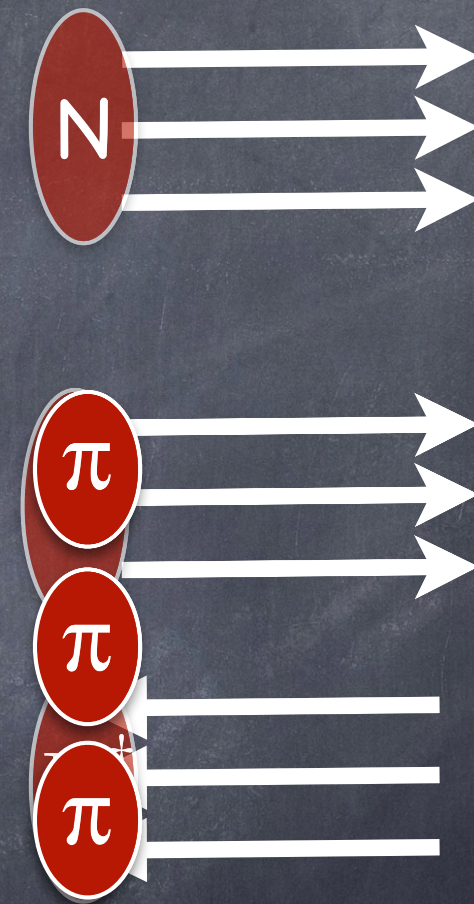
## Proton

$$\text{Signal} \sim \langle C \rangle \propto \mathrm{e}^{-M_P x_0}$$

$$\text{Variance} \sim \langle C\ C^\dagger \rangle - |\langle C \rangle|^2$$

$$\text{noise} \sim \sqrt{\langle C\ C^\dagger \rangle} \propto \mathrm{e}^{-\frac{3}{2} M_\pi x_0}$$

$$\frac{\text{noise}}{\text{signal}} \propto \mathrm{e}^{\left(M_P - \frac{3}{2} M_\pi\right) x_0}$$




Lepage

## For a nucleus A

$$\frac{\text{noise}}{\text{signal}} \propto \mathrm{e}^{A\left(M_P - \frac{3}{2} M_\pi\right) x_0}$$

# Renormalization

The difficult task is a->0 (Continuum limit)

Divergences amount to redefinitions of the parameters and fields in the action

$$\phi_{\mathrm{R}} = \lim_{a \to 0} Z_\phi(a)\phi(a) \qquad m_{\mathrm{R}} = \lim_{a \to 0} Z_m(a)m(a) \qquad g_{\mathrm{R}} = \lim_{a \to 0} Z_g(a)g(a)$$

Short-distance singularities $\lim_{x \to y} \phi(x)\phi(y) = ???$

Symmetries of the regulated theory dictates the mixing pattern

$$[\mathcal{O}_i]_{\mathrm{R}} = \lim_{a \to 0} Z_{ij}(a)\mathcal{O}_j(a)$$

Narayanan, Neuberger: 2006

# Gradient flow

Lüscher 2010-2013

Lüscher, Weisz 2011

$$\partial_{t_f} B_\mu(x,t_f) = D_\nu(t_f) G_{\nu\mu}(x,t_f)$$

$$B_\mu(x,t_f)\big|_{t_f=0} = A_\mu(x)$$

$$D_\nu(t_f) = \partial_\nu + [B_\nu(x,t_f),\cdot]$$

$x_\mu = (t,\underline{x})$ $\quad t_f = \text{flow-time} \quad [t_f] = -2$

$$G_{\mu\nu}(x,t_f) = \partial_\mu B_\nu(x,t_f) - \partial_\nu B_\mu(x,t_f) + [B_\mu, B_\nu]$$

Scale setting

Lüscher: 2010
BMW:2012

Quasi-PDF

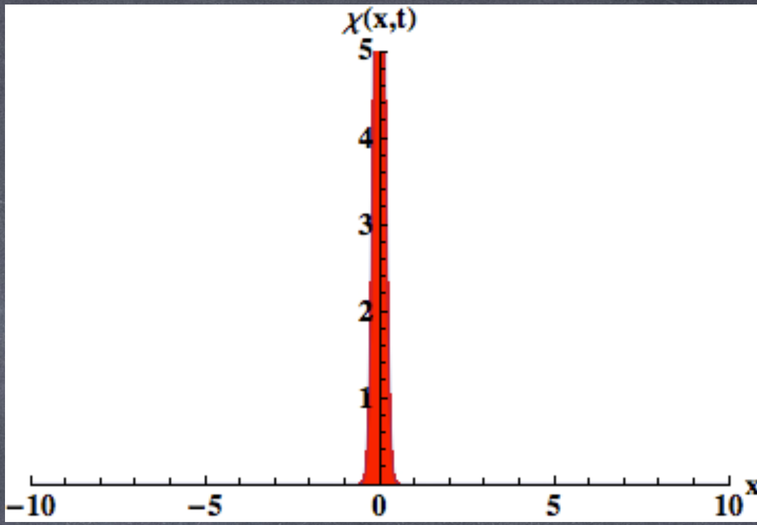
Monahan, Orginos: 2015-2017

Energy-momentum tensor

Del Debbio, Patella, Rago:2013

Continuous form of stout-smearing

Morningstar, Peardon: 2004




$$\partial_t B_\mu = \partial_\nu \partial_\nu B_\mu$$

$$B_\mu(x,t) = \int d^4y \; K(x-y;t) A_\mu(y)$$

$$K(x;t) = \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{ipx} \mathrm{e}^{-tp^2} = \frac{\mathrm{e}^{-x^2/4t}}{(4\pi t)^2}$$

- Gaussian damping at large momenta
- Smoothing at short distance over a range $\sqrt{8t}$

$B_\mu(x,t)$ $\quad t > 0 \quad$ finite

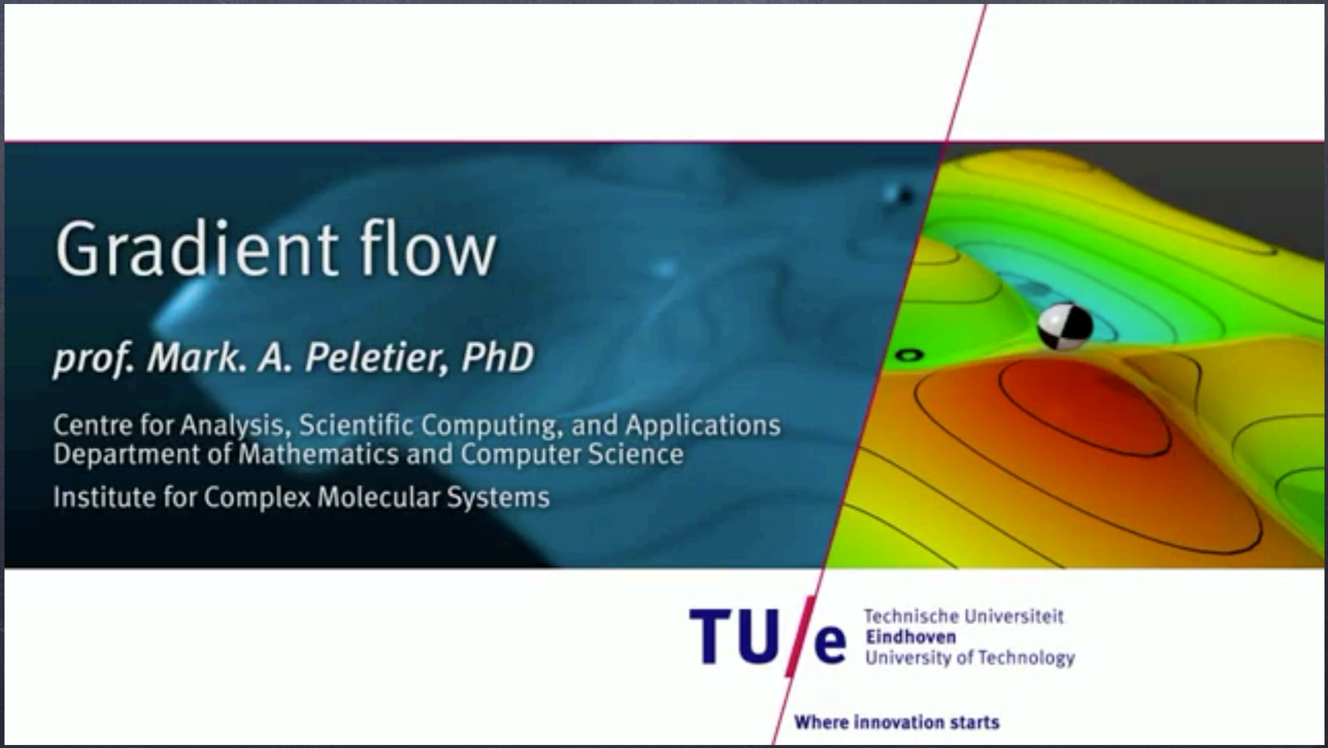
No additional renormalization

Lüscher, Weisz: 2013

# Steepest descent

$$D_{\nu,t} G_{\nu\mu,t} \sim \frac{\delta S}{\delta B_{\mu,t}} \qquad \frac{\Delta x}{\Delta t} = -\frac{\Delta S}{\Delta x} \Rightarrow x_1 = x_0 - \Delta t \left.\frac{\Delta S}{\Delta x}\right|_0$$

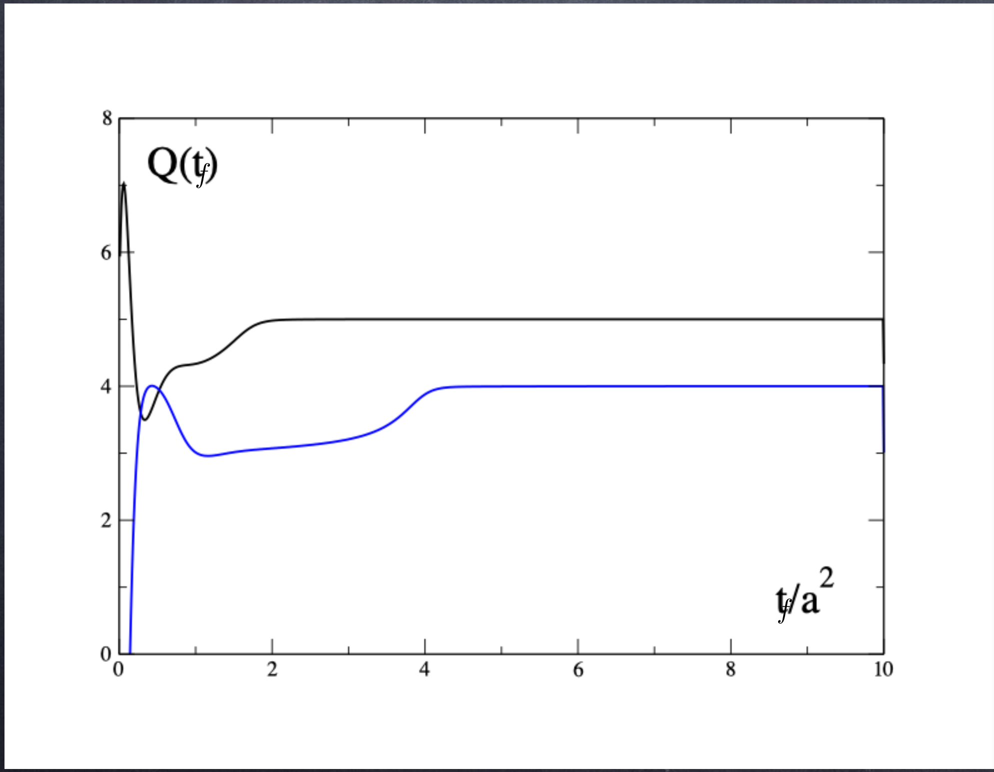
$$S(x_1) = S(x_0) - \Delta t \left( \left.\frac{\Delta S}{\Delta x}\right|_0 \right)^2$$

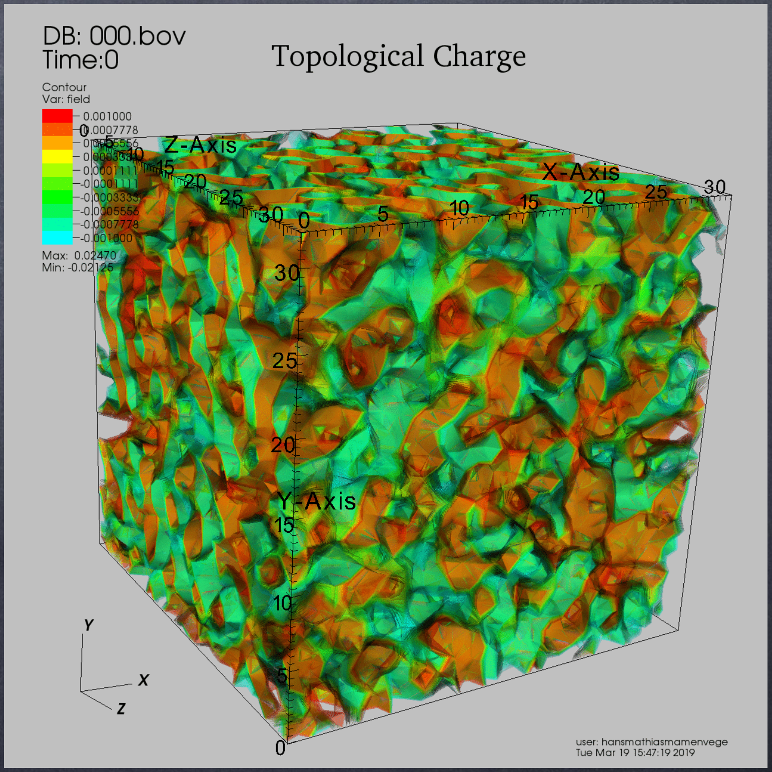


$$q(x, t_f) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\left\{G_{\mu\nu}(x, t_f) G_{\rho\sigma}(x, t_f)\right\}$$

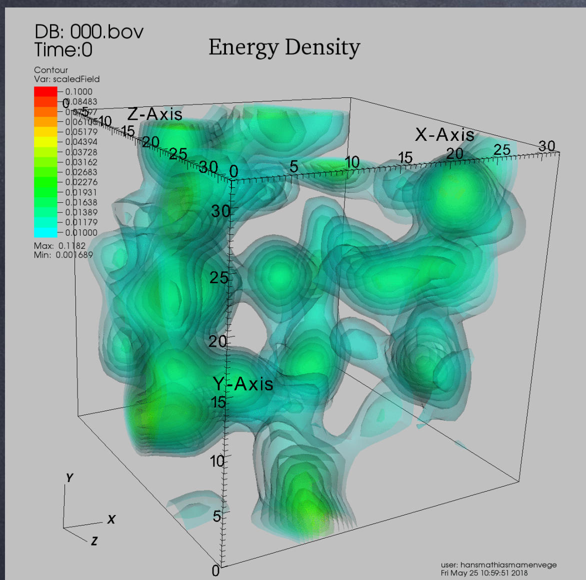
$$Q(t_f) = \int d^4x \; q(x, t_f)$$

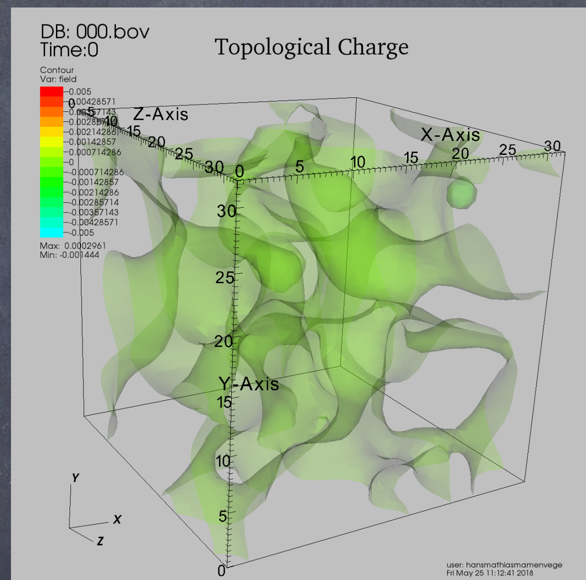
$$S[A] \geq \frac{8\pi^2}{g_0^2} |Q[A]|$$

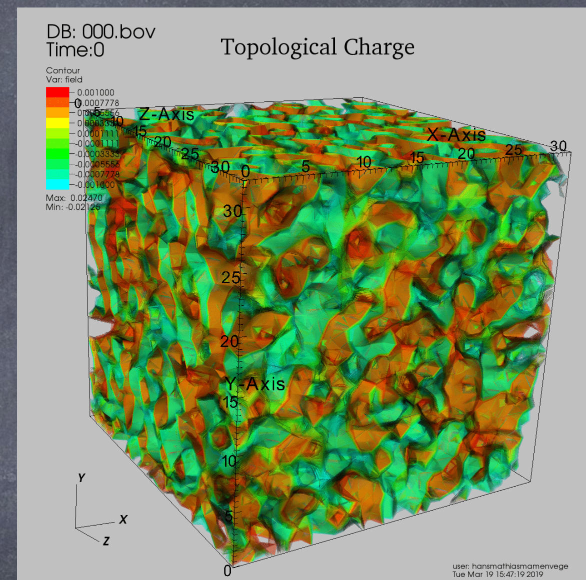


# LatViz



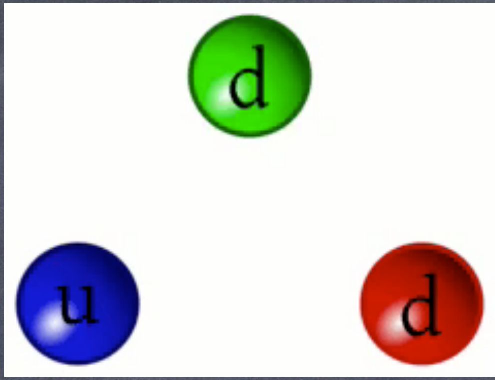



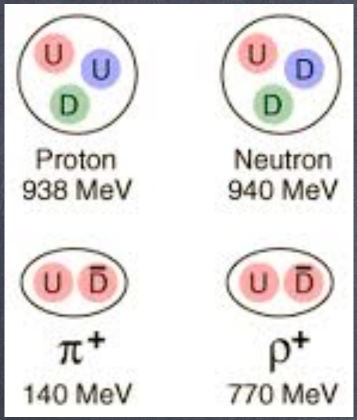



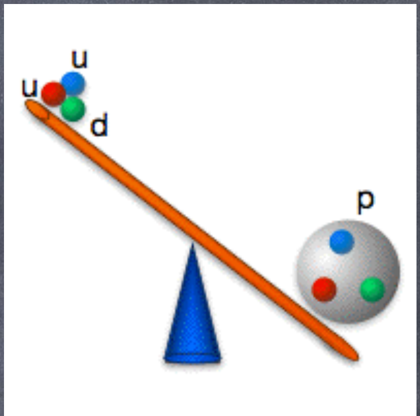

Pederiva, Vege: 2018

# From lecture 1

# Quarks and glue




$$m_u \simeq 2 - 3 \text{ MeV}$$

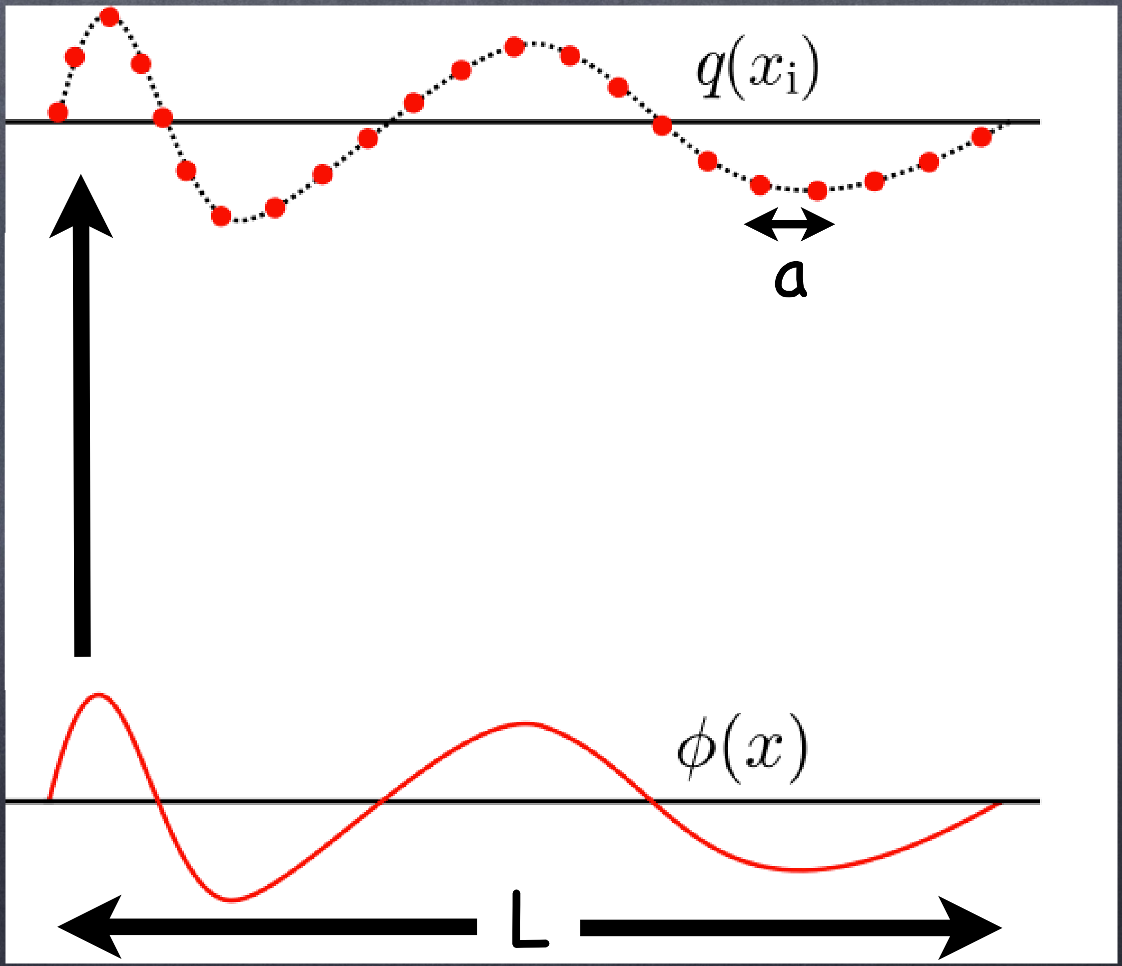
$$m_d \simeq 4 - 5 \text{ MeV}$$

$$\lim_{m_u, m_d \to 0} m_P \simeq 900 \text{ MeV}$$

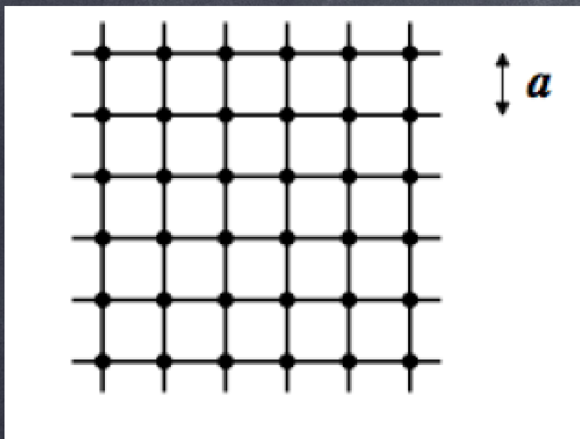
# “Be wise...discretise”

Kac

Discretise space time on finite lattice with spacing a and finite size L



# Functional integral




$$\mathcal{D}[\phi] = \prod_{x_\mu} d\phi(x) = \prod_{x_\mu} d\phi_{x_\mu} \qquad x_\mu = a n_\mu$$

$$64 \times 64 \times 64 \times 64 = 2^{24}$$

$$\int dy_1 \; dy_2 \; \cdots dy_{2^{24}}$$

- Not factorizable. $n^{2^{24}}$ terms in the sums AAAARRRRRGGGGHHHH!!!!!
- Statistical approach => importance sampling

$$\{\phi_i\} \qquad \langle \mathcal{O} \rangle = \frac{1}{N_{\mathrm{cf}}} \sum_{i=1}^{N_{\mathrm{cf}}} \mathcal{O}[\phi_i] + O\left(\frac{1}{\sqrt{N_{\mathrm{cf}}}}\right)$$

# Lattice QCD

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A_\mu, \overline{\psi}, \psi\right] \mathcal{O}\left[A_\mu, \overline{\psi}, \psi\right] \ \mathrm{e}^{-S_F\left[A_\mu, \overline{\psi}, \psi\right] - S_G\left[A_\mu\right]}$$

$$S_F = \int d^4x \ \overline{\psi}(x) D_W\left[A_\mu\right] \psi(x)$$

Capacity

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\left[A_\mu\right] \mathcal{O}\left[A_\mu\right] \ \mathrm{e}^{-S_G\left[A_\mu\right]} \left\{\det \ \left(D_W\left[A_\mu\right]\right)\right\}^{N_f}$$

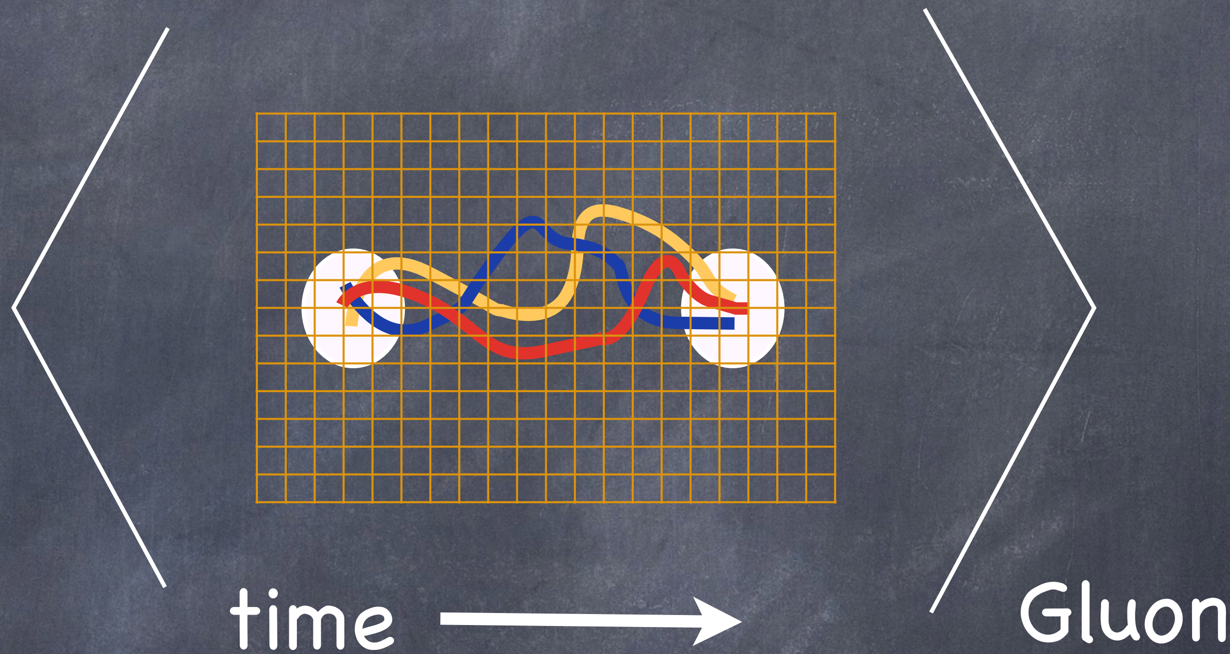
Capability

Large computing resources needed to generate representative ensembles and compute observables

$$\langle \mathcal{O} \rangle = \lim_{N_{\mathrm{cf}} \to \infty} \frac{1}{N_{\mathrm{cf}}} \sum_{\mathrm{gauge\ cf}} \mathcal{O}\left[A_\mu\right]$$

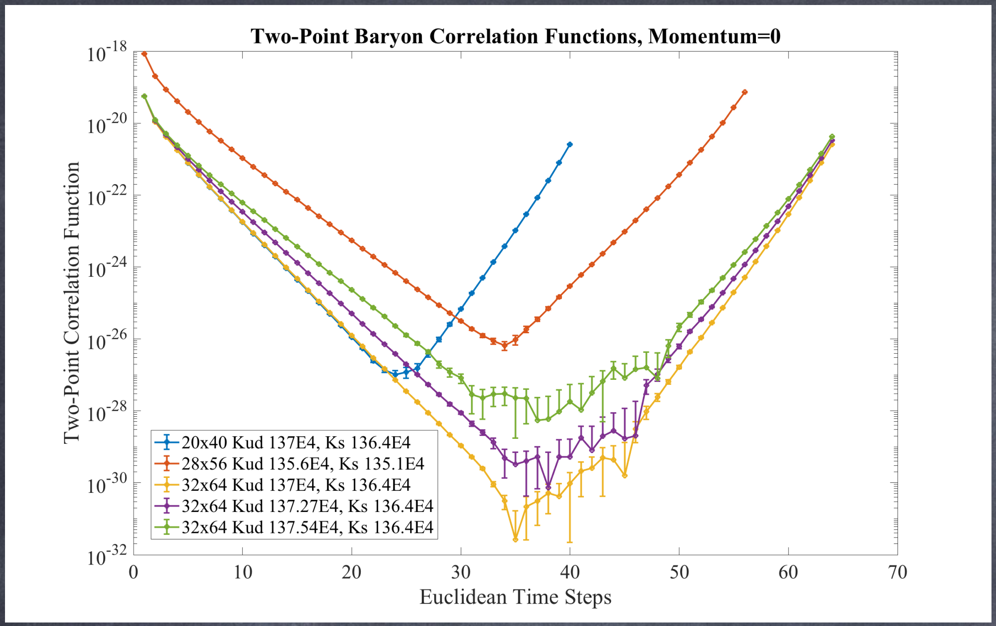
# Hadron masses

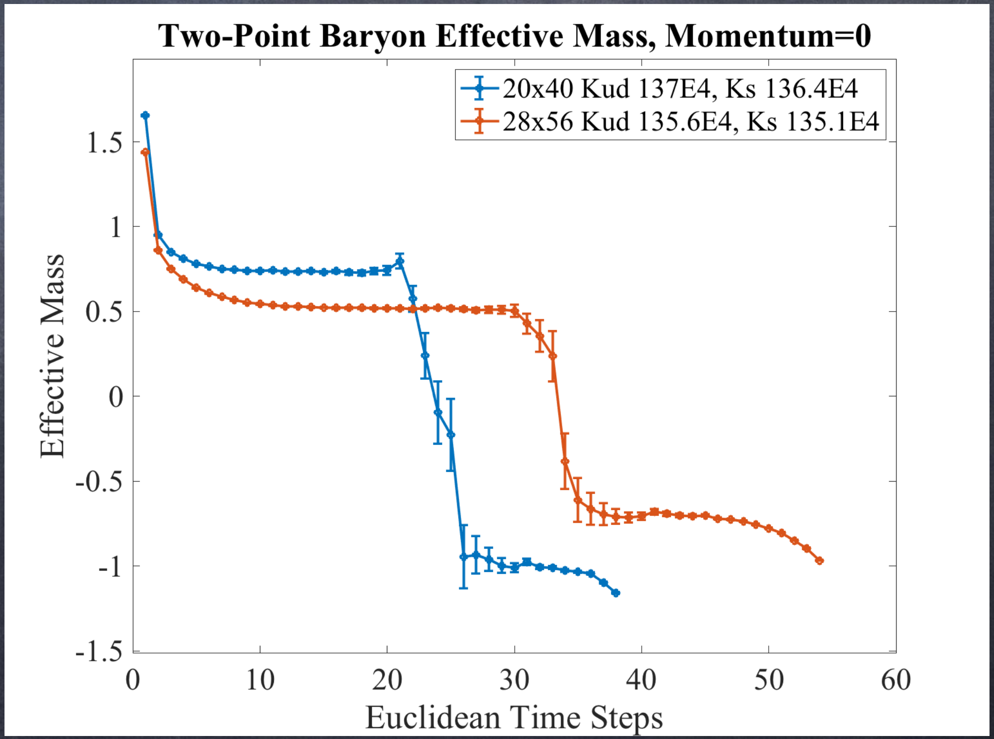
- Create three quarks at source and annihilate three quarks at sink
- Ensembles add virtual gluons and quark loops
- Only correct quantum numbers propagate




$$C(t) = \sum_n Z_n \, \mathrm{e}^{-E_n x_0} \xrightarrow[x_0 \to \infty]{} Z_0 \, \mathrm{e}^{-E_0 x_0}$$

Effective mass


Two-Point Baryon Correlation Functions, Momentum=0
Two-Point Correlation Function
Euclidean Time Steps
20x40 Kud 137E4, Ks 136.4E4
28x56 Kud 135.6E4, Ks 135.1E4
32x64 Kud 137E4, Ks 136.4E4
32x64 Kud 137.27E4, Ks 136.4E4
32x64 Kud 137.54E4, Ks 136.4E4


Two-Point Baryon Effective Mass, Momentum=0
Effective Mass
Euclidean Time Steps
20x40 Kud 137E4, Ks 136.4E4
28x56 Kud 135.6E4, Ks 135.1E4

Shaw

# Thank you!

Tonight pizza at 8pm.
"Antico Pozzo"