# Perturbative Gravity and the link to Gauge Theory

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- Two expansions: Post-Newtonian and Post-Minkowskian

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- This talk is about *weak coupling perturbation theory*
- String theory gives relations between gravity and gauge amplitudes
- Classical general relativity from quantum amplitudes
- Two expansions: Post-Newtonian and Post-Minkowskian
- From one-loop to all-loop: the scheme explained

# Color-ordered Yang-Mills Amplitudes

Tree-level amplitudes with $n$ particles in the adjoint rep.

$$\mathcal{A}(1,2,\ldots,n) = \sum_{P(2,3,\ldots,n)} \mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})A(1,2,\ldots,n)$$

Lots of identities involving $A(1,2,\ldots,n)$

Examples: simple identities like cyclicity and reflections:

$$\begin{aligned} A(1,2,\ldots,n) &= A(2,3,\ldots,n,1) \\ A(1,2,\ldots,n) &= (-1)^n A(n,n-1,\ldots,1) \end{aligned}$$

- plus many more.

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- plus many more. The basis of operators is only of size $(n-3)!$

# The Kleiss-Kuijf Relations

How many color-ordered amplitudes should be computed?

Kleiss and Kuijf suggested that the number of basis amplitudes can be reduced from $(n-1)!$ to $(n-2)!$ because of a highly non-trivial identity:

$$A(\beta_1, \ldots, \beta_r, 1, \alpha_1, \ldots, \alpha_s, n) = (-1)^r \sum_{\sigma \subset \mathrm{OP}\{\alpha\} \cup \{\beta^T\}} A(1, \sigma, n)$$

where the sum runs over "Ordered Permutations" $\mathrm{OP}\{\alpha\} \cup \{\beta^T\}$ that *maintain the order of individual elements in each set within the joint set*

Example,

$$A(\{3,2\}, 1, \{4\}, 5) = A(1,2,3,4,5) + A(1,2,4,3,5) + A(1,4,2,3,5)$$

This reduces the basis of amplitudes to $(n-2)!$

# BCJ Relations

Recently, Bern, Carrasco and Johansson (BCJ) made some surprising observations and a conjecture.

Consider the 4-point function:

$$A(1,2,3,4) + A(1,3,4,2) + A(1,4,2,3) \; = \; 0$$

They argue as follows: This can only be satisfied because of cancellations due to the amplitudes' kinematic invariants.

Then the above should be equivalent to

$$s + t + u \; = \; 0$$

In other words,

$$A(1,2,3,4) + A(1,3,4,2) + A(1,4,2,3) \;=\; (s+t+u)\chi$$

where $\chi$ is a universal function. Because $A(1,2,3,4)$ is symmetric in $s$ and $t$,

$$A(1,2,3,4) \;=\; u\chi$$

and similarly for the other amplitudes. Solving the equations, one gets

$$\begin{aligned} tA(1,2,3,4) &= uA(1,3,4,2) \\ sA(1,2,3,4) &= uA(1,4,2,3) \\ tA(1,4,2,3) &= sA(1,3,4,2) \end{aligned}$$

Checked to hold in explicit helicity amplitudes!

# Jacobi-like identities

Do such relations generalize to higher $n$-point functions?

Intriguing observation: Write the 4-point amplitudes in terms of the poles that can appear:

$$\begin{aligned} A(1,2,3,4) &= \frac{n_s}{s} + \frac{n_t}{t} \\ A(1,3,4,2) &= -\frac{n_u}{u} - \frac{n_s}{s} \\ A(1,4,2,3) &= -\frac{n_t}{t} + \frac{n_u}{u} \end{aligned}$$

Then

$$n_u = n_s - n_t$$

This is like a Jacobi identity for the kinematic factors!

Compare color

$$c_u = f^{a_4 a_2 b} f^{b a_3 a_1} = f^{a_1 a_2 b} f^{b a_3 a_4} - f^{a_2 a_3 b} f^{b a_4 a_1} = c_s - c_t$$

Full color-dressed amplitude:

$$\mathcal{A} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

Symmetry between color and kinematics.

# Higher $n$-point functions

Does the Jacobi-like identity in 4-point functions generalize?

Bern, Carrasco and Johansson take as *ansatz* that for every color Jacobi identity

$$c_\alpha = c_\beta - c_\gamma$$

there is a kinematical Jacobi identity

$$n_\alpha = n_\beta - n_\gamma$$

Much more complicated now!

Consider 5-point example: Two poles in generalized Mandelstam variables $s_{ij} \equiv (k_i + k_j)^2$,

$$A(1,2,3,4,5) = \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}$$

Using Kleiss-Kuijf there are (5-2)! $= 6$ basis amplitudes.

In total 15 numerator factors $n_i$.

There are 9 (color) Jacobi identities, 4 $n_i$'s can be set to zero, $i.e.$ only (15 - 9 - 4) $= 2$ *independent amplitudes*.

New identities!

Example:

$$A(1,3,4,2,5) = \frac{-s_{12}s_{45}A(1,2,3,4,5) + s_{14}(s_{34}+s_{25})A(1,4,3,2,5)}{s_{13}s_{24}}$$

All 5-point amplitudes expressed in terms of, say, $A(1,2,3,4,5)$ and $A(1,4,3,2,5)$.

# An Algebra of Amplitudes?

Bern, Carrasco and Johansson generalized the Jacobi-like construction to any $n$. Checked explicitly for various helicities up to 8 points. But: The general formula remained at the level of a conjecture.

Proof of BCJ-relations from string theory (Bjerrum-Bohr, PD, Vanhove, 2010)

Existence of $n$'s satisfying Jacobi proven (Bjerrum-Bohr, PD, T. Sondergaard, Vanhove, 2011)

The Jacobi-like identities hint at an **algebra of amplitudes**.

Intriguing brand-new work by O'Connell and Monteiro, May 2011).

# String Theory

In open string theory one naturally considers color-ordered amplitudes

$$\mathcal{A}(1,2,\ldots,n) = \sum_{P(2,3,\ldots,n)} \mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})A(1,2,\ldots,n)$$

where color is supplied by the Chan-Paton factors.

Koba-Nielsen measure

$$\begin{aligned} A(a_1,\cdots,a_n) &= \int \prod_{i=1}^{n} dz_i \frac{|z_{ab}\, z_{ac}\, z_{bc}|}{dz_a dz_b dz_c} \prod_{i=1}^{n-1} H(x_{a_{i+1}} - x_{a_i}) \\ &\times \prod_{1\le i<j\le n} |x_i - x_j|^{2\alpha' k_i\cdot k_j}\, F_n \end{aligned}$$

$$
\begin{aligned}
dz_i &= dx_i \qquad z_{ij} = x_i - x_j \qquad \text{(bosonic)} \\
dz_i &= dx_i d\theta_i \qquad z_{ij} = x_i - x_j + \theta_i\theta_j \qquad \text{(supersymmetric)}
\end{aligned}
$$

All helicity dependence of external states contained in $F_n$.

Measure only defined after fixing 3 points, traditionally taken to be $x_1 = 0, x_{n-1} = 1$ and $x_n = \infty$.

3 turns out to be the magic number!

# Explicit Amplitudes

Consider the 4-point amplitude for tachyons $(x_1 = 0, x_3 = 1, x_4 = +\infty)$. Let $x \equiv x_2$:

$$
\begin{aligned}
A(1,2,3,4) &= \int_0^1 dx\ x^{2\alpha' k_1 \cdot k_2} (1-x)^{2\alpha' k_2 \cdot k_3} \\
&= \frac{\Gamma[-2\alpha' k_1 \cdot k_2]\Gamma[1 + 2\alpha' k_2 \cdot k_3]}{\Gamma[2 + 2\alpha' k_1 \cdot k_2 + 2\alpha' k_2 \cdot k_3]}
\end{aligned}
$$

where $s = -(k_1 + k_2)^2$ etc. and

$$
s + t + u \ = \ -4/\alpha'
$$

We can get the other color-orderings by permutations, $e.g.$,

$$A(2,1,3,4) = \frac{\Gamma[-2\alpha' k_1 \cdot k_2]\Gamma[1+2\alpha' k_1 \cdot k_3]}{\Gamma[2+2\alpha' k_1 \cdot k_2 + 2\alpha' k_1 \cdot k_3]}$$

Now use $\Gamma[-x]\Gamma[1+x] = -\pi/\sin(\pi x)$,

$$A(2,1,3,4) = \frac{\sin(2\alpha'\pi\, k_2 \cdot k_3)}{\sin(2\alpha'\pi\, k_1 \cdot k_3)}\, A(1,2,3,4)$$

An exact string-theory identity! In the limit $\alpha' \to 0$,

$$uA(2,1,3,4) \;=\; tA(1,2,3,4)$$

– one of the field-theory identities noted by Bern, Carrasco and Johansson.

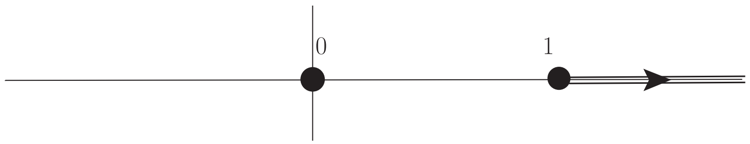
# Monodromy

How does this generalize?

Consider the three different orderings

$$\begin{aligned} A(1,2,3,4) &= \int_0^1 dx\; x^{2\alpha' k_1\cdot k_2}(1-x)^{2\alpha' k_2\cdot k_3} \\ A(1,3,2,4) &= \int_1^\infty dx\; x^{2\alpha' k_1\cdot k_2}(x-1)^{2\alpha' k_2\cdot k_3}\,, \\ A(2,1,3,4) &= \int_{-\infty}^0 dx\; (-x)^{2\alpha' k_1\cdot k_2}(1-x)^{2\alpha' k_2\cdot k_3} \end{aligned}$$

The second integral is

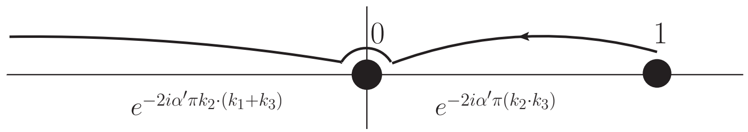


Now deform the contour so that we instead integrate from $-\infty$ to 1.

Careful at branch points!

$$(x-y)^\alpha = (y-x)^\alpha \times \begin{cases} e^{+i\pi\,\alpha} & \text{for clockwise rotation}\,, \\ e^{-i\pi\,\alpha} & \text{for counterclockwise rotation}\,. \end{cases}$$

New contour:



# A first bite at general relations

Because $A(1,3,2,4)$ is real,

$$A(1,3,2,4) = -\Re\mathrm{e}\Big(e^{-2i\alpha'\pi\, k_2\cdot k_3}\, A(1,2,3,4) + e^{-2i\alpha'\pi\, k_2\cdot(k_1+k_3)}\, A(2,1,3,4)\Big)$$

and the imaginary part vanishes:

$$0 = \Im\mathrm{m}\Big(e^{-2i\alpha'\pi\, k_2\cdot k_3}\, A(1,2,3,4) + e^{-2i\alpha'\pi\, k_2\cdot(k_1+k_3)}\, A(2,1,3,4)\Big)\ . \tag{1}$$

Solving this system of equations,

$$\begin{aligned} A(1,3,2,4) &= \frac{\sin(2\alpha'\pi\, k_1\cdot k_2)}{\sin(2\alpha'\pi\, k_2\cdot k_4)}\, A(1,2,3,4)\,, \\ A(2,1,3,4) &= \frac{\sin(2\alpha'\pi\, k_2\cdot k_3)}{\sin(2\alpha'\pi\, k_2\cdot k_4)}\, A(1,2,3,4)\,, \end{aligned} \tag{2}$$

which reduce to the Bern-Carrasco-Johansson identities when $\alpha' \to 0$.

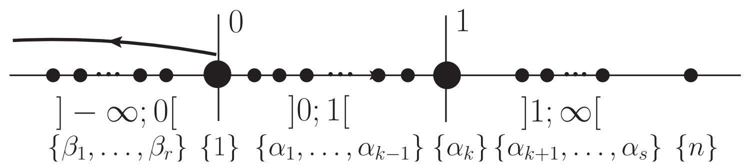
Note: These relations hold for all 4-point amplitudes of any statistics and spin. The integrals change in such a way as to restore the relations.

# String Theory Kleiss-Kuijf Relations I

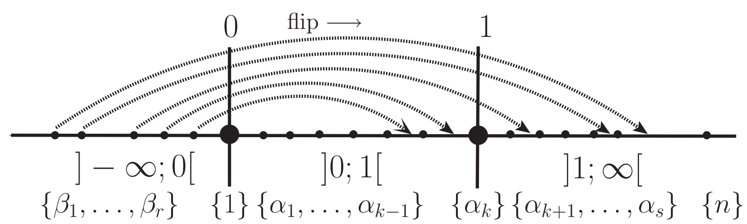
Consider a most general $n$-point amplitude.

We fix $x_1 = 0, x_{\alpha_k} = 1, x_n = \infty$:

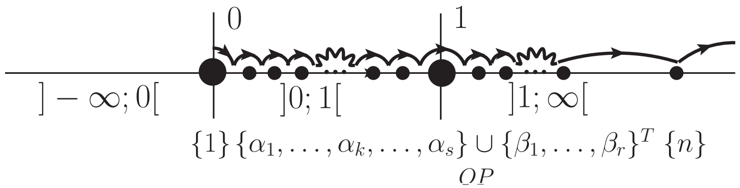
$$A(\beta_1, \ldots, \beta_r, 0, \alpha_1, \ldots, \alpha_s, n) \qquad s \geq k$$




By analytic continuation, flip all $\beta$-integrations in one go:



This relates the original amplitude with integrations on $[-\infty, 0[$ to a sum of integrations in the complementary region $[0, \infty]$



## String Theory Kleiss-Kuijf Relations II

Define $e^{(\alpha,\beta)} \equiv e^{2i\pi\alpha'(k_\alpha \cdot k_\beta)}$ if $x_\beta > x_\alpha$ and 1 otherwise.

Taking the real part:

$$\begin{aligned} A_n(\beta_1,\ldots,\beta_r,0,\alpha_1,\ldots,\alpha_s,n) &= (-1)^r \times \\ & \Re\mathrm{e}\Big[\prod_{1\le i<j\le r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_j)} A_n(0,\sigma,n)\Big], \end{aligned}$$

These are "stringy Kleiss-Kuijf relations. When $\alpha' \to 0$ the reduce to the original relations:

$$A_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} A_n(0,\sigma,n)\,, \quad (3)$$

# String Theory Bern-Carrasco-Johansson Relations

The Kleiss-Kuijf relations reduce the basis to $(n-2)!$ because 2 points are fixed here: 0 and $\infty$. All permutations are allowed inside this interval.

Let us now take the imaginary part,

$$0 = \Im\mathrm{m}\Big[\prod_{1\leq i<j\leq r} e^{2i\pi\alpha'(k_{\beta_i}\cdot k_{\beta_j})} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=0}^{s}\prod_{j=1}^{r} e^{(\alpha_i,\beta_j)} A_n(1,\sigma,n)\Big]. \qquad (4)$$

By systematically using these relations, we reduce to a *minimal basis* of size $(n-3)!$ because we can force $(n-3)$ points to lie in the interval $]0,1[$.

Proof:

- Eliminate all amplitudes with points between $]-\infty, 0[$ in favor of amplitudes with legs in the interval $]0, +\infty[$, using the stringy Kleiss-Kuijf relation.

- Rewrite amplitudes $\mathcal{A}_n(1, \alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_{n-2-k}, n)$ in terms of amplitudes with at least one $\gamma_i$ among the set $\{\alpha_1, \ldots, \alpha_k\}$ using the stringy Bern-Carrasco-Johansson relation.

- There are then at most $n - 3 - k$ elements between $]1, +\infty[$.

- Repeat this procedure until amplitudes with points in the interval $]1, \infty]$ have been expressed in terms of $(n-3)!$ amplitudes restricted to the interval $]0, 1[$.

In the $\alpha' \to 0$ limit this proves the conjecture of Bern, Carrasco and Johansson.

# We learn more from string theory

Some of the most amazing relations between gravity and gauge theory amplitudes were derived by Kawai, Lewellen and Tye (KLT) from string theory.

A closed string amplitude factorizes into a product of two open string amplitudes.

In the field theory limit this translates into relations between gravity amplitudes and Yang-Mills amplitudes.

# KLT-relations: Examples

Let an $n$-point gravity amplitude be denoted by $M(1,2,\ldots,n)$.

For 4-point amplitudes (let $s_{12} = (p_1 + p_2)^2$ etc.):

$$M(1,2,3,4) \;=\; s_{12} A(1,2,3,4) A(3,4,2,1)$$

For higher $n$ one gets a *sum* of terms on the right hand side. Only a conjectured form was known for arbitrary $n$.

One would like to prove these relations directly from field theory!

# New identities in gauge theories

Recently we have proven

- The general KLT-relations for any $n$

- A series of surprising identities in gauge theories

The proof is technical, but conceptually simple.

It uses *on-shell recursion*: If we know $(n-1)$–point amplitudes we can generate $n$–point amplitudes.

# The precise statement

Define

$$X_n^{(n_+,n_-)} = \sum_{\gamma,\beta \in S_{n-3}} A(1,\beta_{2,n-2},n-1,n)\widetilde{\mathcal{S}}[\beta_{2,n-2}|\gamma_{2,n-2}]\widetilde{A}(1,n-1,\gamma_{2,n-2},n)$$

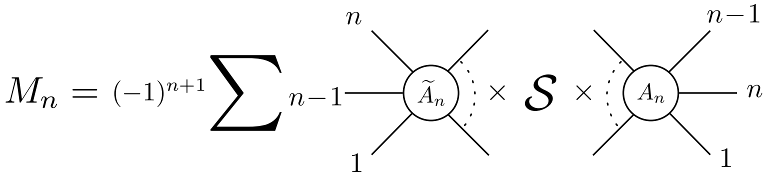
where $\widetilde{\mathcal{S}}$ is a 'kernel' depending on external momenta $s_{ij}$.

$n_+$ ($n_-$) denotes the number of positive (negative) helicity legs in $A$ which is changed to negative (positive) helicity legs in $\widetilde{A}$.

- When $n_+ = n_- = 0,\ X_n^{(0,0)} = M(1,2,\ldots,n)$.
- When $n_+ \neq n_-,\ X_n^{(n_+,n_-)} = 0$.

# First: Gravity from Gauge Theory

Pictorially:




It is as if two gluons of helicity $+1$ generate one graviton of helicity $+2$.

The 'kernel' in the middle miraculously cancels all unwanted double poles.

# It's even more amazing

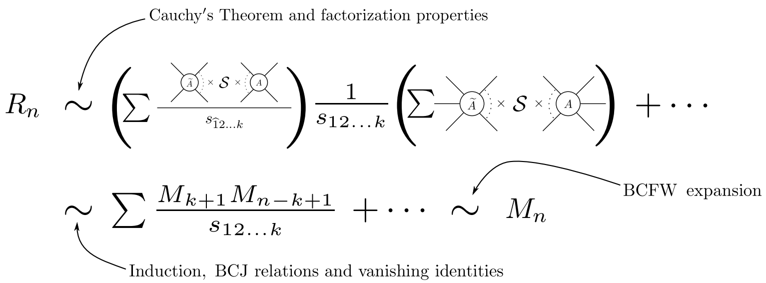
The same kernel that glues two Yang-Mills amplitudes together to give a gravity amplitude is the **generator of BCJ-relations**:

$$0 = \sum_{\sigma \in S_{n-2}} \widetilde{\mathcal{S}}[\beta_{2,n-1}|\sigma_{2,n-1}]\widetilde{A}(1, \sigma_{2,n-1}, n)$$

after summing over one more leg.

# Proof of gravity-gauge relations

Pictorially:




This uses on-shell recursion relations.

Curiously: for this to hold we need quadratic *vanishing identities* in gauge theory.

Those we prove separately.

## Examples

Consider a 4-point amplitude with $(n_+, n_-) = (0, 1)$:

$$0 = s_{12} A(1^-, 2^-, 3^+, 4^+) \widetilde{A}(3^+, 4^+, 2^+, 1^-).$$

Actually, this reproduces a well-known 'MHV rule'.

A new and non-trivial 5-point example with $(n_+, n_-) = (1, 0)$:

$$\begin{aligned} 0 = {} & s_{12} A(1^-, 2^-, 3^+, 4^+, 5^+) \big[ s_{13} \widetilde{A}(4^+, 5^+, 2^-, 3^-, 1^-) \\ & + (s_{13} + s_{23}) \widetilde{A}(4^+, 5^+, 3^-, 2^-, 1^-) \big] \\ & + s_{13} A(1^-, 3^+, 2^-, 4^+, 5^+) \big[ s_{12} \widetilde{A}(4^+, 5^+, 3^-, 2^-, 1^-) \\ & + (s_{12} + s_{23}) \widetilde{A}(4^+, 5^+, 2^-, 3^-, 1^-) \big]. \end{aligned}$$

# A physical interpretation

Every time we have

$$X_n^{(n_+,n_-)} = 0$$

we have a new non-linear identity among gauge theory amplitudes.

How can we understand these new identities?

A flipped helicity on an external leg produces $(+\ 1\ \text{-}\ 1 = 0)$ a *scalar* leg. This corresponds to gravity amplitudes with a single scalar: **it vanishes**.

In this way the gravity – gauge theory relation can be used to deduce identities in Yang-Mills theory alone!

## Examples

When we flip **one plus helicity** and **one minus helicity**, i.e.

$$(n_+, n_-) = (1, 1)$$

we get a gravity amplitude with two external scalars. It is non-zero, and we get no new Yang-Mills identity.

When we flip only **one plus helicity** and **no minus helicity**, i.e.

$$(n_+, n_-) = (1, 0)$$

we get a gravity amplitude with *one* external scalar. It vanishes, and we get a new Yang-Mills identity.

All of this generalizes to any $n$.

# Generalizations

All of this generalizes to the full multiplet of $\mathcal{N} = 4$ Super Yang-Mills.

Then one gets new gauge theory identities with also fermions and scalars.

The helicity selection rules can be understood in terms of conservations of $R$-symmetry charges.

# Conclusion: The message

- Sometimes hadronic physics gets input from odd directions

- String theory can guide us to new insight in field theory

- We have proven directly in field theory the KLT gravity-gauge connection

- As a by-product we have discovered new identities among Yang-Mills amplitudes