Pavel Buividovich

(Regensburg, Germany)

[in collaboration with M. Hanada, A. Schäfer]

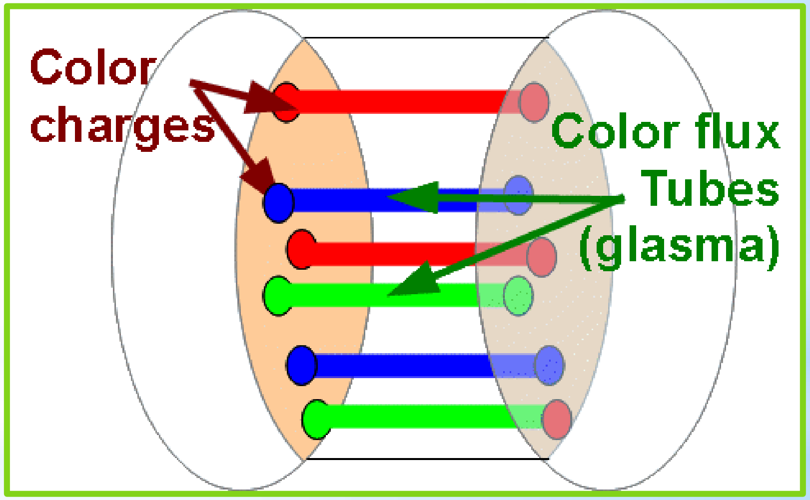
Lyapunov exponents and entanglement generation in realtime simulations of the ungauged BFSS matrix model

Motivation

Glasma state at early stages of HIC
Overpopulated gluon states
Almost “classical” gauge fields

Chaotic Classical Dynamics
[Saviddy,Susskind…]

- Positive Lyapunov exponents
- Gauge fields forget initial conditions



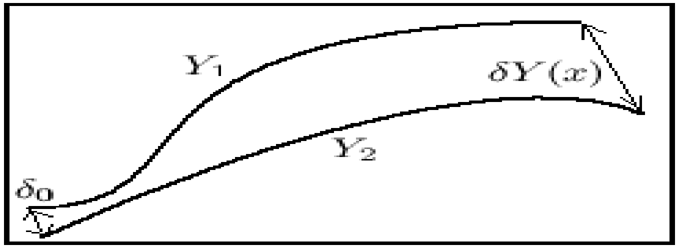
…but is it enough for Thermalization?

Motivation

Thermalization for quantum systems?

- Quantum extension of Lyapunov exponents - OTOCs $\langle [P(0),X(t)]^2 \rangle$
- Generation of entanglement between subsystems

$$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$$



Timescales: quantum vs classical?

☹ QFT tools extremely limited beyond strong-field classic regime…

☺ …Holography provides intuition

Bounds on chaos

Reasonable physical assumptions
Analyticity of OTOCs ⇒

$$\boxed{\lambda_L < 2\pi T}$$

(QGP ~0.1 fm/c)

[Maldacena Shenker Stanford'15]

- **Holographic models with black holes saturate the bound(e.g. SYK)**
- **In contrast, for classical YM** $\boxed{\lambda_L \sim T^{1/4}}$

What happens at low T ???

BFSS Model: Classically chaotic system with a holographic dual

N=1 Supersymmetric Yang-Mills in D=1+9:

Reduce to a single point = BFSS matrix model [Banks, Fischler, Shenker, Susskind'1997]

$$L = \frac{1}{2g}\left[\mathrm{tr}\dot{X}^i\dot{X}^i + 2\theta^T\dot{\theta} - \frac{1}{2}\mathrm{tr}[X^i, X^j]^2 - 2\theta^T\gamma_i[\theta, X^i]\right]$$

N x N hermitian matrices

Majorana-Weyl fermions, N x N hermitian

System of *N* D0 branes joined by open strings [Witten'96]:

- X^{ii}_{μ} = D0 brane positions
- X^{ij}_{μ} = open string excitations

Classical chaos and BH physics

Stringy interpretation:

- Dynamics of gravitating D0 branes
- Thermalized state = black hole
- Classical chaos = info scrambling

Expected to saturate the MSS bound at low temperatures!

In this talk:

Numerical attempt to look at the real-time dynamics of BFSS and bosonic matrix models

↓

Of course, not an exact simulation, but should be good at early times

↓

Approximating all states by Gaussians

Gaussian state approximation

Simple example:
Double-well potential

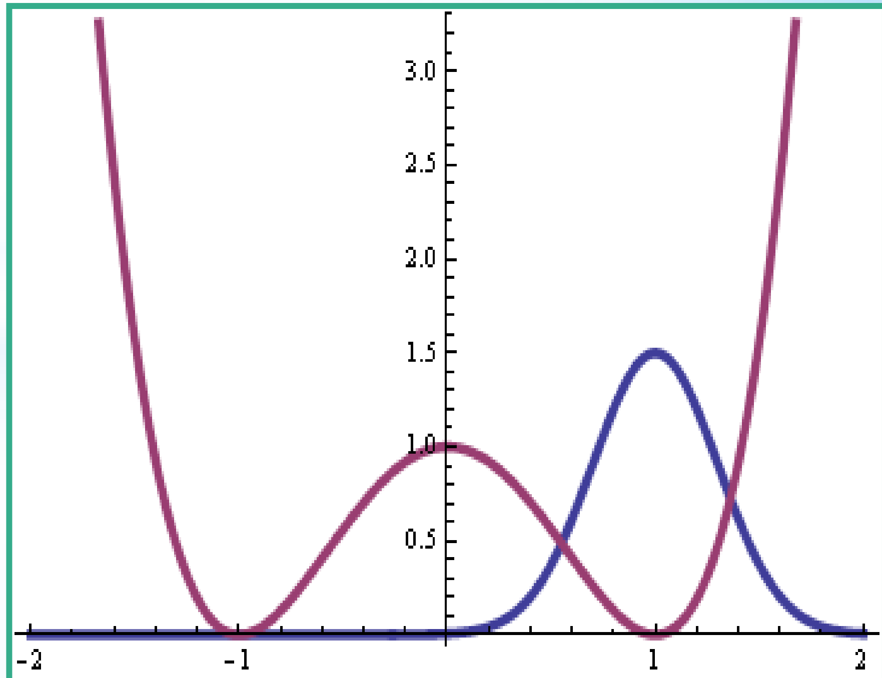
$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{a\hat{x}^2}{2} + \frac{b\hat{x}^3}{3} + \frac{c\hat{x}^4}{4}$$

Heisenberg equations of motion

$$\partial_t \hat{x} = \hat{p},$$
$$\partial_t \hat{p} = -a\hat{x} - b\hat{x}^2 - c\hat{x}^3$$

Also, for example

$$\partial_t \left(\hat{x}\hat{x}\right) = \hat{x}\hat{p} + \hat{p}\hat{x}$$



Next step: Gaussian Wigner function

Assume Gaussian wave function at any t

Simpler: Gaussian Wigner function

$$\langle \hat{x}^2 \rangle = x^2 + \sigma_{xx},$$
$$\langle \hat{p}^2 \rangle = p^2 + \sigma_{pp},$$
$$\langle \frac{\hat{x}\hat{p}+\hat{p}\hat{x}}{2} \rangle = xp + \sigma_{xp}$$

For other correlators: use Wick theorem!

$$\langle \hat{x}^4 \rangle = x^4 + 6x^2\sigma_{xx} + 3\sigma_x x^2,$$
$$\langle \hat{x}^2\hat{p} \rangle = x^2 p + 2x\sigma_{xp} + p\sigma_{xx}$$

Derive closed equations for

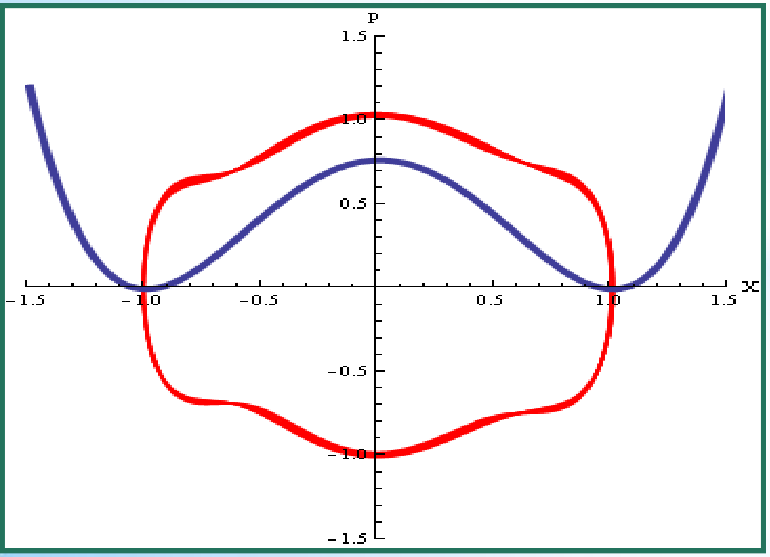
x, p, σ_{xx} , σ_{xp} , σ_{pp}

Origin of tunnelling

$$\partial_t p = -ax - bx^2 - cx^3 - b\sigma_{xx} - 3cx\sigma_{xx},$$
$$\partial_t x = p$$

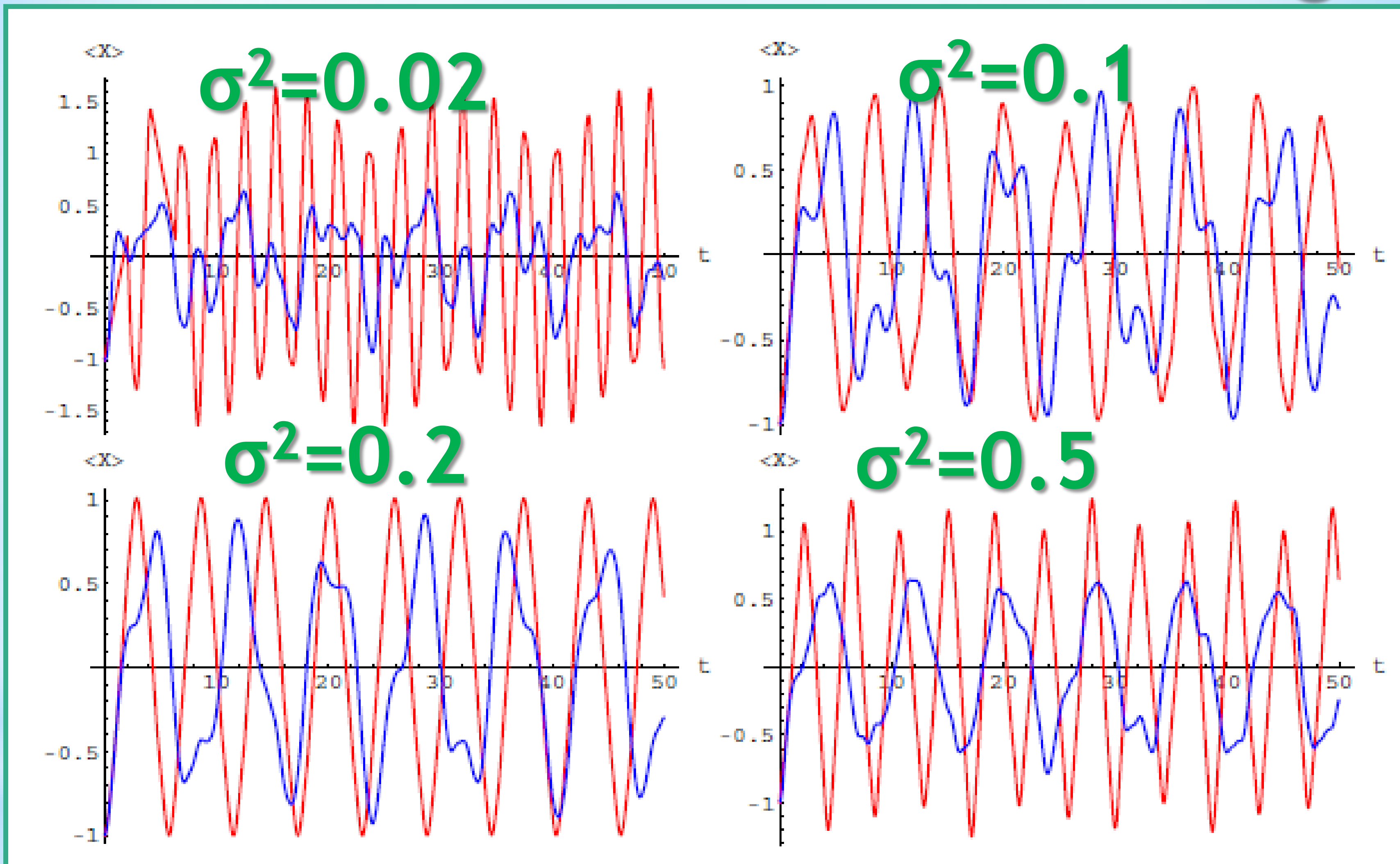
Positive force even at *x=0* (classical minimum)

$$\partial_t \sigma_{xx} = 2\sigma_{xp},$$
$$\partial_t \sigma_{xp} = \sigma_{pp} - a\sigma_{xx} - 2bx\sigma_{xx} - 3cx^2\sigma_{xx} - 3c\sigma_{xx}^2,$$
$$\partial_t \sigma_{pp} = -2\left(a\sigma_{xp} + 2bx\sigma_{xp} + 3cx^2\sigma_{xp} + 3c\sigma_{xx}\sigma_{xp}\right)$$



Quantum force causes classical trajectory to leave classical minimum

Gaussian state vs exact Schrödinger



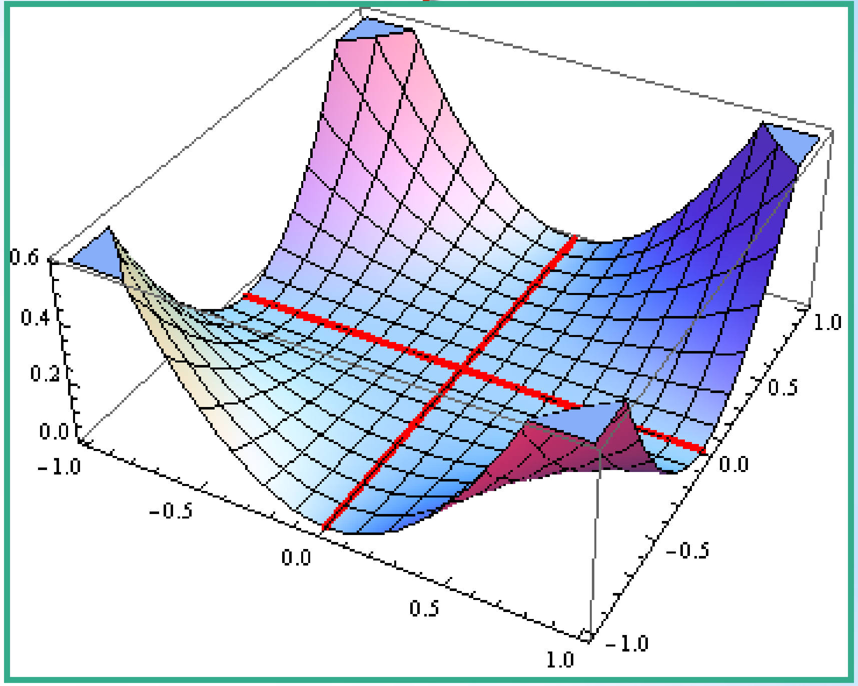
- Early-time evolution OK
- Tunnelling period qualitatively OK

2D potential with flat directions
(closer to BFSS model)

$$\hat{H} = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_y^2}{2} + \frac{\kappa}{2}\hat{x}^2\hat{y}^2$$

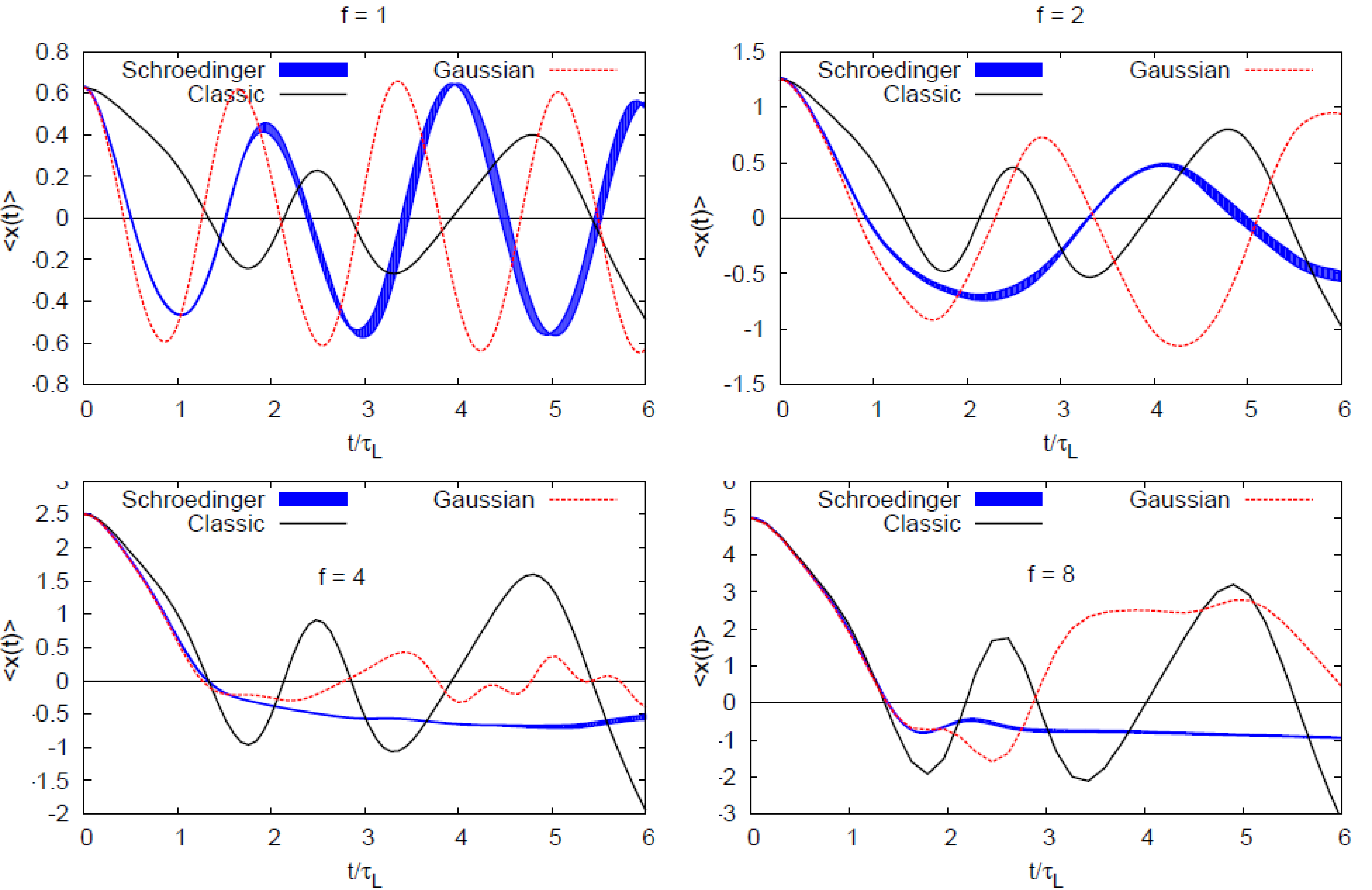
Classic runaway along *x=0 or y=0*

Classically chaotic!



We start with a Gaussian wave packet at distance *f* from the origin (away from flat directions)

Gaussian state vs exact Schrödinger



Gaussian state approximation

- ✓ Is good for at least two classical Lyapunov times
- ✓ Maps pure states to pure states [discussion follows below]
- ✓ Allows to study entanglement
- ✓ Closely related to semiclassics
- ✓ Is better for chaotic than for regular systems [nlin/0406054]
- ✓ Is likely safe in the large-N limit
- X Is not a unitary evolution

BFSS matrix model: Hamiltonian formulation

$$\hat{H} = \frac{1}{2}\hat{P}_i^a \hat{P}_i^b + \frac{1}{4} C_{abc} C_{ade} \hat{X}_i^b \hat{X}_j^c \hat{X}_i^d \hat{X}_j^e + \frac{i}{2} C_{abc} \hat{\psi}_\alpha^a \left[\sigma_i\right]_{\alpha\beta} \hat{X}_i^b \hat{\psi}_\beta^c,$$

a,b,c - su(N) Lie algebra indices

Heisenberg equations of motion

$$\partial_t \hat{X}_i^a = \hat{P}_i^a$$

$$\partial_t \hat{P}_i^a = -C_{abc} C_{cde} \hat{X}_j^b \hat{X}_i^d \hat{X}_j^e - \frac{i}{2} C_{bac} \sigma^i_{\alpha\beta} \hat{\psi}_\alpha^b \hat{\psi}_\beta^c,$$

$$\partial_t \hat{\psi}_\alpha^a = C_{abc} X_i^b \sigma^i_{\alpha\beta} \hat{\psi}_\beta^c$$

GS approximatio for BFSS model

$$\partial_t P^a_i = -C_{abc}C_{cde}X^b_j X^d_i X^e_j - \frac{i}{2}C_{bac}\sigma^i_{\alpha\beta}\langle \psi^b_\alpha \psi^c_\beta \rangle -$$
$$- C_{abc}C_{cde}X^b_j [XX]^{de}_{ij} - C_{abc}C_{cde}[XX]^{be}_{jj}X^d_i - C_{abc}C_{cde}[XX]^{bd}_{ji}X^e_j$$

$$\partial_t [XX]^{ab}_{ij} = [XP]^{ab}_{ij} + [XP]^{ba}_{ji},$$
$$\partial_t [XP]^{af}_{ik} = [PP]^{af}_{ik} - C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XX]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XX]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XX]^{ef}_{jk},$$
$$\partial_t [PP]^{af}_{ik} = -C_{abc}C_{cde}\left(X^d_i X^e_j + [XX]^{de}_{ij}\right)[XP]^{bf}_{jk} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^e_j + [XX]^{be}_{jj}\right)[XP]^{df}_{ik} -$$
$$- C_{abc}C_{cde}\left(X^b_j X^d_i + [XX]^{bd}_{ji}\right)[XP]^{ef}_{jk} + (\{a,i\} \leftrightarrow \{f,k\})$$

- **CPU time ~ *N^5* (double commutators)**
- **RAM memory ~ *N^4***
- ***SUSY broken, unfortunately …***

Ungauging the BFSS model

- Gauge constraints

$$\hat{J}_a = C_{abc}\hat{X}_i^b \hat{P}_i^c - \frac{i}{2} C_{abc}\hat{\psi}_\alpha^b \hat{\psi}_\alpha^c \qquad \hat{J}_a |\psi\rangle = 0$$

- For Gaussian states we can only have a weaker constraint $\langle\psi| \hat{J}_a |\psi\rangle = 0$
- We work with ungauged model [Maldacena,Milekhin'1802.00428] (e.g. LGT with unit Polyakov loops)
- Ungauging preserves most of the features of the original model [Berkowitz,Hanada, Rinaldi, Vranas 1802.02985]

Fate of supersymmetry

16 supercharges in BFSS model:

$$\hat{Q}_\alpha = \hat{P}_i^a [\sigma_i]_{\alpha\beta} \hat{\psi}_\beta^a - \frac{1}{4} C_{abc} \hat{X}_i^b \hat{X}_j^c [\sigma_{ij}]_{\alpha\beta} \hat{\psi}_\beta^a$$

$$\sigma_{ij} \equiv \sigma_i \sigma_j - \sigma_j \sigma_i$$

$$\left\{ \hat{Q}_\alpha, \hat{Q}_\beta \right\} = 2\delta_{\alpha\beta} \hat{H} - 2 (\sigma_i)_{\alpha\beta} \hat{X}_i^a \hat{J}^a$$

$$\left[\hat{H}, \hat{Q}_\gamma \right] = -i \hat{\psi}_\gamma^a \hat{J}^a$$

Gauge transformations

$$\hat{J}^a = C_{abc} \hat{X}_i^b \hat{P}_i^c - \frac{i}{2} C_{abc} \hat{\psi}_\alpha^b \hat{\psi}_\alpha^c$$

Fate of supersymmetry

In full quantum theory

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) = 0$$

Fierz identity (cyclic shift of indices):

$$\left[\sigma_i\right]_{\alpha\beta} \left[\sigma_i\right]_{\gamma\delta} + \left[\sigma_i\right]_{\alpha\gamma} \left[\sigma_i\right]_{\beta\delta} + \left[\sigma_i\right]_{\alpha\delta} \left[\sigma_i\right]_{\gamma\beta} = \\ = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}$$

In CSFT approximation

$$\partial_t \hat{Q}_\delta = \frac{i}{2} C_{abc} \langle \hat{\psi}^a_\alpha \hat{\psi}^b_\beta \rangle \hat{\psi}^c_\gamma \left(\sigma^i_{\alpha\beta} \sigma^i_{\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \right) \neq 0$$

Fermionic 3pt function seems necessary!

Equation of state and temperature

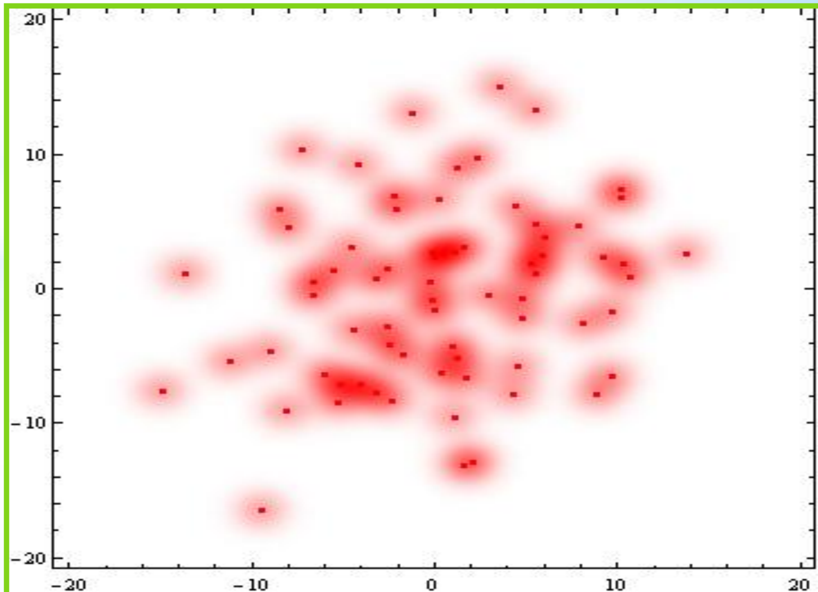
- Consider **mixed Gaussian states** with fixed energy $E = \langle H \rangle$
- *Maximize entropy w.r.t.* $\langle xx \rangle, \langle pp \rangle$
- Calculate temperature using

$$\boxed{T^{-1} = \frac{\partial S}{\partial E}}$$

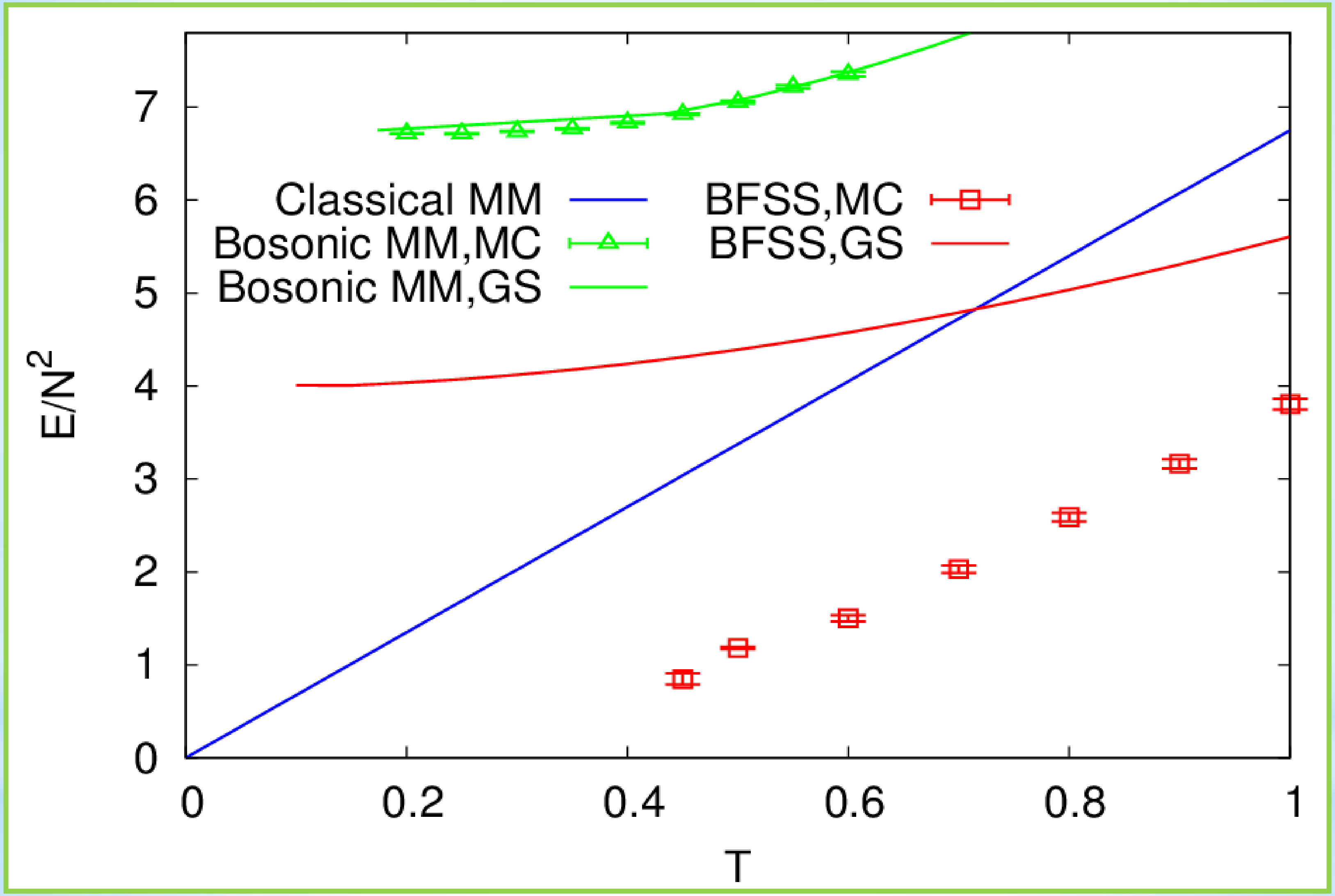
- Can be done analytically using rotational and SU(N) symmetries

"Thermal" initial conditions

- At $T=0$ pure "ground" state with minimal $<pp>,<xx>$
- At $T>0$ mixed states, interpret as mixture of pure states, shifted by "classical" coordinates with dispersion $<xx>-<xx>_0$
- Makes difference for non-unitary evolution
- Fermions in ground state at fixed classical coordinates

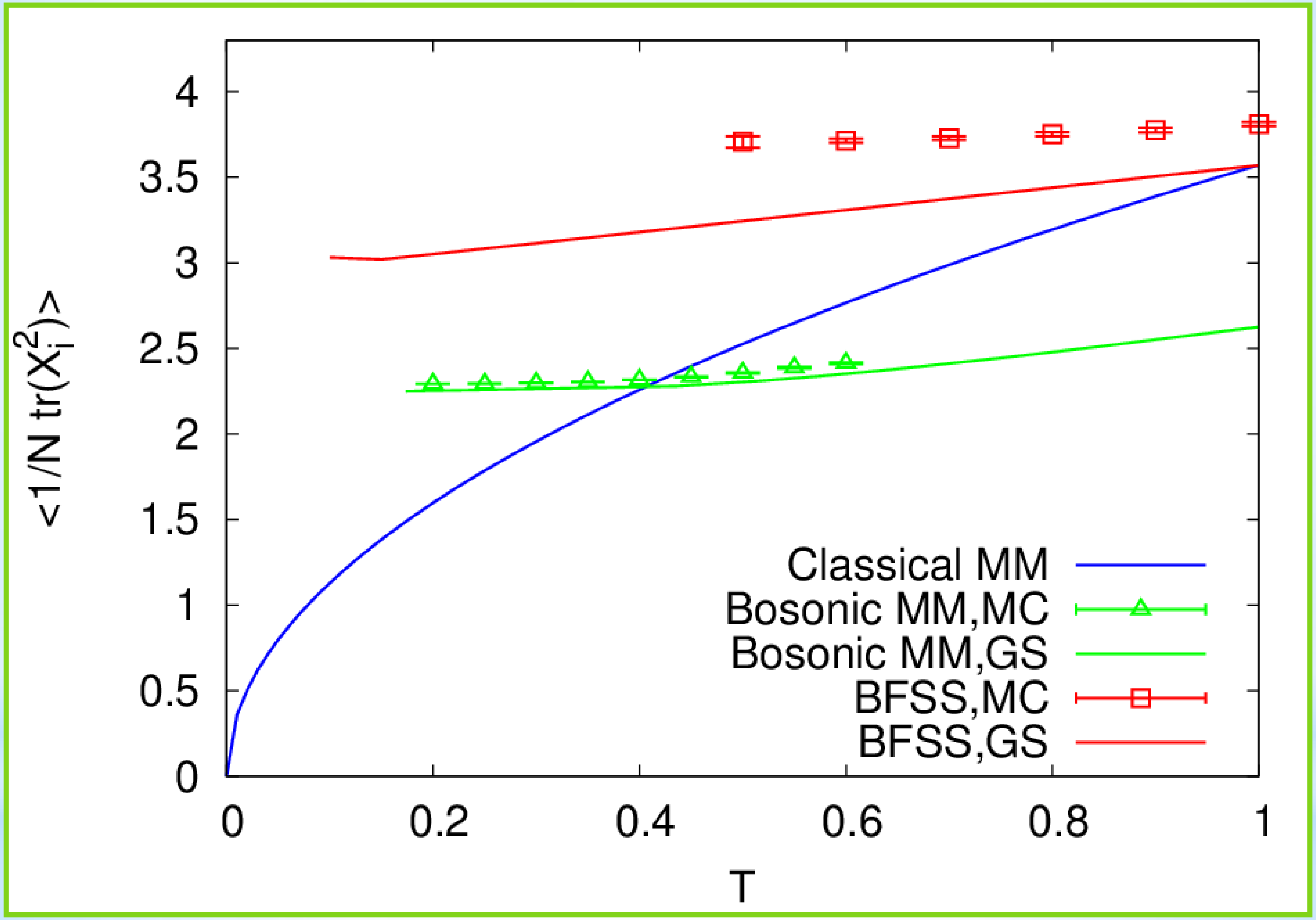


Energy vs temperature



MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], **we agree well for pure gauge**

<1/N $Tr(X_i^2)$> vs temperature

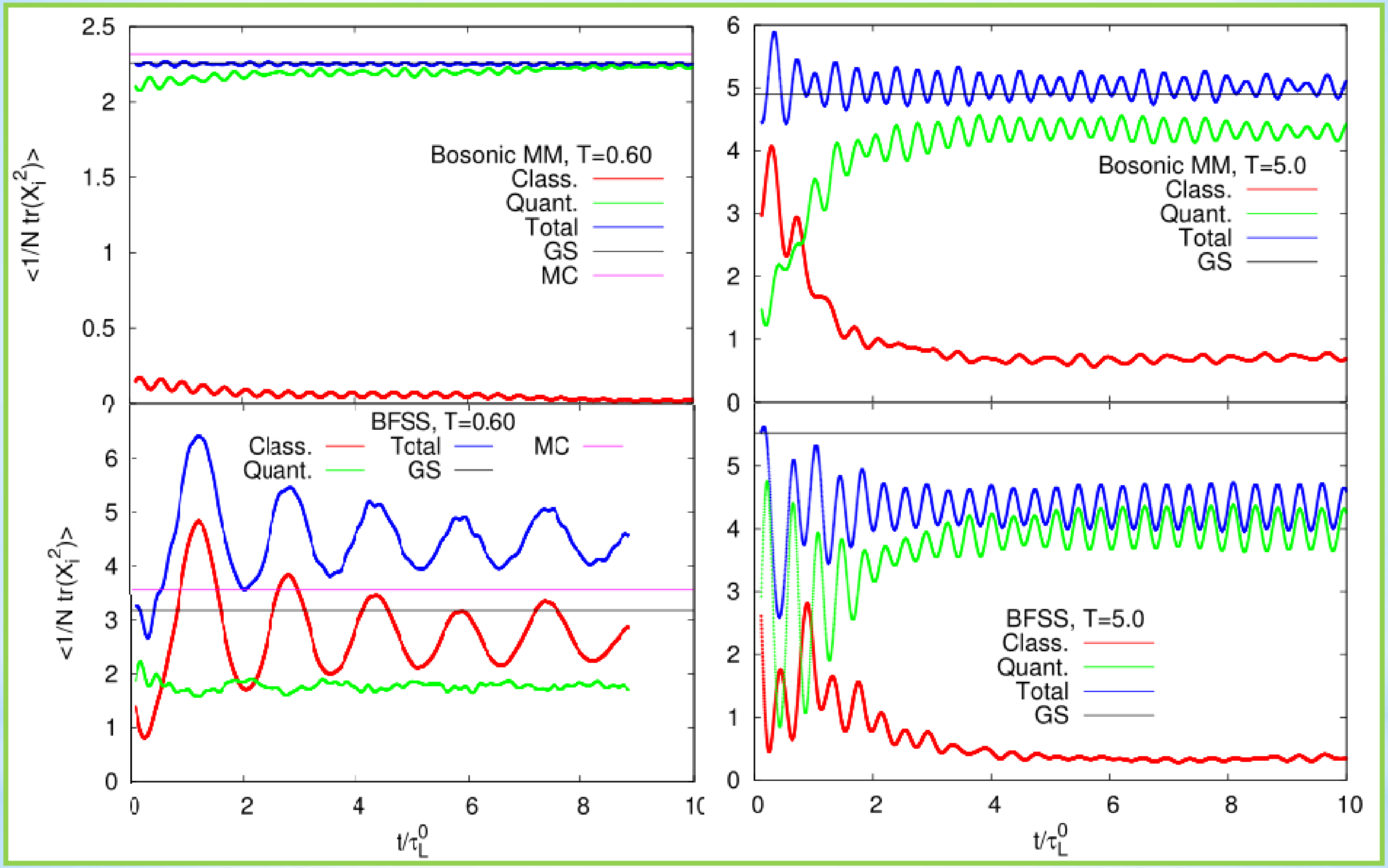


MC data from [Berkowitz,Hanada, Rinaldi, Vranas, 1802.02985], we agree well for pure gauge

Real-time evolution

- Thermal initial conditions
- Randomly shifted Gaussian wave functions
- Only a few instances of random initial conditions
- Good self-averaging at sufficiently large N
- Numerically solving the evolution equations for X, P, $\langle XX \rangle$, $\langle XP \rangle$, $\langle PP \rangle$, $\langle \psi\psi \rangle$
- We use $N=5$ and $N=7$ (remember N^5 scaling of CPU time)

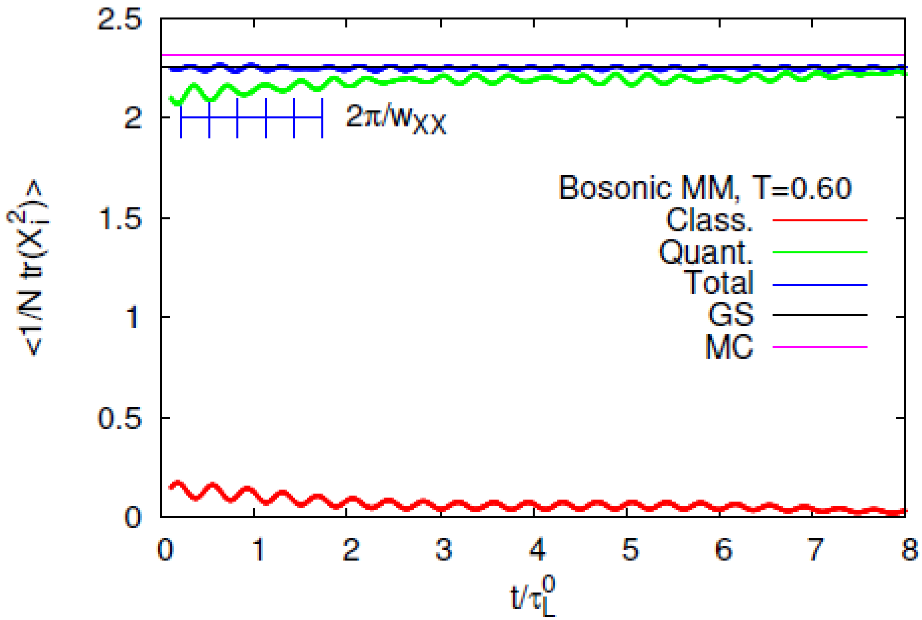
Real-time evolution: $\langle 1/N\, Tr(X_i^2) \rangle$

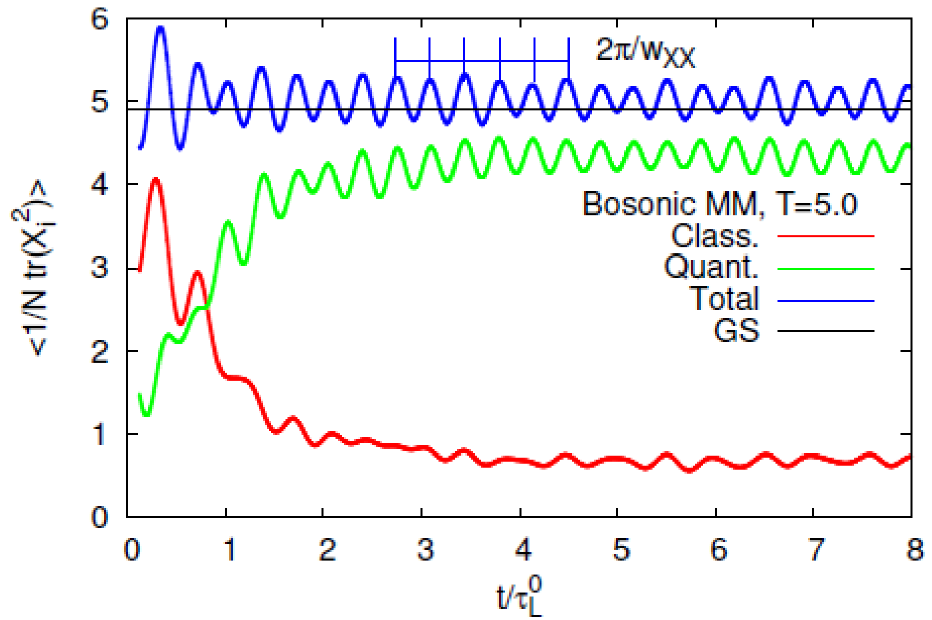


Wavepacket spread vs classical shrinking

For BFSS $\langle 1/N\, Tr(X_i^2) \rangle$ grows, instability?

Quasinormal ringing I





Linearizing equations of motion around thermal equilibrium, we get oscillations with frequencies:

- $w_X = (2d-2)/d <1/N\ tr(X_i^2)>$ (X and P)
- $w_{XX} = 6\ w_X^2$ (XX, XP and PP)

To-be quasinormal modes!

OTOCs and Lyapunov distances

OTOC in an overfull basis of Gaussian states:

$$
\begin{aligned}
&\mathrm{Tr}\left(\hat{\rho}\left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right]^2\right) = \\
&\int dX'\, dP' \left\langle \langle X, P| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X', P'\rangle \times \right. \\
&\left. \times \langle X', P'| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X, P\rangle \right\rangle_c .
\end{aligned}
$$

Saturated by saddle point at *X=X'*, *P=P'* !!!

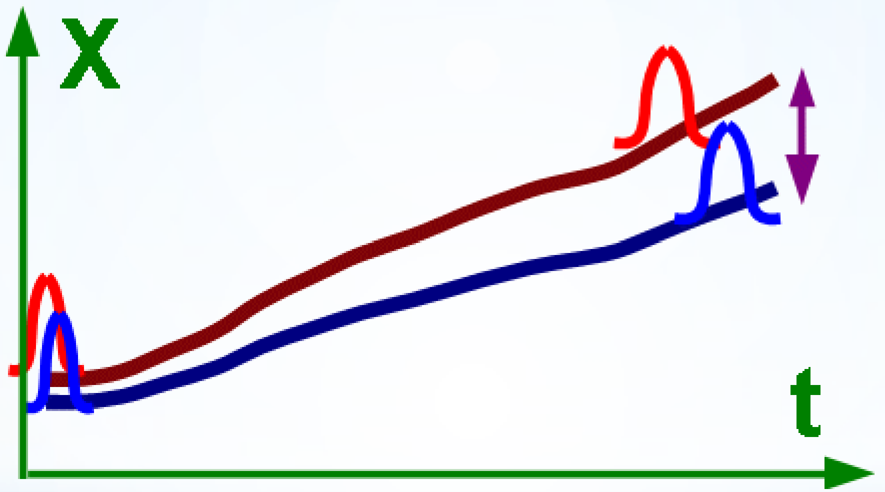
OTOC in terms of infinitesimal shift:

$$
\begin{aligned}
&\langle X, P| \left[\hat{X}_i^a(t), \hat{P}_j^b(0)\right] |X', P'\rangle = \\
&= -i\frac{\partial}{\partial \epsilon_j^b} \langle X, P| e^{i\epsilon \hat{P}(0)} \hat{X}_i^a(t)\, e^{-i\epsilon \hat{P}(0)} |X', P'\rangle,
\end{aligned}
$$

Very similar to classical Lyapunov distance!!!

OTOCs and Lyapunov distances

Our approximation for OTOC *$[X(t),P(0)]^2$*: Distance between centers of slightly shifted wave packets

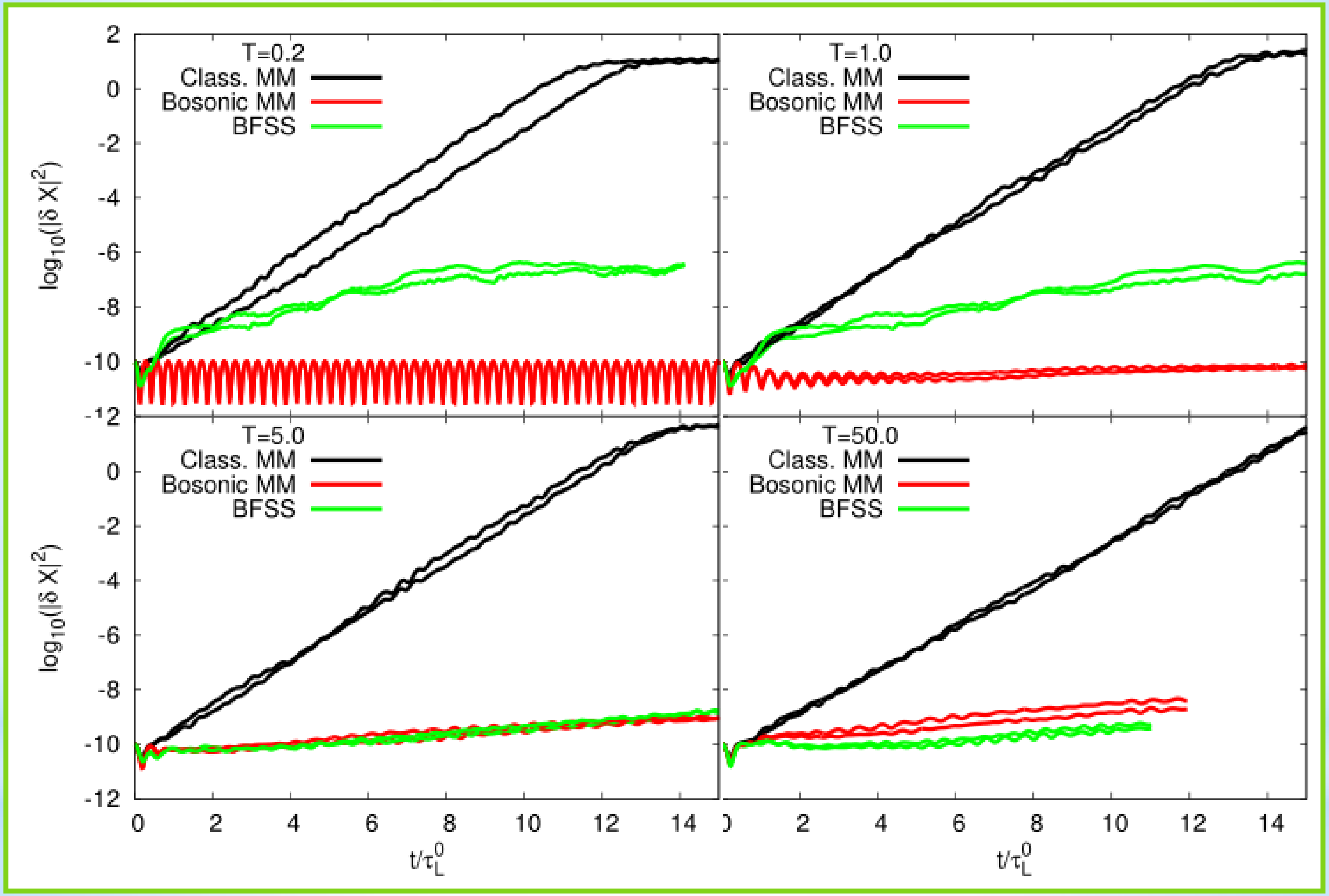


Difference between X^a_i coordinates

Two initial conditions with X^a_i shifted by random

$|\varepsilon^a_i| \sim 10^{-5}$

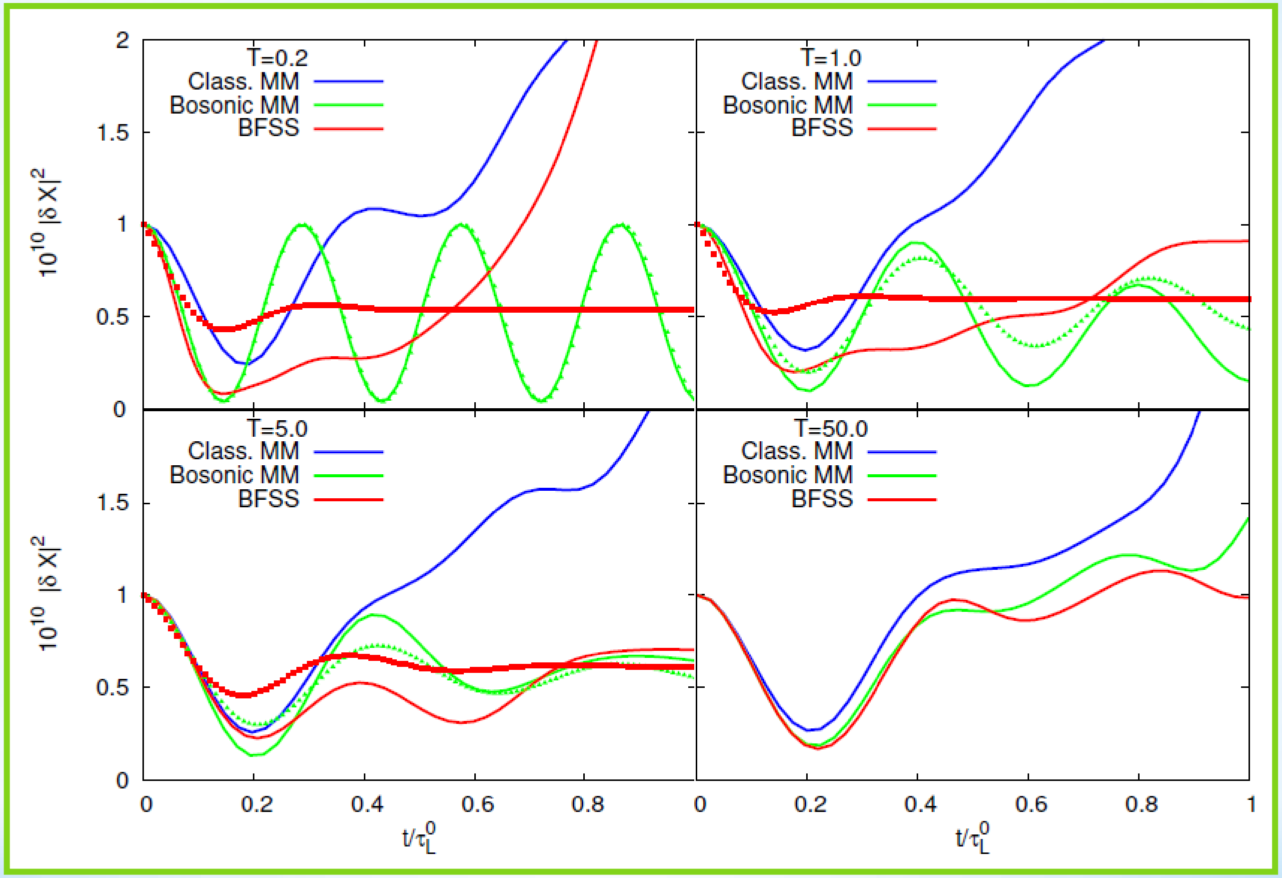
Lyapunov distances



Early times: Very similar to classical dynamics

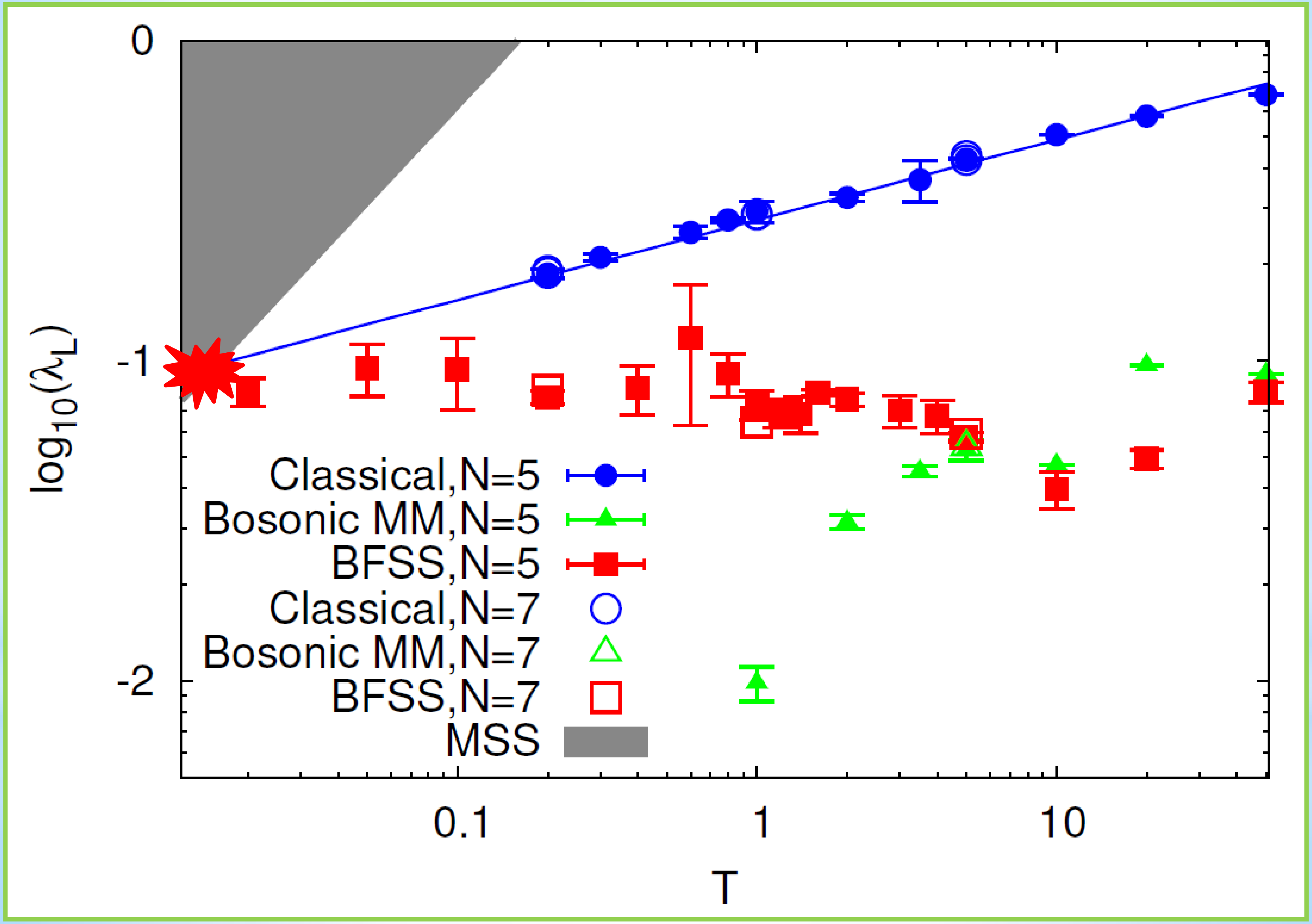
Late times: significantly slower growth

Dissipation and quasinormal ringing II

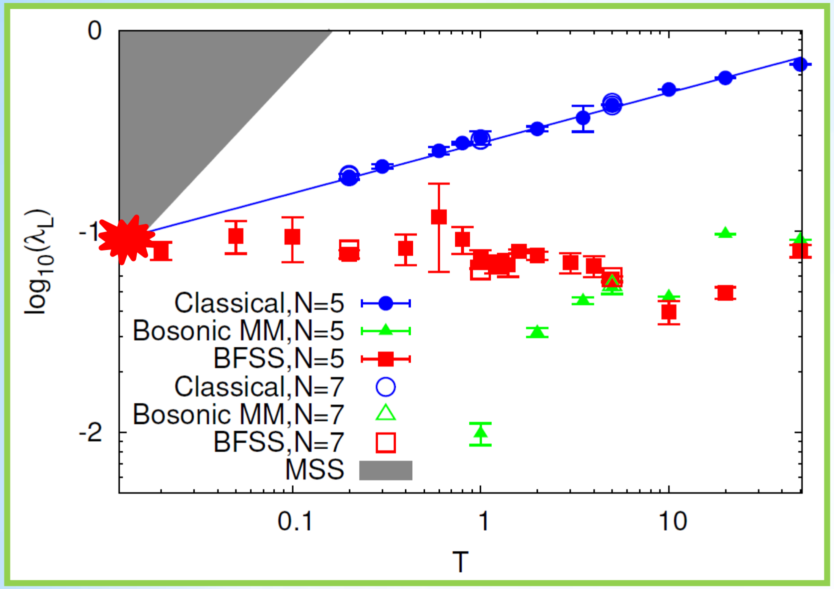


- Quasinormal ringing at early times
- Exponential growth at late times
- Lyapunov time happens to be larger than dissipation time

Lyapunov exponents and MSS bound



Lyapunov exponents and MSS bound



$$\boxed{\lambda_L < 2\pi T}$$

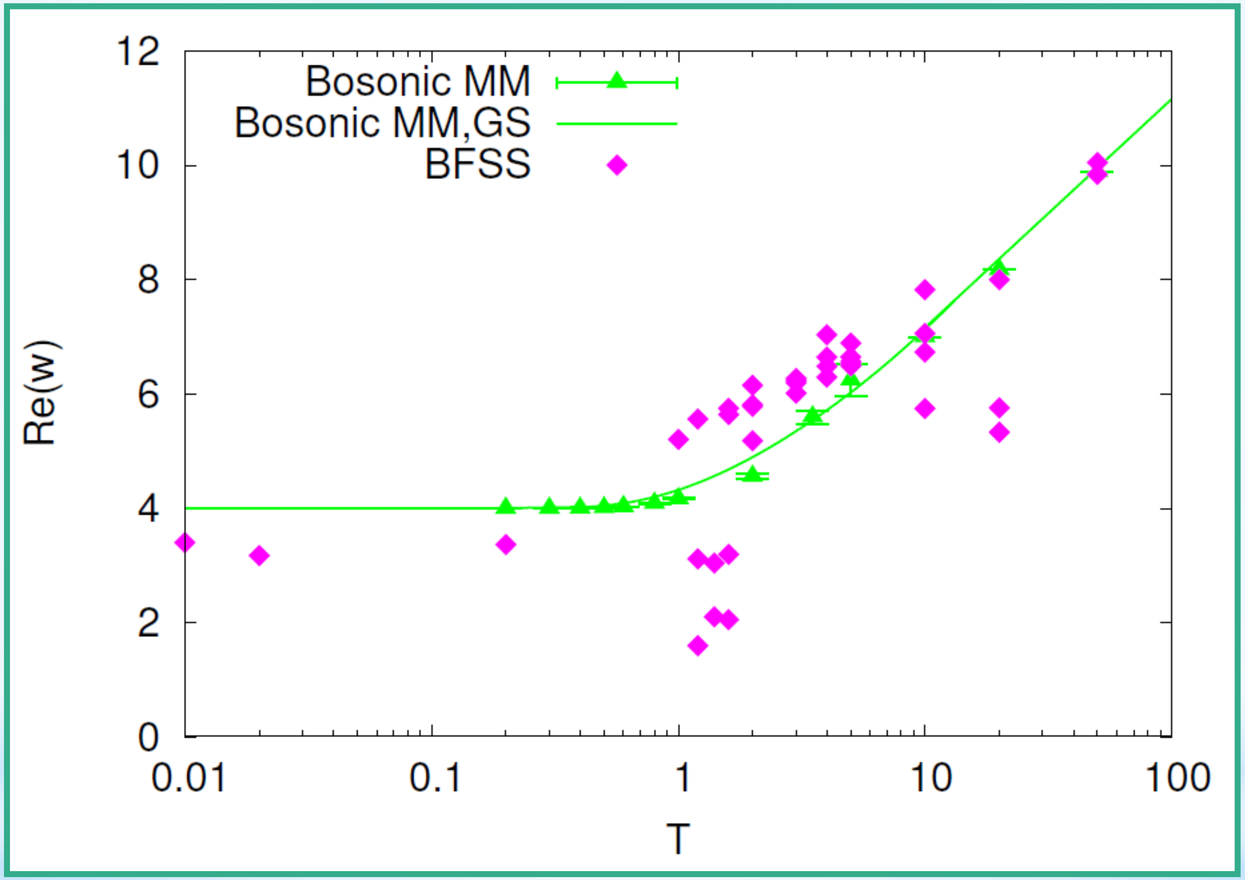
VS

$$\boxed{\lambda_L \sim T^{1/4}}$$

(for classical dynamics)

- **Lyapunov exponents vanish in confinement-like regime at low temperatures**
- **Full BFSS model: Lyapunov exponents finite at all T, might saturate the MSS bound**

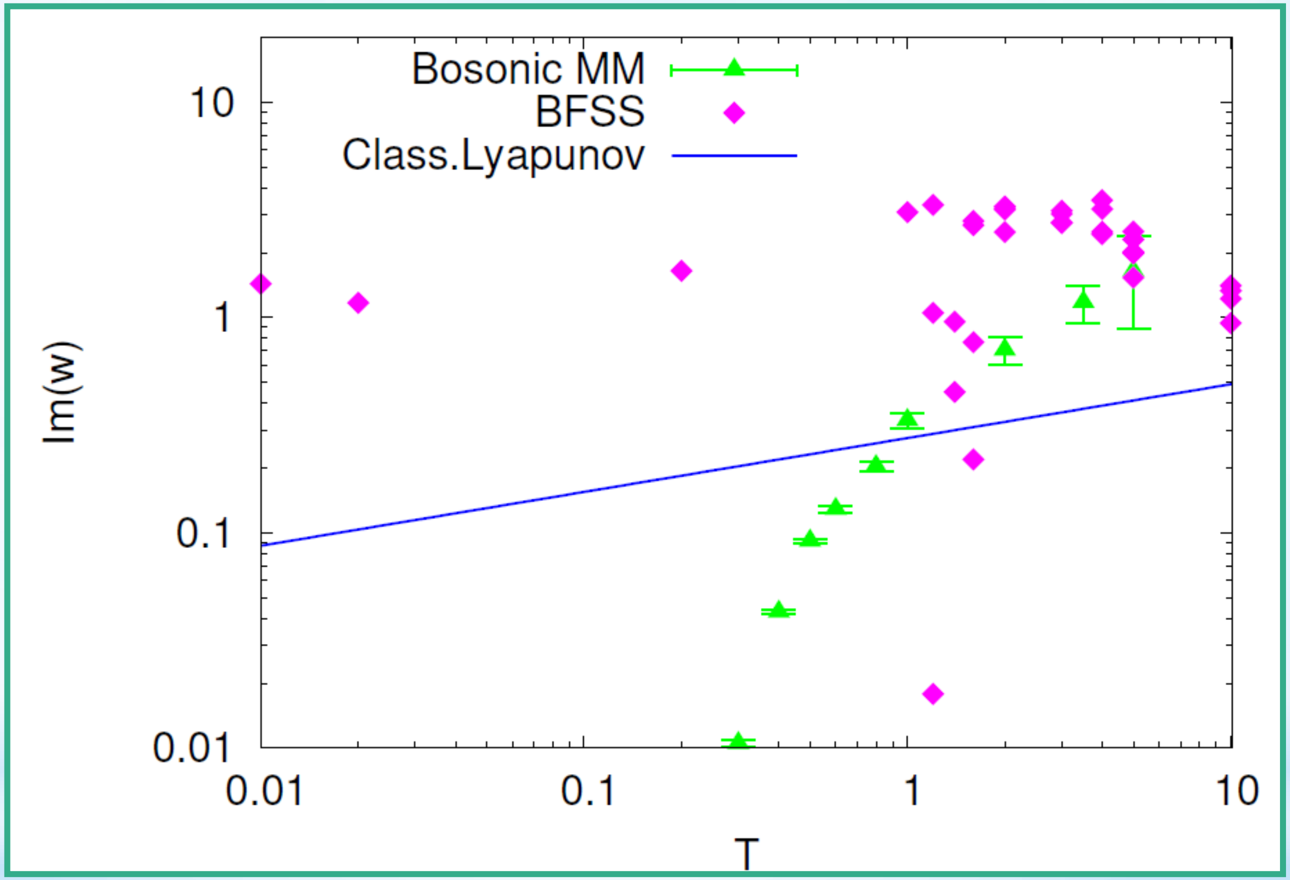
Quasinormal ringing
Re(w) vs Temperature



High-T scaling: $w_{xx} = 4.89\ T^{1/4}$

vs. $w_{xx} = 5.15\ T^{1/4}$ [Romatschke, Hanada]

Quasinormal ringing
Im(w) vs Temperature



Dissipation rate vanishes in the confinement regime, in contrast to BFSS

Entanglement entropy: Gaussian states

$$\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|$$

- **Reduced density matrix is also Gaussian!**
- **Entropy of (mixed) Gaussian state:**

$$S = -\mathrm{tr}\,(\hat{\rho}\log\hat{\rho})$$

$$S = \sum_k \left(f_k + \frac{1}{2}\right)\ln\left(f_k + \frac{1}{2}\right) - \\ -\sum_k \left(f_k - \frac{1}{2}\right)\ln\left(f_k - \frac{1}{2}\right).$$

- ***f_k* are symplectic eigenvalues of the block matrix**

$$\Delta = \begin{pmatrix} \langle\langle \hat{X}_i^a \hat{X}_j^b \rangle\rangle & \langle\langle \hat{X}_i^a \hat{P}_j^b \rangle\rangle \\ \langle\langle \hat{X}_j^b \hat{P}_i^a \rangle\rangle & \langle\langle \hat{P}_i^a \hat{P}_j^b \rangle\rangle \end{pmatrix}$$

- **Uncertainty principle: *$f_k \geq 1/2$***

Gaussian states: symplectic structure

- Symplectic eigenvalues of *(2 N)x(2 N)* real, symmetric, positive-definite matrix *A*:
  Eigenvalues of *Ω A*,
  Pairs ± *i* f_k

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- For the correlator block matrix of entire system, our evolution equations can be written as

$$\partial_t (\Delta\Omega) = \Upsilon (\Delta\Omega) - (\Delta\Omega) \Upsilon .$$

$$\Upsilon = \begin{pmatrix} 0 & I \\ -V & 0 \end{pmatrix}$$

$$V_{AB} = 3V_{ABCD} \langle \hat{X}_C \hat{X}_D \rangle$$

$$\hat{H} = \hat{P}_A^2/2 + V_{ABCD} \hat{X}_A \hat{X}_B \hat{X}_C \hat{X}_D / 4$$

- Symplectic eigenvalues are conserved
- Pure states are mapped to pure states

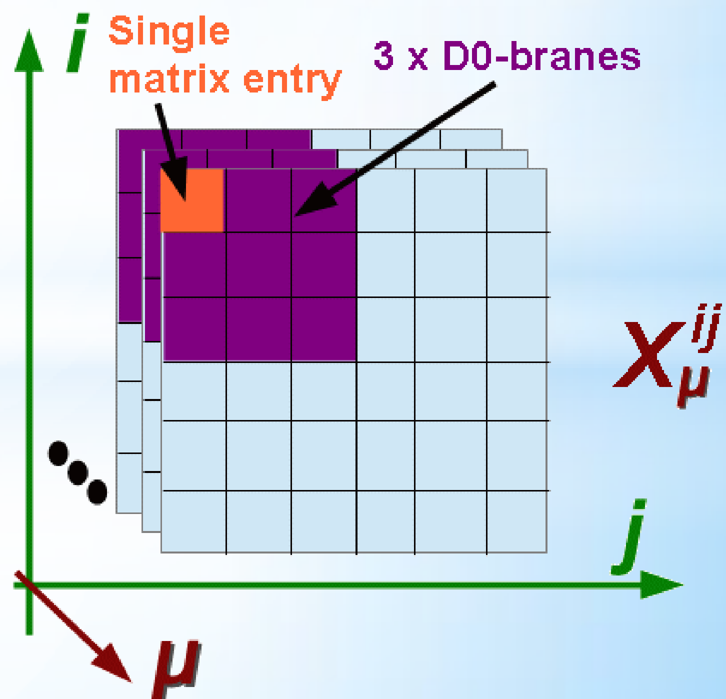
Entanglement entropy

$$\boxed{\rho_A = \mathrm{Tr}_B\, |\Psi\rangle\langle\Psi|}$$

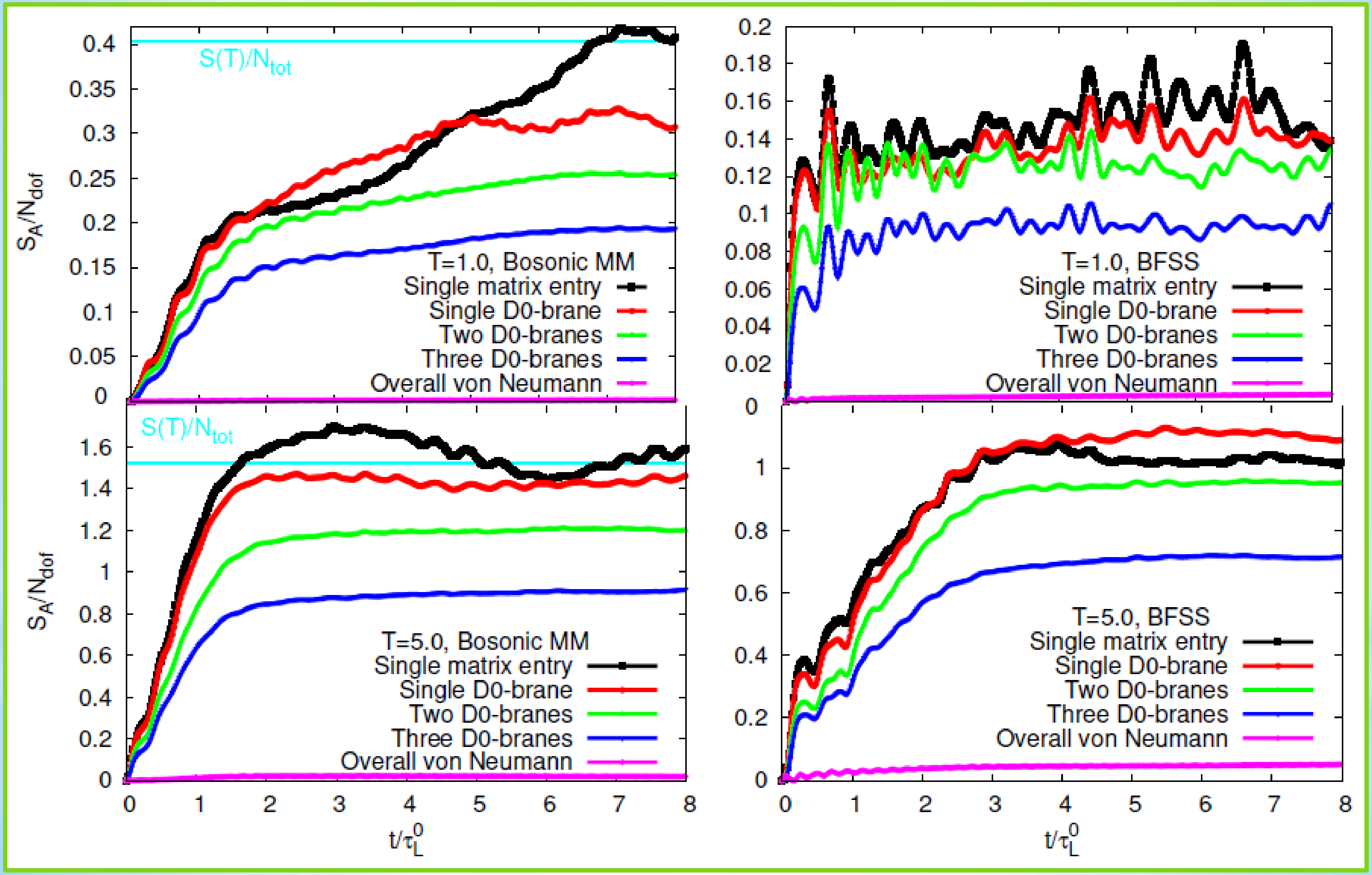
$$\boxed{S_A = -\mathrm{Tr}\left(\rho_A \log\left(\rho_A\right)\right)}$$

- Chaotic systems are expected to entangle A and B ➡ Entanglement entropy saturates

- Subsystem A is a matrix block
- Gauge constraints are anyway irrelevant due to ungauging

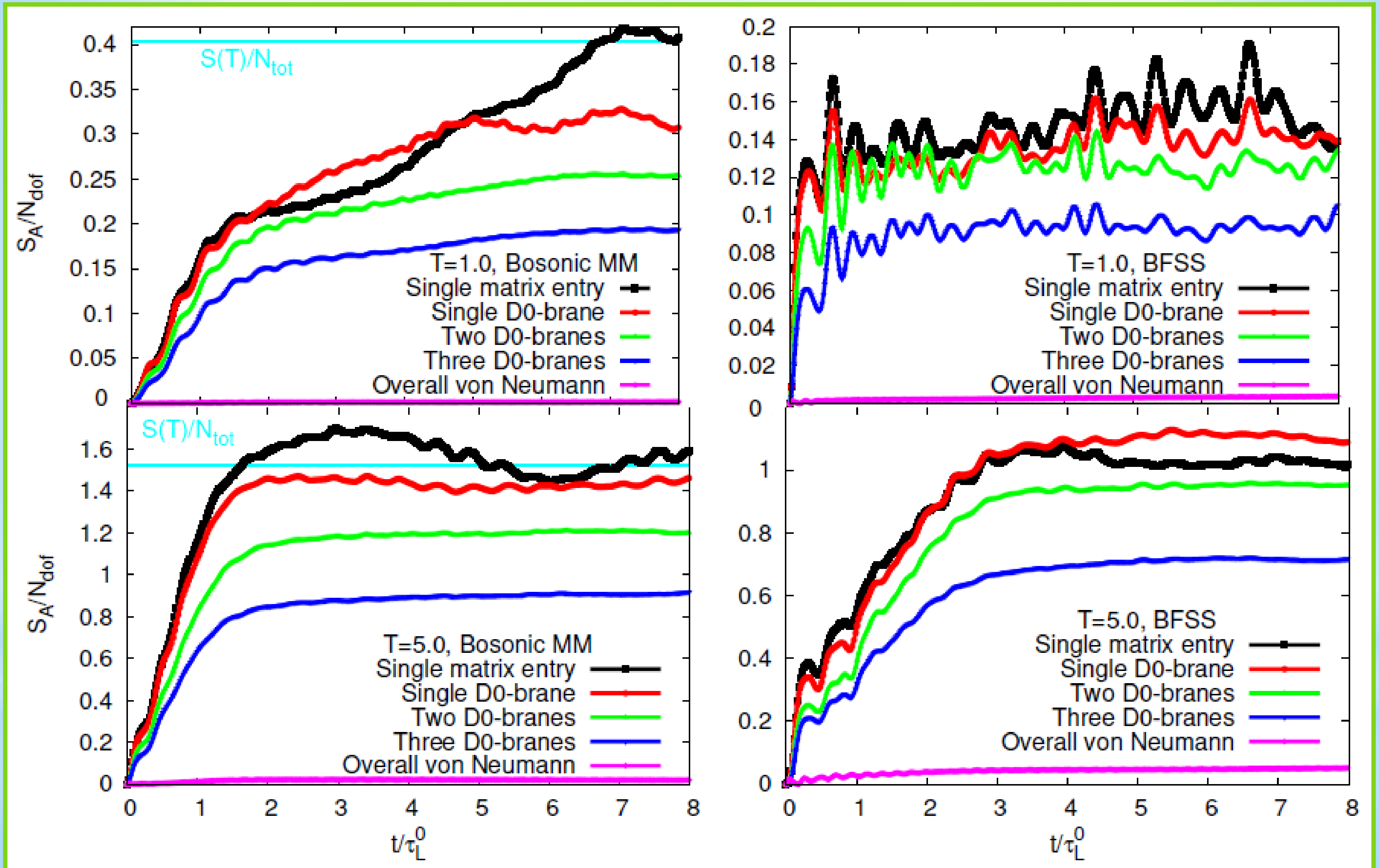


Entanglement vs time



Late-time saturation = information scrambling

Micro- vs. Macro-canonical ensemble



Late-time saturation value?

Micro- vs. Macro-canonical ensemble

- For pure gauge BFSS,
- For sufficiently small subsystem $N_{dof} << N_{tot}$

Saturation value of entanglement entropy is:

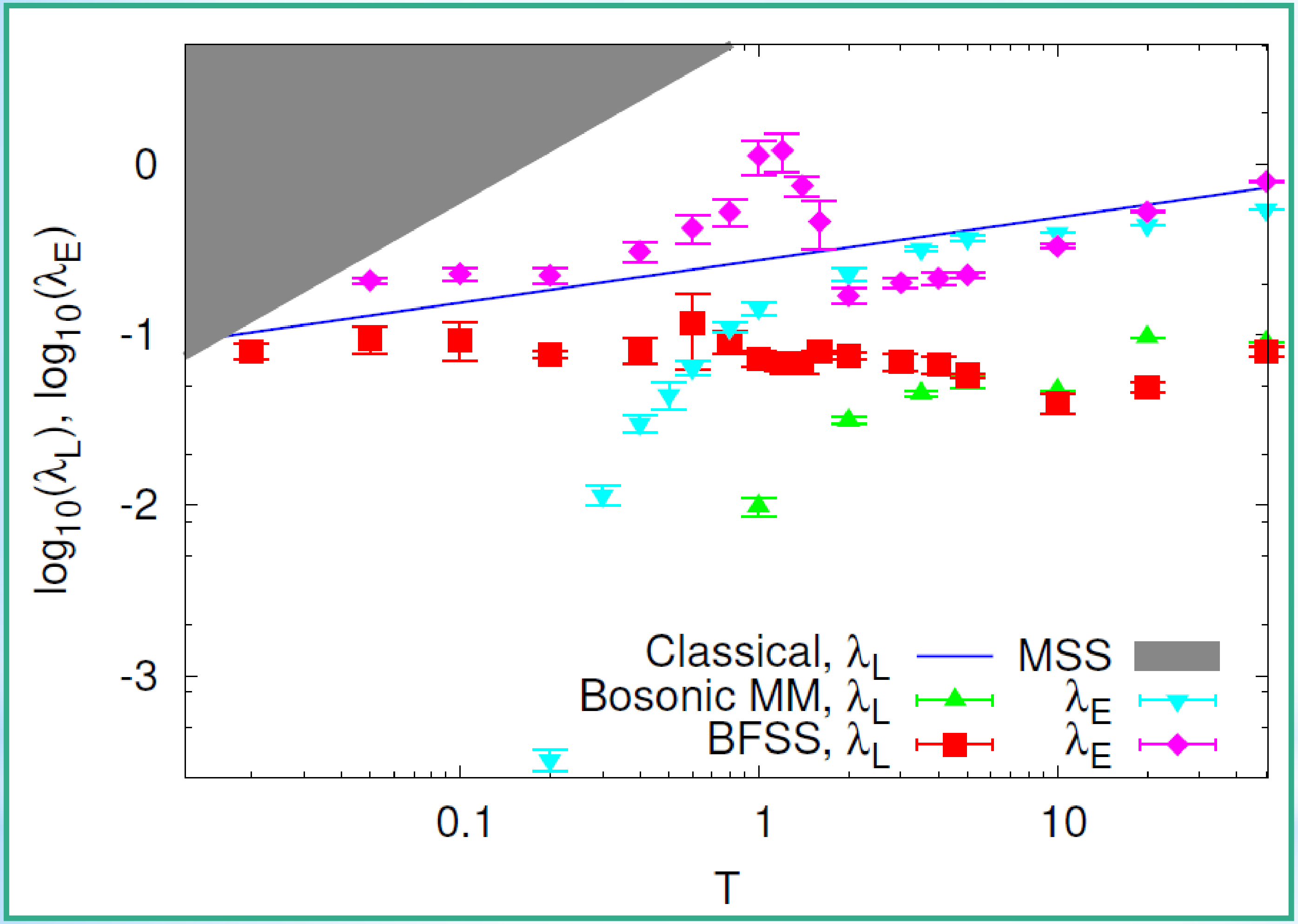
$$S_{max} \approx S(T)\, N_{dof} / N_{tot}$$

Entanglement entropy of a pure state

Von Neumann entropy of a thermal state, defines EoS and T

- Entanglement entropy is locally indistinguishable from thermal entropy
- Real-time thermalization of microcanonical ensemble

Entanglement saturation time
(vs Lyapunov exponents)



Fit $S_E(t)$ as

$A\ Tanh(t/\tau_E)$

Entanglement saturates much faster than Lyapunov time, at high T - classical Lyapunov

Summary: Lyapunov exponents

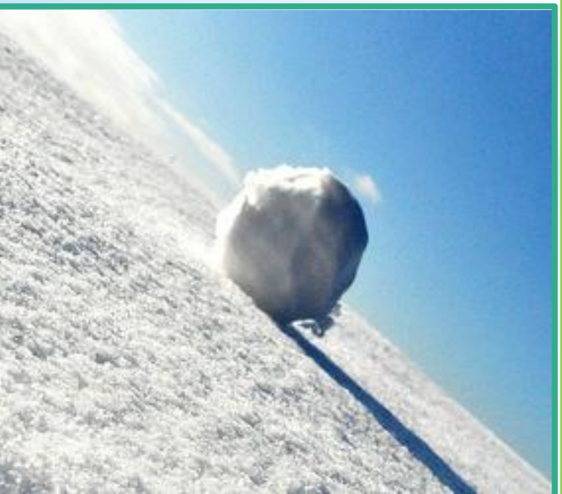
- Longer quantum Lyapunov times vs. classical, important for MSS bound
- "Confining" regime non-chaotic
- Full BFSS model chaotic at all T
- Lyapunov time longer than dissipation time
- Potential bias, since Lyapunov growth at late times, approximation might fail

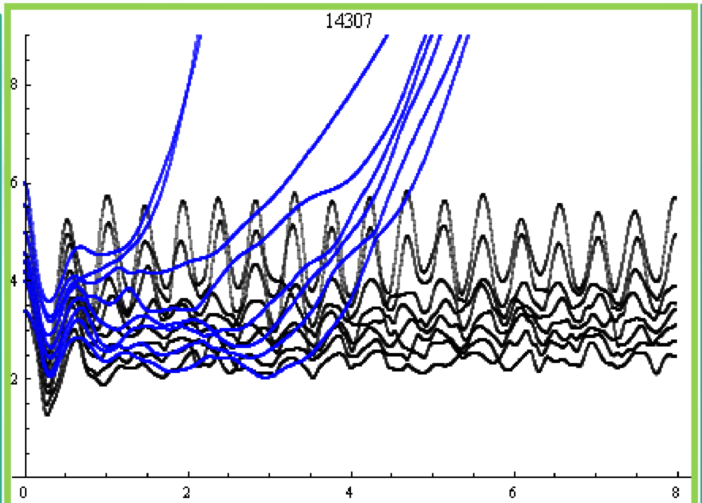
Summary: Entanglement

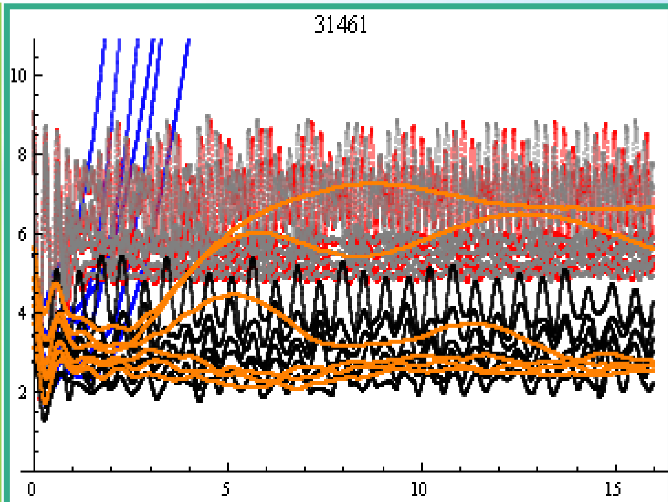
- "Scrambling" behavior for entanglement entropy
- Entanglement saturation timescale is the shortest
- Saturation value given by thermal entropy, Evidence for real-time thermalization!
- At high T governed by classic, rather than quantum Lyapunov
- Entanglement entropy is the best short-time probe of thermalization in our simulations

Summary: Outlook

- Hawking radiation of D0 branes conjectured
- We do see it if quantum bosonic corrections are omitted
- Bosonic quantum corrections remove the instability   imperfect cancellation because of broken SUSY?







Summary

- Gaussian state approximation: ~V^2 scaling of CPU time for QCD/ Yang-Mills
- Feasible on moderately large lattices
- Quantum effects on thermalization?
- Topological transitions in real time?