Calculating Entanglement entropy on the lattice

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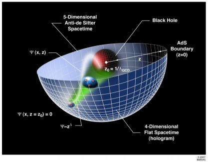
Universität Regensburg

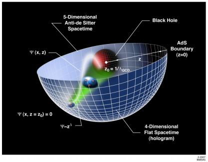


[Brodsky '08]

Weak form of AdS/CFT

Classical SUGRA, $g_s \to 0,\ \frac{\alpha'}{L^2} \to 0$	$\leftrightarrow$	$\mathcal{N} = 4$ YM $N_c \to \infty,\ \lambda$ large

Motivation



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- QCD is not SUSY, has $N_c = 3$, λ finite, ...

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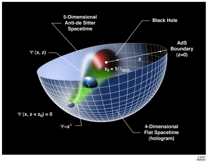

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$\rightarrow$ Calculate corrections [This workshop: Waeber]

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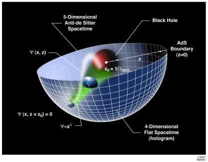

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Weak form of AdS/CFT

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- QCD is not SUSY, has $N_c = 3$, λ finite, ...

$\rightarrow$ Calculate corrections [This workshop: Waeber]

$\rightarrow$ Calculate observable in both theories and compare, e.g. Entanglement Entropy

1. Motivation
2. Entanglement entropy
3. Holographic Entanglement Entropy
4. Entanglement entropy on the lattice

Outline

1. Motivation
2. Entanglement entropy
3. Holographic Entanglement Entropy
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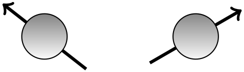
- 2-Qubit-system

$$\mathcal{H} = \mathcal{H}_{\text{qubit}} \otimes \mathcal{H}_{\text{qubit}} = \text{span}\left\{ |\uparrow\uparrow\rangle\,, |\uparrow\downarrow\rangle\,, |\downarrow\uparrow\rangle\,, |\downarrow\downarrow\rangle \right\}$$



Introductory example

- 2-Qubit-system

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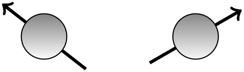
- Basis states are clearly separable



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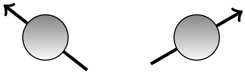
- Basis states are clearly separable
- But EPR/Bell/cat state

$$\frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right)$$

not separable



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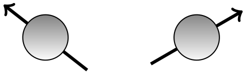
- Basis states are clearly separable $\rightarrow$ not entangled
- But EPR/Bell/cat state

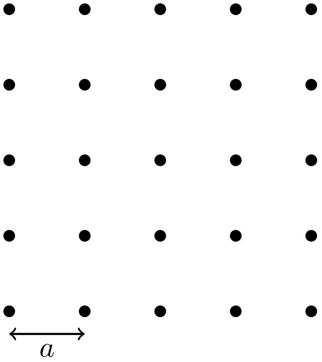
$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

not separable $\rightarrow$ entangled



Entanglement entropy on a lattice

- Lattice system with $\mathcal{H} = \otimes_\alpha \mathcal{H}_\alpha$



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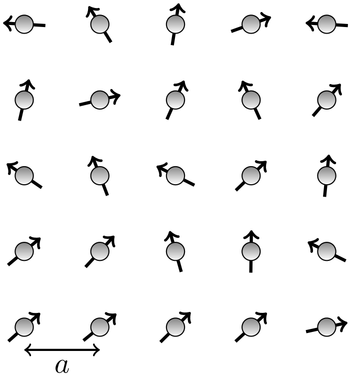

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- Lattice system with $\mathcal{H} = \otimes_\alpha \mathcal{H}_\alpha$
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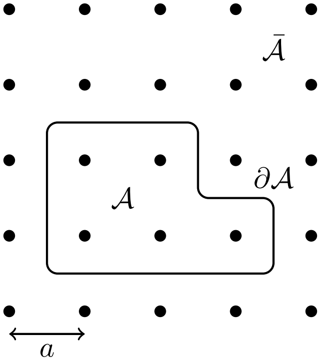

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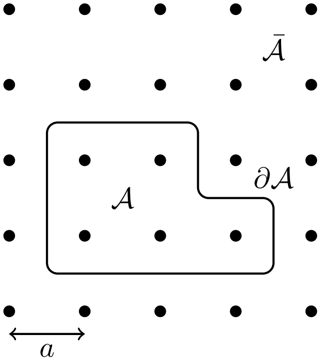

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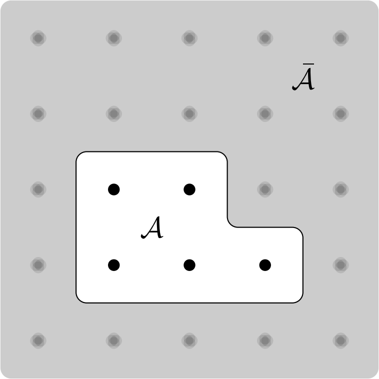
$$\rho_\mathcal{A} = \operatorname{tr}_{\bar{\mathcal{A}}} \rho$$



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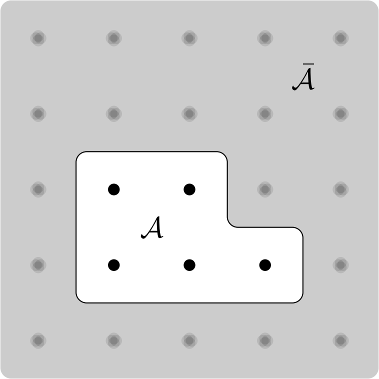
- Quantification of entanglement

$$S_{\mathsf{EE}} = -\,\mathrm{tr}_\mathcal{A} \left(\rho_\mathcal{A} \log(\rho_\mathcal{A})\right)$$



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$$S^{(q)} = \frac{1}{q-1}\log\left(\operatorname{tr}_{\mathcal{A}}\left(\rho_{\mathcal{A}}^{q}\right)\right) \qquad q \in \mathbb{Z}_{+}$$

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Analytic continuation $q \in \mathbb{R}_{+}$

$$\lim_{q \to 1} S^{(q)} = S_{\mathsf{EE}}$$

2-Qubit-system

- Non-entangled state

$$|\psi\rangle = |\uparrow\uparrow\rangle$$

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\begin{aligned}
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\rho_{\mathcal{A}} &= {}_{\bar{\mathcal{A}}}\langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle_{\bar{\mathcal{A}}} + {}_{\bar{\mathcal{A}}}\langle\downarrow|\psi\rangle\langle\psi|\downarrow\rangle_{\bar{\mathcal{A}}} \\
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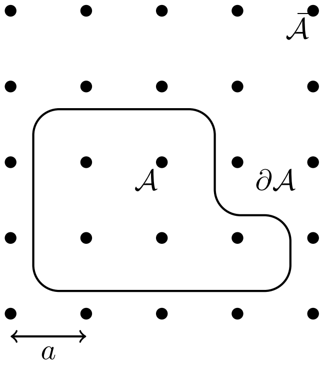
|\psi\rangle &= \frac{1}{\sqrt{2}}\Big(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \Big) \\
\rho_A &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\
S_A &= \log 2
\end{aligned}
$$

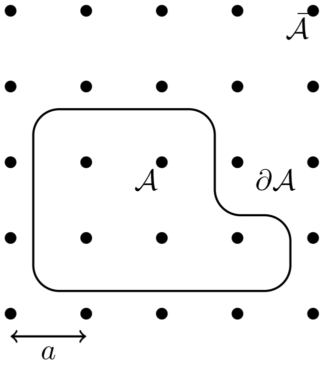

$\bar{\mathcal{A}}$
$\mathcal{A}$
$\partial\mathcal{A}$
a

- Lattice spacing:

$$a \to 0$$

Hilbertspace:

discrete $\to$ non-compact



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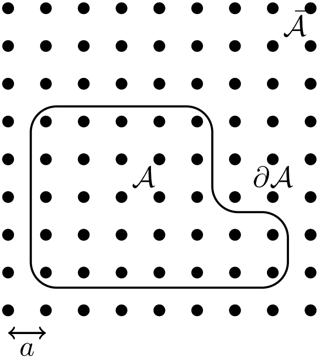

Entanglement Entropy in continuum QFT

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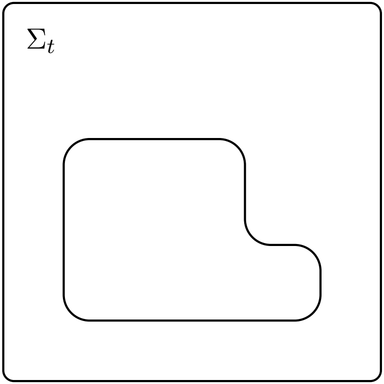

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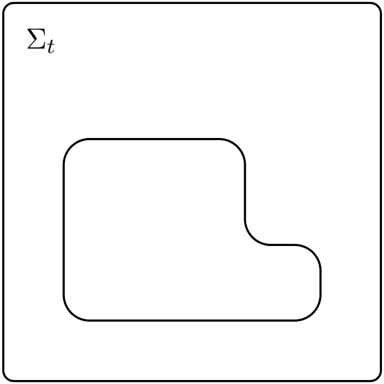

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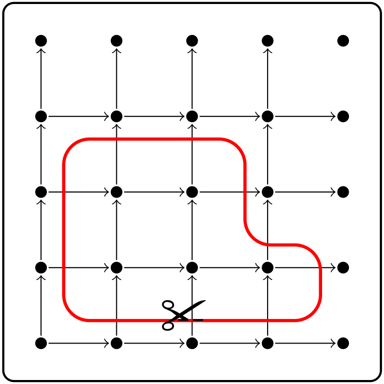

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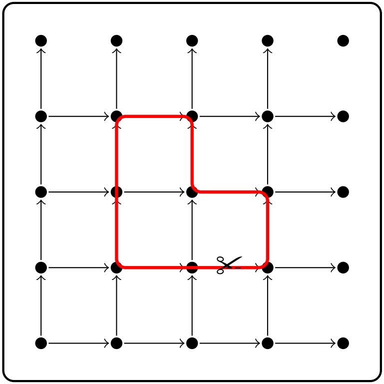

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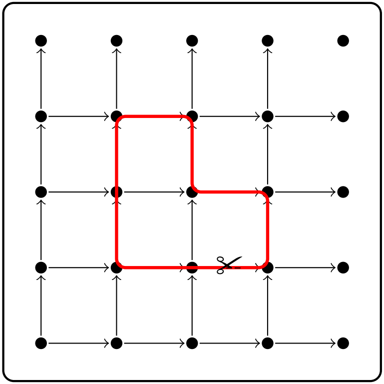
- UV limit $a \to 0$

$$S_{\text{EE}} = \frac{\text{Area}\,(\partial\mathcal{A})}{a_{UV}^{d-2}} + \dots$$



Relevance of entanglement entropy

- Quantum information theory (related entanglement measures)
- Universal order parameter for quantum phase transitions
- Holography: simple prescription via minimal surfaces
- Black hole entropy conjectured to be EE across horizon
- Experimentally measured [Islam, Ma, Preiss, Tai, Lukin, Rispoli, Greiner '15]
- ...

Outline

1 Motivation

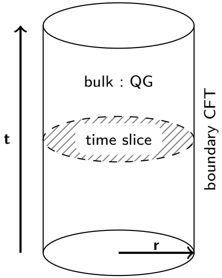
2 Entanglement entropy

3 Holographic Entanglement Entropy

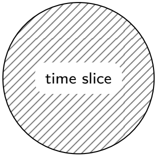
4 Entanglement entropy on the lattice

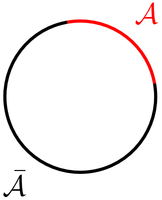
Holography

Quantum Gravity on $AdS_{d+1} \times \mathcal{M} \Longleftrightarrow$ CFT on $\mathbb{R} \times \mathbb{S}^{d-1}$



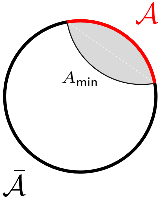
- Several examples
- Strong evidence in classical limit
- Some evidence beyond


time slice


$\mathcal{A}$
$\bar{\mathcal{A}}$

Ryu-Takayanagi formula:

$$S_{\mathsf{EE}} = \frac{A_{\mathsf{min}}}{4G\hbar}$$ large N_c and large 't Hooft coupling λ



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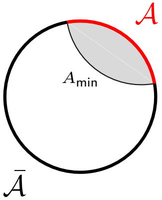
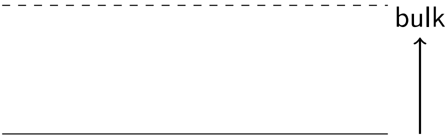


- Passes many test [Ryu, Takayanagi '06, . . .]
- Derived from existing AdS/CFT dictionary [Lewkowycz, Maldacena '13]
- Generalises to Rényi entropies [Dong '16]

Ryu-Takayanagi using a confining background [Klebanov, Kutasov, Murugan '07]



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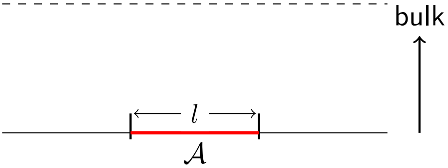

HEE in confining background

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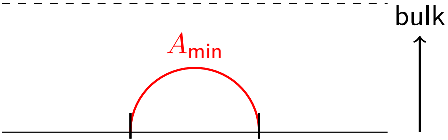

$l \ll l_{\text{crit.}}$

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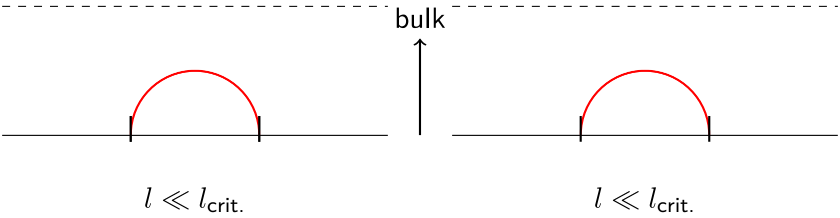

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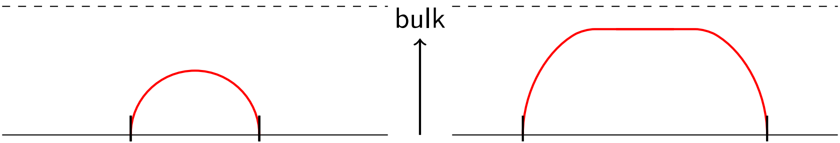

$l \ll l_{\text{crit.}}$ $l \approx l_{\text{crit.}}$

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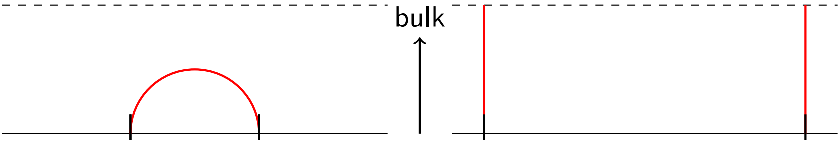

$l \ll l_{\text{crit.}}$ $l \gg l_{\text{crit.}}$

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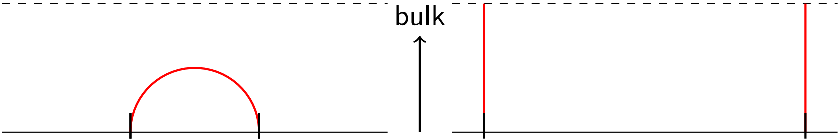

$l \ll l_{\text{crit.}}$ $\quad$ $l \gg l_{\text{crit.}}$

Holographic Entanglement Entropy

- $l \gg l_{\text{crit.}} \rightarrow$ color-neutral states $\rightarrow S_{\text{EE}} \sim N_c^0$, $\partial_l S_{\text{EE}} = 0$
- $l \ll l_{\text{crit.}} \rightarrow$ free gluons $\rightarrow S_{\text{EE}} \sim N_c^2$

Entropic C-function

- UV divergence for $d = 4$

$$S_{\mathsf{EE}}(l) = \frac{\mathsf{Area}\,(\partial\mathcal{A})}{a_{UV}^2} + \frac{C}{l^2} + \ldots$$

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- UV finite observable [Nishioka, Takayanagi '06]

$$C(l) = \frac{l^3}{|\partial\mathcal{A}|}\frac{\partial}{\partial l}S_{\text{EE}}(l)$$

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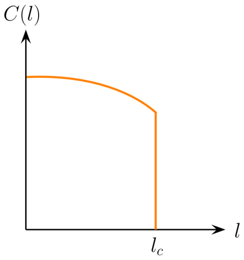
$$S_{\mathsf{EE}}(l) = \frac{\text{Area}\,(\partial\mathcal{A})}{a_{UV}^2} + \frac{C}{l^2} + \ldots$$

- UV finite observable [Nishioka, Takayanagi '06]

$$C(l) = \frac{l^3}{|\partial\mathcal{A}|}\frac{\partial}{\partial l} S_{\mathsf{EE}}(l)$$

- Dimensionless
- Decreases for increasing l [Zamolodchikov '86]
- AdS/CFT: $C(l)$ jumps at $l = l_{\text{crit.}}$

[Klebanov, Kutasov, Murugan '07]



[Nishioka, Ryu, Takayanagi '09]

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Lattice results

- Relation of EE to free energies / path integral
  [Calabrese, Cardy, '04, '05]
- Prescription for free energy measurements
  [Fodor '07; Endrödi, Fodor, Katz, Szabo '07]
- Pure SU(2) Yang-Mills [Buividovich, Polikarpov '08]
- Pure SU(3) Yang-Mills [Itou, Nagata, Nakagawa, Nakamura, Zakharov '15]

- Relation of EE to free energies / path integral
  [Calabrese, Cardy, '04, '05]
- Prescription for free energy measurements
  [Fodor '07; Endrödi, Fodor, Katz, Szabo '07]
- Pure SU(2) Yang-Mills [Buividovich, Polikarpov '08]
- Pure SU(3) Yang-Mills [Itou, Nagata, Nakagawa, Nakamura, Zakharov '15]
- Pure SU(3) Yang-Mills (more statistics) [This work]
- Pure SU(4) Yang-Mills [This work]

Entanglement entropy

$$S_{\mathsf{EE}} = \lim_{q\to 1} S^{(q)} = \lim_{q\to 1} \frac{1}{q-1} \log\left(\mathrm{tr}_A\left(\rho_A^q\right)\right) \approx S^{(2)}$$

Calculation of Entanglement entropy on the lattice

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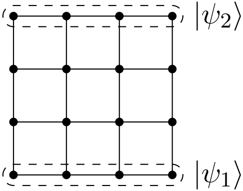
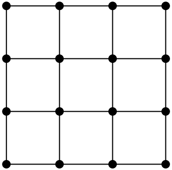
Density matrix $\hat{\rho}$

$$\langle\psi_1|\,\hat{\rho}\,|\psi_2\rangle = \frac{1}{Z}\langle\psi_1|\,e^{-\beta\hat{H}}\,|\psi_2\rangle$$





Reduced density matrix $\hat{\rho}_A$

$$\langle\psi_1|\,\mathrm{tr}_{\bar{\mathcal{A}}}\hat{\rho}\,|\psi_2\rangle = \frac{1}{Z}\,\langle\psi_1|\,\mathrm{tr}_{\bar{\mathcal{A}}}e^{-\beta\hat{H}}\,|\psi_2\rangle$$

Reduced density matrix on the lattice

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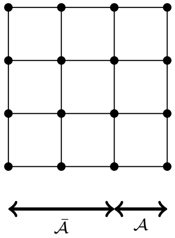

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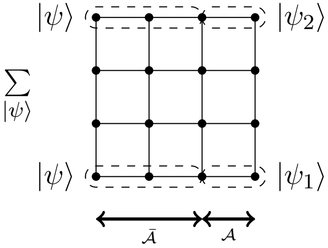

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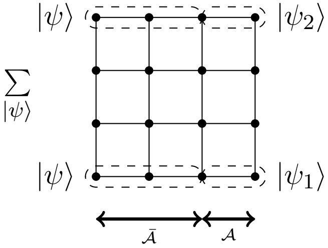

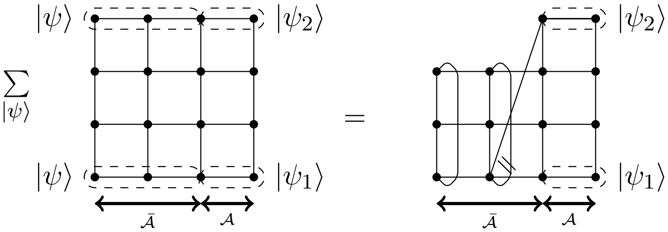

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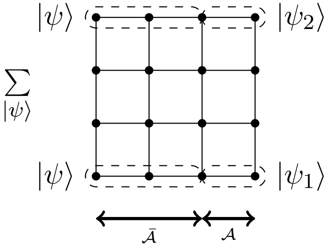

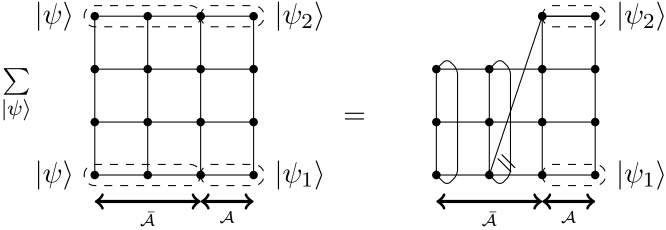

$\boxed{\text{partial trace} = \text{lattice deformation}}$

Powers of the reduced density matrix

$$\langle\psi_1|\,(\mathrm{tr}_{\bar{\mathcal{A}}}\hat{\rho})^2\,|\psi_2\rangle$$

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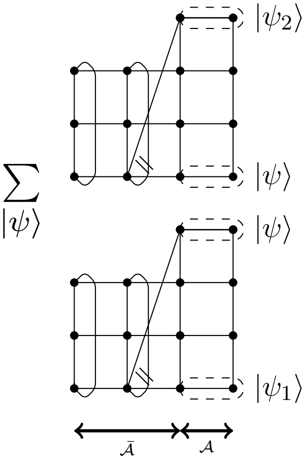

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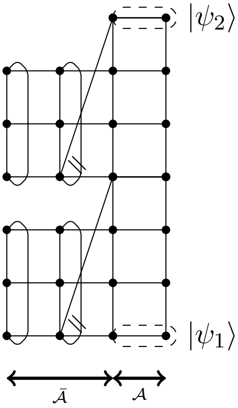

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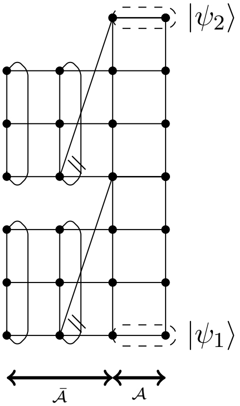

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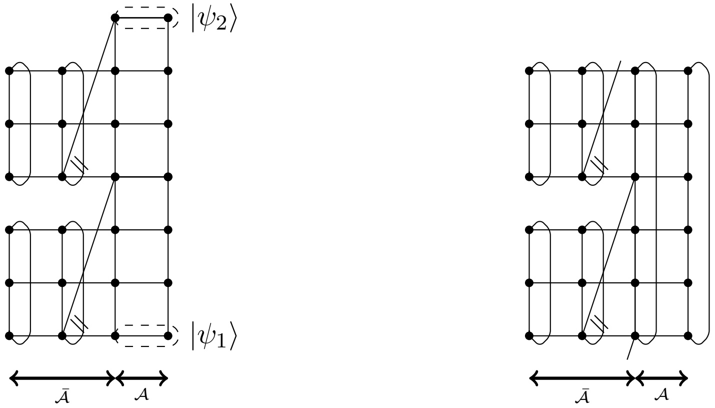
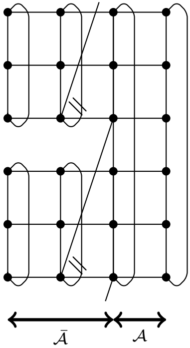


Generalization to arbitrary powers

$$\mathrm{tr}_{\mathcal{A}}\left(\hat{\rho}_{\mathcal{A}}^{q}\right) = \frac{Z[\mathcal{A},q,T]}{Z^{q}[T]}$$



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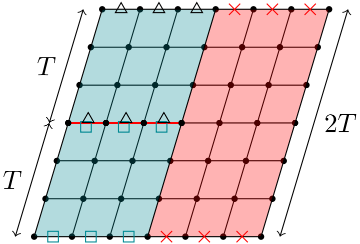

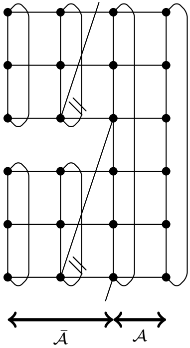

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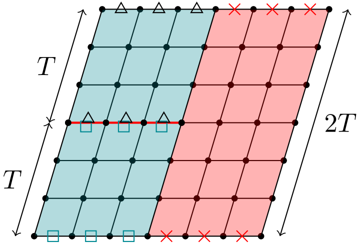

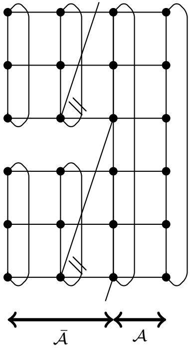

Rényi entropy

$$S^{(q)} = -\frac{1}{q-1}\log\left(\mathrm{tr}_{\mathcal{A}}\hat{\rho}_{\mathcal{A}}^{q}\right) = \frac{1}{q-1}\underbrace{F[\mathcal{A}, q, T]}_{q-1 \text{ cuts}} - \frac{q}{q-1}\underbrace{F[T]}_{0 \text{ cuts}}$$

Entanglement entropy and entropic C function

Approximate Entanglement entropy by 2nd Rényi entropy

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$\Rightarrow$ Have to calculate difference of free energies on the lattice

Calculating differences of free energy on the lattice

Difference of free energies F_1 and F_2

$$F_2 - F_1 = -\log Z_2 + \log Z_1$$

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$$Z(\alpha) = \int D\phi \exp\left(-(1-\alpha)S_1[\phi] - \alpha S_2[\phi]\right)$$

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Entropic C function:

$$\boxed{C(l - {}^{a}\!/_{2}) = \frac{(l - {}^{a}\!/_{2})^3}{L^2} \int_0^1 d\alpha \, \langle S_{l+1} - S_l \rangle_\alpha}$$

Simulation plan

$$\boxed{C(l - a/2) = \frac{(l - a/2)^3}{L^2} \int_0^1 d\alpha \left\langle S_{l+1} - S_l \right\rangle_\alpha}$$

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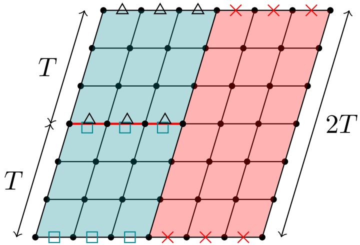

- Generate configurations with interpolating action

$$S_{\text{int}} = (1 - \alpha) S_l[U] + \alpha S_{l+1}[U]$$

with $S_l[U]$ and $S_{l+1}[U]$ action on deformed geometry.

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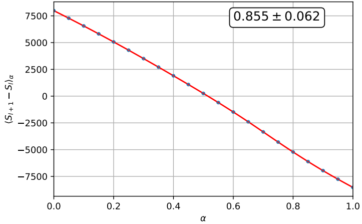

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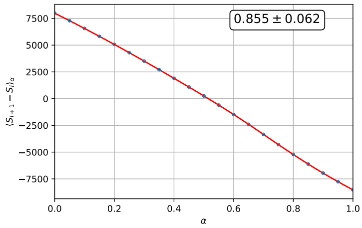

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- Measure $S_{l+1} - S_l$ and generate bootstrap samples
- Integrate over α using cubic spline interpolation

Simulation details

- Use standard Wilson action

$$S[U] = \frac{\beta}{N_c} \sum_{U_{\mu\nu}} \mathrm{Re}\ \mathrm{tr}\left(1 - U_{\mu\nu}\right)$$

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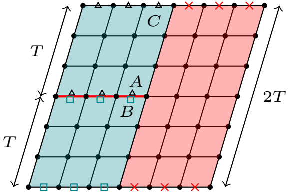
- Be careful closing plaquettes



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- Be careful closing plaquettes
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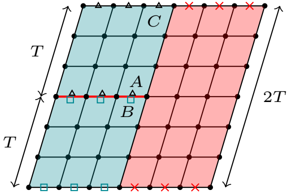

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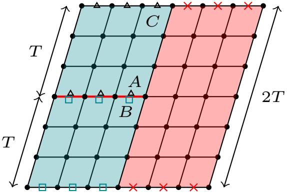
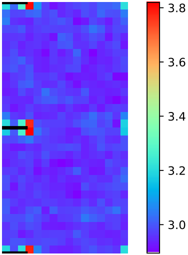
- Perform 100 sweeps to avoid autocorrelation
- Reunitarize link variables
- Use lattice of size $16^3 \times (2 \times 16)$





Simulation Parameters

- Need to compare different spacings $\rightarrow$ string tension $\sqrt{\sigma}$

  [Lucini, Teper, Wenger '05]
- Take shorter cut $\rightarrow$ "lower temperature"
- High statistics required, more than $300,000$ configs

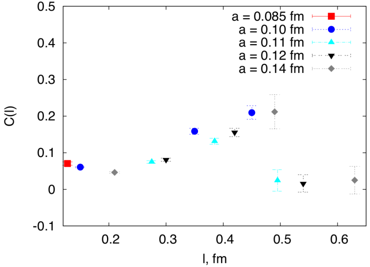
Configurations $\mathbf{SU(3)}$

β/l	2	3	4	5	6	7
5.700	44,976	44,976	130,656	253,056	218,328	230,568
5.720			34,728			
5.740			34,728			
5.750	34,728	34,728	34,728	120,408	208,080	134,640
5.770			34,728			
5.780			34,728			
5.800	34,728	34,728	34,728	120,408	255,048	134,640

Configurations $\mathbf{SU(4)}$

β/l	2	3	4	5	6	7
11.000	24,480	70,560	93,600	93,600	69,120	184,320
11.004						57,600
11.008						30,744
11.058		26,650				
11.075					160,221	115,200
11.100				253,440	253,440	253,440
11.112					30,744	
11.156						57,600
11.192				33,961		
11.200		103,680	103,680	149,760	357,120	357,120
11.300		11,520	11,520	126,720	126,720	264,960
11.398			31,704			

$SU(2)$ [Buividovich, Polikarpov '08]



Numerical results for pure Yang Mills Theory

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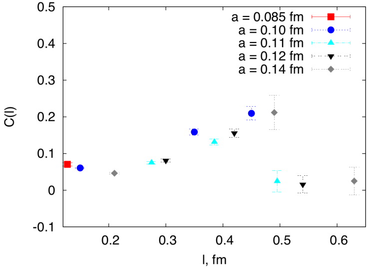

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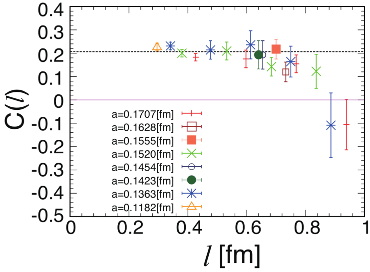

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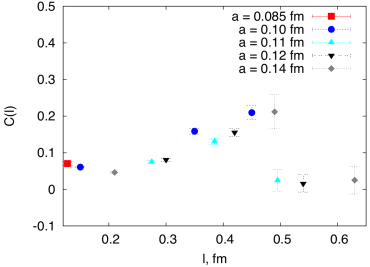

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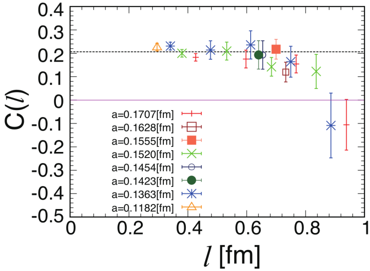
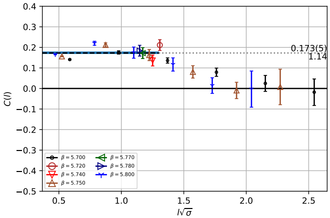


$SU(3)$ [This work]



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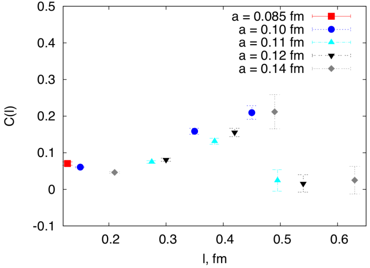

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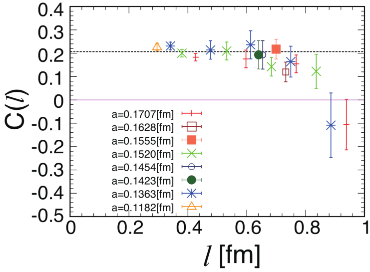

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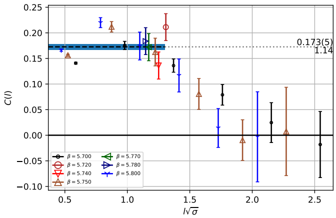

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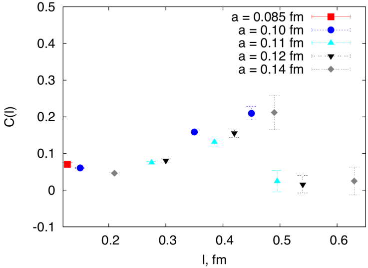

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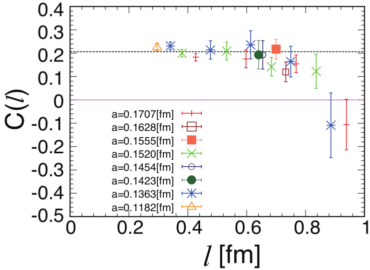

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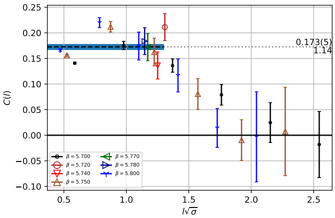
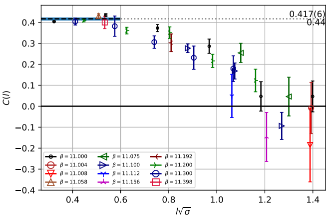


$SU(4)$ [This work]



N_c comparison

At small distance: $N_c^2 - 1$ color degrees of freedom

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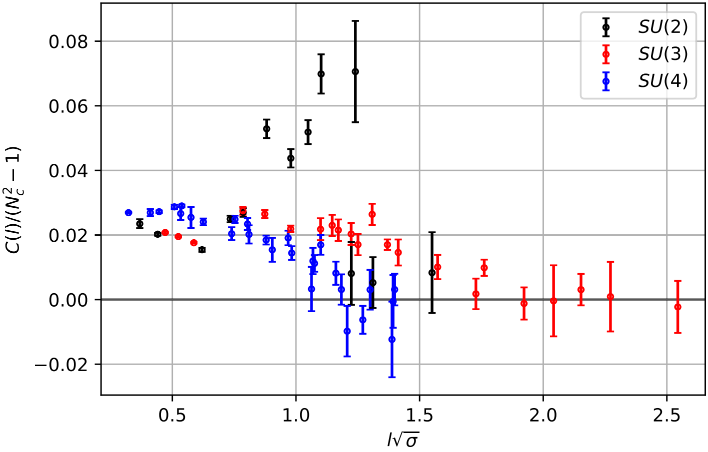

N_c comparison

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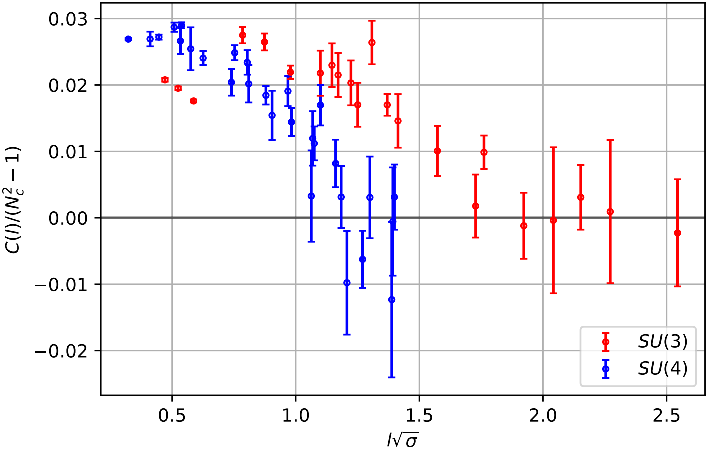

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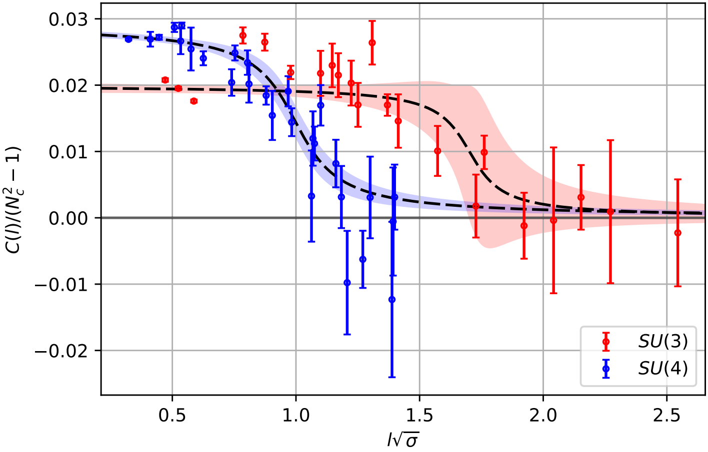

$$A\left[\frac{\pi}{2} - \arctan\left(B\left(x - C\right)\right)\right]$$

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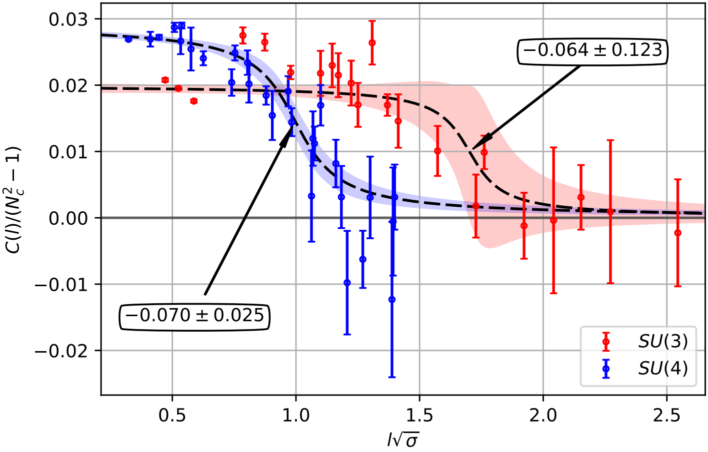
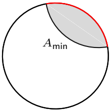


$$A\left[\frac{\pi}{2} - \arctan\left(B\left(x - C\right)\right)\right]$$

- EE calculatable using holography and on lattice

SUSY
$N_c = \infty$:



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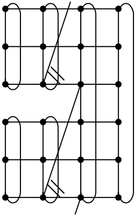

Outlook/Summary

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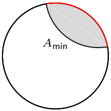

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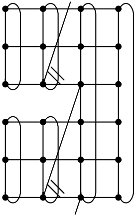
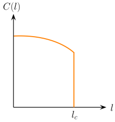


- AdS/CFT jump in $C(l) \longrightarrow$ smooth transition for $N_c \ll \infty$



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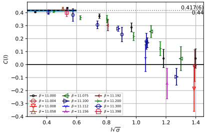

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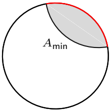

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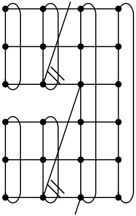

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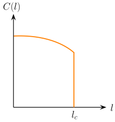

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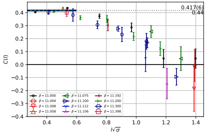

- Higher Rényi entropies possible

Outlook/Summary

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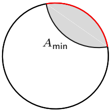
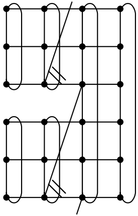
SUSY $N_c = \infty$: 

YM $N_c \ll \infty$: 

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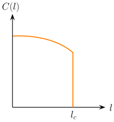

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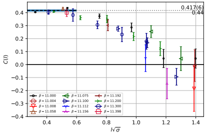

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Outlook/Summary

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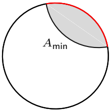

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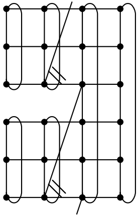

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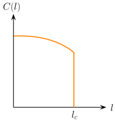

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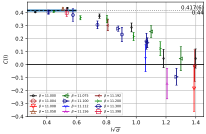

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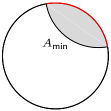

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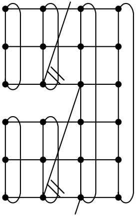

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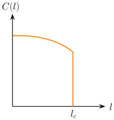

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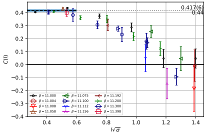

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Thank you for your attention