Fermions on the lattice

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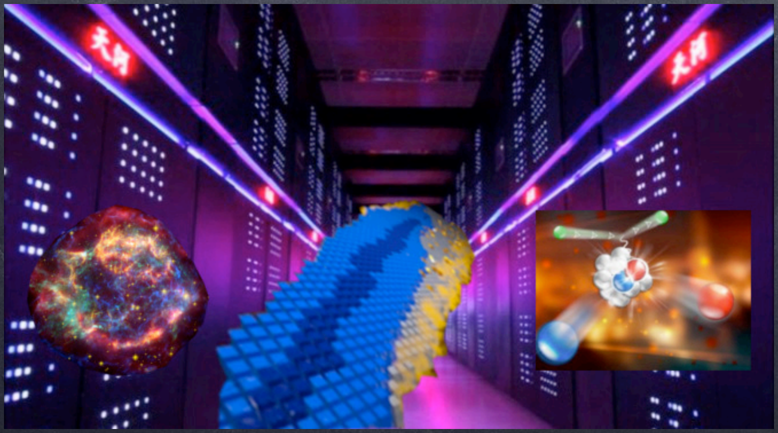
TALENT School - From quarks and gluons to nuclear forces and structures



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Quantization of LQCD

Gauge invariant correlation function

$$\langle O_1(x_1)\ldots O_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U]\int \mathcal{D}[\psi,\overline{\psi}]$$
$$\times \quad O_1(x_1)\ldots O_n(x_n)\exp\left\{-S_F[\psi,\overline{\psi},U]-S_G[U]\right\}$$

Lattice action

$$\mathcal{Z}=\int \mathcal{D}[U]\int \mathcal{D}[\psi,\overline{\psi}]\exp\left\{-S_F[\psi,\overline{\psi},U]-S_G[U]\right\}$$

QCD path integral

$$\mathcal{D}[\psi,\overline{\psi}] = \prod_{x\in\Lambda}\prod_{f,\alpha,A} d\psi^A_{f\alpha}(x)\, d\overline{\psi}^A_{f\alpha}(x)$$

Grassmann numbers

$$\mathcal{D}[U] = \prod_{x\in\Lambda}\prod_{\mu=1}^{4} dU(x,\mu)$$

Haar measure

Fermi statistics and Grassmann numbers

$$\langle O \rangle = \langle [O]_F \rangle_G$$

$$[O]_F = \frac{1}{Z_F} \int \mathcal{D}[\psi, \overline{\psi}] \mathrm{e}^{-S_F[\psi, \overline{\psi}, U]} O[\psi, \overline{\psi}, U]$$

$$X[U] = [O]_F$$

$$Z_F = \int \mathcal{D}[\psi, \overline{\psi}] \mathrm{e}^{-S_F[\psi, \overline{\psi}, U]}$$

Fermion determinant

$$\langle X \rangle_G = \frac{1}{Z} \int \mathcal{D}[U] \mathrm{e}^{-S_G[U]} Z_F[U] X[U]$$

$$Z = \int \mathcal{D}[\psi, \overline{\psi}, U] \mathrm{e}^{-S_G[U] - S_F[\psi, \overline{\psi}, U]} = \int \mathcal{D}[U] \mathrm{e}^{-S_G[U]} Z_F[U]$$

Fermi statistics and Grassmann numbers

Fermi statistics <—> antisymmetry
interchange quantum numbers

$$\left\langle \psi^{A_1}_{f_1,\alpha_1}(x_1)\psi^{A_2}_{f_2,\alpha_2}(x_2)\overline{\psi}^{B_1}_{g_1,\beta_1}(y_1)\cdots\overline{\psi}^{B_m}_{g_m,\beta_m}(y_m)\right\rangle$$

$f_1 \leftrightarrow f_2 \qquad x_1 \leftrightarrow x_2 \qquad \alpha_1 \leftrightarrow \alpha_2 \qquad A_1 \leftrightarrow A_2$

$$\psi^{A}_{f,\alpha}(x)\psi^{A'}_{f',\alpha'}(x') = -\psi^{A'}_{f',\alpha'}(x')\psi^{A}_{f,\alpha}(x)$$

$$\overline{\psi}^{A}_{f,\alpha}(x)\overline{\psi}^{A'}_{f',\alpha'}(x') = -\overline{\psi}^{A'}_{f',\alpha'}(x')\overline{\psi}^{A}_{f,\alpha}(x)$$

Smit

$$\psi^{A}_{f,\alpha}(x)\overline{\psi}^{A'}_{f',\alpha'}(x') = -\overline{\psi}^{A'}_{f',\alpha'}(x')\psi^{A}_{f,\alpha}(x)$$

Grassmann numbers

$$\eta_i\eta_j = -\eta_j\eta_i \quad \forall i,j \quad i,j = 1,\cdots,N \qquad \Rightarrow \eta_i^2 = 0$$

Nilpotent

Any $f(\underline{\eta})$ is a polynomial

$$f(\eta_1,\ldots,\eta_N) = a + \sum_i a_i\eta_i + \sum_{i<j} a_{ij}\eta_i\eta_j + \sum_{i<j<k} a_{ijk}\eta_i\eta_j\eta_k + \cdots a_{12\cdots N}\eta_1\cdots\eta_N$$

$$a, a_i, a_{ij}, \ldots = \mathbb{C}$$

Zinn-Justin: Quantum Field Theory and Critical Phenomena

Derivatives

$$f(\eta_1, \eta_2) = a + a_1\eta_1 + a_2\eta_2 + a_{12}\eta_1\eta_2$$

$$\frac{\partial}{\partial \eta_1} f(\eta_1, \eta_2) \equiv a_1 + a_{12}\eta_2$$

$$f(\eta_1, \eta_2) = a + a_1\eta_1 + a_2\eta_2 - a_{12}\eta_2\eta_1 \quad \Rightarrow \quad \frac{\partial}{\partial \eta_1}\eta_2 = -\eta_2 \frac{\partial}{\partial \eta_1}$$

$$\frac{\partial}{\partial \eta_2}\frac{\partial}{\partial \eta_1} = -\frac{\partial}{\partial \eta_1}\frac{\partial}{\partial \eta_2}$$

$$\frac{\partial}{\partial \eta_i} 1 = 0 \qquad \frac{\partial}{\partial \eta_i}\eta_i = 1 \qquad \frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \eta_j} = -\frac{\partial}{\partial \eta_j}\frac{\partial}{\partial \eta_i} \qquad \frac{\partial}{\partial \eta_i}\eta_j = -\eta_j \frac{\partial}{\partial \eta_i} \qquad i \neq j$$

Integration

Keep properties of integration in $\mathbb{R}^N$ $\quad \Omega \subset \mathbb{R}^N \quad f(\underline{x}) = 0 \quad \underline{x} \in \partial\Omega$

$$\int_\Omega f(\underline{x}) = \int dx_1 \cdots dx_N f(x_1, \dots, x_N)$$ Linear function of f

$$\int d^N \eta f(\underline{\eta}) \in \mathbb{C}$$ and linear

1. $$\int d^N \eta (\lambda_1 f_1(\underline{\eta}) + \lambda_2 f_2(\underline{\eta})) = \lambda_1 \int d^N \eta f_1(\underline{\eta}) + \lambda_2 \int d^N \eta f_2(\underline{\eta})$$

2. $$\int d^N \eta \frac{\partial}{\partial \eta_i} f(\underline{\eta}) = 0$$

2.1 $\int d^N \eta f(\underline{\eta}) = a_{12\cdots N} \quad \int d^N \eta \eta_1 \cdots \eta_N = 1$

Integration

Keep properties of integration in $\mathbb{R}^N$ $\quad \Omega \subset \mathbb{R}^N \quad f(\underline{x}) = 0 \quad \underline{x} \in \partial\Omega$

$$d^N\eta = d\eta_N d\eta_{N-1} \cdots d\eta_1$$

$$\int d\eta_i = 0 \qquad \int d\eta_i \eta_i = 1 \qquad d\eta_i d\eta_j = -d\eta_j d\eta_i$$

$$\int d\eta_i \Leftrightarrow \frac{\partial}{\partial \eta_i}$$

Linear change of variables

$$\eta'_i = M_{ij}\eta_j \quad M_{ij} \in \mathbb{C} \quad N \times N$$

$$\int d^N\eta \; \eta_1 \cdots \eta_N = \int d^N\eta' \; \eta'_1 \cdots \eta'_N = \int d^N\eta' \sum_{i_1,\cdots,i_N} M_{1i_1} \cdots M_{Ni_N} \eta_{i_1} \cdots \eta_{i_N}$$

$$\int d^N\eta' \sum_{i_1,\cdots,i_N} \epsilon_{i_1 \cdots i_N} M_{1i_1} \cdots M_{Ni_N} \eta_1 \cdots \eta_N = \det[M] \int d^N\eta' \; \eta_1 \cdots \eta_N$$

$$d^N\eta = \det M d^N\eta' \qquad d^N x = \frac{1}{\det M} d^N x'$$

Gaussian Integrals

2N generators $\eta_i, \overline{\eta}_i \quad i = 1, \ldots, N$

$$\eta_i \eta_j = -\eta_j \eta_i \quad \overline{\eta}_i \overline{\eta}_j = -\overline{\eta}_j \overline{\eta}_i \quad \eta_i \overline{\eta}_j = -\overline{\eta}_j \eta_i$$

$$Z_F = \int d\eta_N \ d\overline{\eta}_N \cdots d\eta_1 d\overline{\eta}_1 \exp\left[\overline{\eta}_i M_{ij} \eta_j\right] = \det[M]$$

$$\eta_i, \overline{\eta}_i \quad \theta_i, \overline{\theta}_i \quad i = 1, \ldots, N$$

Gaussian Integrals

$$\eta_i, \overline{\eta}_i \quad \theta_i, \overline{\theta}_i \quad i = 1, \ldots, N$$

Sources

$$W[\theta, \overline{\theta}] = \int \left(\prod_{i=1}^{N} d\eta_i \; d\overline{\eta}_i \right) \exp\left\{ \overline{\eta}_k M_{kl} \eta_l + \overline{\theta}_i \eta_i + \overline{\eta}_i \theta_i \right\}$$

$$\boxed{W[\theta, \overline{\theta}] = \det[M] \exp\left\{ -\overline{\theta}_k (M^{-1})_{kl} \theta_l \right\}}$$

Wick's theorem

$$\left[\eta_{i_1}\overline{\eta}_{j_1}\cdots\eta_{i_N}\overline{\eta}_{j_N}\right]_F = \frac{1}{Z_F}\int\left(\prod_{k=1}^{N} d\eta_k\ d\overline{\eta}_k\right)\eta_{i_1}\overline{\eta}_{j_1}\cdots\eta_{i_N}\overline{\eta}_{j_N}\times\exp\left\{\overline{\eta}_i M_{ij}\eta_j\right\} =$$

$$= (-1)^N \sum_{P(1,2,\cdots,N)} \mathrm{sign}(P)(M^{-1})_{i_1 j_{P_1}}\cdots(M^{-1})_{i_N j_{P_N}}$$

$$\left[\eta_{i_1}\overline{\eta}_{j_1}\cdots\eta_{i_N}\overline{\eta}_{j_N}\right]_F = \frac{1}{Z_F}\frac{\partial}{\partial\theta_{j_1}}\frac{\partial}{\partial\overline{\theta}_{i_1}}\cdots\frac{\partial}{\partial\theta_{j_N}}\frac{\partial}{\partial\overline{\theta}_{i_N}}W[\theta,\overline{\theta}]\Big|_{\theta,\overline{\theta}=0}$$

Fermion action

Wilson 1974

$$S = S_G + S_F \qquad S_G = \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu} P_{\mu\nu}(x)$$

$$S_F = a^4 \sum_x \overline{\psi}(x) \left[\gamma_\mu \frac{1}{2} \left(\nabla_\mu + \nabla_\mu^* \right) + m \right] \psi(x)$$

Covariant derivatives

$$\nabla_\mu \psi(x) = \frac{1}{a} \left[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x) \right]$$

$$\nabla_\mu^* \psi(x) = \frac{1}{a} \left[\psi(x) - U(x-a\hat{\mu},\mu)^{-1}\psi(x-a\hat{\mu}) \right]$$

Fermion action

Wilson 1974

$$S_F = a^4 \sum_x \overline{\psi}(x) D \psi(x) \qquad D = \left[\gamma_\mu \frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right) + m\right]$$

$$U(x;\mu) = 1 \qquad \frac{1}{2}\left(\partial_\mu^* + \partial_\mu\right) \longrightarrow \frac{i}{a}\sin\left(ap_\mu\right) \equiv i\mathring{p}_\mu \qquad \mathring{p}_\mu = p_\mu + \mathrm{O}(a^2)$$

$$\widetilde{D}(p) = i\gamma_\mu \mathring{p}_\mu + m \qquad \widetilde{D}(p)^{-1} = \frac{m - i\gamma_\mu \mathring{p}_\mu}{m^2 + \mathring{p}^2} \qquad \mathring{p}^2 = \frac{1}{a^2}\mathring{p}_\mu \mathring{p}_\mu$$

Quark propagator

$$m = 0 \qquad \widetilde{D}(p)^{-1}\Big|_{m=0} = \frac{-i\gamma_\mu \mathring{p}_\mu}{\mathring{p}^2} \xrightarrow{a\to 0} \frac{-i\gamma_\mu p_\mu}{p^2}$$

16 poles

1 pole

Wilson fermions

Wilson 1974

$$S_F = a^4 \sum_x \overline{\psi}(x) \left[\gamma_\mu \frac{1}{2} \left(\nabla_\mu + \nabla_\mu^*\right) + m\right] \psi(x) \quad \Rightarrow \quad S_F = a^4 \sum_x \overline{\psi}(x) \left[D_W + m\right] \psi(x)$$

$$D_W = \frac{1}{2} \left[\gamma_\mu \left(\nabla_\mu + \nabla_\mu^*\right) - \underline{a \nabla_\mu^* \nabla_\mu}\right]$$

$$\nabla_\mu \psi(x) = \frac{1}{a} \left[U(x,\mu)\psi(x + a\hat{\mu}) - \psi(x)\right]$$

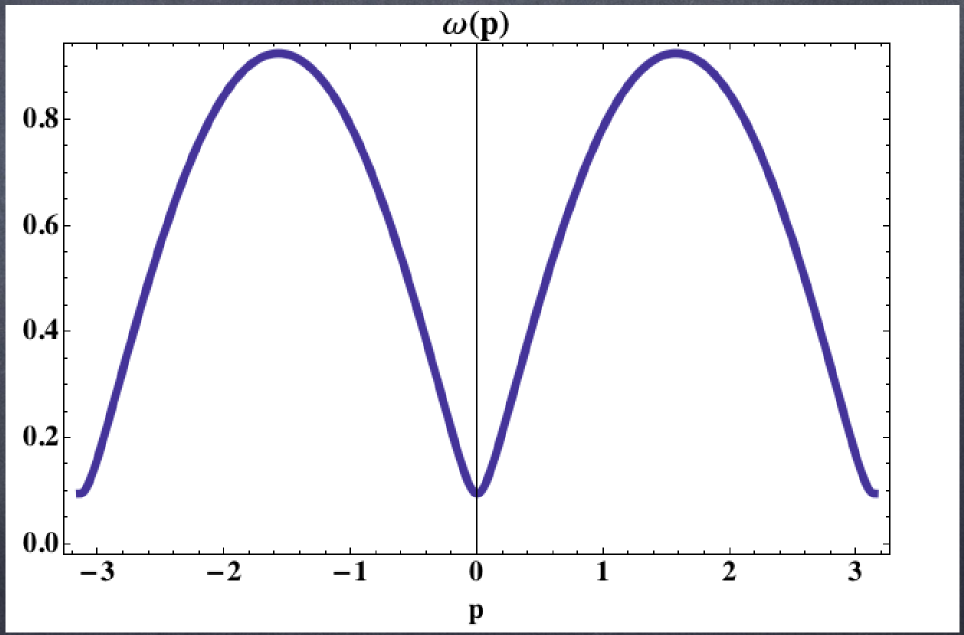
$$\nabla_\mu^* \psi(x) = \frac{1}{a} \left[\psi(x) - U(x - a\hat{\mu}, \mu)^{-1} \psi(x - a\hat{\mu})\right]$$

Doublers

$$\frac{1}{i\,\mathring{\not{p}}+m} = \frac{-i\,\mathring{\not{p}}+m}{\mathring{p}^2+m^2}$$

Poles $\quad p_0 = \pm i\omega(\underline{p})$

$$\omega(\underline{p}) = \frac{2}{a}\,\mathrm{asinh}\left\{\frac{a}{2}\sqrt{\frac{\underline{\mathring{p}}^2+m^2}{1+am}}\right\}$$



Additional states with energy = mass

Wilson–Dirac operator

$$D_W = \sum_{\mu=0}^{3} \frac{1}{2} \left[\gamma_\mu \left(\partial_\mu^* + \partial_\mu \right) - \underline{a \partial_\mu^* \partial_\mu} \right] \longrightarrow i \not{p} + \underline{\frac{1}{2} a \hat{p}^2}$$

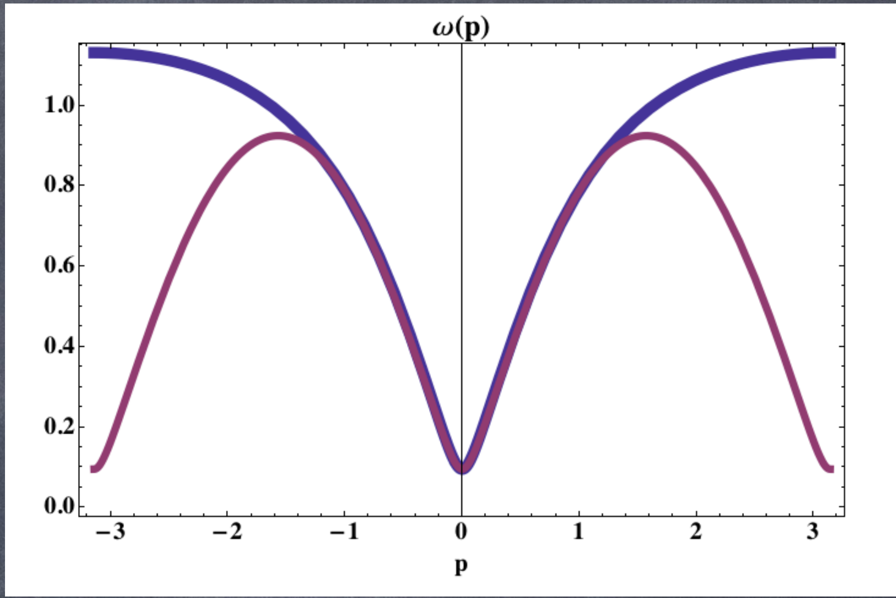
Free-quark two-point functions

$$\left(D_W + m \right) \psi(x) \overline{\psi}(0) = \frac{1}{a^4} \delta_{x,0}$$

$$\psi(x) \overline{\psi}(0) = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{e}^{ipx}}{i \not{p} + \frac{1}{2} a \hat{p}^2 + m}$$

Energy

$$\frac{1}{i\,\mathring{\not{p}} + \frac{1}{2}a\hat{p}^2 + m} = \frac{-i\,\mathring{\not{p}} + M(p)}{\mathring{p}^2 + M(p)^2}$$



Poles $\quad p_0 = \pm i\omega(\underline{p})$

$$\omega(\underline{p}) = \frac{2}{a}\text{asinh}\left\{\frac{a}{2}\sqrt{\frac{\underline{\mathring{p}}^2 + M(\underline{p})^2}{1 + aM(\underline{p})}}\right\}$$

$$M(\underline{p}) \equiv m + \frac{1}{2}a\underline{\hat{p}}^2$$

Wilson fermions

Wilson 1974

$$S_F = a^4 \sum_x \overline{\psi}(x) \left\{ \frac{1}{2} \left[\gamma_\mu \left(\nabla_\mu + \nabla_\mu^* \right) - a \nabla_\mu^* \nabla_\mu + m \right] \right\} \psi(x)$$

$$\nabla_\mu^* \nabla_\mu \psi(x) = \frac{1}{a^2} \sum_{\mu=1}^{4} \left[U(x;\mu)\psi(x+a\hat{\mu}) + U(x-a\hat{\mu};\mu)^{-1}\psi(x-a\hat{\mu}) - 2\psi(x) \right]$$

$$\overline{\psi}(x) \left[D_W + m \right] \psi(x) = \overline{\psi}(x) \left\{ \frac{1}{2a} \sum_{\mu=1}^{4} \gamma_\mu \left[U(x;\mu)\psi(x+a\hat{\mu}) - U(x-a\hat{\mu};\mu)^{-1}\psi(x-a\hat{\mu}) \right] \right.$$

$$- \frac{1}{2a} \sum_{\mu=1}^{4} \left[U(x;\mu)\psi(x+a\hat{\mu}) + U(x-a\hat{\mu};\mu)^{-1}\psi(x-a\hat{\mu}) \right]$$

$$\left. + \left(m + \frac{4}{a} \right) \psi(x) \right\}$$

Wilson fermions

Wilson 1974

$$\overline{\psi}(x)\left[D_W+m\right]\psi(x)=\overline{\psi}(x)\left\{-\frac{1}{2a}\sum_{\mu=1}^{4}\left[(1-\gamma_\mu)U(x;\mu)\psi(x+a\hat{\mu})+(1+\gamma_\mu)U(x-a\hat{\mu};\mu)^{-1}\psi(x-a\hat{\mu})\right]+\right.$$

$$\left.+\left(m+\frac{4}{a}\right)\psi(x)\right\}$$

$$=\left(m+\frac{4}{a}\right)\overline{\psi}(x)\left[\psi(x)-\frac{1}{2am+8}\sum_{\mu=1}^{4}(1-\gamma_\mu)U(x;\mu)\psi(x+a\hat{\mu})+(1+\gamma_\mu)U(x-a\hat{\mu};\mu)^{-1}\psi(x-a\hat{\mu})\right]$$

$$\kappa=\frac{1}{2am+8}$$

Hopping parameter

$$\psi(x)\to\sqrt{2a\kappa}\;\psi(x)$$

$$\overline{\psi}(x)\to\overline{\psi}(x)\sqrt{2a\kappa}$$

$$\overline{\psi}(x)\left[D_W+m\right]\psi(x)\to\overline{\psi}(x)\left[1-\kappa\cdot H\right]\psi(x)$$

Lattice fermions

Incomplete list

- Clover fermions

  Sheikholeslami, Wohlert: 1985

  Lüscher et al.: 1996,1997

- Staggered fermions

  Kogut, Susskind: 1975

- Wilson twisted mass fermions

  Frezzotti, Grassi, Sint, Weisz: 2001
  A.S.: 2007

- Ginsparg-Wilson fermions: Neuberger, domain-walls, perfect action, chirally improved

  Narayanan, Neuberger: 1992-1994
  Kaplan: 1992 - Shamir: 1993
  Neuberger: 1998 - Lüscher: 1998
  Hasenfratz, Niedermeyer: 1993- Hasenfratz:1998