

# Superstrings on the Lattice

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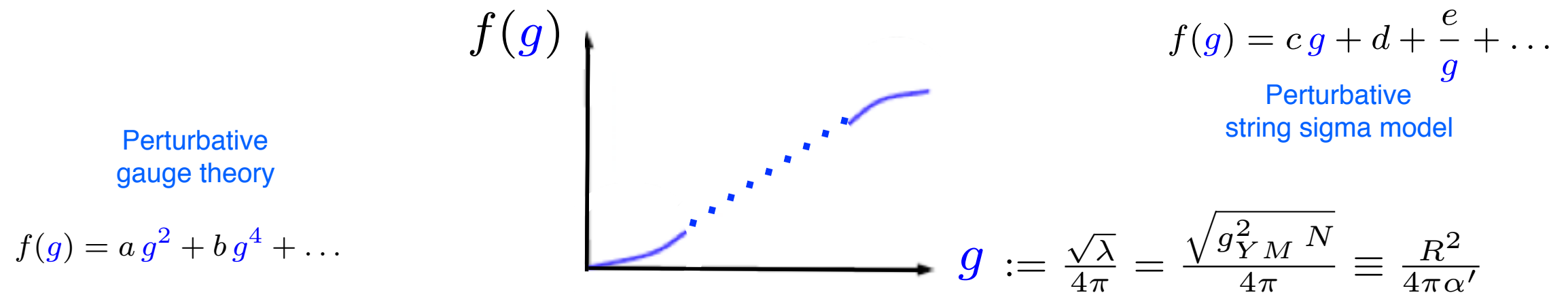
**& Humboldt University Berlin**



**Quantum Gravity meets Lattice QCD**  
**ECT\*, Trento, September 2018**

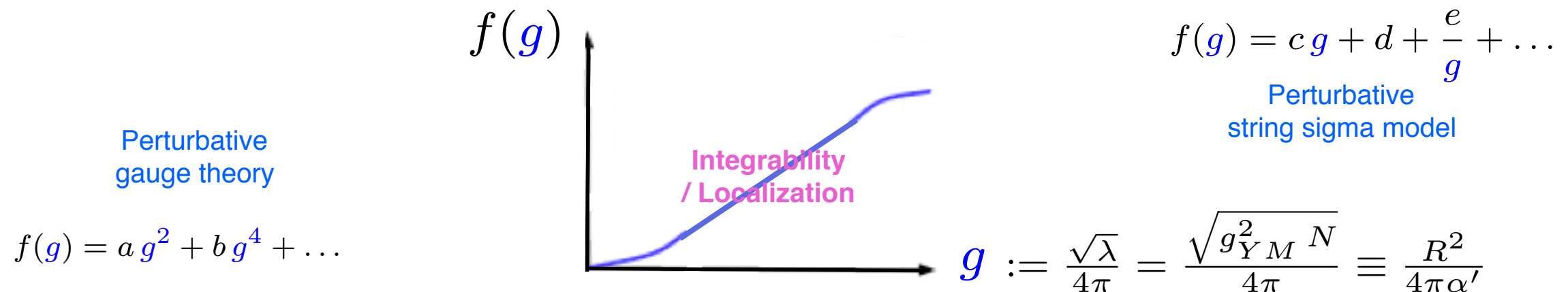
# Exact results in AdS/CFT

Beautiful progress in obtaining within AdS/CFT results **exact** in the coupling



# Exact results in AdS/CFT

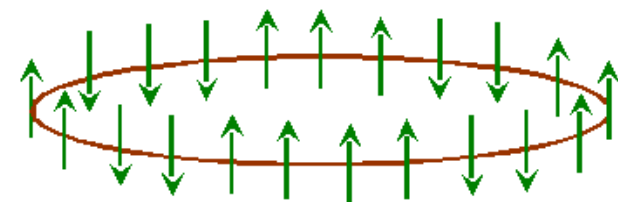
Beautiful progress in obtaining within AdS/CFT results **exact** in the coupling



- ▶ from integrability
- ▶ from supersymmetric localization

Gauge theory **spectrum** (set of dimensions of local operators): eigenvalue problem for a spin chain hamiltonian

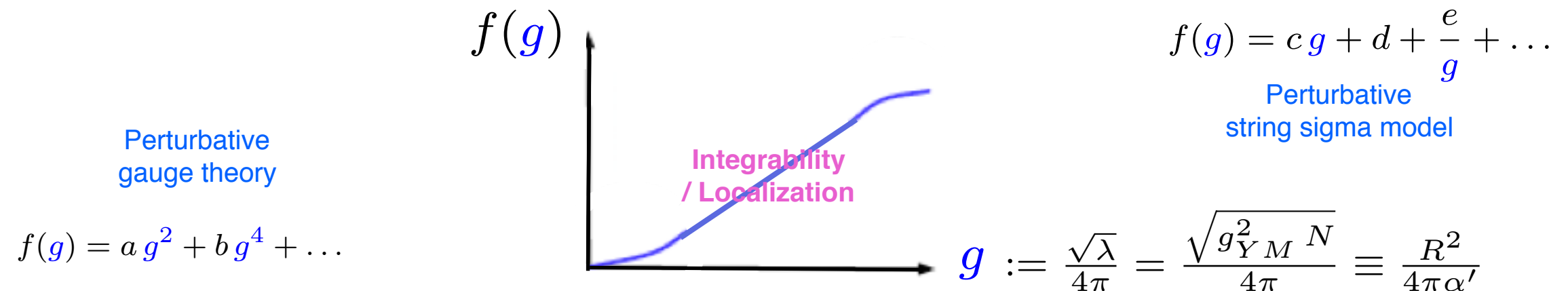
It allows **exact** evaluation of path integrals in supersymmetric gauge theories



[Minahan Zarembo, 02] [Beisert Staudacher 05] [...]

# Motivation

Beautiful progress in obtaining within AdS/CFT results **exact** in the coupling



- ▶ from integrability (**assumed**)
- ▶ from supersymmetric localization (**supersymmetric observables**)

In the **world-sheet** string theory **integrability only classically**, **localization not formulated**.

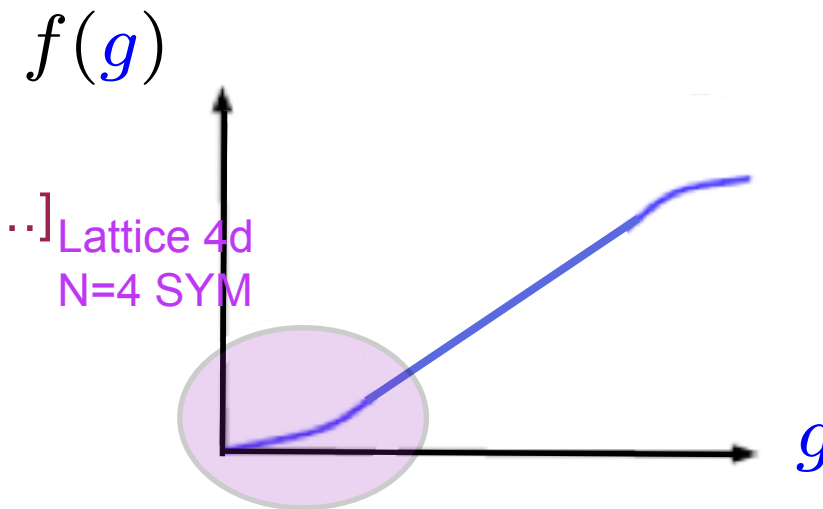
Green-Schwarz superstring in  $AdS$  backgrounds with RR fluxes: complicated interacting 2d field theory which **has subtleties also perturbatively**.

Call for **genuine 2d QFT** to cover the **finite-coupling region**.

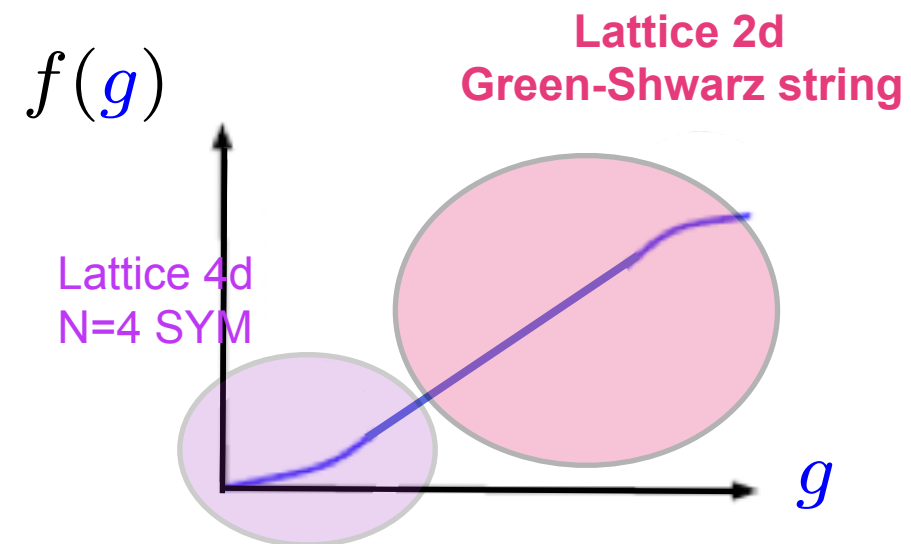


# Lattice techniques in AdS/CFT (part of) David talk later!

Consolidated program on 4d CFT side,  
[Catterall, Damgaard, DeGrand, Giedt, Schaich...]  
**(part of) David talk later!**  
control on the perturbative region.



# Lattice techniques in AdS/CFT



[previous study: Roiban McKeown 2013]

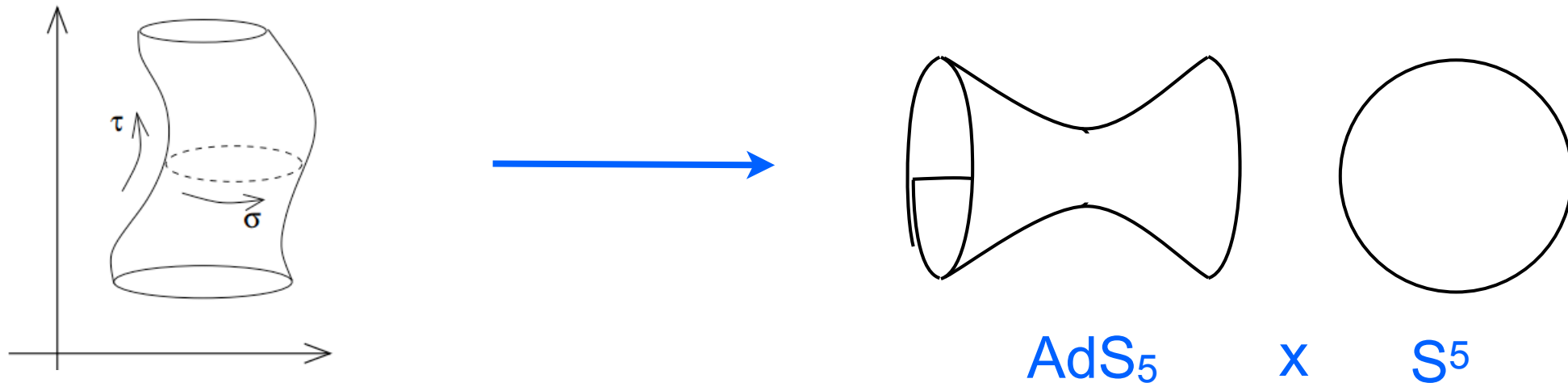
Features:

- ▶ **2d**: computationally **cheap**
- ▶ **no supersymmetry** (only as flavour symmetry, Green-Schwarz)
- ▶ **all gauge symmetries are fixed** (no formulation à la Wilson), only **scalar** fields (some of which anti-commuting)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

# The model in perturbation theory

# Green-Schwarz string in $AdS_5 \times S^5$



[Metsaev Tseytlin 1998]

$$S = g \int d\tau d\sigma \left[ \partial_a X^\mu \partial^a X^\nu G_{\mu\nu} + \bar{\theta} \Gamma (D + F_5) \theta \partial X + \bar{\theta} \partial \theta \bar{\theta} \partial \theta + \dots \right]$$

Symmetries:

- ▶ **global**  $PSU(2, 2|4)$ , **local** bosonic (diffeomorphism) and fermionic ( $\kappa$ -symmetry)
- ▶ **classical** integrability

manifest when written as sigma-model action on  $G/H = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ .

# Green-Schwarz string in $AdS_5 \times S^5$ + RR flux perturbatively

Highly non-linear, to quantize it use semiclassical methods

$$X = X_{cl} + \tilde{X} \longrightarrow \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \dots \right]$$

- General analysis of fluctuations in terms of background geometry,

[Drukker Gross Tseytlin 00] [Buchbinder Tseytlin 14] [VF Giangreco Griguolo Seminara Vescovi 15]

- Explicit analytic form of one-loop partition function  $Z = \det O_F / \sqrt{\det O_B}$  for a class of effectively one-dimensional problems.

Several “vacua” (GKP string, quark-antiquark potential, generalized cusp)  
“solved” this way at one loop, they agree with predictions.

[Drukker Gross Tseytlin Frolov VF Beccaria Dunne Giangreco, Ohlson Sax, Griguolo Seminara Vescovi ....]

In supersymmetric cases – e.g. strings dual to circular Wilson loop –  
much more care needed (measure ambiguities, zero modes, etc.)

[Kruczenski Tirziu 08] [Kristjansen Makeenko 12] [Buchbinder Tseytlin 14]

[VF, Giangreco, Griguolo, Seminara, Vescovi 15] [Pando-Zayas Trancanelli et al.16]

[VF, Vescovi, Tseytlin 17] [Cagnazzo, Medina-Rincon, Zarembo 17] [Medina-Rincon, Tseytlin, Zarembo 18]

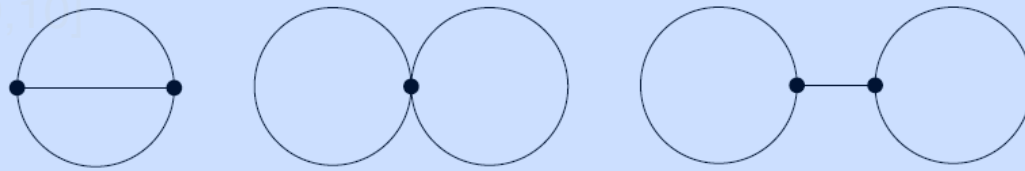
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2 loops is current limit: “homogenous” configs, “AdS light-cone” gauge-fixing

[Uvarov, 09,10][Uvarov, 09,10]



[Giombi Ricci Roiban Tseytlin 09] [Bianchi<sup>2</sup> Bres VF Vescovi 14]

Check of exact predictions based on integrability and localization [Gromov, Syzov 14]

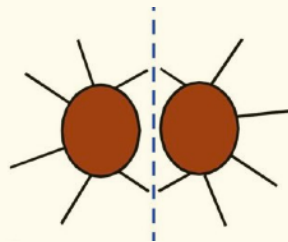
and check of quantum consistency (UV finiteness) of certain string actions. [Uvarov 09,10]

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Highly non-linear, to quantize it use semiclassical methods

$$X = X_{cl} + \tilde{X} \longrightarrow \Gamma = g \left[ \Gamma_0 + \frac{\Gamma_1}{g} + \frac{\Gamma_2}{g^2} + \dots \right]$$

Efficient alternative to Feynman diagrams for on-shell objects (worldsheet S-matrix)



unitarity cuts (on-shell methods) in d=2

[Bianchi **VF** Hoare 2013][Engelund Roiban 2013] [Bianchi Hoare 14]

# Beyond perturbation theory

with L. Bianchi, M. S. Bianchi, B. Leder, P. Töpfer, E. Vescovi



# The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

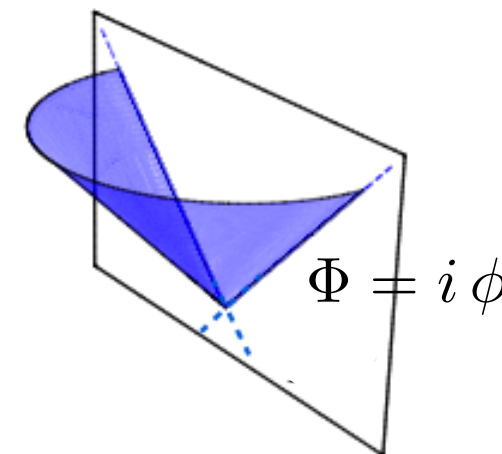
Completely solved via integrability. [Beisert Eden Staudacher 2006]

Expectation value of a light-like cusped Wilson loop

AdS/CFT

$$\langle W[C_{\text{cusp}}] \rangle \sim e^{-f(g) \phi \ln \frac{L_{\text{IR}}}{\epsilon_{\text{UV}}}}$$

$$Z_{\text{cusp}} = \int [D\delta X][D\delta\theta] e^{-S_{\text{IIB}}(X_{\text{cusp}} + \delta X, \delta\theta)}$$



String partition function with “cusp” boundary conditions.

$X_{\text{cusp}}$  is the minimal surface

[Giombi Ricci Roiban Tseytlin 2009]

$$ds^2_{AdS_5} = \frac{dz^2 + dx^+ dx^- + dx^* dx}{z^2} \quad x^\pm = x^3 \pm x^0 \quad x = x^1 + i x^2$$

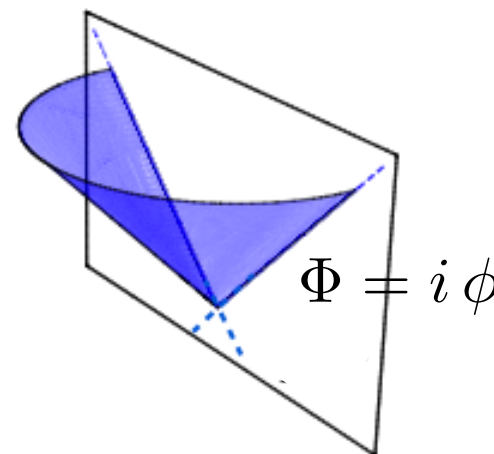
$$z = \sqrt{\frac{\tau}{\sigma}} \quad x^+ = \tau \quad x^- = -\frac{1}{2\sigma} \quad x^+ x^- = -\frac{1}{2} z^2$$

ending on a null cusp, since  $x^+ x^- = 0$  at the boundary  $z = 0$ .

# The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

Completely solved via integrability. [Beisert Eden Staudacher 2006]

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$$Z_{\text{cusp}} = \int [D\delta X][D\delta\theta] e^{-S_{\text{IIB}}(X_{\text{cusp}} + \delta X, \delta\theta)} = e^{-\Gamma_{\text{eff}}} \equiv e^{-f(g) V_2}$$

String partition function with “cusp” boundary conditions.

Perturbatively

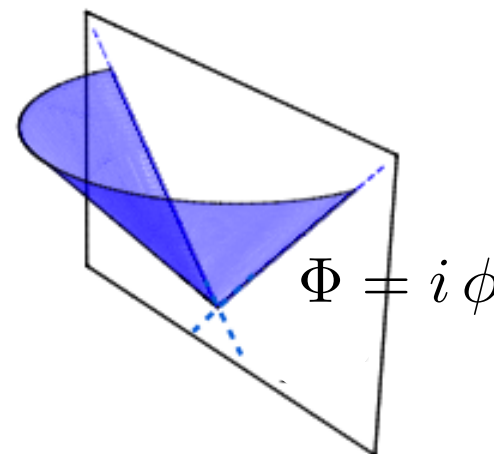
$$f(g)|_{g \rightarrow 0} = 8g^2 \left[ 1 - \frac{\pi^2}{3}g^2 + \frac{11\pi^4}{45}g^4 - \left( \frac{73}{315} + 8\zeta_3 \right) g^6 + \dots \right] \quad [\text{Bern et al. 2006}]$$

$$f(g)|_{g \rightarrow \infty} = 4g \left[ 1 - \frac{3 \ln 2}{4\pi} \frac{1}{g} - \frac{K}{16\pi^2} \frac{1}{g^2} + \dots \right] \quad \begin{array}{l} [\text{Gubser Klebanov Polyakov 02}] \\ [\text{Frolov Tseytlin 02}][\text{Giombi et al. 2009}] \end{array}$$

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$$\langle W[C_{\text{cusp}}] \rangle \sim e^{-f(g) \phi \ln \frac{L_{\text{IR}}}{\epsilon_{\text{UV}}}}$$

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String partition function with “cusp” boundary conditions.

A lattice approach prefers **expectation values**

$$\langle S_{\text{cusp}} \rangle = \frac{\int [D\delta X][D\delta\Psi] S_{\text{cusp}} e^{-S_{\text{cusp}}}}{\int [D\delta X][D\delta\Psi] e^{-S_{\text{cusp}}}} = -g \frac{d \ln Z_{\text{cusp}}}{dg} \equiv g \frac{V_2}{8} f'(g)$$

$\downarrow$   
 $S_{\text{cusp}} = g \int \mathcal{L}_{\text{cusp}}$

# Green-Schwarz string in the null cusp background

The (AdS lightcone) gauge-fixed action for fluctuations above the null cusp is

$$S_{\text{cusp}} = g \int dt ds \mathcal{L}_{\text{cusp}}$$

[Giombi Ricci Roiban Tseytlin 2009]

$$\begin{aligned} \mathcal{L}_{\text{cusp}} = & |\partial_t x + \tfrac{1}{2}x|^2 + \tfrac{1}{z^4} |\partial_s x - \tfrac{1}{2}x|^2 + \left( \partial_t z^M + \tfrac{1}{2}z^M + \tfrac{i}{z^2} z_N \eta_i (\rho^{MN})^i_j \eta^j \right)^2 + \tfrac{1}{z^4} (\partial_s z^M - \tfrac{1}{2}z^M)^2 \\ & + i (\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i) - \tfrac{1}{z^2} (\eta^i \eta_i)^2 \\ & + 2i \left[ \tfrac{1}{z^3} z^M \eta^i (\rho^M)_{ij} (\partial_s \theta^j - \tfrac{1}{2}\theta^j - \tfrac{i}{z} \eta^j (\partial_s x - \tfrac{1}{2}x)) + \tfrac{1}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij} (\partial_s \theta_j - \tfrac{1}{2}\theta_j + \tfrac{i}{z} \eta_j (\partial_s x - \tfrac{1}{2}x)^*) \right] \end{aligned}$$

- ▶ 8 bosons:  $x, x^*, z^M$  ( $M = 1, \dots, 6$ ),  $z = \sqrt{z_M z^M}$ ;
- ▶ 8 fermions:  $\theta^i = (\theta_i)^\dagger, \eta^i = (\eta_i)^\dagger, i = 1, 2, 3, 4$ , complex Grassmann;
- ▶  $\rho^M$  are off-diagonal blocks of  $SO(6)$  Dirac matrices
- ▶  $(\rho^{MN})^i_j$  are the  $SO(6)$  generators

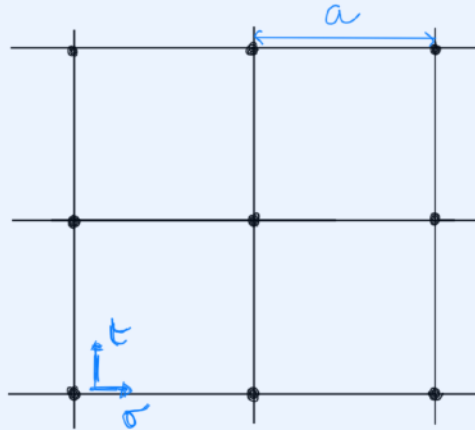
Remnant global symmetry is  $SO(6) \times SO(2)$ .

Fermionic interactions at most quartic.

# Lattice QFT basics

Discretize Euclidean worldsheet in a grid of lattice spacing  $a$ , size  $L = N a$ .

Fields  $\phi \equiv \phi_n$  defined at  $\xi = (an_1, an_2) \equiv a n$ .



**Graßmann-odd** fields are formally integrated out:  $P[\Phi_i] = \frac{e^{-S_E[\Phi_i]} \det \mathcal{O}_F}{Z}$

- action must be **quadratic** in fermions (linearization via auxiliary fields):

$$\text{X} \equiv \text{>---<} \quad \text{Introduce auxiliary fields (complex bosons)}$$

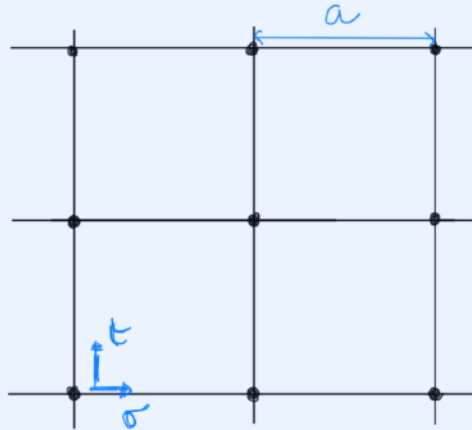
- determinant must be **positive definite**

$$\det O_F \longrightarrow \sqrt{\det(\mathcal{O}_F \mathcal{O}_F^\dagger)} = \int D\zeta D\bar{\zeta} e^{-\int d^2\xi \bar{\zeta}(\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}} \zeta}$$

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- determinant must be **positive definite**

$$\text{Pf } O_F \longrightarrow (\det O_F^\dagger O_F)^{\frac{1}{4}} \equiv \int D\zeta D\bar{\zeta} e^{-\int d^2\xi \bar{\zeta} (O_F^\dagger O_F)^{-\frac{1}{4}} \zeta}$$

# Linearization

# Four-fermion interactions

Linearization via Hubbard-Stratonovich transformation

$$\exp \left\{ -g \int dt ds \mathcal{L}_4 \right\} \sim \int d\phi d\phi^M \exp \left\{ -g \int dt ds \mathcal{L}_{\text{aux}} \right\}$$

$$\begin{aligned} & \exp \left\{ -g \int dt ds \left[ -\frac{1}{z^2} (\eta^i \eta_i)^2 + \left( \frac{i}{z^2} z_N \eta_i \rho^{MNi}{}_j \eta^j \right)^2 \right] \right\} \\ & \sim \int D\phi D\phi^M \exp \left\{ -g \int dt ds \left[ \frac{1}{2} \phi^2 + \frac{\sqrt{2}}{z} \phi \eta^2 + \frac{1}{2} (\phi_M)^2 - i \frac{\sqrt{2}}{z^2} \phi^M \left( \frac{i}{z^2} z_N \eta_i \rho^{MNi}{}_j \eta^j \right) \right] \right\} . \end{aligned}$$

- +7 bosonic auxiliary fields  $\phi, \phi^M$  ( $M = 1, \dots, 6$ )



# Four-fermion interactions

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hermitian

- ▶ +7 bosonic auxiliary fields  $\phi, \phi^M$  ( $M = 1, \dots, 6$ )
- ▶  $\mathcal{L}_{\text{aux}}$  is not hermitian,  $e^{-\frac{b^2}{4a}} = \int dx e^{-a x^2 + i b x}$ ,  $b \in \mathbb{R}$ .

# Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ( $m \sim P_+$ )

$$\begin{aligned} \mathcal{L}_{\text{cusp}} = & \left| \partial_t x + \frac{m}{2} x \right|^2 + \frac{1}{z^4} \left| \partial_s x - \frac{m}{2} x \right|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4} (\partial_s z^M - \frac{m}{2} z^M)^2 \\ & + \frac{1}{2} \phi^2 + \frac{1}{2} (\phi_M)^2 + \psi^T O_F \psi, \end{aligned}$$

where  $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$  and

$$O_F = \begin{pmatrix} 0 & i\partial_t & -i\rho^M \left(\partial_s + \frac{m}{2}\right) \frac{z^M}{z^3} & 0 \\ i\partial_t & 0 & 0 & -i\rho_M^\dagger \left(\partial_s + \frac{m}{2}\right) \frac{z^M}{z^3} \\ i\frac{z^M}{z^3} \rho^M \left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4} \rho^M \left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & i\frac{z^M}{z^3} \rho_M^\dagger \left(\partial_s - \frac{m}{2}\right) & i\partial_t + A & -2\frac{z^M}{z^4} \rho_M^\dagger \left(\partial_s x^* - m\frac{x^*}{2}\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2} \phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z} \phi + i \frac{z_N}{z^2} \rho^{MN} \partial_t z^M$$

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As  $A^\dagger \neq A$ , Pfaffian is complex:  $\text{Pf}(\mathcal{O}_F) = e^{i\theta} (O_F O_F^\dagger)^{\frac{1}{4}}$ .

# Phase problem

Even with  $\text{Pf}(\mathcal{O}_F) = e^{i\theta} (O_F O_F^\dagger)^{\frac{1}{4}}$ , vev's can be still obtained via **reweighting**:

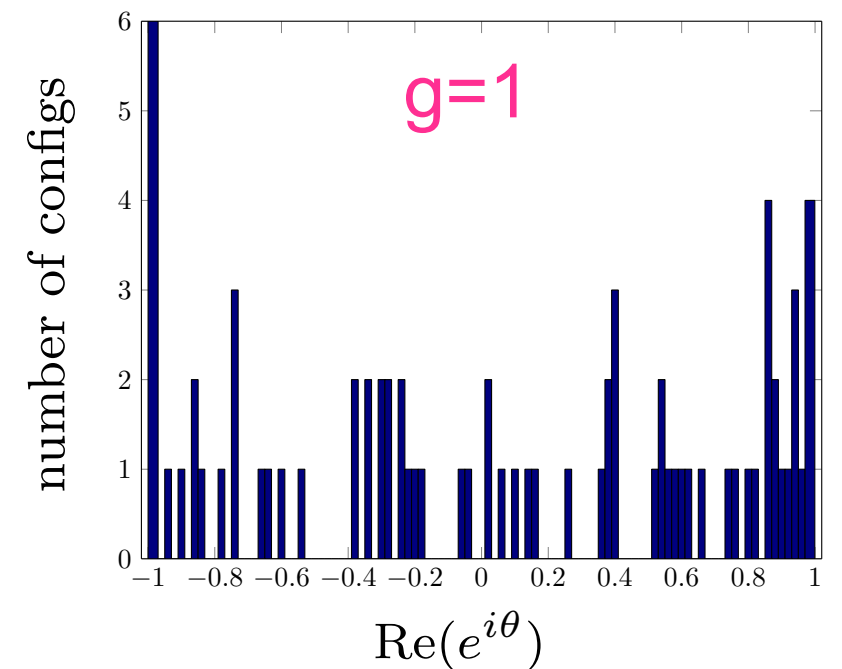
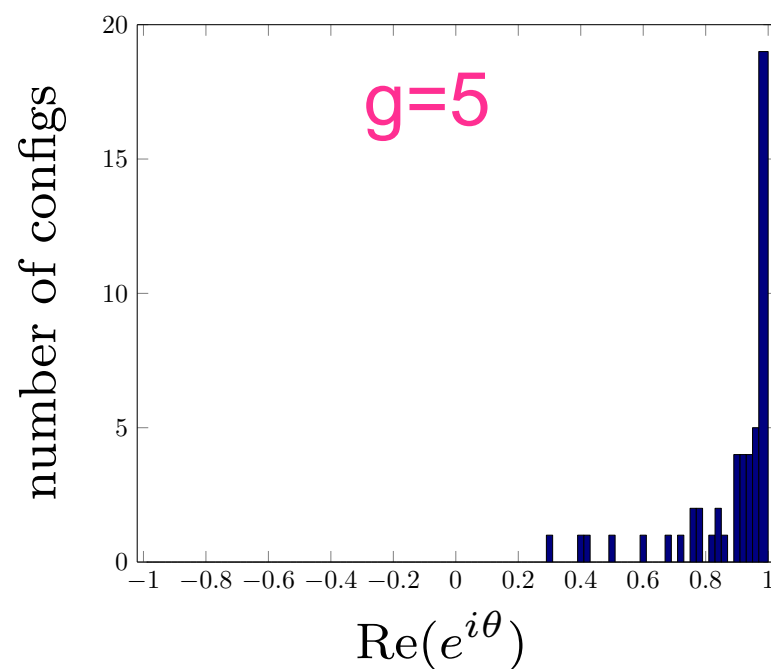
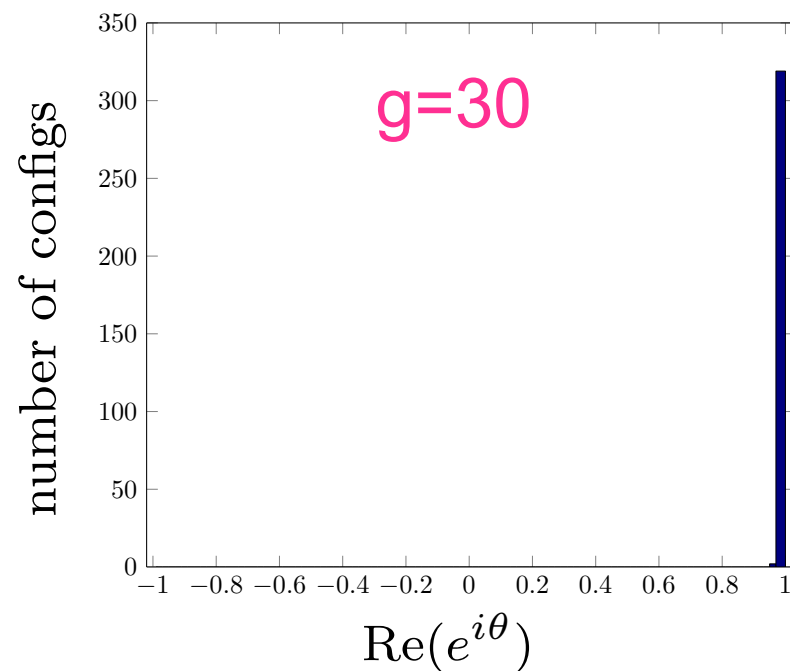
$$\begin{aligned}\langle \mathcal{A} \rangle &= \frac{\int D\Phi \mathcal{A} \text{Pf}(O_F) e^{-S[\Phi]}}{\int D\Phi \text{Pf}(O_F) e^{-S[\Phi]}} \\ &= \frac{\int D\Phi D\zeta D\bar{\zeta} \mathcal{A} e^{i\theta} e^{-S[\Phi] - \int d^2\xi \bar{\zeta} (\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}} \zeta}}{\int D\Phi D\zeta D\bar{\zeta} e^{i\theta} e^{-S[\Phi] - \int d^2\xi \bar{\zeta} (\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}} \zeta}} = \frac{\langle \mathcal{A} e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}\end{aligned}$$

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However, the **phase averages to zero**:



# Alternative linearization

The phase is implicit in the linearization, like  $e^{-\frac{b^2}{4a}} = \int dx e^{-a x^2 + i b x}$

Consider a simple SO(4) invariant four-fermion interaction

[Catterall 2015]

$$\mathcal{L}_{4F} = \frac{1}{2} \epsilon_{abcd} \psi^a(x) \psi^b(x) \psi^c(x) \psi^d(x) \equiv \Sigma^{ab} \tilde{\Sigma}^{ab}$$

where  $\Sigma^{ab} = \psi^a \psi^b$ ,  $\tilde{\Sigma}^{ab} = \frac{1}{2} \epsilon_{abcd} \psi^c \psi^d$ . Introducing  $\Sigma_{\pm}^{ab} = \frac{1}{2} (\Sigma^{ab} \pm \tilde{\Sigma}^{cd})$ , rewrite

$$\mathcal{L}_{4F} = \pm 2 \left( \Sigma_{\pm}^{ab} \right)^2$$

just exploiting the Grassmann character of the underlying fermions.

$$\begin{aligned} \pm \Sigma_{\pm}^{ab} \Sigma_{\pm}^{ab} &= \pm \frac{1}{4} \left[ \Sigma^{ab} \pm \frac{1}{2} \epsilon_{abcd} \Sigma^{cd} \right] \left[ \Sigma^{ab} \pm \frac{1}{2} \epsilon_{abef} \Sigma^{ef} \right] \\ &= \pm \frac{1}{4} \left[ \cancel{\Sigma^{ab} \Sigma^{ab}} \pm \frac{1}{2} \epsilon_{abcd} (\Sigma^{ab} \Sigma^{cd} + \Sigma^{cd} \Sigma^{ab}) + \frac{1}{4} \epsilon_{abcd} \epsilon_{abef} \cancel{\Sigma^{cd} \Sigma^{ef}} \right] \\ &\quad \text{since } \psi^a \psi^b \psi^a \psi^b = 0 \\ &= \frac{1}{4} \left( \Sigma^{ab} \tilde{\Sigma}^{ab} + \tilde{\Sigma}^{ab} \Sigma^{ab} \right) = \frac{1}{2} \Sigma^{ab} \tilde{\Sigma}^{ab} \end{aligned}$$

# Alternative linearization

In our case,  $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 + \frac{1}{z^2} \left( i \eta_i (\rho^{MN})^i_j n^N \eta^j \right)^2$$

# Alternative linearization

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$$\Sigma_i^j = \eta_i \eta^j, \quad \tilde{\Sigma}_j^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l, \quad \Sigma_{\pm}^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

Choosing the **good** sign ( $-$ ), new set of **1** + **16** real auxiliary fields

$$\mathcal{L}_{\text{aux}} = \frac{12}{z} \eta^2 \phi + 6\phi^2 + \frac{2}{z} \Sigma_{\pm}^i_j \phi_i^j + \phi_j^i \phi_i^j \quad \mathcal{L}_{\text{aux}}^\dagger = \mathcal{L}_{\text{aux}}$$

Antisymmetry and  $\Gamma_5$ -hermiticity ( $\Gamma_5^\dagger \Gamma_5 = \mathbb{1}$ ,  $\Gamma_5^\dagger = -\Gamma_5$ )

$$O_F^\dagger = \Gamma_5 O_F \Gamma_5, \quad O_F^T = -O_F$$

ensure positive-definite **determinant**  $(\text{Pf} O_F)^2 = \det O_F \geq 0$ , and a **real Pfaffian**.



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$$O_F^\dagger = \Gamma_5 O_F \Gamma_5, \quad O_F^T = -O_F$$

ensure positive-definite **determinant**  $(\text{Pf} O_F)^2 = \det O_F \geq 0$ , and a **real Pfaffian**.

Gain in computational costs, but  $\text{Pf} O_F = \pm \sqrt{\det O_F}$ .

# Alternative linearization

In our case,  $(\rho^M)^{im}(\rho^M)^{kn} = 2\epsilon^{imkn}$ , we analogously rewrite

$$\mathcal{L}_{F4} = -\frac{1}{z^2}(\eta^2)^2 + \frac{1}{z^2} \left( i \eta_i (\rho^{MN})^i_j n^N \eta^j \right)^2$$

$$\Sigma_i^j = \eta_i \eta^j, \quad \tilde{\Sigma}_j^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l, \quad \Sigma_{\pm}^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

Choosing the **good** sign ( $-$ ), new set of **1 + 16** real auxiliary fields

$$\mathcal{L}_{\text{aux}} = \frac{12}{z} \eta^2 \phi + 6\phi^2 + \frac{2}{z} \Sigma_{\pm}^i \phi_i^j + \phi_j^i \phi_i^j \quad \mathcal{L}_{\text{aux}}^\dagger = \mathcal{L}_{\text{aux}}$$

Antisymmetry and  $\Gamma_5$ -hermiticity ( $\Gamma_5^\dagger \Gamma_5 = \mathbb{1}$ ,  $\Gamma_5^\dagger = -\Gamma_5$ )

$$O_F^\dagger = \Gamma_5 O_F \Gamma_5, \quad O_F^T = -O_F$$

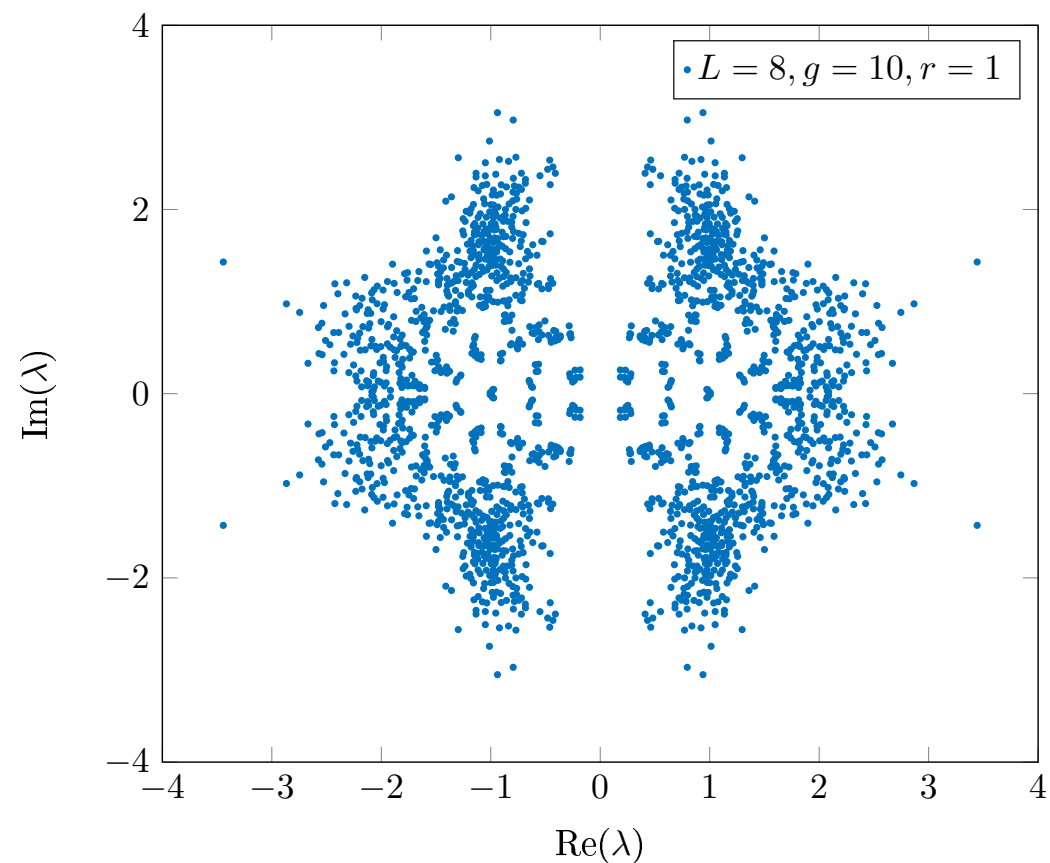
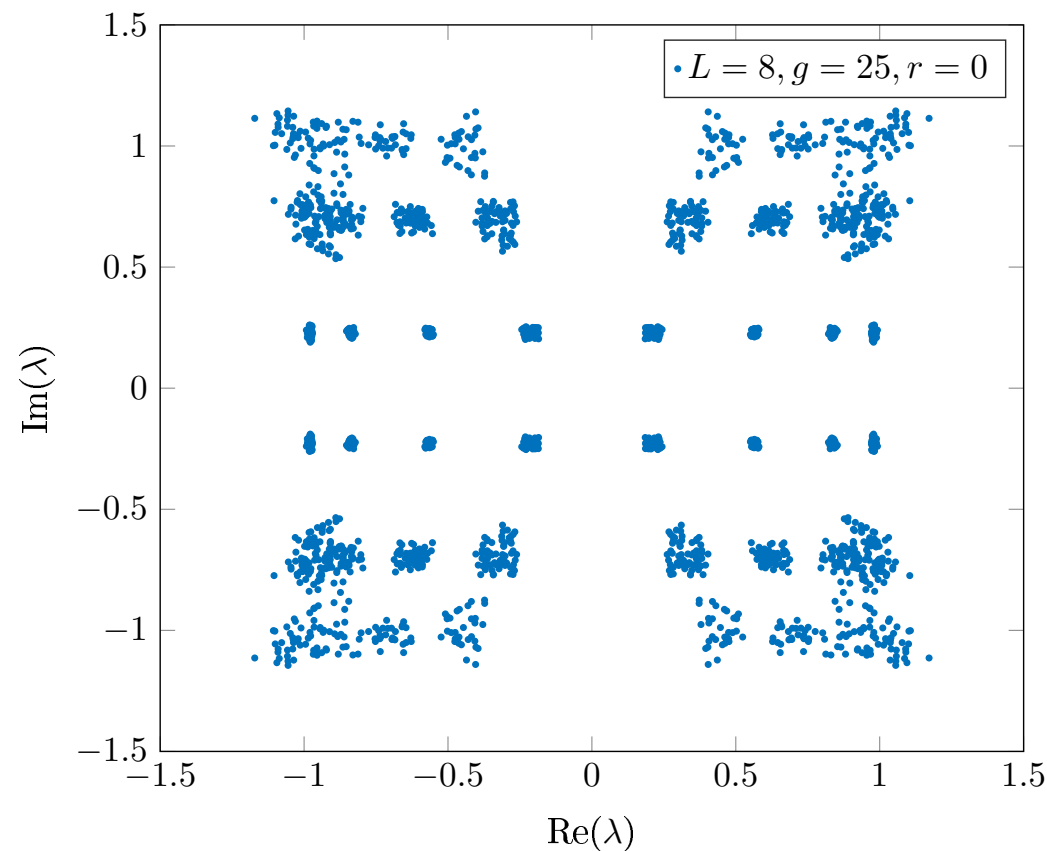
ensure positive-definite **determinant**  $(\text{Pf} O_F)^2 = \det O_F \geq 0$ , and a **real Pfaffian**.

In simpler models with four-fermion interactions, similar manipulations ensure a definite positive Pfaffian. There real, antisymmetric operator with **doubly degenerate** eigenvalues: quartets  $(ia, ia, -ia, -ia)$ ,  $a \in \mathbb{R}$ .

# Spectrum of $O_F$

From  $\Gamma_5$ -hermiticity and antisymmetry,

$$\begin{aligned}\mathcal{P}(\lambda) &= \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5 (O_F - \lambda \mathbb{1}) \Gamma_5) \\ &= \det(O_F^\dagger + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*\end{aligned}$$



Spectrum characterized by **quartets**  $\{\lambda, -\lambda^*, -\lambda, \lambda^*\}$ .

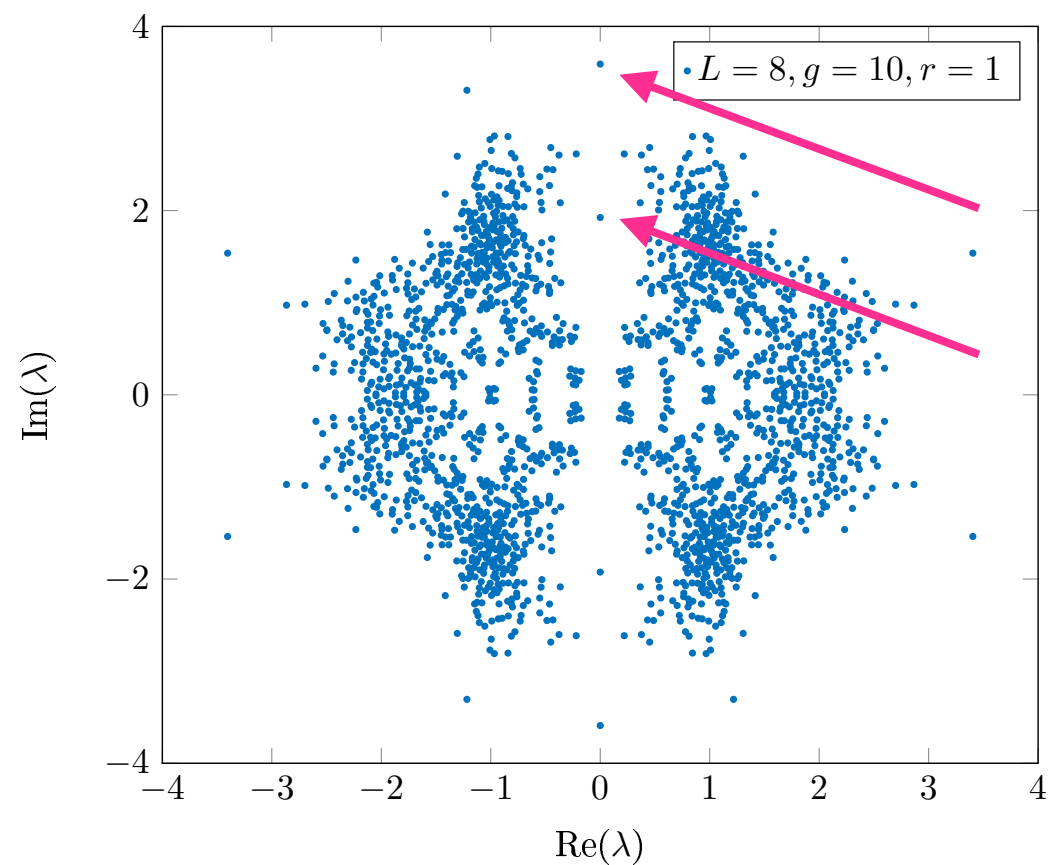
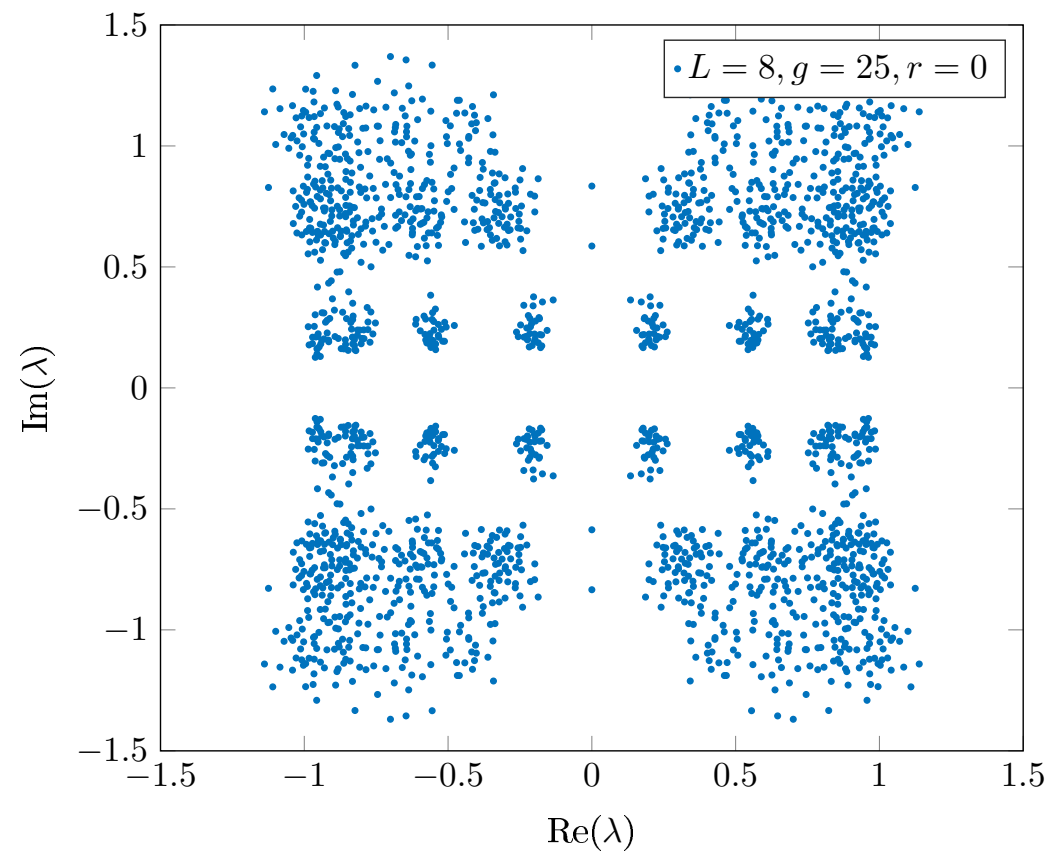
$$\det O_F = \prod_i |\lambda_i|^2 |\lambda_i|^2 \quad \longrightarrow \quad \text{Pf}(O_F) = \pm \prod_i |\lambda_i|^2$$

Choosing a starting configuration with positive Pfaffian, no sign change possible.

# Spectrum of $O_F$

From  $\Gamma_5$ -hermiticity and antisymmetry,

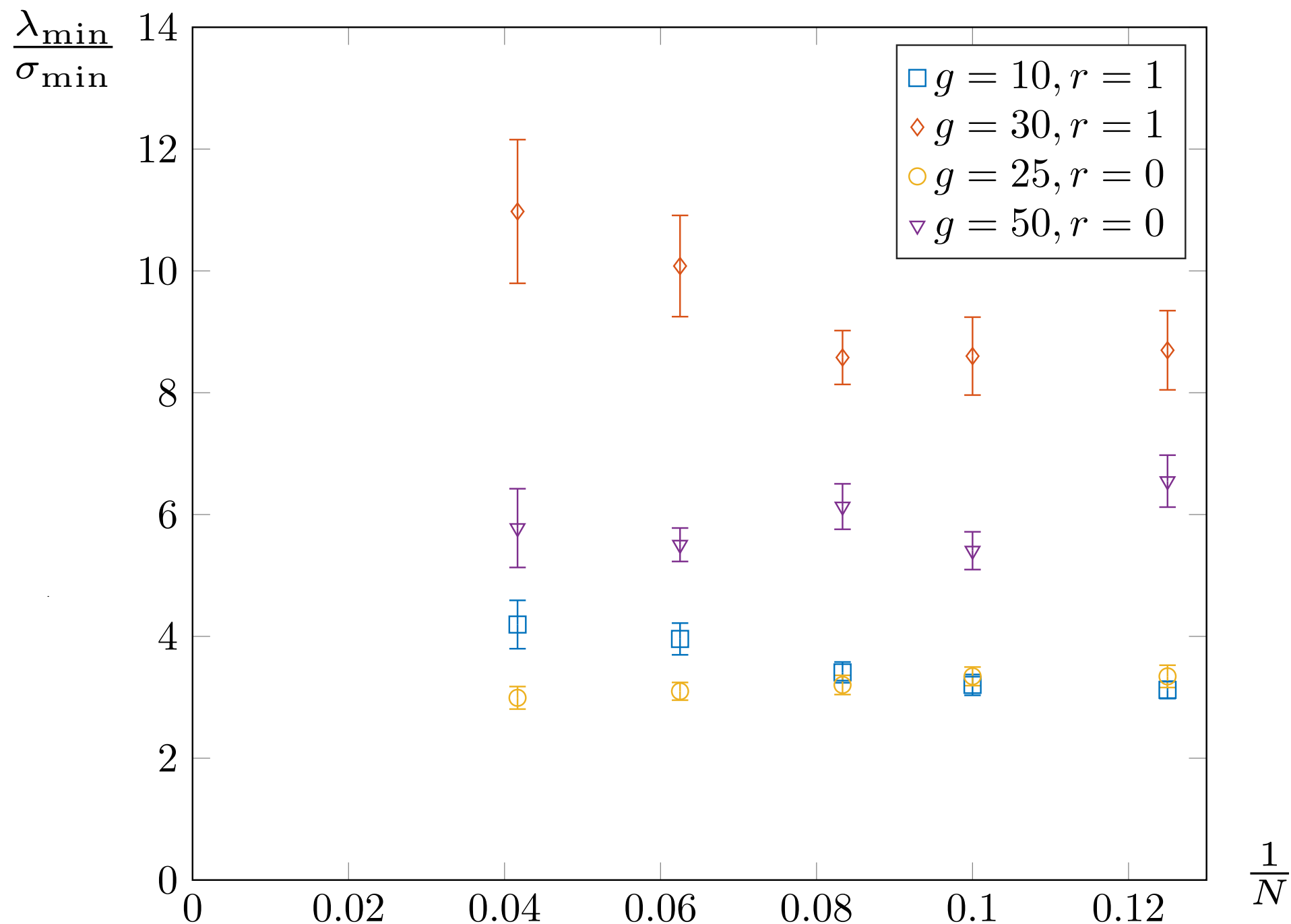
$$\begin{aligned}\mathcal{P}(\lambda) &= \det(O_F - \lambda \mathbb{1}) = \det(\Gamma_5 (O_F - \lambda \mathbb{1}) \Gamma_5) \\ &= \det(O_F^\dagger + \lambda \mathbb{1}) = \det(O_F + \lambda^* \mathbb{1})^* = \mathcal{P}(-\lambda^*)^*\end{aligned}$$



For  $\lambda = \pm \lambda^*$ , no four-fold property: due to **zero crossings**, Pfaffian may change sign.

Purely imaginary eigenvalues correspond to Yukawa-terms, even those present in the original Lagrangian: no “suitable enough” choice of auxiliary fields.

# Where are we sign-problem free?



Eigenvalue distribution of fermionic operators well separated from zero,  
no sign problem for  $g \geq 10$ , where nonperturbative physics is captured.

# Discretization

# Guiding lines for discretization

- ▶ Lattice perturbation theory  $\xrightarrow{a \rightarrow 0}$  continuum perturbation theory
- ▶ Preserve the symmetries of the model
- ▶ No complex phases

# Guiding lines for discretization

- ▶ Lattice perturbation theory  $\xrightarrow{a \rightarrow 0}$  continuum perturbation theory

In the continuum, the free kinetic part of the fermionic operator

$$K_F = \begin{pmatrix} 0 & -p_0 \mathbb{1} & (p_1 - i \frac{m}{2}) \rho^M u_M & 0 \\ -p_0 \mathbb{1} & 0 & 0 & (p_1 - i \frac{m}{2}) \rho_M^\dagger u^M \\ -(p_1 + i \frac{m}{2}) \rho^M u_M & 0 & 0 & -p_0 \mathbb{1} \\ 0 & -(p_1 + i \frac{m}{2}) \rho_M^\dagger u^M & -p_0 \mathbb{1} & 0 \end{pmatrix}$$

gives the contribution  $\det K_F = \left(p_0^2 + p_1^2 + \frac{m^2}{4}\right)^8$  to the one-loop partition function

$$\begin{aligned} \Gamma^{(1)} &= -\ln Z^{(1)} = \frac{V_2}{a^2} \frac{1}{2} \int_{-\pi}^{\pi} \frac{dp_0 dp_1}{(2\pi)^2} \ln \left[ \frac{(p_0^2 + p_1^2 + m^2)(p_0^2 + p_1^2 + \frac{m^2}{2})^2 (p_0^2 + p_1^2)^5}{(p_0^2 + p_1^2 + \frac{m^2}{4})^8} \right] \\ &= -\frac{3 \ln 2}{8\pi} m^2 V_2 \end{aligned}$$



# Guiding lines for discretization

- Lattice perturbation theory  $\xrightarrow{a \rightarrow 0}$  continuum perturbation theory

A naive discretization  $p_\mu \rightarrow \mathring{p}_\mu \equiv \frac{1}{a} \sin(a p_\mu)$  leads to **fermion doublers**,

$$K_F = \begin{pmatrix} 0 & -\mathring{p}_0 \mathbb{1} & (\mathring{p}_1 - i \frac{m}{2}) \rho^M u_M & 0 \\ -\mathring{p}_0 \mathbb{1} & 0 & 0 & (\mathring{p}_1 - i \frac{m}{2}) \rho_M^\dagger u^M \\ -(\mathring{p}_1 + i \frac{m}{2}) \rho^M u_M & 0 & 0 & -\mathring{p}_0 \mathbb{1} \\ 0 & -(\mathring{p}_1 + i \frac{m}{2}) \rho_M^\dagger u^M & -\mathring{p}_0 \mathbb{1} & 0 \end{pmatrix}$$

spoiling UV finiteness (effective 2d supersymmetry).

# A Wilson-like fermion discretization

- ▶ Lattice perturbation theory  $\xrightarrow{a \rightarrow 0}$  continuum perturbation theory
- ▶ Preserve  $SO(6)$ , breaks  $U(1) \sim SO(2)$
- ▶ No complex phases:  $(O_F^W)^\dagger = \Gamma_5 O_F^W \Gamma_5$ ,  $(O_F^W)^T = -O_F^W$

Add to the action a “Wilson term”,  $K_F + W \equiv K_F^W$

$$K_F^W = \begin{pmatrix} W_+ & -\not{p}_0 \mathbb{1} & (\not{p}_1 - i \frac{m}{2}) \rho^M u_M & 0 \\ -\not{p}_0 \mathbb{1} & -W_+^\dagger & 0 & (\not{p}_1 - i \frac{m}{2}) \rho_M^\dagger u^M \\ -(\not{p}_1 + i \frac{m}{2}) \rho^M u_M & 0 & W_- & -\not{p}_0 \mathbb{1} \\ 0 & -(\not{p}_1 + i \frac{m}{2}) \rho_M^\dagger u^M & -\not{p}_0 \mathbb{1} & -W_-^\dagger \end{pmatrix}$$

where  $W_\pm = \frac{r}{2} (\hat{p}_0^2 \pm i \hat{p}_1^2) \rho^M u_M$ ,  $|r| = 1$ , and  $\hat{p}_\mu \equiv \frac{2}{a} \sin \frac{p_\mu a}{2}$ , leads to

$$\Gamma_{\text{LAT}}^{(1)} = \frac{V_2}{2 a^2} \int_{-\pi}^{+\pi} \frac{d^2 p}{(2\pi)^2} \ln \left[ \frac{4^8 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2})^5 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2} + \frac{M^2}{8})^2 (\sin^2 \frac{p_0}{2} + \sin^2 \frac{p_1}{2} + \frac{M^2}{4})}{(\sin^2 p_0 + \sin^2 p_1 + \frac{M^2}{4} + 4 \sin^4 \frac{p_0}{2} + 4 \sin^4 \frac{p_1}{2})^8} \right]$$

$$\xrightarrow{a \rightarrow 0} -\frac{3 \ln 2}{8\pi} V_2 m^2, \quad \text{cusp anomaly at strong coupling} \quad (|r| = 1, M = m a.)$$

Simulations, continuum limit: measurements

# Parameter space, continuum limit ( $a \rightarrow 0$ )

- ▶ Two bare parameters,  $g = \frac{\sqrt{\lambda}}{4\pi}$  and  $P^+ \sim m$ , assume the only additional scale is  $a$

$$F_{\text{LAT}} = F_{\text{LAT}}(g, M, N) \quad M = m a, \quad N = \frac{L}{a}$$

- ▶ The continuum limit must be taken along a **line of constant physics**: curve in  $\{g, M, N\}$  where **physical** quantities are kept fixed as  $a \rightarrow 0$ .

$$\begin{aligned} \text{E.g.} \quad m_x^2 &= \frac{m^2}{2} \left( 1 - \frac{1}{8g} + \mathcal{O}(g^{-2}) \right) \\ L^2 m_x^2 &= \text{const} \quad \longrightarrow \quad (L m)^2 \equiv (N M)^2 = \text{const}. \end{aligned}$$

- ▶ For a generic observable

finite lattice spacing  
( $\sim a$ ) effects

finite volume  
( $\sim m L$ ) effects

$$F_{\text{LAT}} = F_{\text{LAT}}(g, M, N) = F(g) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}\left(e^{-MN}\right)$$

Recipe: fix  $g$ , fix  $MN$  large enough, evaluate  $F_{\text{LAT}}$  for  $N = 6, 8, 10, 12, 16, \dots$ ;

Obtain  $F(g)$  extrapolating to  $N \rightarrow \infty$ .

# Measurement I: $\langle x, x^* \rangle$ correlator

From the correlator of the  $x$  fields

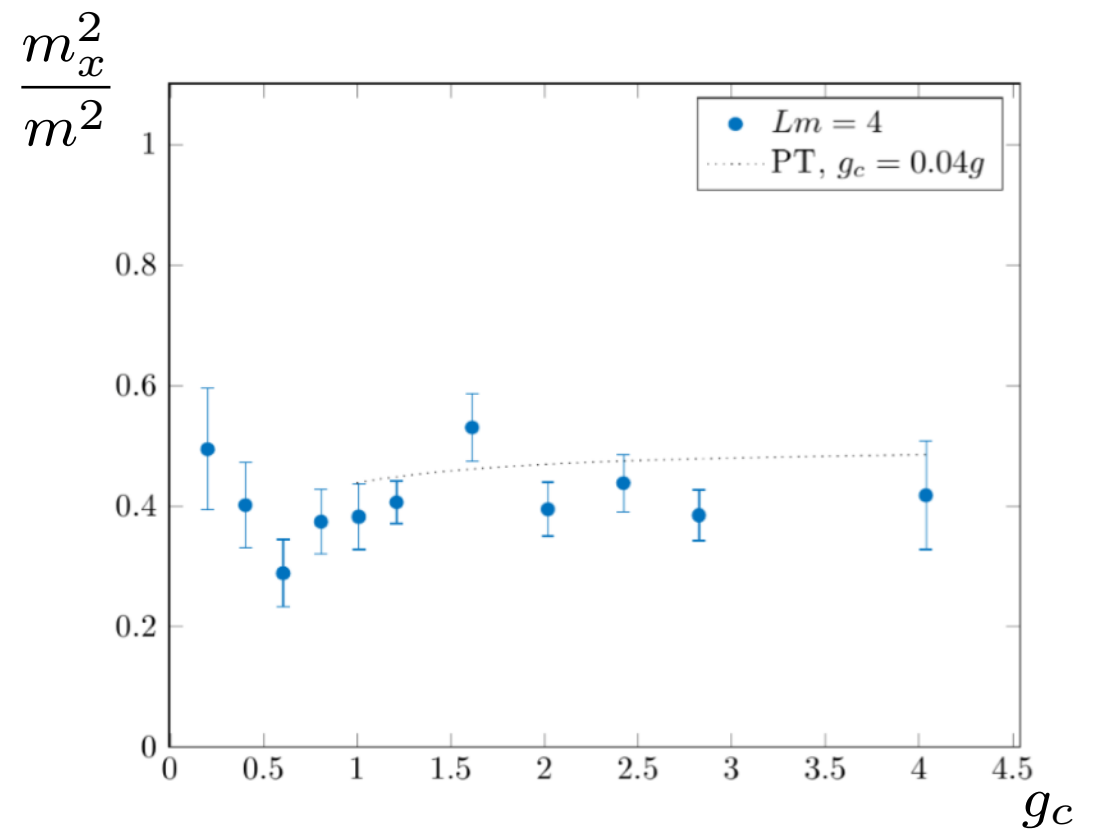
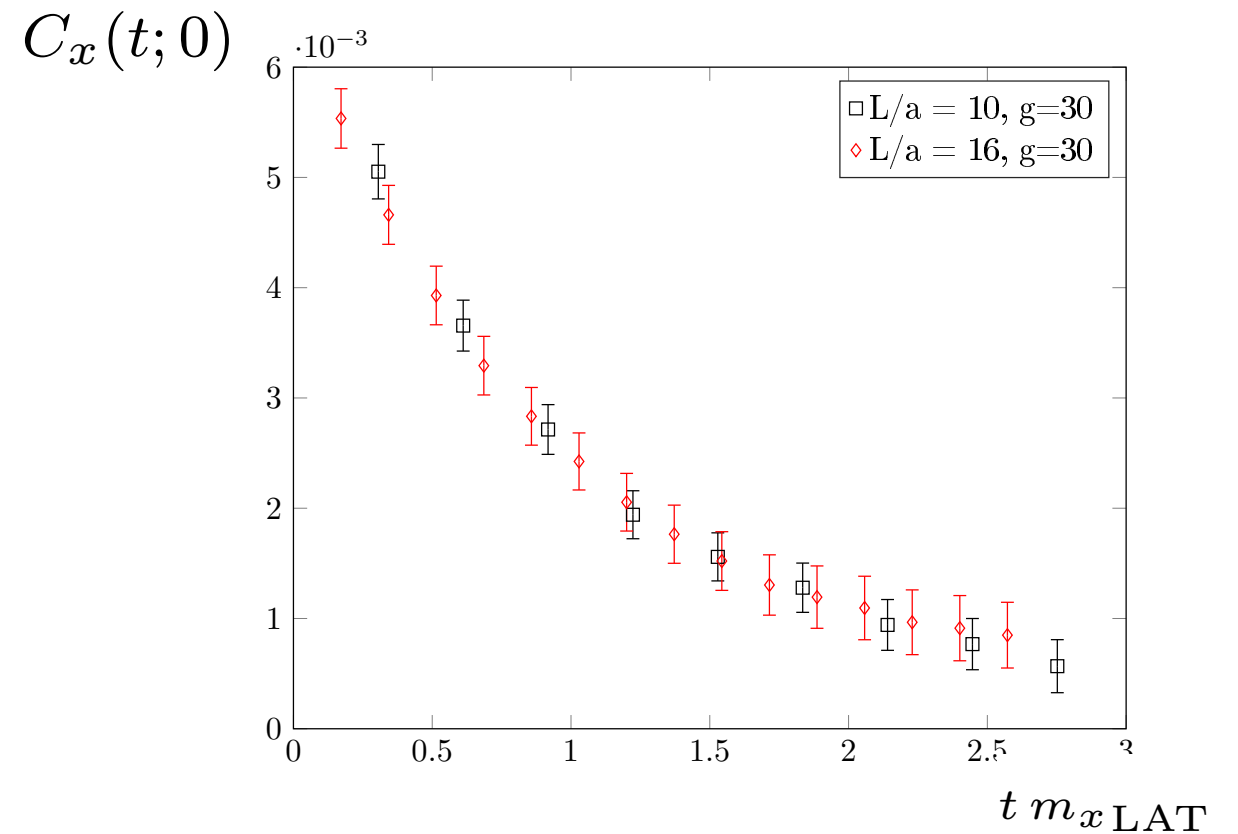
$$C_x(t; 0) = \sum_{s_1, s_2} \langle x(t, s_1) x^*(0, s_2) \rangle$$

$$t \gg 1 \quad e^{-t m_{x \text{ LAT}}}$$

extract the  $x$ -mass

$$m_{x \text{ LAT}} = \lim_{t \rightarrow \infty} m_x^{\text{eff}}$$

$$\equiv \lim_{t \rightarrow \infty} \frac{1}{a} \log \frac{C_x(t; 0)}{C_x(t + a; 0)}$$

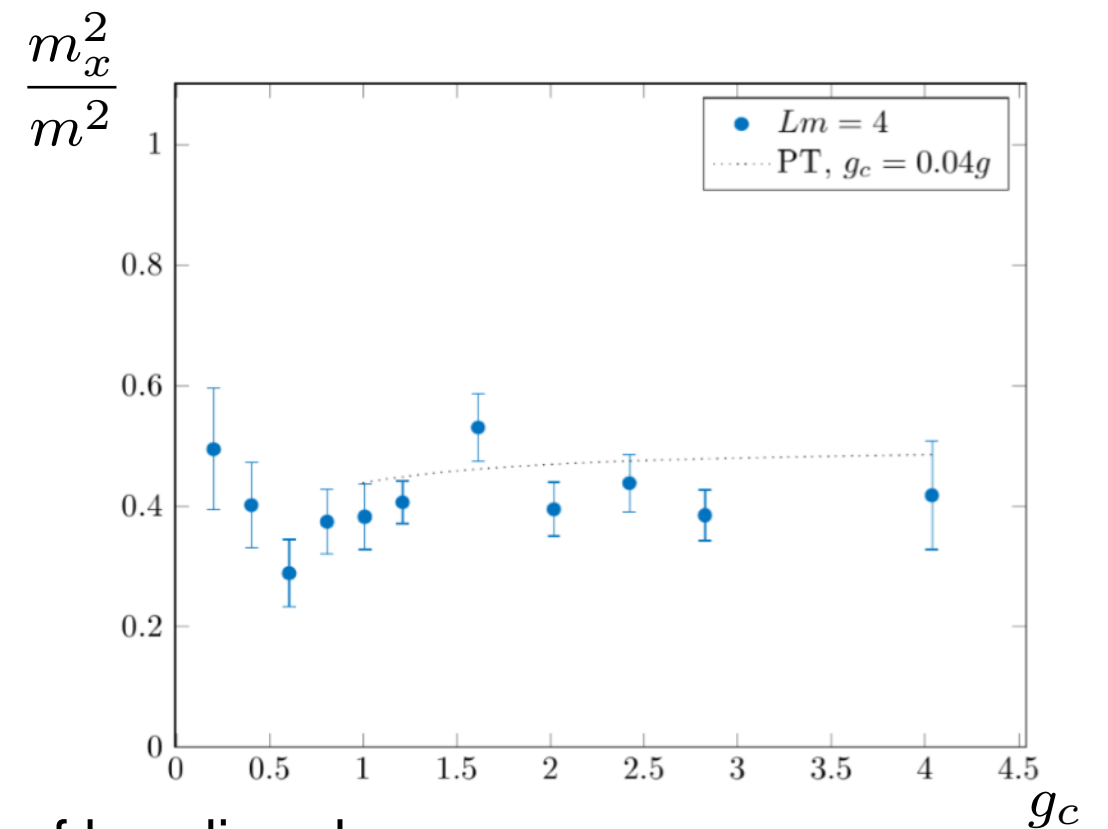
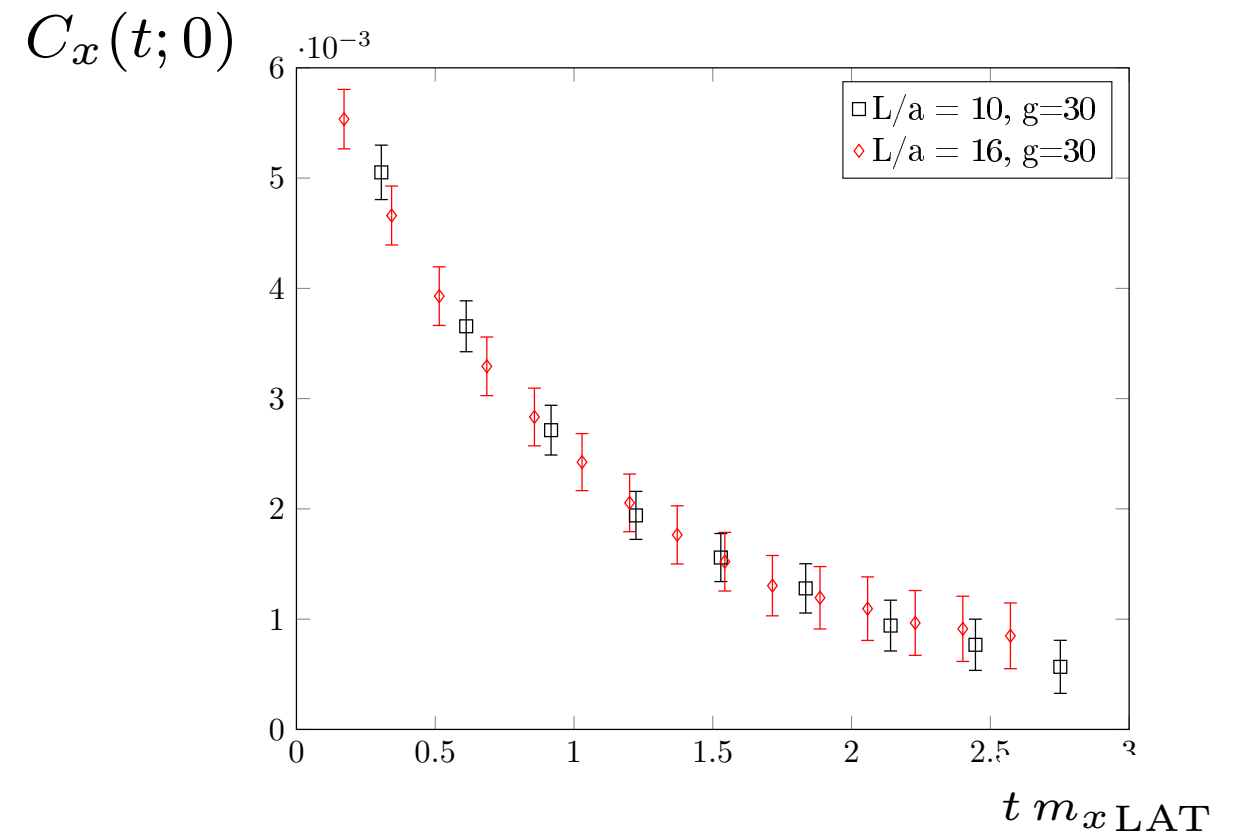
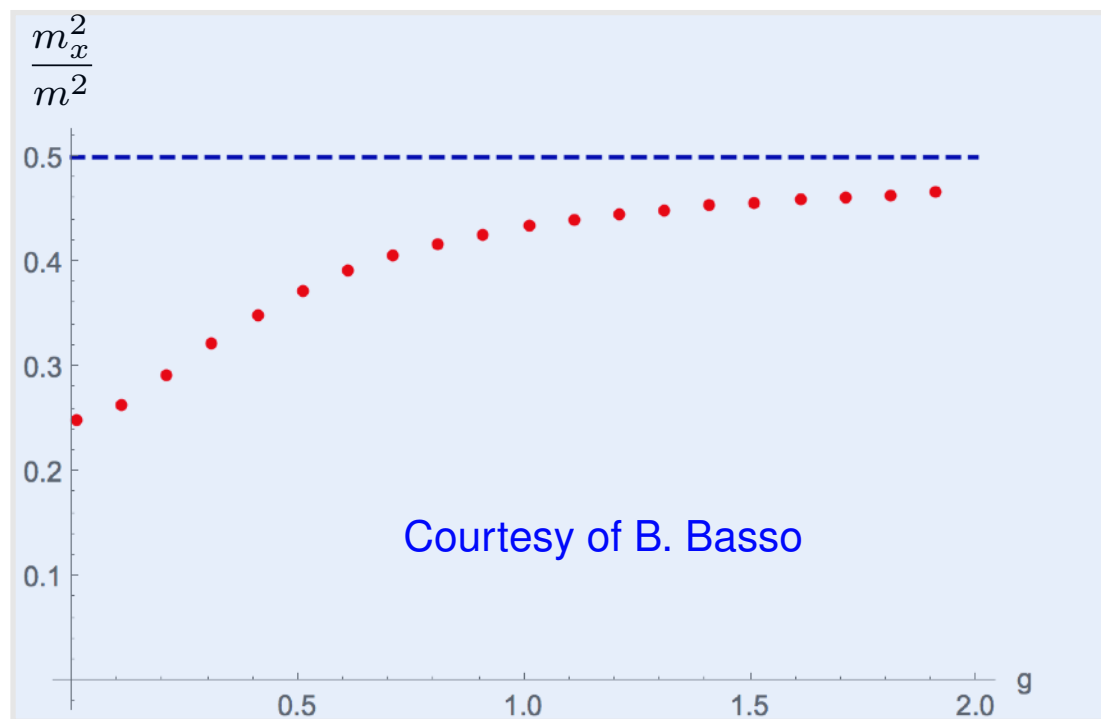


# Measurement I: $\langle x, x^* \rangle$ correlator

From the correlator of the  $x$  fields

$$C_x(t; 0) = \sum_{s_1, s_2} \langle x(t, s_1) x^*(0, s_2) \rangle$$

$$t \gg 1 \quad e^{-t m_{x \text{ LAT}}}$$



Consistent with large  $g$  prediction, no clear signal of bending down.

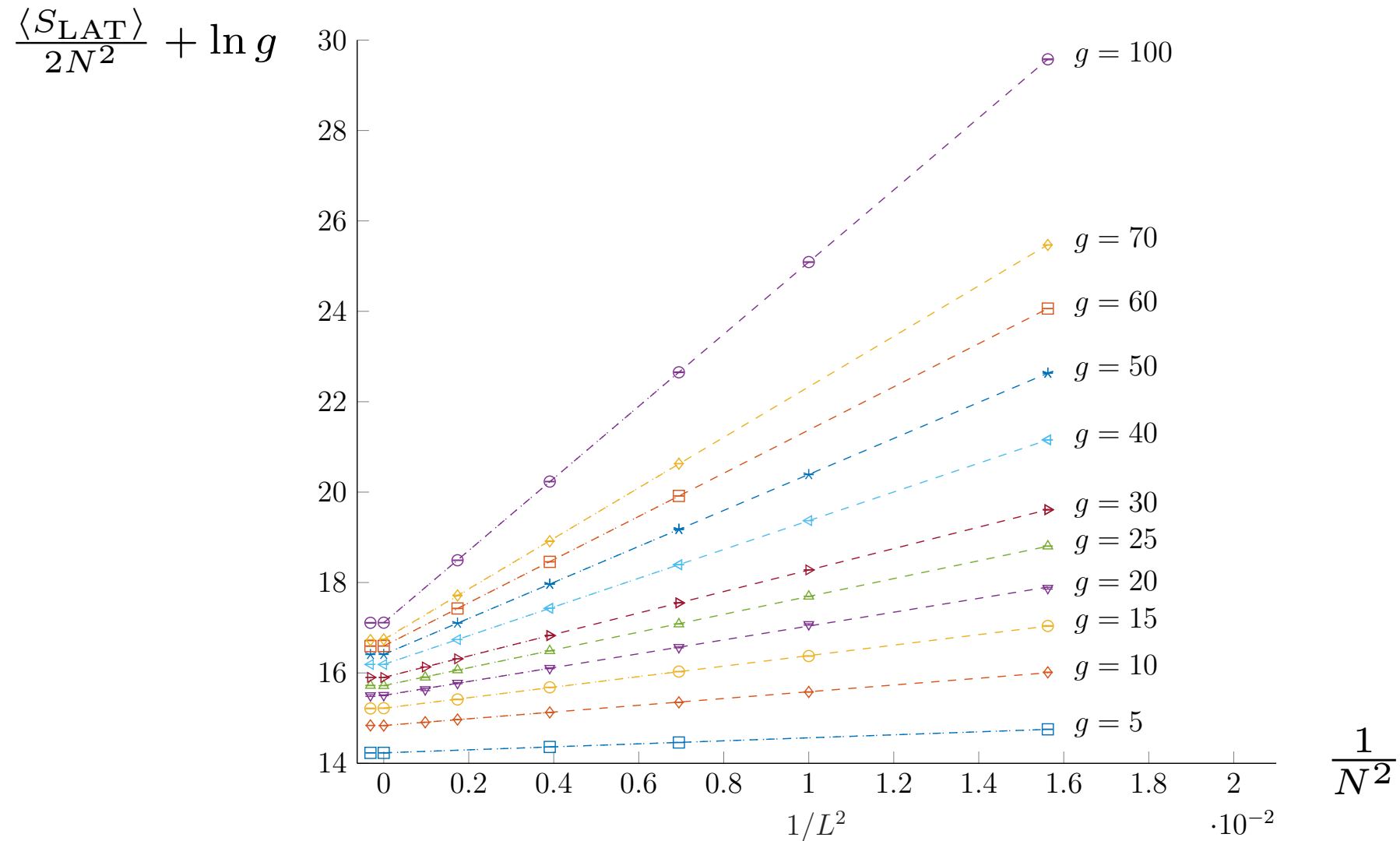
No infinite renormalization occurring.

## Measurement II: (derivative of the) cusp anomaly

We measure  $\langle S_{\text{cusp}} \rangle \equiv g \frac{V_2 m^2}{8} f'(g)$ . At large  $g$ ,

$$\langle S_{\text{LAT}} \rangle \equiv g \frac{N^2 M^2}{4} 4 + \frac{c}{2} (2N^2)$$

quadratic divergences appear, with  $c = n_{\text{bos}} = 8 + 17 = 25$ .

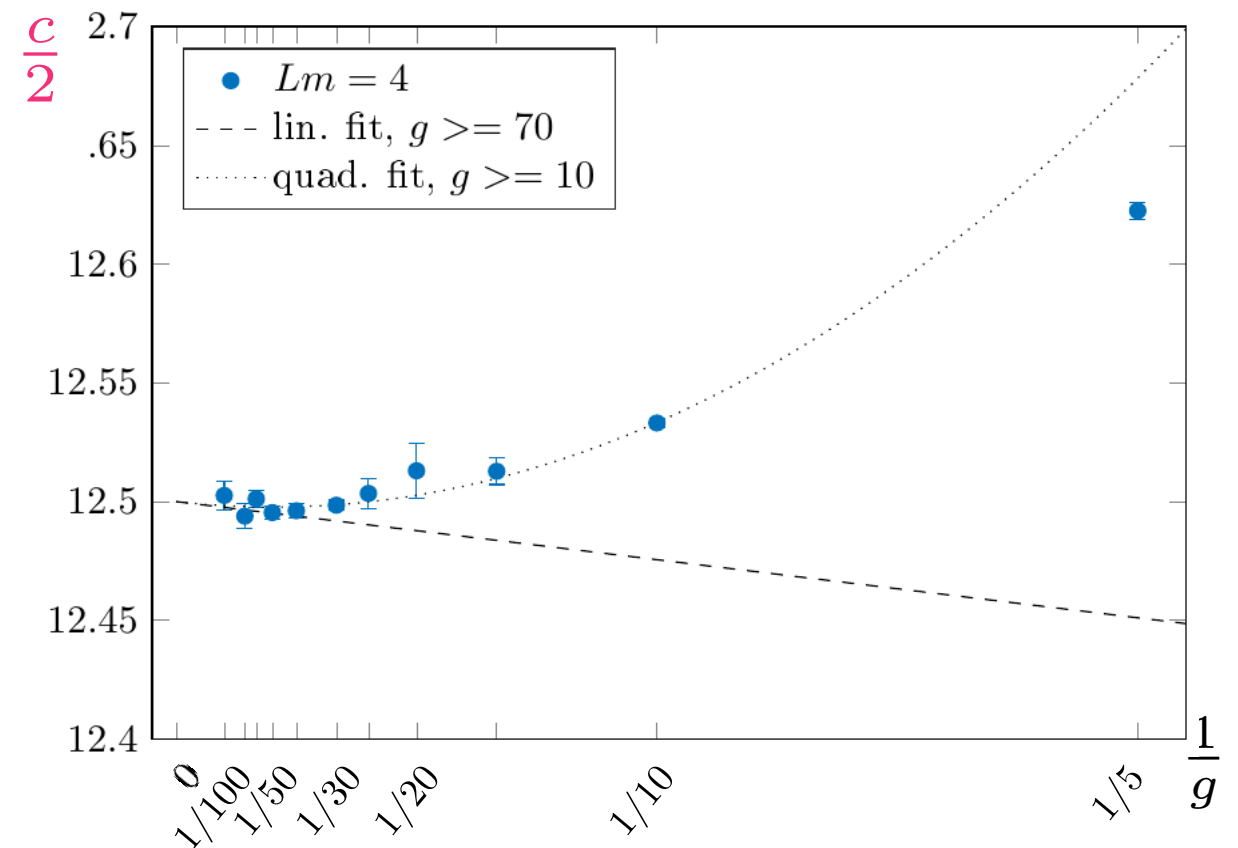
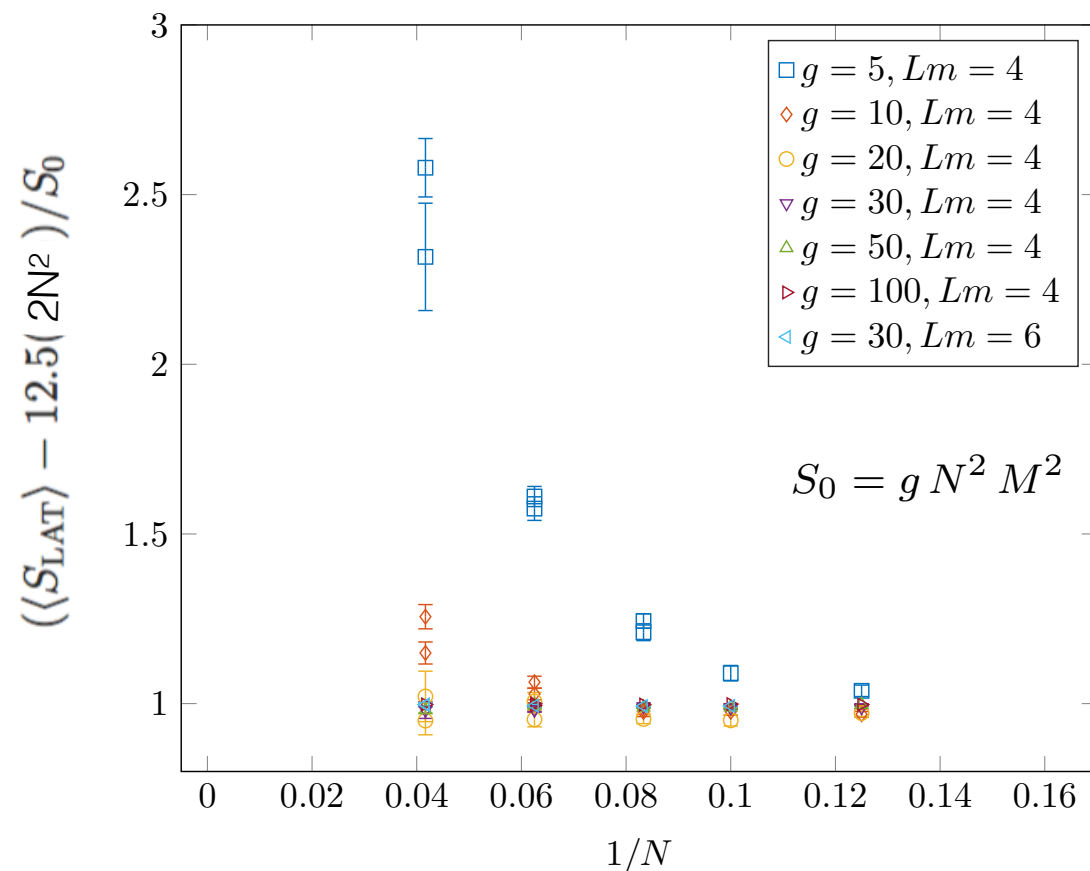


Indeed,  $\langle S \rangle = -\frac{\partial \ln Z}{\partial \ln g}$  and  $Z \sim \Pi_{n_{\text{bos}}} (\det g \mathcal{O})^{-\frac{1}{2}}$ , so for each bosonic species there is a factor  $\sim g^{-\frac{(2N^2)}{2}}$ . In lattice codes, coupling omitted from fermionic part.

# Measurement II: (derivative of the) cusp anomaly

We measure  $\langle S_{\text{cusp}} \rangle \equiv g \frac{V_2 m^2}{8} f'(g)$ . At **finite**  $g$ ,

$$\langle S_{\text{LAT}} \rangle \equiv g \frac{N^2 M^2}{4} f'_{\text{LAT}}(g) + \frac{c(g)}{2} (2N^2)$$



In **continuum**, existing power divergences are set to zero (**dim. reg.**)

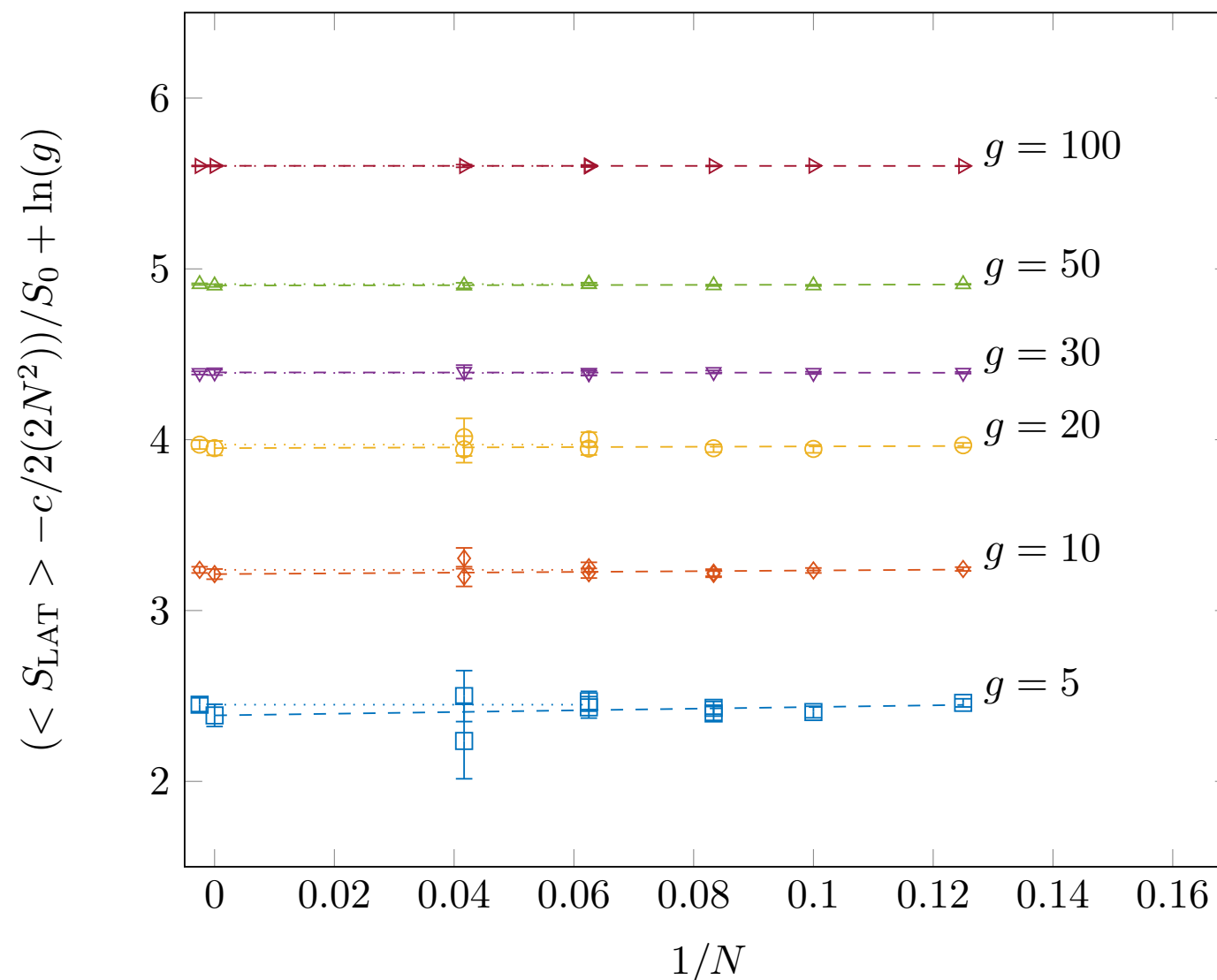
**Here**, expected mixing of the Lagrangian with lower dimension operator

$$\mathcal{O}(\phi(s))_r = \sum_{\alpha: [\mathcal{O}_\alpha] \leq D} Z_\alpha \mathcal{O}_\alpha(\phi(x)), \quad Z_\alpha \sim \Lambda^{(D - [\mathcal{O}_\alpha])} \sim a^{-(D - [\mathcal{O}_\alpha])}$$



## Measurement II: (derivative of the) cusp anomaly

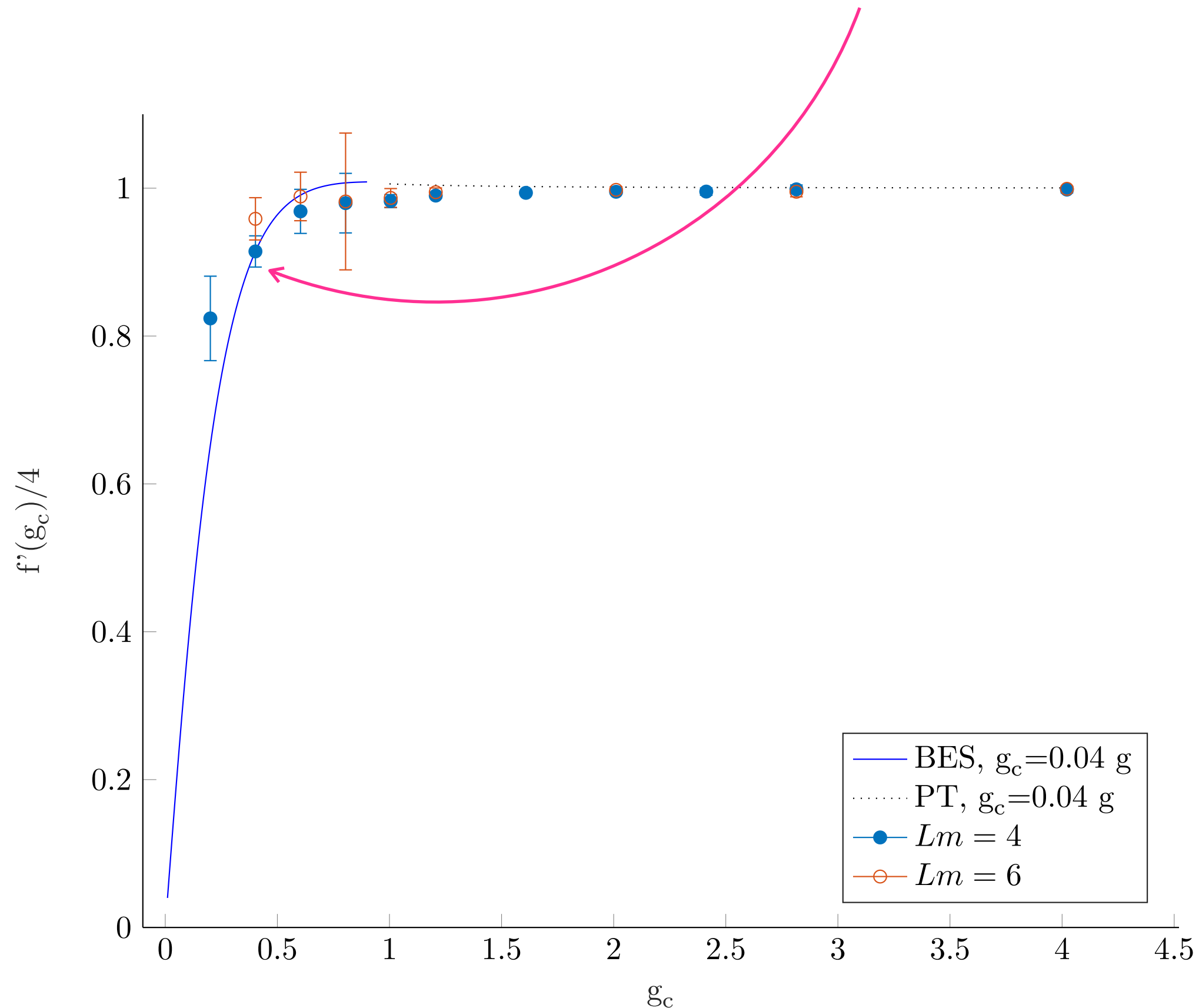
We proceed **subtracting** the continuum extrapolation of  $\frac{c}{2}$  multiplied by  $N^2$ :  
divergences appear to be completely subtracted, confirming their quadratic nature.  
Errors are small, and do not diverge for  $N \rightarrow \infty$ .  
Flatness of data points indicates very small lattice artifacts.



We can thus extrapolate at infinite  $N$  to show the continuum limit.

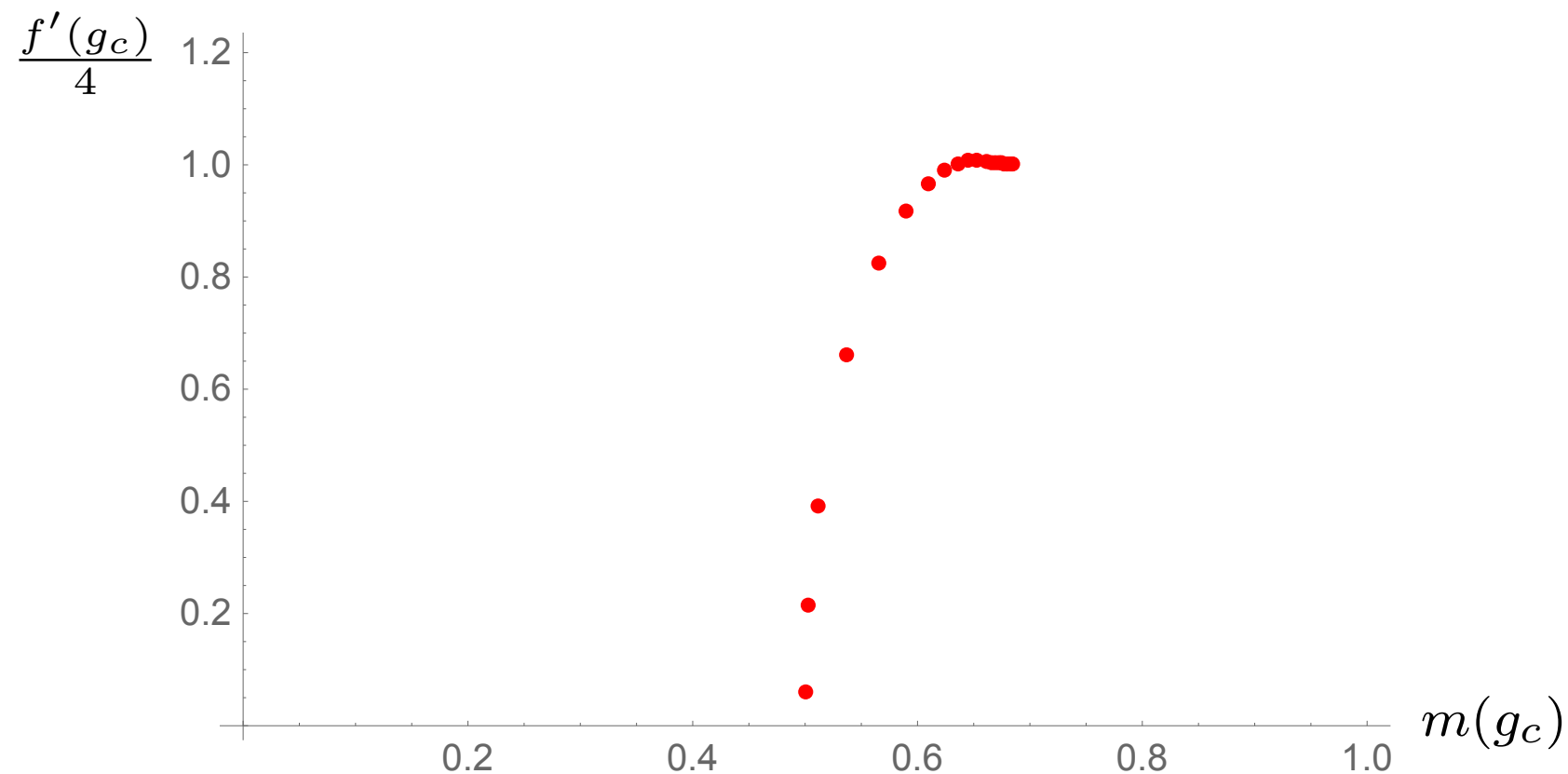
# Measurement II: (derivative of the) cusp anomaly

To compare, **assume**  $g = \alpha g_c$ : then from  $f'(g) = f'(g_c)_c$  is  $g_c = 0.04g$ .



## In progress

- The relation among  $g_c$  and  $g$  may be non-trivial. Then the cusp may be “declared” as the coupling, and e.g. mass measurements plotted against it.



- We are observing an unexpected splitting in the fermionic masses ( $m_F^2 = \frac{1}{2}$ ) related to the  $U(1)$ -breaking of the discretization. The corresponding Ward identity may be used as renormalization condition, a single tuning is expected.
- We are extending our simulations to  $g \leq 5$ .

# Conclusions

- ▶ I presented a study of **lattice field theory methods** for *gauge-fixed* string  $\sigma$ -models relevant in AdS/CFT: address **ab initio**, non-perturbative calculations within them.
  - ▶ The model – GS string on GKP vacuum – is **amenable** to study using standard techniques (Wilson-like fermion discretizations, RHMC algorithm).
  - ▶ We observe **good agreement** with expectation **at large  $g$** , and indications of **non-perturbative** physics;

Ongoing work on several open questions, which include the proper continuum limit.

2019-2022: **EU-funded ITN EuroPLEx - European network for Particle physics, Lattice field theory and Extreme computing**

(Parma, Berlin, Bielefeld, Dublin, Edinburgh, Madrid, Odense, Regensburg, Swansea)

- ▶ Future: different backgrounds/gauge-fixing/observables . . .
- ▶ Non-perturbative definition of string theory?  
For sure, suitable framework for first principle statements (proofs of AdS/CFT) and (potentially) very efficient tool in numerical holography.

Thanks for your attention.

Extra-slides

# Numerical setup

- > Rational Hybrid Mont-Carlo
- > Conjugate Gradient multi-shift solver
- > Fortran and Matlab code

The rational approximation for the inverse fractional power is of degree 15 (accuracy is always better than  $10^{-3}$ ).

$g$	$T/a \times L/a$	$Lm$	$am$	$\tau_{\text{int}}^S$	$\tau_{\text{int}}^{m_x}$	statistics [MDU]
5	$16 \times 8$	4	0.50000	0.8	2.2	900
	$20 \times 10$	4	0.40000	0.9	2.6	900
	$24 \times 12$	4	0.33333	0.7	4.6	900,1000
	$32 \times 16$	4	0.25000	0.7	4.4	850,1000
	$48 \times 24$	4	0.16667	1.1	3.0	92,265
10	$16 \times 8$	4	0.50000	0.9	2.1	1000
	$20 \times 10$	4	0.40000	0.9	2.1	1000
	$24 \times 12$	4	0.33333	1.0	2.5	1000,1000
	$32 \times 16$	4	0.25000	1.0	2.7	900,1000
	$48 \times 24$	4	0.16667	1.1	3.9	594,564
20	$16 \times 8$	4	0.50000	5.4	1.9	1000
	$20 \times 10$	4	0.40000	9.9	1.8	1000
	$24 \times 12$	4	0.33333	4.4	2.0	850
	$32 \times 16$	4	0.25000	7.4	2.3	850,1000
	$48 \times 24$	4	0.16667	8.4	3.6	264,580
30	$20 \times 10$	6	0.60000	1.3	2.9	950
	$24 \times 12$	6	0.50000	1.3	2.4	950
	$32 \times 16$	6	0.37500	1.7	2.3	975
	$48 \times 24$	6	0.25000	1.5	2.3	533,652
	$16 \times 8$	4	0.50000	1.4	1.9	1000
	$20 \times 10$	4	0.40000	1.2	2.7	950
	$24 \times 12$	4	0.33333	1.2	2.1	900
	$32 \times 16$	4	0.25000	1.3	1.8	900,1000
	$48 \times 24$	4	0.16667	1.3	4.3	150
50	$16 \times 8$	4	0.50000	1.1	1.8	1000
	$20 \times 10$	4	0.40000	1.2	1.8	1000
	$24 \times 12$	4	0.33333	0.8	2.0	1000
	$32 \times 16$	4	0.25000	1.3	2.0	900,1000
	$48 \times 24$	4	0.16667	1.2	2.3	412
100	$16 \times 8$	4	0.50000	1.4	2.7	1000
	$20 \times 10$	4	0.40000	1.4	4.2	1000
	$24 \times 12$	4	0.33333	1.3	1.8	1000
	$32 \times 16$	4	0.25000	1.3	2.0	950,1000
	$48 \times 24$	4	0.16667	1.4	2.4	541

Table 1: Parameters of the simulations: the coupling  $g$ , the temporal ( $T$ ) and spatial ( $L$ ) extent of the lattice in units of the lattice spacing  $a$ , the line of constant physics fixed by  $Lm$  and the mass parameter  $M = am$ . The size of the statistics after thermalization is given in the last column in terms of Molecular Dynamic Units (MDU), which equals an HMC trajectory of length one. In the case of multiple replica the statistics for each replica is given separately. The auto-correlation times  $\tau$  of our main observables  $m_x$  and  $S$  are also given in the same units.

# Boundary conditions

Fluctuations must vanish at the AdS boundary (two sides of the grid)

$$\tilde{X}(t = -\infty, s) = 0 = \tilde{X}(t, s = +\infty)$$

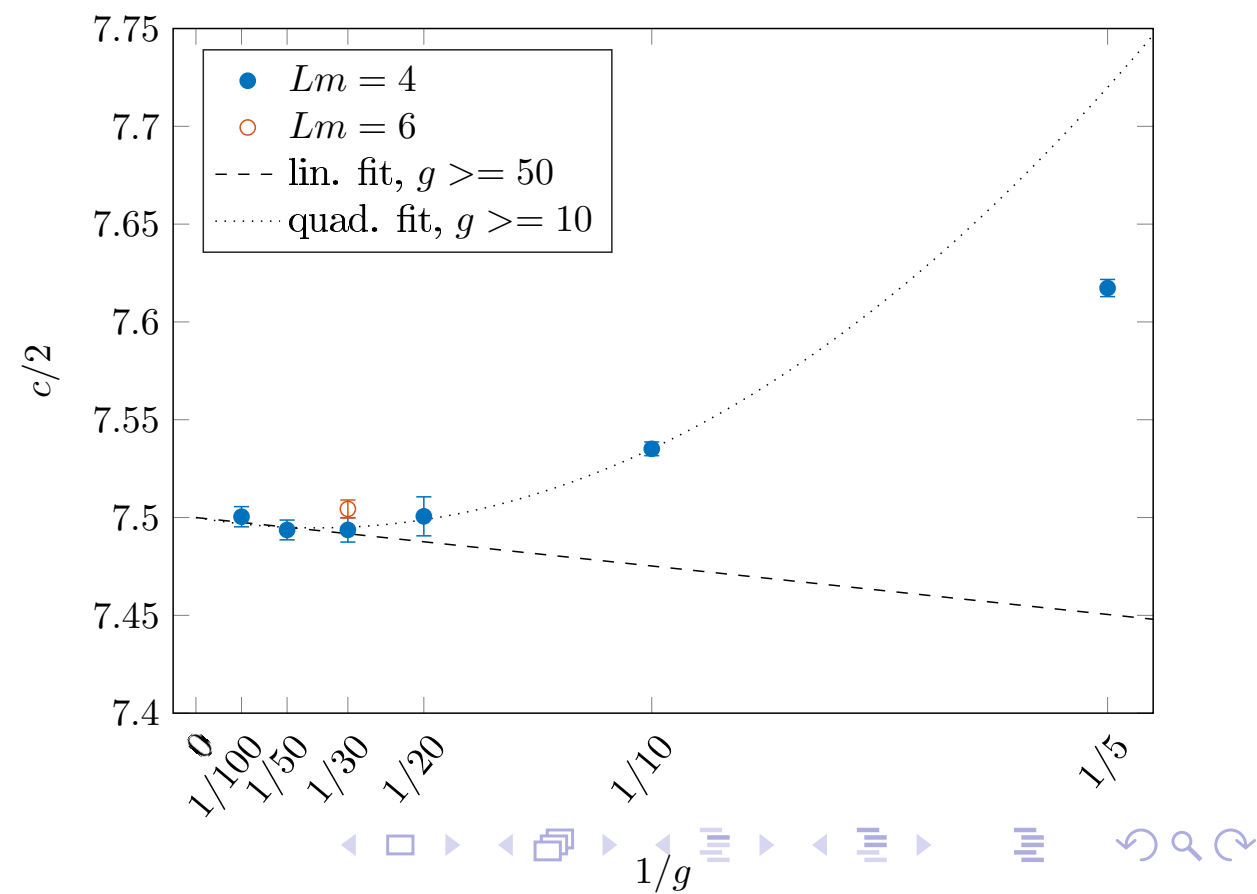
and be free to fluctuate elsewhere. Field redefinitions adopted in the continuum lead to exotic (unstable) boundary conditions.

So far we used **periodic** BC for all the fields (antiperiodic temporal BC for fermions).

and evaluated finite volume effects  $\sim e^{-m L} \equiv e^{-M N}$ .

Most run are done at  $M N = 4$  ( $e^{-4} \simeq 0.02$ ),  
some at  $M N = 6$  ( $e^{-6} \simeq 0.002$ ).

Appear to play a role only in evaluating  
the coefficient of divergences.





## A remark on numerics

The most difficult part of the algorithm is the **inversion** of the fermionic matrix

$$|\text{Pf } O_F| \equiv (\det O_F^\dagger O_F)^{\frac{1}{4}} \equiv \int d\zeta d\bar{\zeta} e^{-\int d^2\xi \bar{\zeta} (O_F^\dagger O_F)^{-\frac{1}{4}} \zeta}.$$

The RHMC (Rational Hybrid Montecarlo) uses a rational approximation

$$\bar{\zeta} (O_F^\dagger O_F)^{-\frac{1}{4}} \zeta = \alpha_0 \bar{\zeta} \zeta + \sum_{i=1}^P \bar{\zeta} \frac{\alpha_i}{O_F^\dagger O_F + \beta_i} \zeta$$

with  $\alpha_i$  and  $\beta_i$  tuned by the range of eigenvalues of  $O_F$ .

Defining  $s_i \equiv \frac{1}{O_F^\dagger O_F + \beta_i} \zeta$ , one solves

$$(O_F^\dagger O_F + \beta_i) s_i = \zeta, \quad i = 1, \dots, P.$$

with a (multi-shift conjugate) solver for which

$$\text{number of iterations} \sim \lambda_{\min}^{-1}$$

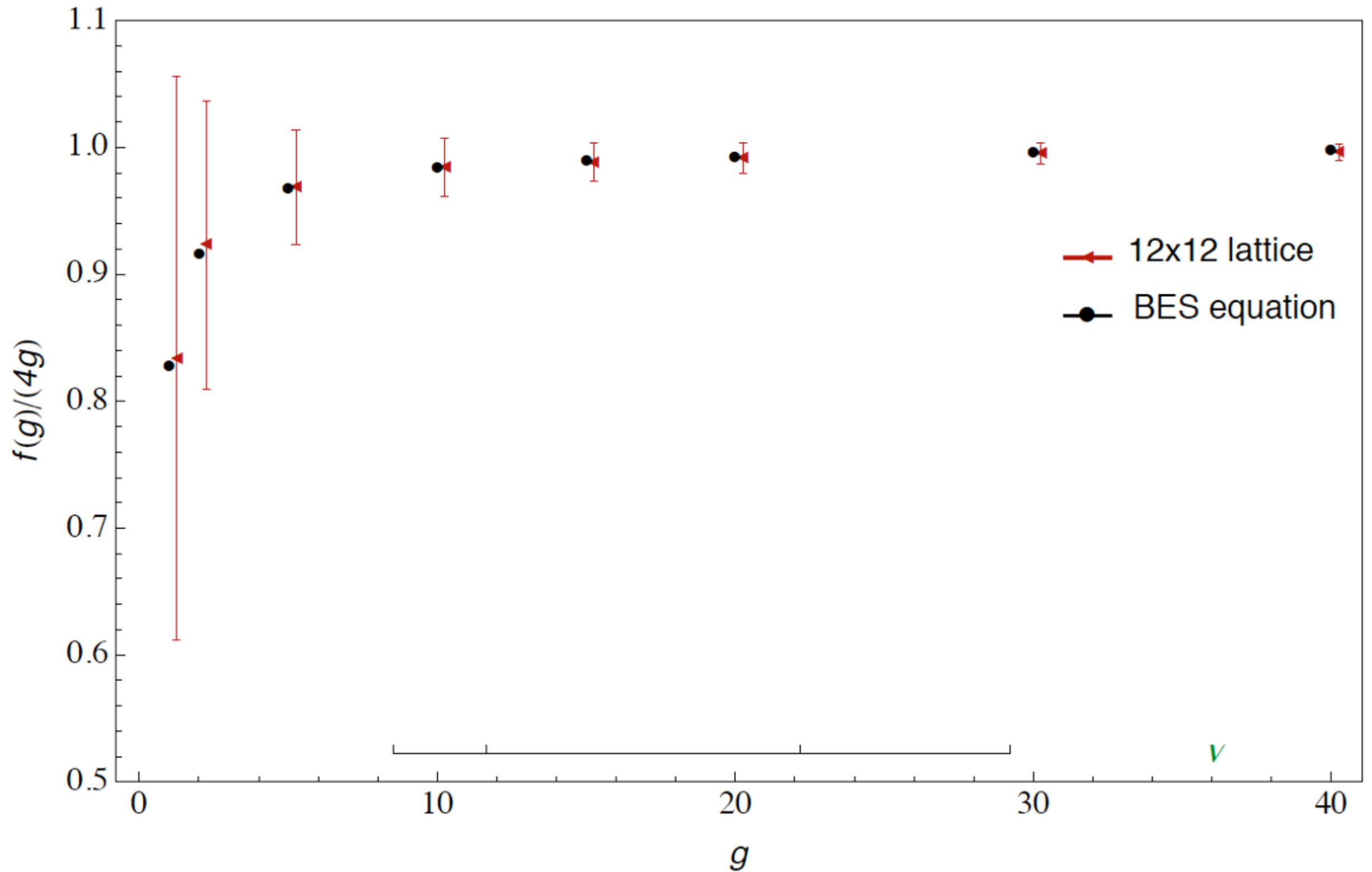
In our case the spectrum of  $O_F$  has **very small eigenvalues**.

And:

$$O_F = \begin{pmatrix} i\partial_t & \\ i\frac{z^M}{z^3} \rho^M (\partial_s - \frac{m}{2}) & \end{pmatrix}$$

## Previous study

[McKeown Roiban, arXiv: 1308.4875]



## Remark

Completely solved via integrability.

- ▶ In general, **no quest** here for **integrability-preserving discretization**.  
We use an integrable model for establishing a **benchmark** of the method,  
(we'll actually break manifest symmetries, let alone hidden ones!)  
the integrability prediction as final **check** for standard lattice field theory methods.
- ▶ That the dispersion relation for string world-sheet excitations on a BMN vacuum

$$\epsilon^2 = 1 + 16 g^2 \sin^2 \left( \frac{p}{4g} \right)$$

is **lattice-like** plays no role here.