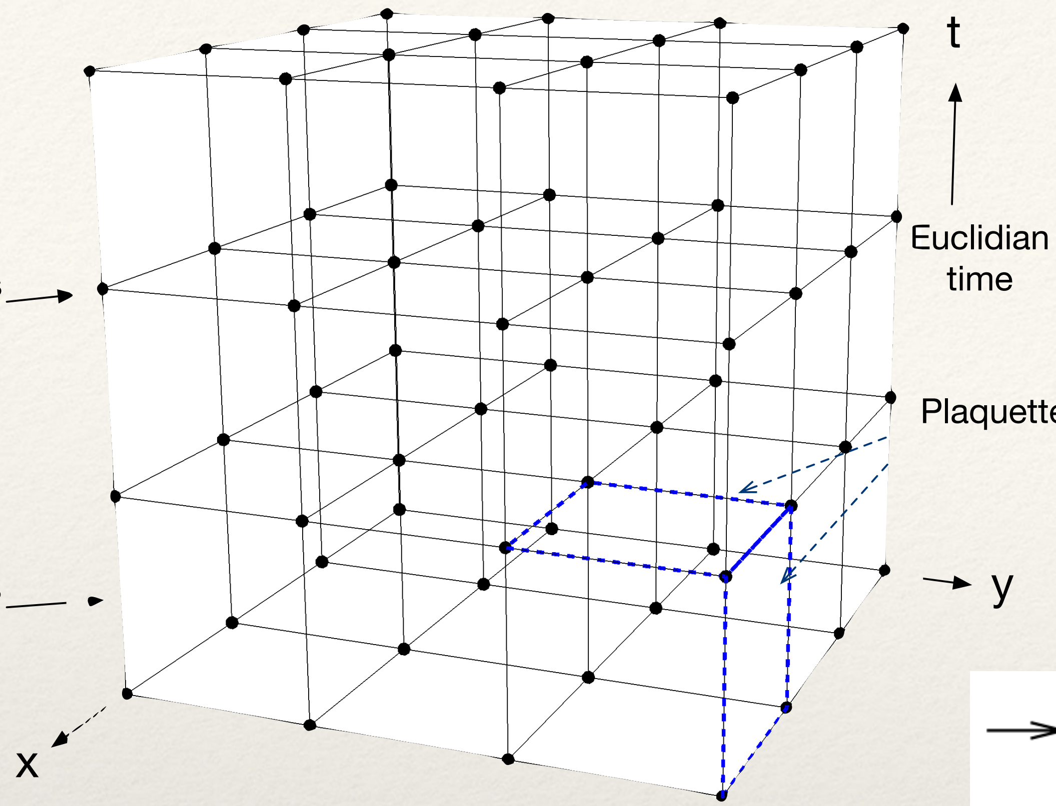
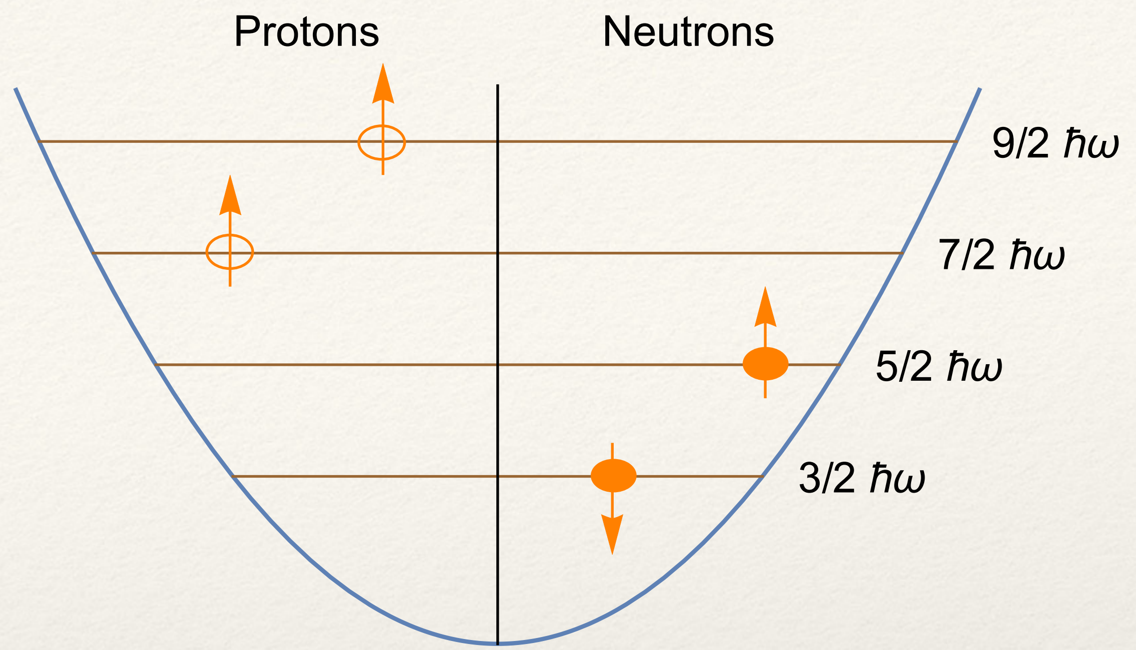
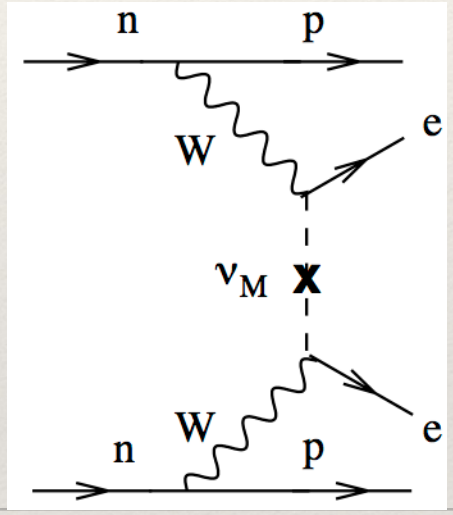






In collaboration with Wick Haxton

HOBET: QCD to Effective Interaction and Effective Operators

Kenneth S. McElvain

Berkeley
UNIVERSITY OF CALIFORNIA

BERKELEY LAB

Two Paths to Connect QCD to the Effective Interaction

- HOBET (Harmonic-Oscillator-Based Effective Theory)
- Compute phase shifts in LQCD - Fit HOBET LECs
    - See arXiv:1511.02262 (CalLat LQCD)+ arXiv: 1902.03543 (McElvain, Haxton)
- Compute nucleon scattering spectrum in LQCD - Fit HOBET LECs directly to spectrum + periodic boundary conditions.

The Bloch-Horowitz Equation

- For practical calculation reasons we often want to work in a subspace of the full Hilbert space.
  - P projects the subspace and Q=1-P gets the rest.
- The BH equation is the answer to the question: Does there exist an operator H^{eff} that lives in P with the same eigenvalues and projected eigenvectors of the full H.

$$Insert\ (P+Q)\ in\ H|\psi_i\rangle = E_i|\psi_i\rangle$$

$$PHP|\psi_i\rangle + PHQ|\psi_i\rangle = E_i P|\psi_i\rangle$$
$$QHP|\psi_i\rangle + QHQ|\psi_i\rangle = E_i Q|\psi_i\rangle$$

$$Q|\psi_i\rangle = \frac{1}{E_i - QH} QHP|\psi_i\rangle$$

$$P\left(H + H\frac{1}{E_i - QH}QH\right)P|\psi_i\rangle = E_i P|\psi_i\rangle$$

$$H^{eff}(E)P|\psi_i\rangle = PH\frac{1}{E_i - QH}P|\psi_i\rangle = E_i P|\psi_i\rangle$$

BH Characteristics

- Eigenstates of $H^{eff}(E)$ are projections with the same eigenvalues.
  - All eigenstates that overlap P are included!
  - True even if P projects a finite number of states.
- It is continuous in energy, including across E=0. An effective theory based on the BH equation can be fit in the continuum and used to find bound states.
- Explicitly energy dependent: Must solve self consistently.
  - Simple fixed point iteration converges rapidly.

HO Effective Theory

- Why the HO basis?
  - Discrete so we can use matrix techniques for solution.
  - Good for confined wave function of nucleus
  - With a consistent A-body quanta cutoff the center of mass is separable.
- In an HO ET with included space projector $P(\Lambda,b)$, both UV and IR are excluded.
- Major Issue
  - The kinetic energy operator T is a hopping operator, strongly connecting P & Q (IR).

HOBET Introduction

arXiv:1902.03543, McElvain & Haxton (2019)

- HOBET is based on a reorganization of the Bloch-Horowitz equation by Haxton and Luu.

$$H^{eff,\Lambda}(E) = P\left[H\frac{E}{E-QH}\right]P = P\frac{E}{E-TQ}\left[T + T\frac{Q}{E}T + V + V\frac{1}{E-QH}QV\right]\frac{E}{E-QT}P$$

- The reorganization isolates the impact of T for analytic calculation to all orders.
- The remaining part is replaced by a long range potential (like an OPEP) plus V_δ, which is a short range expansion around it.

HOBET Introduction

arXiv:1902.03543, McElvain & Haxton (2019)

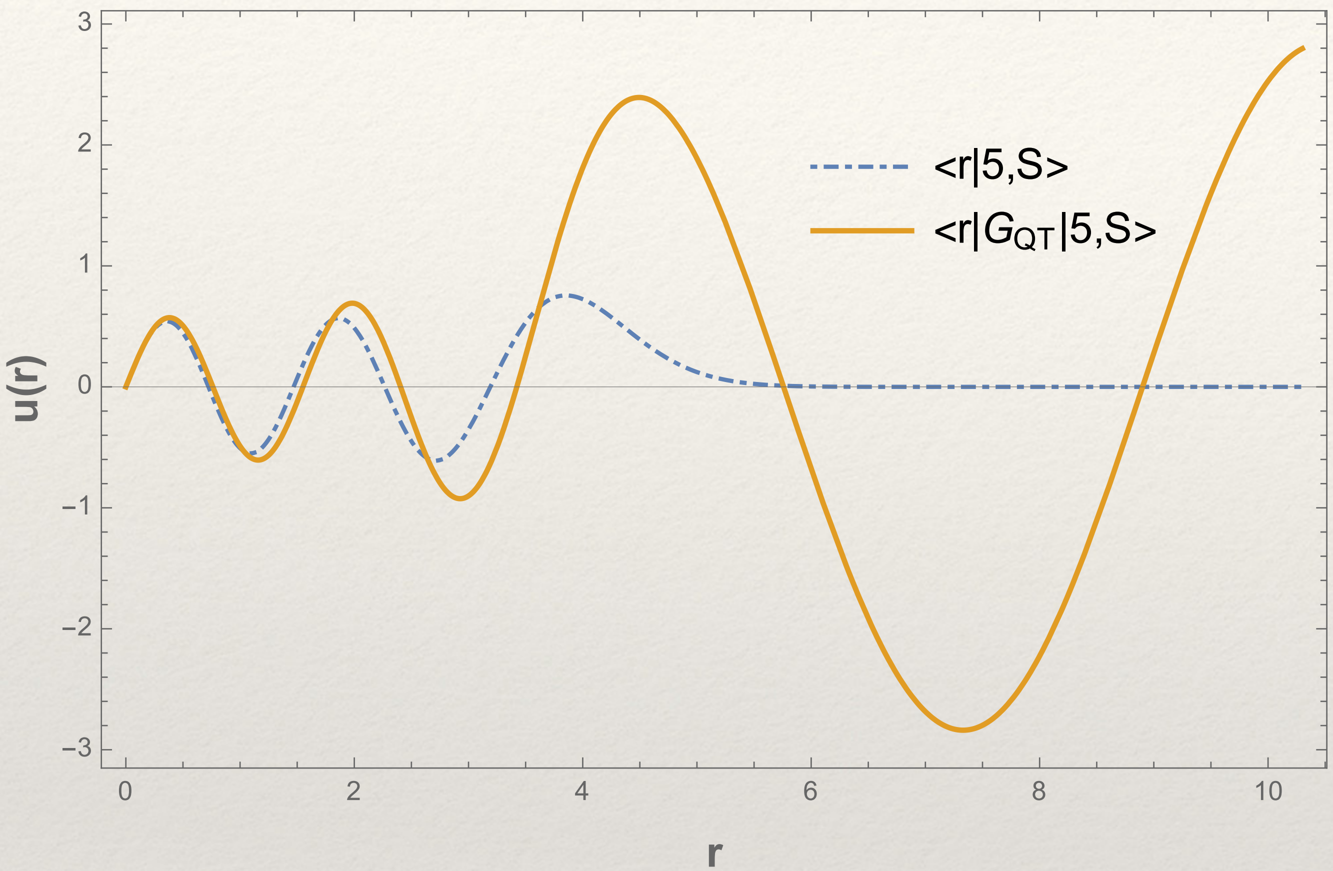
- HOBET is based on a reorganization of the Bloch-Horowitz equation by Haxton and Luu.

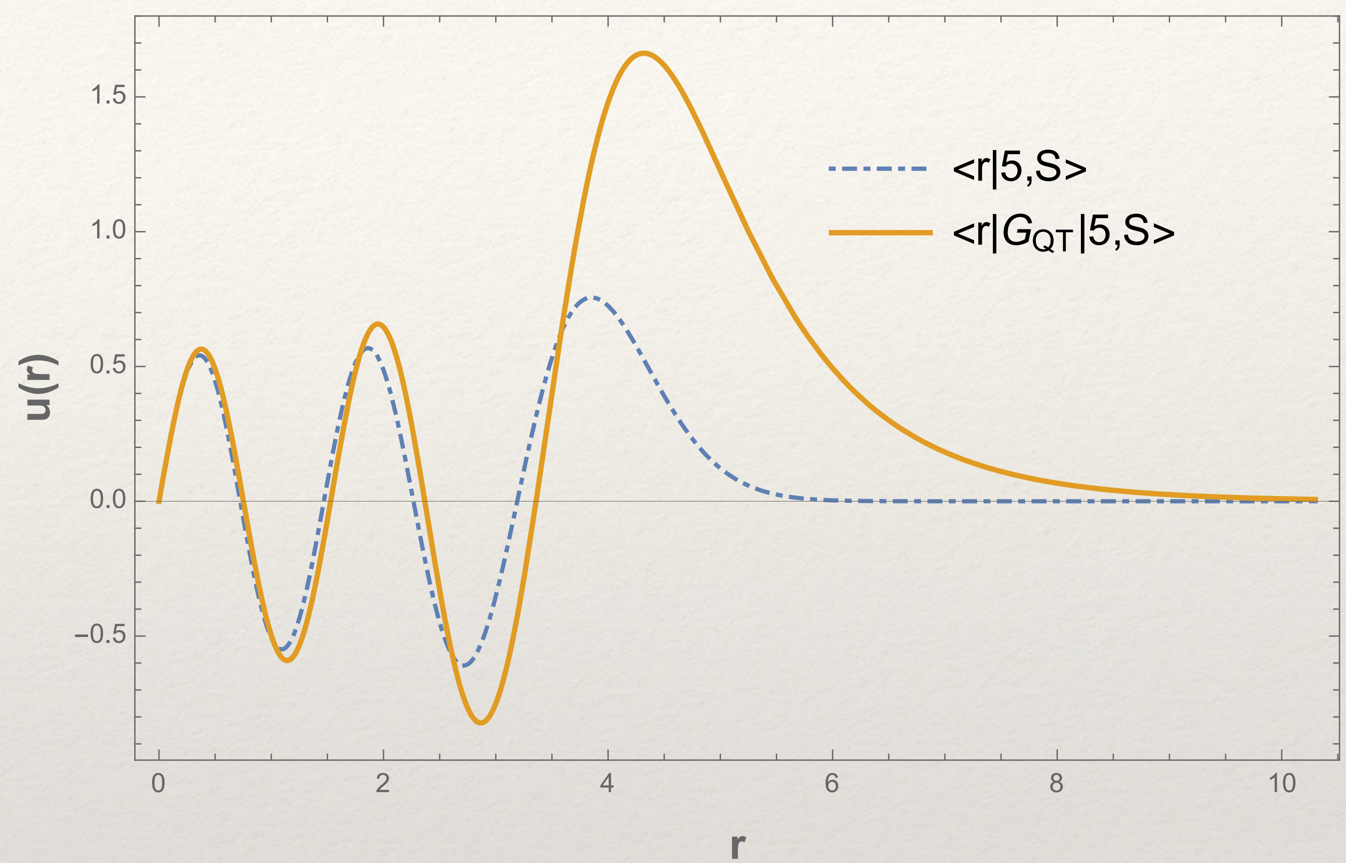
$$H^{eff,\Lambda}(E) = P\left[H\frac{E}{E-QH}\right]P = P\frac{E}{E-TQ}\left[T + T\frac{Q}{E}T + \underbrace{V + V\frac{1}{E-QH}QV}_{\overbrace{V_{IR}+V_{\delta}}}\right]\frac{E}{E-QT}P$$

$V_{IR} + V_{\delta}$ *ET Substitution*

- The reorganization isolates the impact of T for analytic calculation to all orders.
- The remaining part is replaced by a long range potential (like an OPEP) plus V_{δ}, which is a short range expansion around it.

E/(E-QT) Transform of Edge States





- Acting on edge state with $E/\hbar\omega = 1/2$.

  Recovers scattering wave function with phase shift.

- Acting on edge state with $E/\hbar\omega$=-$1/2$.

  Recovers bound state exponential decay from gaussian falloff of HO state.

$$G_{QT}P = \frac{E}{E-QT}P = \frac{E}{E-T}\left\{P\frac{E}{E-T}P\right\}^{-1}, \qquad b_{ij} = \left\{P\frac{E}{E-T}P\right\}^{-1}_{ij}$$

Sum T to All Orders

- T contributions can be summed to all orders.

$$\left\langle j \left| \frac{E}{E-TQ} \left[T - T\frac{Q}{E}T \right] \frac{E}{E-QT} \right| i \right\rangle = E\left(\delta_{ji} - b_{ji}\right)$$

- A surprisingly simple result.
- A non-perturbative sum of kinetic energy scattering is key to a convergent ET expansion of the remaining parts.

The V_δ Expansion

- V_δ is described in terms of HO lowering operators.

$$\hat{C}\ lowers\ L,\ \hat{A}\ lowers\ nodal\ n, \quad \left[\hat{C},\hat{A}\right]=0$$

$$V_\delta^S = a_{LO}^S\delta(r) + a_{NLO}^S\left(\hat{A}^\dagger\delta(r)+\delta(r)\hat{A}\right)+\ldots$$

$$V_\delta^{SD} = a_{NLO}^{SD}\left(\hat{C}^{\dagger 2}\delta(r)+\delta(r)\hat{C}^2\right)+a_{NNLO}^{22,SD}\left(\hat{C}^{\dagger 2}\delta(r)\hat{A}+\hat{A}^\dagger\delta(r)\hat{C}^2\right)$$

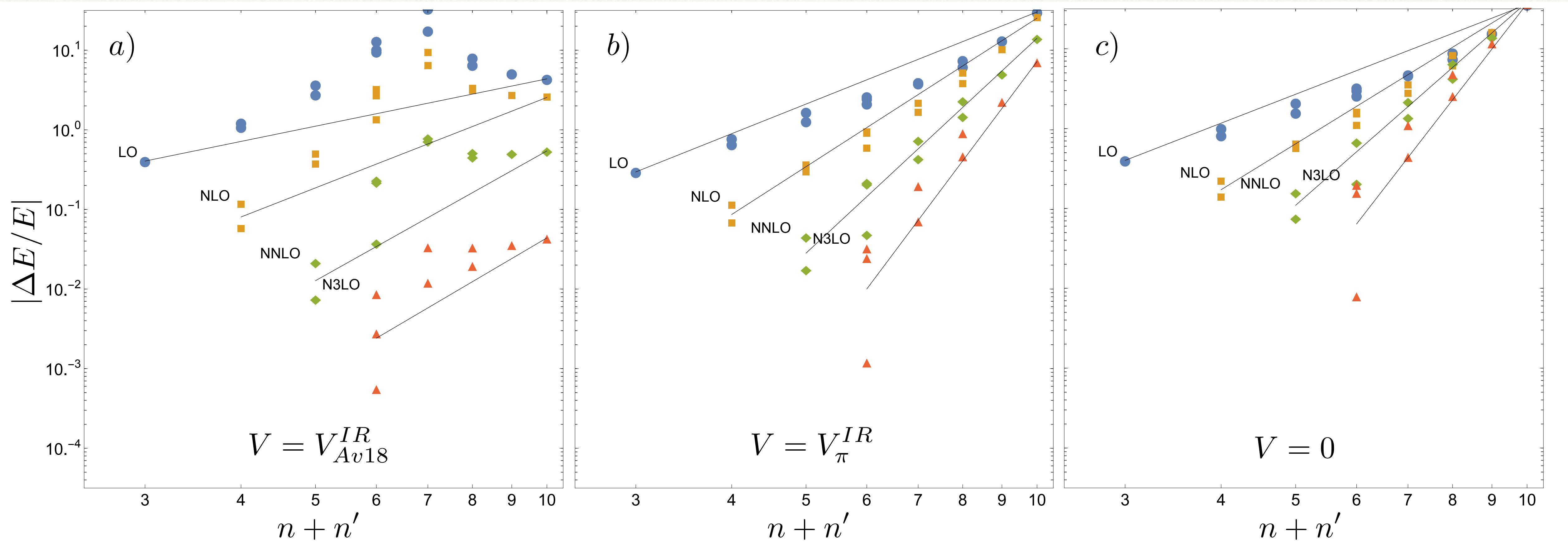
$$+a_{NNLO}^{40,SD}\left(\hat{C}^{\dagger 2}\hat{A}^\dagger\delta(r)+\delta(r)\hat{A}\hat{C}^2\right)+\ldots$$

$$\ldots$$

- This is slightly simplified by absorbing a constant related to coupling spins to angular momentum into the LECs.

- [$\hat{C}^2$ is really $\hat{C}^2 \equiv [\tilde{a}\otimes\tilde{a}]^{(2)}\odot[\vec{\sigma}_1\otimes\vec{\sigma}_2]^{(2)}$, coupling angular momentum to spins, with vector HO lowering op $\tilde{a}$]

HOBET Lepage Plots



- H^{eff} S-D matrix elements in Λ=8 are are directly calculated at E=-2.2245 MeV from HO matrix elements in Λ=400 and LECs fit in a scheme independent way.
- For the middle plot, O(1) errors appear at n′+n=8 making diagonal matrix elements with n=4 unreliable. n=3 at $4\hbar\omega$=60 MeV is a reasonable breakdown scale.
- Convergence is good order by order, but the value of a good long range V^{IR} is clear.

Power Counting

- The expansion is for V_δ, whose range R is shorter than that of V.
- For a known V_δ the LECs are proportional to a non-local Talmi integral.
- The tail overlap shrinks rapidly with order.
- For a short range V_δ the expansion parameter is function of b/R.

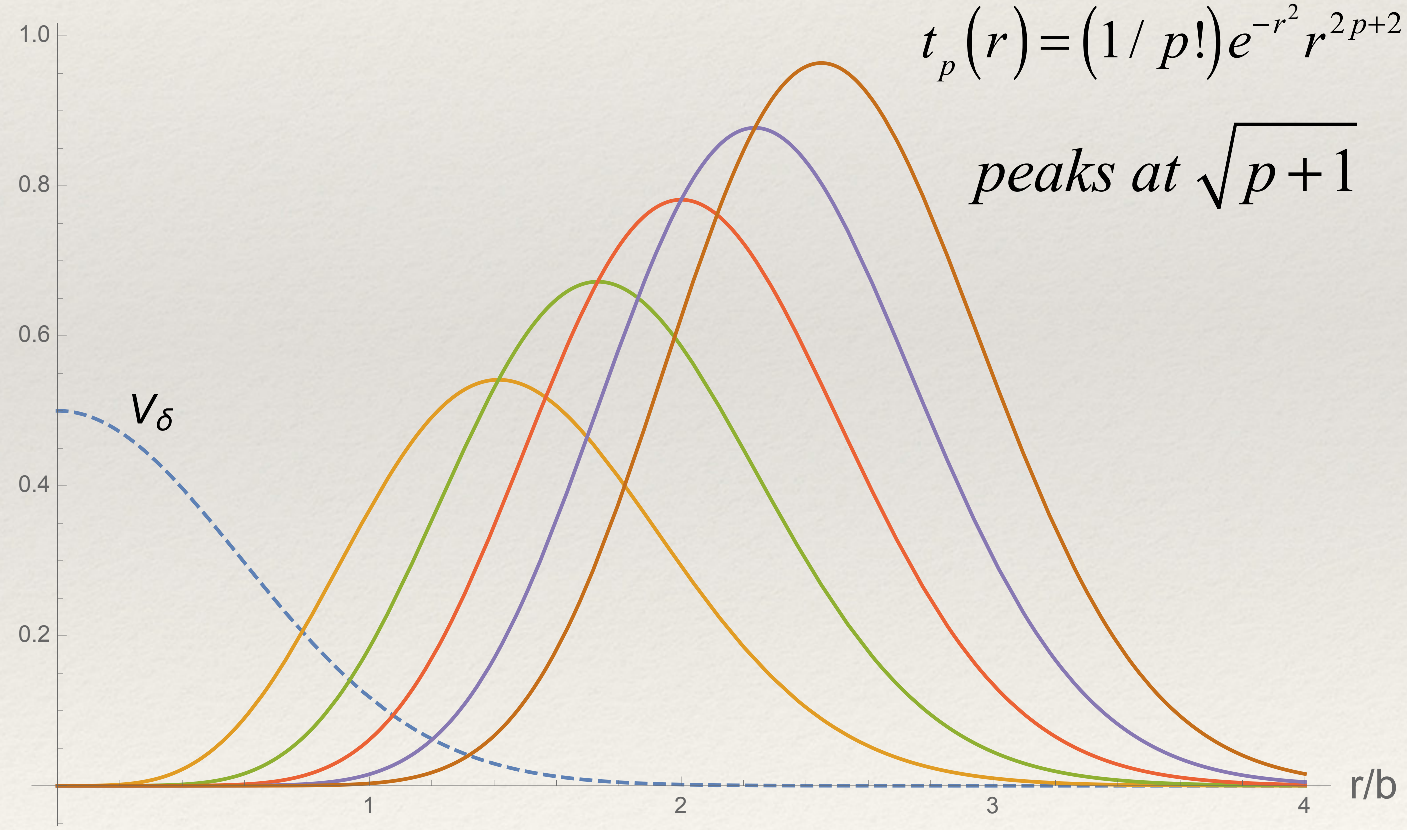
$$V_\delta(r',r) = V - V_{IR} + V\frac{1}{E-QH}QV$$

$$LEC_{n',n} \propto \int r'^2\, dr'\, r^2\, dr\, r'^{2(n'-1)} e^{-r'/2} V_Q e^{-r/2} r^{2(n-1)}$$

$$p = n' + n - 2$$

$$t_p(r) = (1/p!)e^{-r^2} r^{2p+2}$$

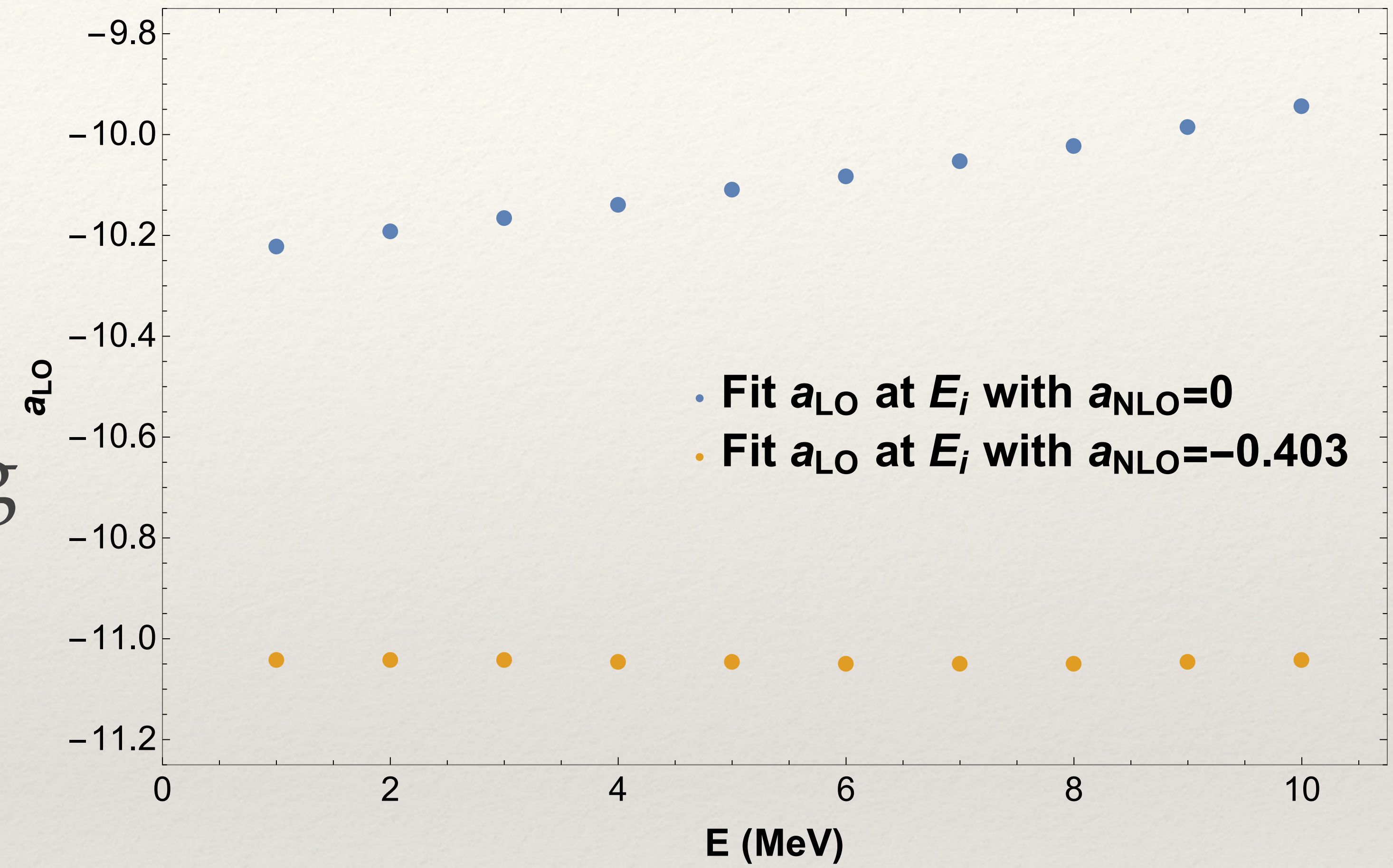
$$peaks\ at\ \sqrt{p+1}$$



Local Talmi basis functions with p ranging from 0 to 4.

Energy Independence of LECs

- Enables fitting to data a range of energies.
- The upper blue dots are the result of solving for a_{LO} at individual samples (E_i, δ_i)



- The lower gold dots represent an NLO fit to data from 1 and 10 MeV to determine a_{NLO} followed by refitting a_{LO} at each energy while holding a_{NLO} constant.
- Conclusion: Energy dependence is adsorbed into higher order operators.

Fitting LECs

- Principle: The BH equation is energy self consistent

$$H_{eff}^{full}\left(E_i\right)P\left|\psi_i\right\rangle = E_i P\left|\psi_i\right\rangle$$

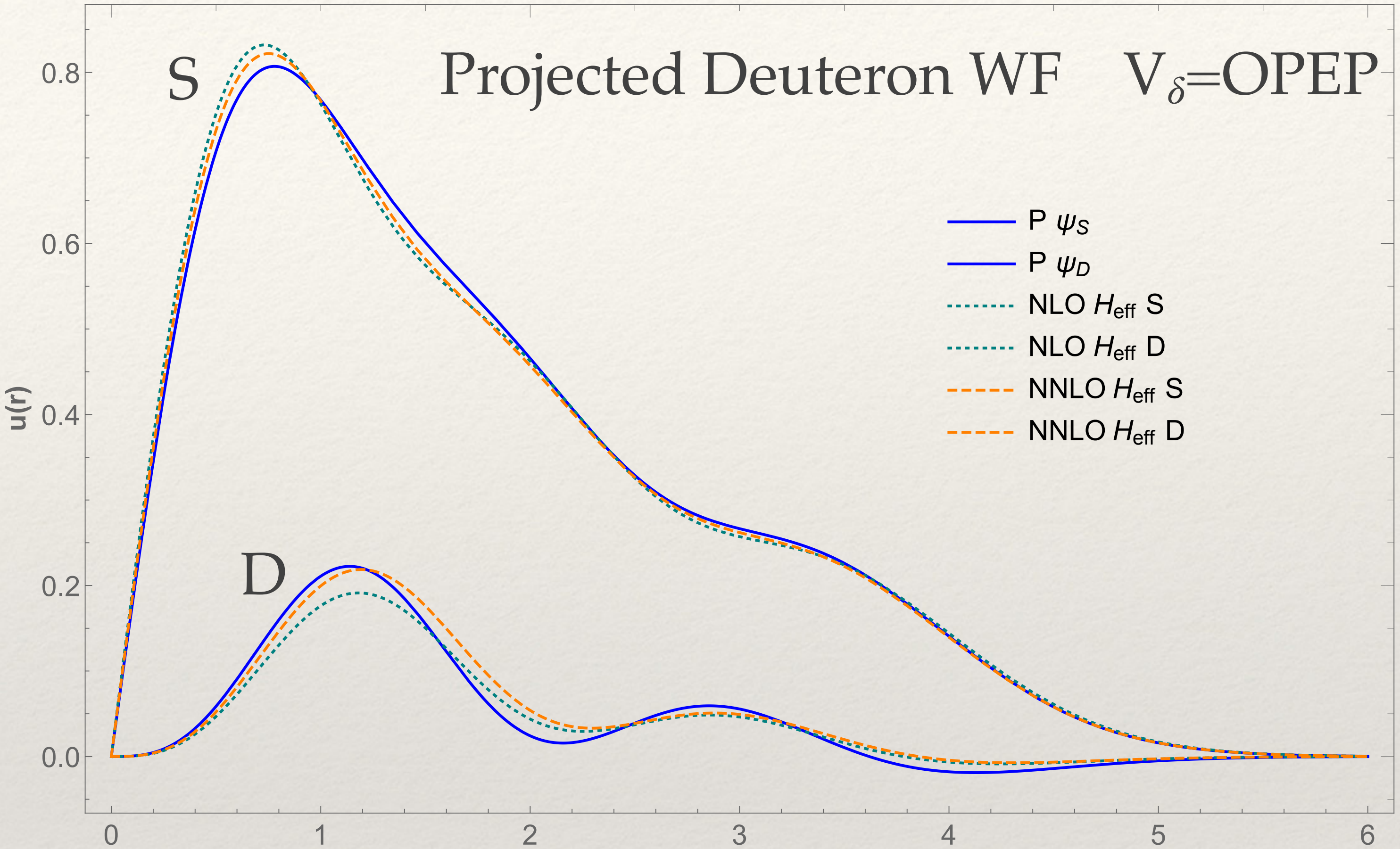
- At fixed order we instead have a nearby eigenstate.

$$H_{eff}\left(E_i, LECs\right)P\left|\psi_i'\right\rangle = \varepsilon_i P\left|\psi_i'\right\rangle$$

- The mismatch must be due to LEC values.
- Repair by minimizing $\sum_{i\in samples}\left(\varepsilon_i - E_i\right)^2/\sigma_i^2$
- The variance for the difference can be estimated from the sensitivity of ε_i to next order LECs, automatically suppressing data outside the validity range for the current LEC order.

Predicting the Deuteron

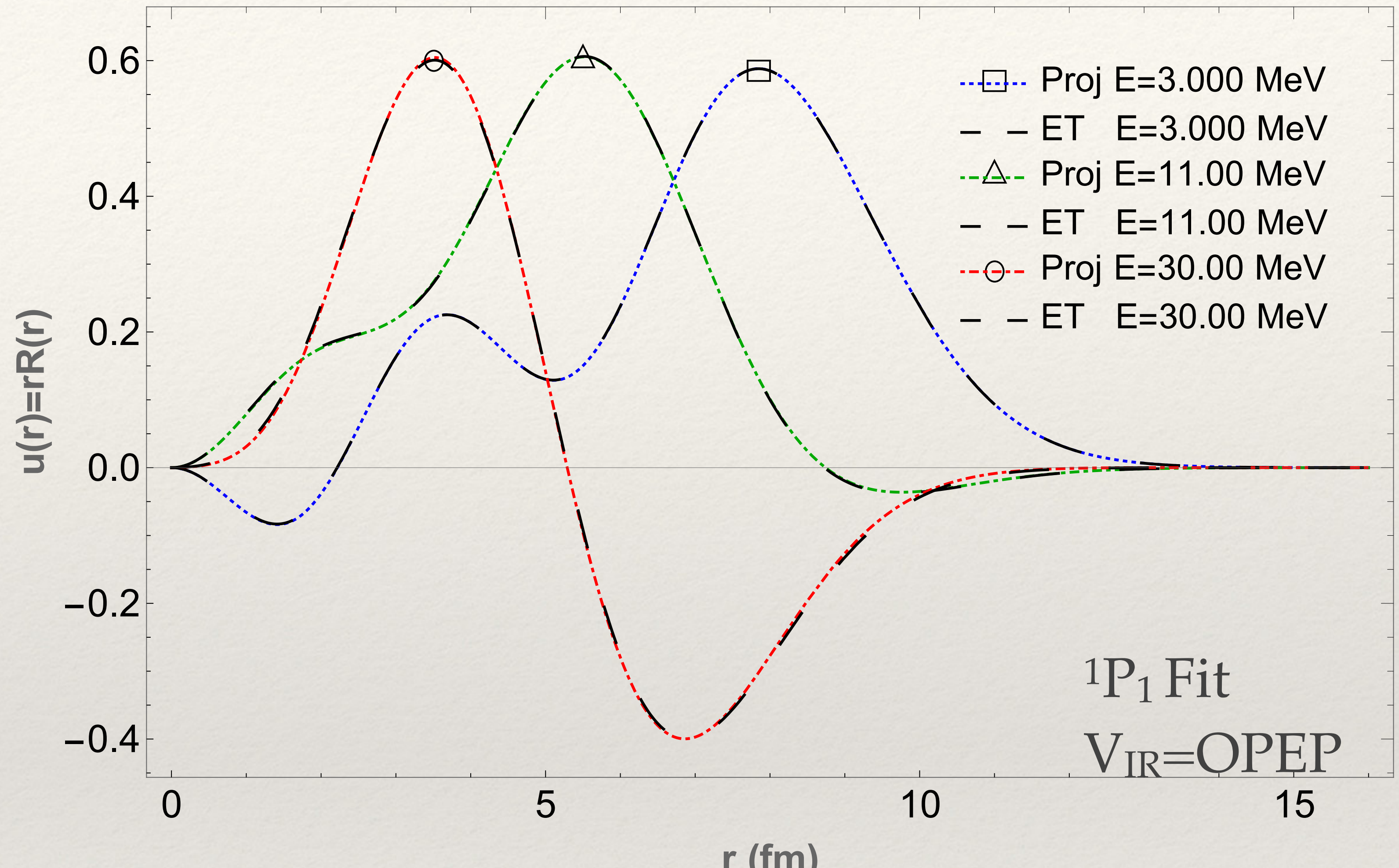
- Prediction of Deuteron WF from phase shift fit.
- ET Wave functions should match projections of numerical solutions with Av_{18} - solid blue lines



- The matrix elements are continuous in energy across Ē=0, one can fit V_δ in the continuum and determine bound states.
- Using the same phase shift data we get
  - With pionful V_{IR}=OPEP, at N3LO $E_{binding}$=-2.2278 MeV
  - With pionless V_{IR}=0, at N3LO $E_{binding}$=-2.0690 MeV

Continuum Wave Functions

- ET Wave functions (long black dashes) should match projections of numerical solutions with Av_{18} (dotted colored lines)



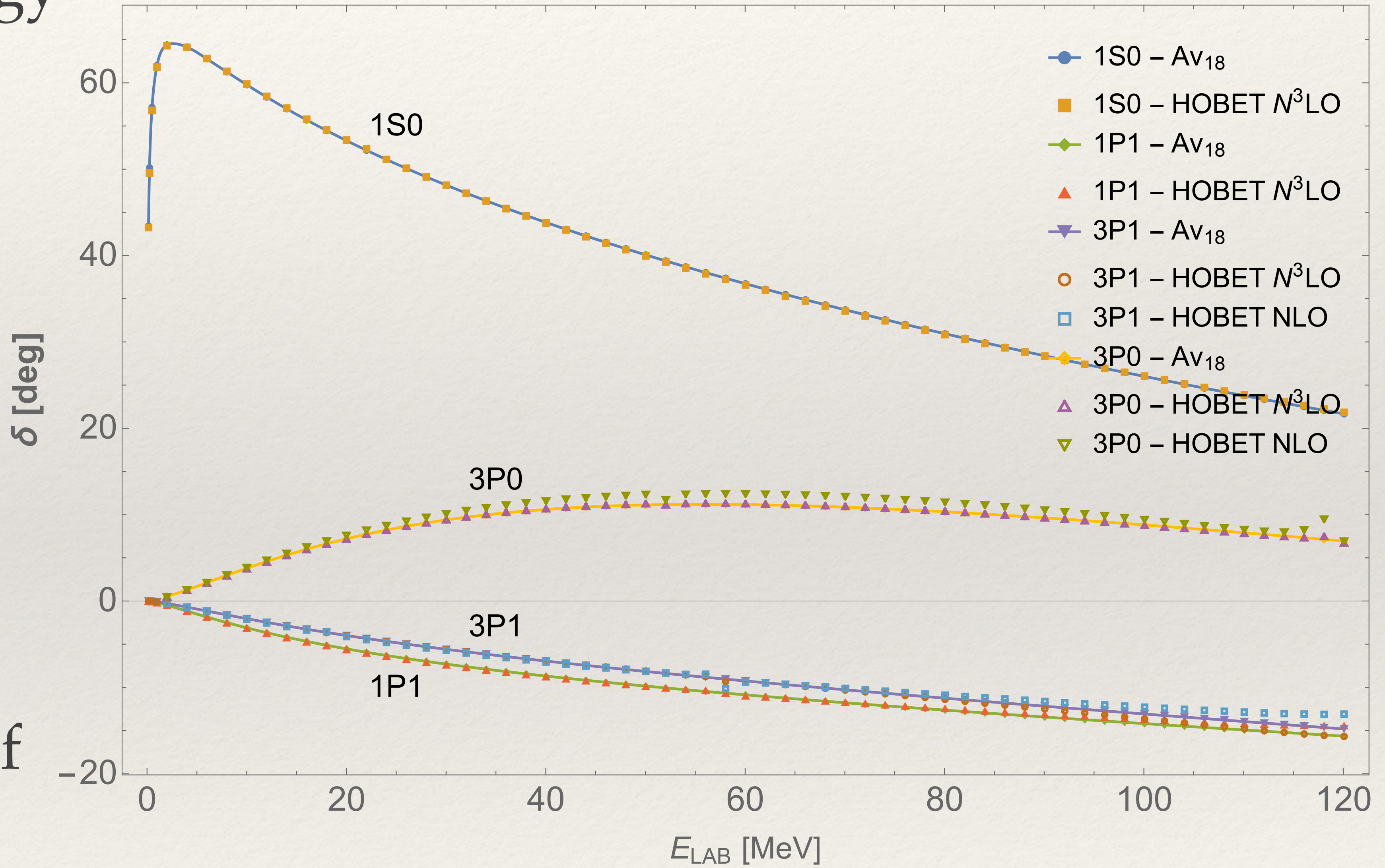
- The energies chosen in the plot are deliberately chosen to be distinct from the (E_i,δ_i) used in fitting the LECs.

- Phase shifts are recovered by solving for δ in

$$H^{eff}\left(E_i, LECs, \delta\right) P|\psi\rangle = E_i P|\psi\rangle.$$

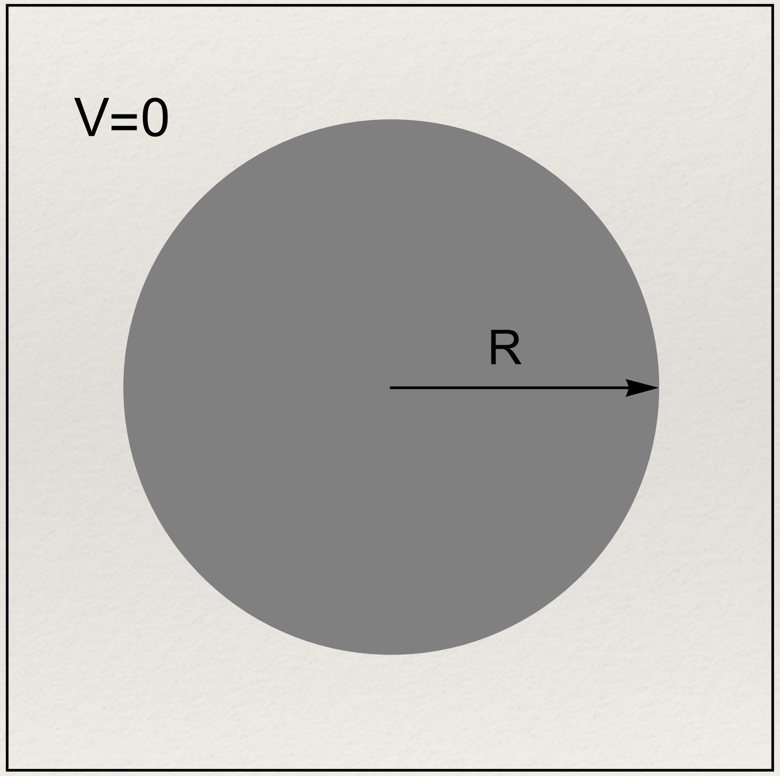
LECs →Phase Shifts

- Use fixed LECs at energy E, dial phase shift produce eigenvalue match to E.
- Even NLO 3P1 fit produces a good reproduction of phase shifts.
- A very small number of LECs reproduce phase shifts. P channel NLO has 1, other N3LO have 4.



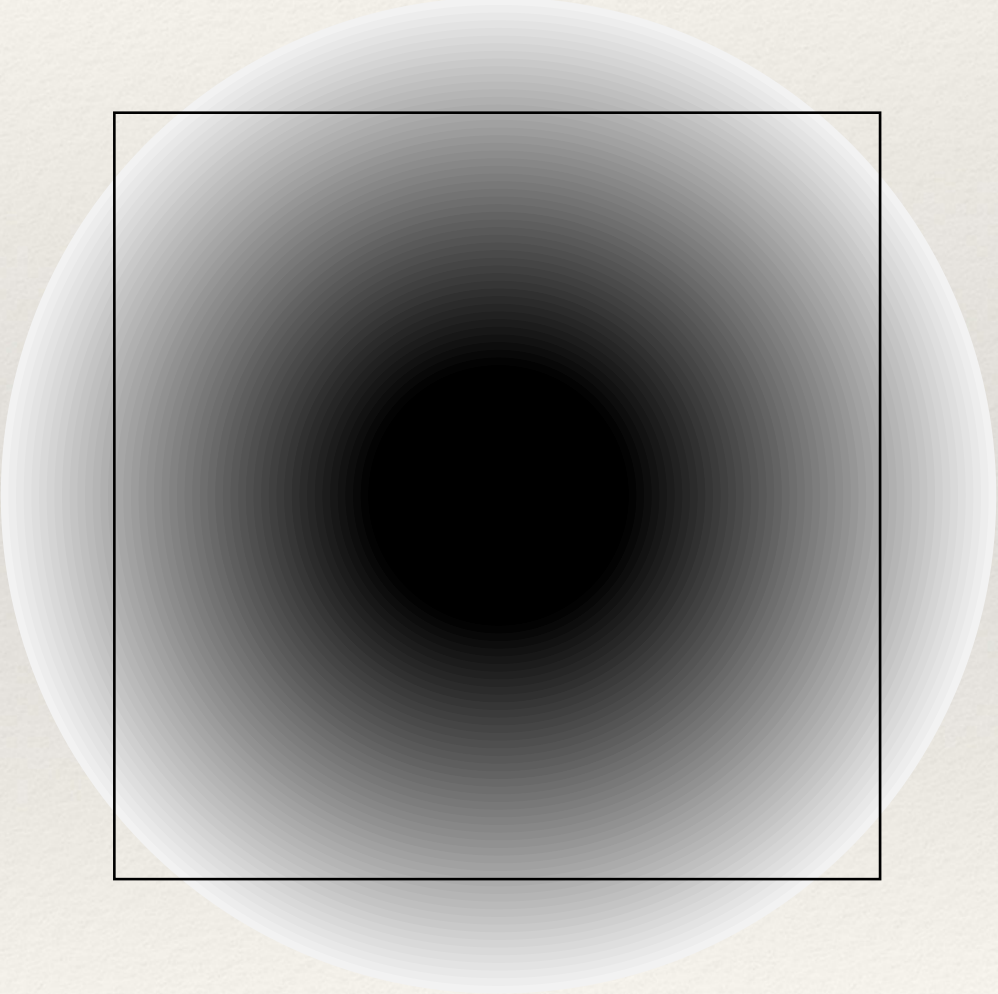
Connecting to LQCD

- Lüscher's method can be used to map the spectrum of two nucleons to phase shifts.
  - Use traditional path: collect enough phase shift data in multiple channels and use it to fit the HOBET effective interaction.
  - **This is the first method of connecting QCD to HOBET.**
- Sources of error
  - Tail of interaction exceeding L/2.
  - Divergences of the zeta function in higher order terms of Lüscher's formula.



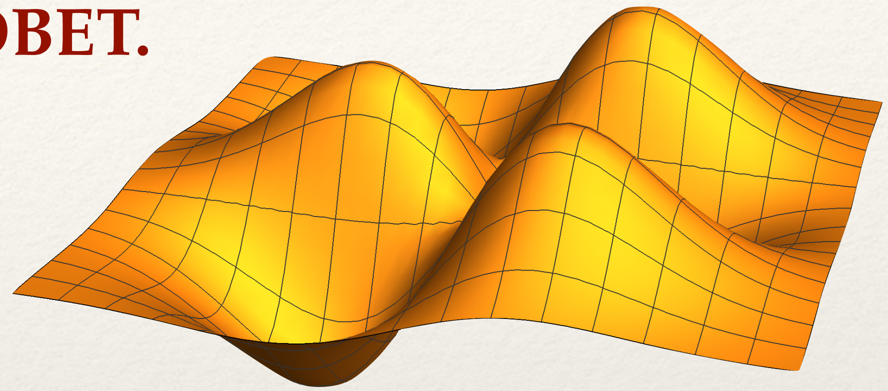
Connecting to LQCD

- Lüscher's method can be used to map the spectrum of two nucleons to phase shifts.
  - Use traditional path: collect enough phase shift data in multiple channels and use it to fit the HOBET effective interaction.
  - **This is the first method of connecting QCD to HOBET.**
- Sources of error
  - Tail of interaction exceeding L/2.
  - Divergences of the zeta function in higher order terms of Lüscher's formula.



HOBET in Periodic Volumes

- **This is the second way to connect QCD to HOBET.**
- Phase shifts as boundary conditions are replaced by periodic boundary conditions.
- Easier to construct in Cartesian HO basis.



Slice of 3D Cartesian State

- Key Observation: V_δ is short range and isolated from the boundary conditions by Green's functions. It is the same object in infinite volume, or periodic volumes.
  - We can use Cartesian-spherical brackets to relate V_δ in both domains. The Cartesian V_δ can be written in terms of the infinite volume spherical LECs!
- If V_{IR} is longer range than L/2, introduce images of V_{IR}.
  - This is a key advantage over Lüscher's method which requires a free propagation region outside the range of V, but inside the volume.

Evaluate by Inserting Periodic Basis

Sum T to all orders: $\left\langle \vec{n}' \left| \frac{E}{E-TQ}\left[T + T\frac{Q}{E}T\right]\frac{E}{E-QT}P \right| \vec{n} \right\rangle = E(\delta_{\vec{n}'\vec{n}} - b_{\vec{n}'\vec{n}})$

$$b_{ij} = \left\{ P\frac{E}{E-T}P \right\}_{ij}^{-1}$$

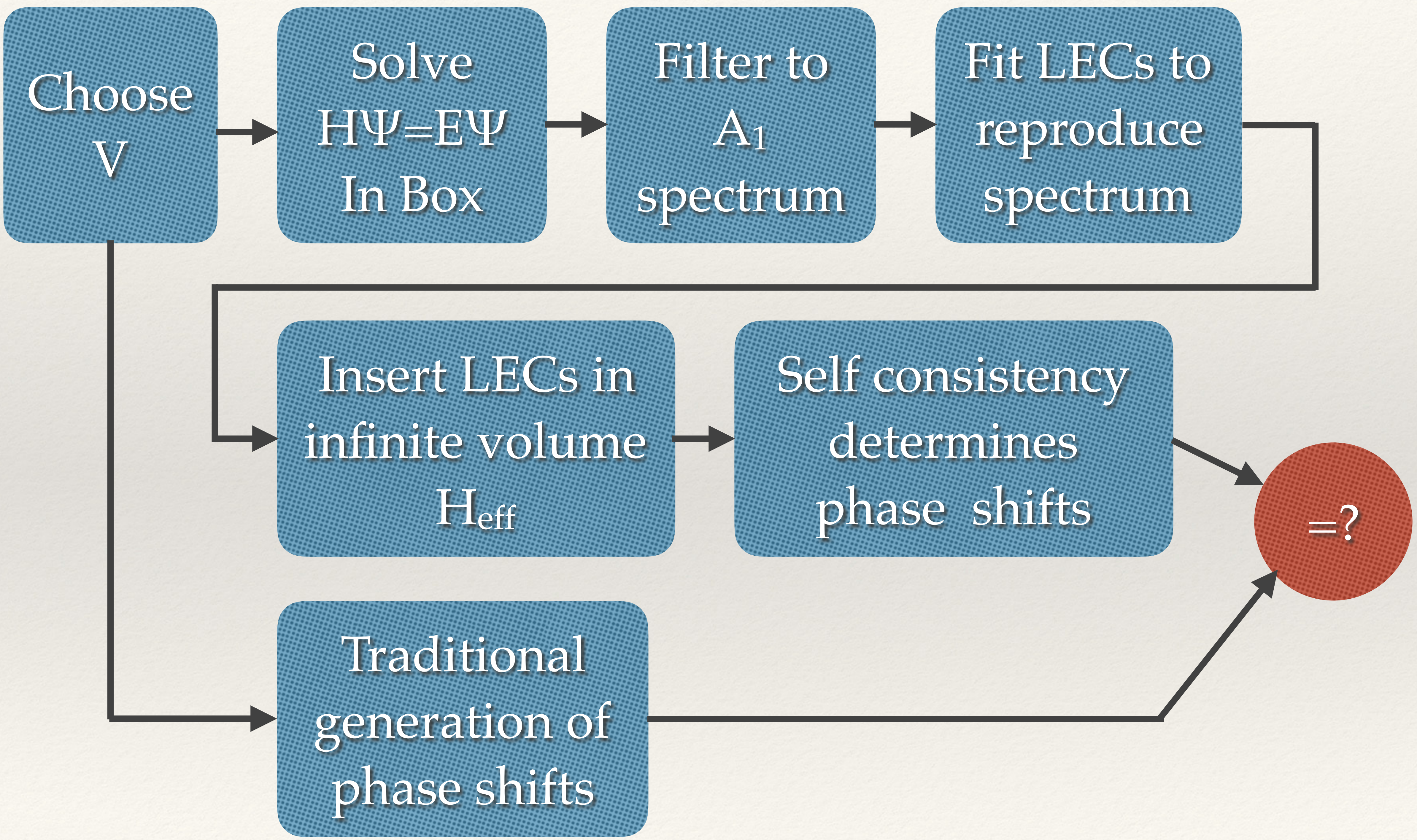
- V_{IR} matrix elements are the most expensive part of H_{eff}

$$\left\langle \vec{n}' \left| G_{TQ} V_{IR} G_{QT} \right| \vec{n} \right\rangle = \sum_{\vec{m}',\vec{m},\vec{s},\vec{t}} b_{\vec{n}',\vec{s}} \frac{E}{E-\lambda_{\vec{m}'}} \langle \vec{s} | \vec{m}' \rangle \langle \vec{m}' | V_{IR} | \vec{m} \rangle \langle \vec{m} | \vec{t} \rangle \frac{E}{E-\lambda_{\vec{m}}} b_{\vec{t},\vec{n}}$$

$\vec{m},\vec{m}'$ *are discrete momentum states;* s,t *are HO states*

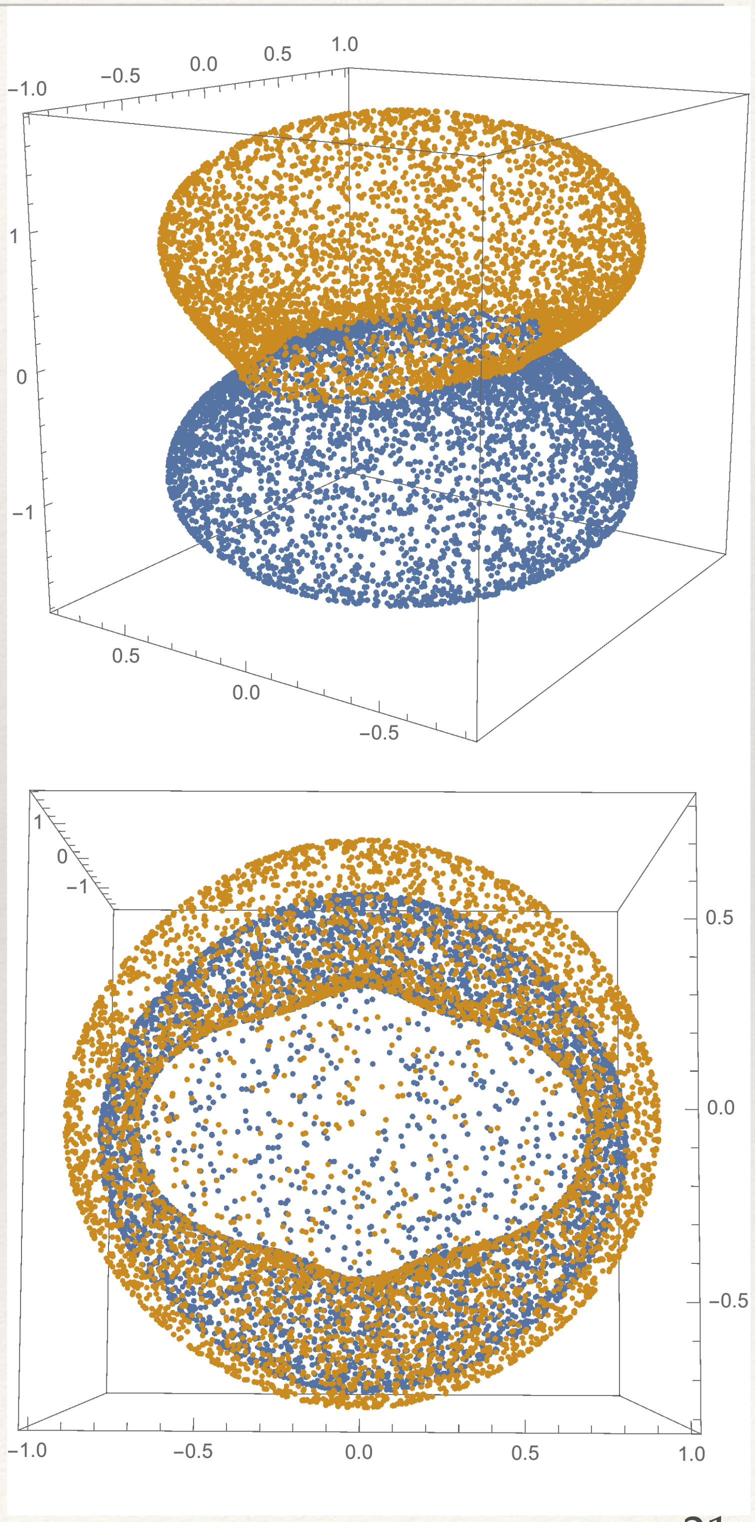
- All pieces are precomputed, but sum is still very large.
- For $\vec{n}',\vec{n} \in P^-$ $G_{QT}=1$, which can be used to check results.

Testing Plan

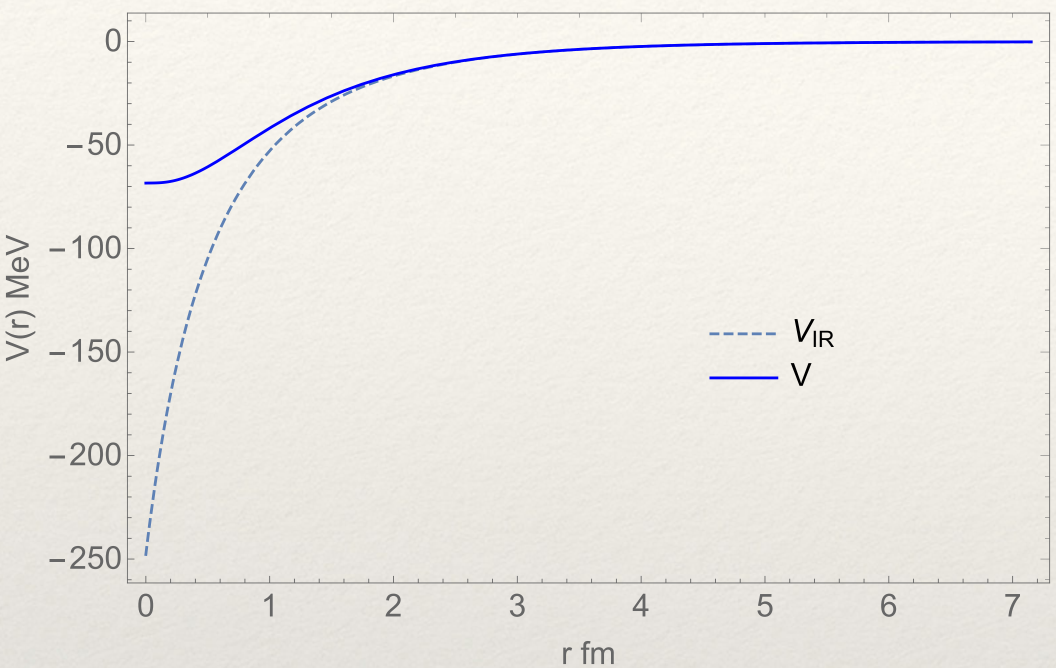


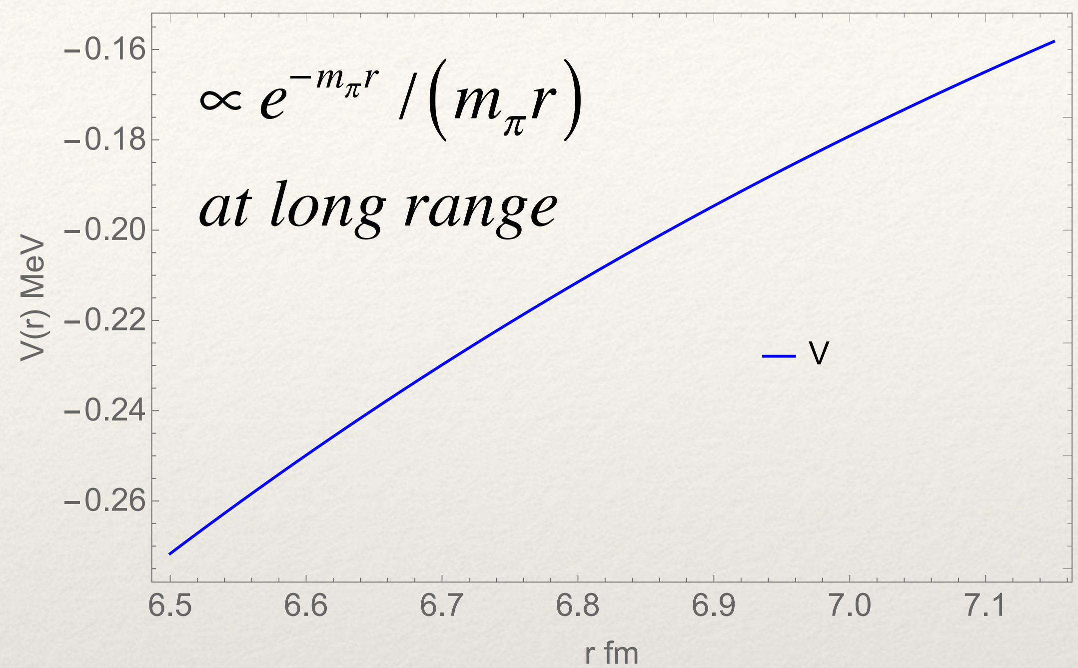
Induced Mixing

- Setup: spherical well potential in a periodic finite volume.
- The wave function is sampled on sphere outside potential and displayed as a radial displacement from a unit sphere.
- Higher order structure induced by periodic boundary conditions is obvious.
- All this mixing is isolated in E/(E-QT) Green's functions.



Test Setup: Range(V)>L/2





$L = 14.3\ fm$

$m_\pi L = 10$

- Periodic images of the potential make a contribution.
- Continuum extrapolation done on N^3 lattice with N={350,400,450}.
- Infinite volume bound state at -4.052 MeV.
- LECs are fit using states with *L*=0 overlap.

Rep	MeV	L=0	L=2	L=4	L=6
A_1^+	-4.4428	0.5	0	0.866	0
A_1^+	2.0314	0.155	0	0.988	0
E^+	7.5995	0	0.424	0.361	0.830
E^+	15.2980	0	0.474	0.393	0.788
A_1^+	21.6167	0.326	0	0.265	0.908
E^+	23.2423	0	0.468	0.597	0.651
A_1^+	29.4041	0.521	0	0.853	0.023
E^+	30.9457	0	0.567	0.428	0.704
A_1^+	35.2449	0.655	0	0.189	0.732
E^+	38.4043	0	0.882	0.176	0.437
A_1^+	45.1402	0.526	0	0.576	0.625

Phase Shift Setup

- Reference phase shifts for L=0 and L=4 are directly calculated from V.
- HOBET S-channel phase shifts are computed from the N3LO LECs that reproduce the spectrum. The phase shift is found by dialing the phase shift to produce energy self consistency.
- Lüscher's method phase shifts come from the formula

$$k \cot \delta_0 = \frac{2}{\sqrt{\pi} L} \mathcal{Z}_{0,0}\left(1;\tilde{k}^2\right) + \frac{12288\pi^7}{7L^{10}} \frac{\mathcal{Z}_{4,0}\left(1;\tilde{k}^2\right)^2}{k^9 \cot \delta_4} + \mathcal{O}\left(\tan^2 \delta_4\right)$$

Luu, Savage, arXiv:1101.3347

- An effective range expansion up to k^6 is used to interpolate.
- For simplicity the second term is evaluated using the L=4 phase shift directly determined from V.

Phase Shift Results

$L = 14.3\ fm$

$m_\pi L = 10$

E MeV	V	HOBET	Leading Lüscher	Next Order Lüscher
1	142.023	141.931	142.552	142.751
2	128.972	128.860	129.571	129.823
4	113.602	113.464	114.205	114.403
8	96.919	96.752	97.575	97.3135
10	91.473	91.296	92.228	91.6403
15	81.672	81.480	82.852	81.3184
20	74.876	74.691	76.667	74.0936

The V column should be considered the reference.

- ET bound state found at -4.066 MeV v.s. -4.052 MeV (directly from V).
- HOBET errors are from PV solution + Momentum basis cutoff.
- Lüscher errors are from Range(V) > L/2 and magnification of errors by Zeta function poles.

Effective Operators

- The Bloch Horowitz equation tell us how to renormalize an operator: $\hat{O}_{ji}^{eff,\Lambda}(E) = P\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}P$
- i,j Label eigenstates of H.
- The Green's functions reconstruct the full wave function from the projection.

$$(P+Q)H|\psi_i\rangle = E_i|\psi_i\rangle$$

$$E_i P|\psi_i\rangle = (E_i - QH)|\psi_i\rangle$$

$$|\psi_i\rangle = \frac{E_i}{E_i - QH}P|\psi_i\rangle$$

- In bound states the boundary condition for E/(E-QH) is an exponential decay outside the range of V.

Operator Expansion

- Short range operators can also be matched to an expansion.

$$\hat{O}^{eff}_{ji} = P\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}P$$

$$= P\frac{E_j}{E_j - TQ}\left[\hat{O} + VQ\frac{E_j}{E_j - HQ}\hat{O} + \hat{O}\frac{E_i}{E_i - QH}QV + VQ\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}QV\right]\frac{E_i}{E_i - QT}P$$

$$\rightarrow P\frac{E_j}{E_j - TQ}\left[\hat{O} + \hat{O}_\delta\right]\frac{E_i}{E_i - QT}P$$

- O_δ has an expansion much like V_δ with an expansion in harmonic oscillator quanta.
- **Key Point:** The LECs $\hat{1}$ of the expansion can be fit to a set of LQCD measurements. The boundary conditions are then replaced in E/(E-QT) with the infinite volume boundary conditions (phase shifts) to give the effective operator in infinite volume.

Operator Expansion

- Short range operators can also be matched to an expansion.

Short Range

$$\hat{O}_{ji}^{eff} = P\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}P$$

$$= P\frac{E_j}{E_j - TQ}\left[\hat{O} + VQ\frac{E_j}{E_j - HQ}\hat{O} + \hat{O}\frac{E_i}{E_i - QH}QV + VQ\frac{E_j}{E_j - HQ}\hat{O}\frac{E_i}{E_i - QH}QV\right]\frac{E_i}{E_i - QT}P$$

$$\rightarrow P\frac{E_j}{E_j - TQ}\left[\hat{O} + \hat{O}_\delta\right]\frac{E_i}{E_i - QT}P$$

- O_δ has an expansion much like V_δ with an expansion in harmonic oscillator quanta.

- **Key Point:** The LECs $\hat{1}$ of the expansion can be fit to a set of LQCD measurements. The boundary conditions are then replaced in E/(E-QT) with the infinite volume boundary conditions (phase shifts) to give the effective operator in infinite volume.

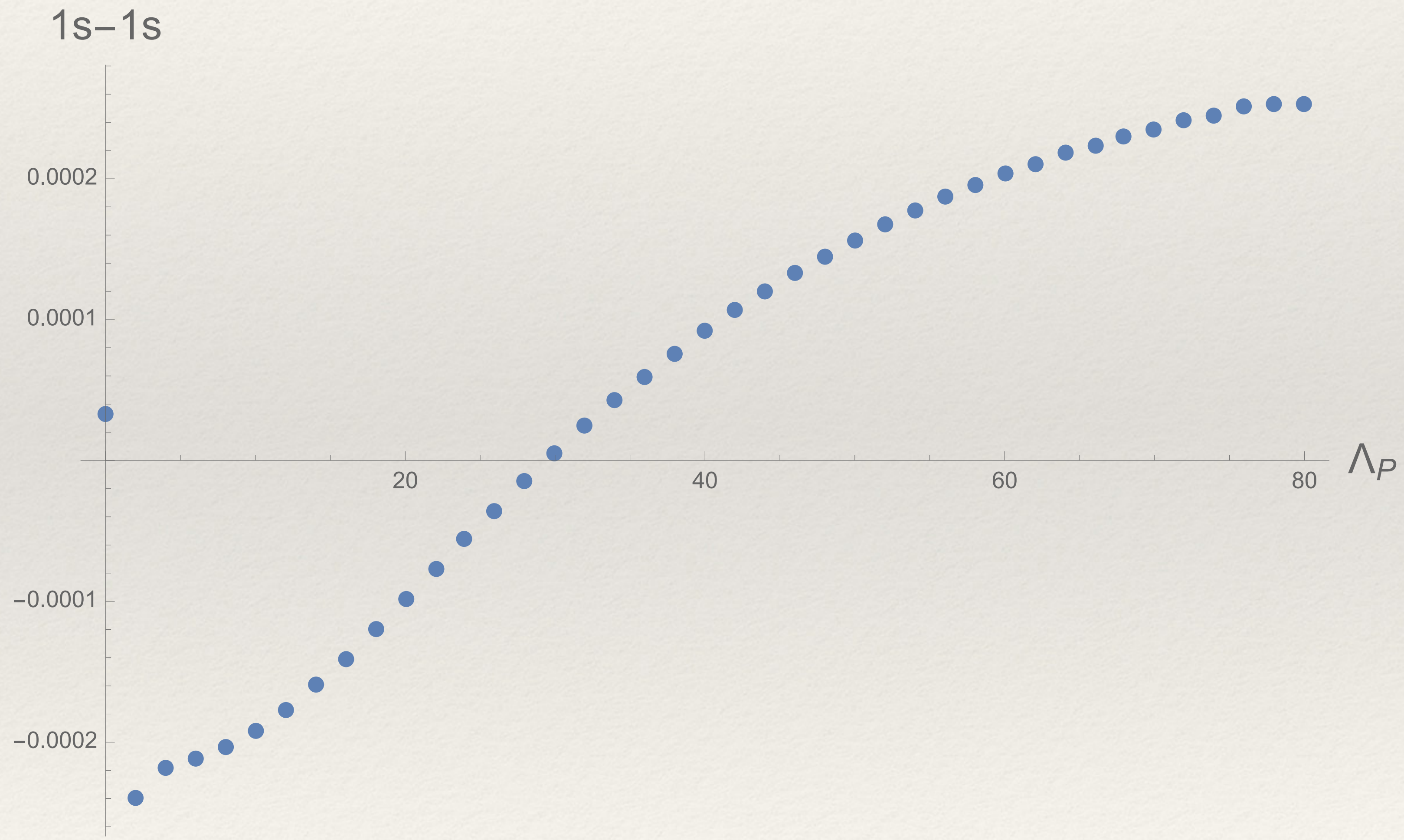
Application to 0νββ Operators

Example: $V_i^{nn \to pp} = -O_i \frac{g_A^2}{4F_\pi^2} \boxed{\tau_1^+ \tau_2^+ \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{q}\, \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{\left(|q|^2 + m_\pi^2\right)^2}}$

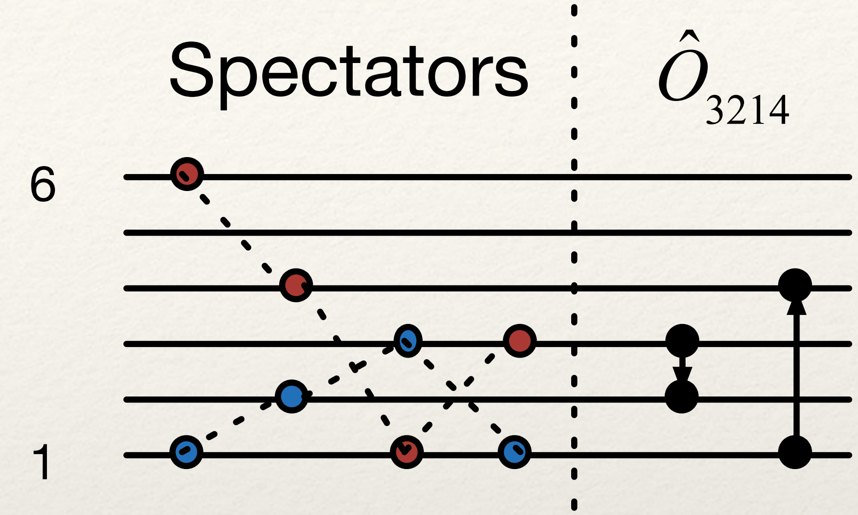
Nicholson et al. Phys. Rev. Lett. **121**, 172501 (2018)

- Boxed part * 10^6.
- HO Length scale b=1.7fm
- Start in Λ_∞ =80 and integrate out shell by shell.
- Note jump when 1S becomes an edge state at Λ_P=0.

Running of 1S−1S Matrix Element, E=−1.961 MeV



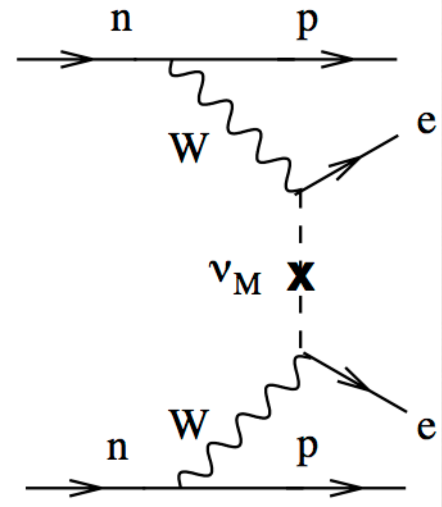
Effective Operators in A-Body Context



- The E in $\hat{O}^{eff}$ is the A-body E.
- Translation invariance requires a total Λ cutoff. If spectators are excited - red dots, then Λ_{12} must be reduced.
- We add a spectator quanta index to the standard density matrix. The interacting particles are then in a 2 particle P space defined by $\Lambda_{12}=\Lambda-\Lambda_S$.
- Matched with this we produce $O_{ij}^{eff,\Lambda_{12}}$ for $\Lambda_{12}=0\ldots\Lambda$.

Implementation with BIGSTICK

- We (Evan Rule erule@berkeley.edu) are constructing a 2-body spectator dependent density matrix for BIGSTICK.
- We will use a realistic potential for H in $\hat{O}^{eff}$.
- Given universality with respect to the starting potential, we hope for the same with $\hat{O}_{ij}^{eff,\Lambda_{12}}$.



- We will test with operators associated with experiments.
- Last we will evaluate various $0\upsilon\beta\beta$ operators.

$$\hat{O} = \tau_1^+ \tau_2^+ \frac{\sigma_1 \cdot \mathbf{q}\, \sigma_2 \cdot \mathbf{q}}{\left(|q|^2 + m_\pi^2\right)^2}$$

Summary

- HOBET can be connected to QCD via LQCD observables, or an LQCD nucleon scattering spectrum in finite volume.
- Operators have an expansion, with LECs isolated from boundary conditions by Green's functions and can be fit to LQCD measurements.
- We have made progress on operator renormalization and evaluation in an A-body context. We hope to have results for $0\upsilon\beta\beta$ soon via a hybrid approach with a standard shell model.
- Longer term we are continuing on a path to a HOBET based shell model code.

End