SU(N) gauge theories beyond large N: universalities of spectral ratios

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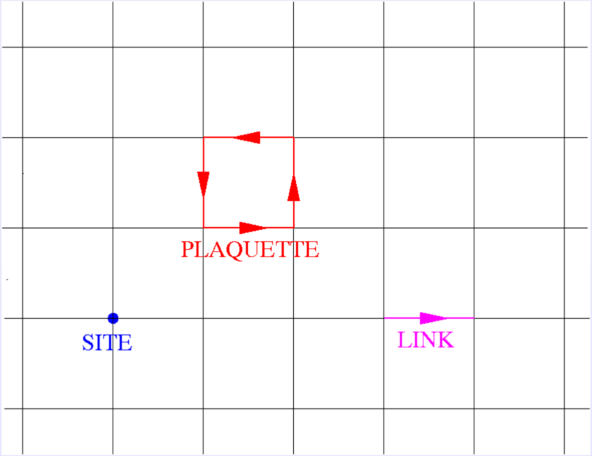
Outline

Outline

The Lattice



Lattice action

Path integral

$$Z = \int (\mathcal{D}U_\mu(i)) \left(\det M(U_\mu)\right)^{N_f} e^{-S_g(U_{\mu\nu}(i))}$$

with

$$U_\mu(i) = Pexp\left(ig\int_i^{i+a\hat{\mu}} A_\mu(x)\mathrm{d}x\right) \ , \ U_{\mu\nu}(i) = U_\mu(i)U_\nu(i+\hat{\mu})U^\dagger_\mu(i+\hat{\nu})U^\dagger_\nu(i)$$

Gauge part

$$S_g = \beta\sum_{i,\mu}\left(1 - \frac{1}{N}\mathcal{R}\mathrm{e}\ \mathrm{Tr}(\mathrm{U}_{\mu\nu}(\mathrm{i}))\right) \quad , \quad with\ \beta = 2N/g_0^2$$

Invariance under SU(N) gauge transformations
$\tilde{U}_\mu(i) = G^\dagger(i)U_\mu(i)G(i+\hat{\mu})$

Wilson fermions

Take the naive Dirac fermions (assumed to be in a representation $\mathcal{R}$) of the gauge group and add an irrelevant term that goes like the Laplacian

$$\begin{aligned} M_{\alpha\beta}(ij) = & \quad (m + 4r)\delta_{ij}\delta_{\alpha\beta} \\ & - \frac{1}{2}\left[(r - \gamma_\mu)_{\alpha\beta} U^R_\mu(i)\delta_{i,j+\mu} + (r + \gamma_\mu)_{\alpha\beta} U^{R\dagger}_\mu(j)\delta_{i,i-\mu}\right] \end{aligned}$$

This formulation breaks explicitly chiral symmetry

Define the hopping parameter

$$\kappa = \frac{1}{2(m + 4r)}$$

Chiral symmetry recovered in the limit $\kappa \to \kappa_c$ (κ_c to be determined numerically)

Quenched approximation

For an observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int (\mathcal{D}U_\mu(i)) \, (\det M(U_\mu))^{N_f} f(M) e^{-S_g(U_{\mu\nu}(i))}}{\int (\mathcal{D}U_\mu(i)) \, (\det M(U_\mu))^{N_f} e^{-S_g(U_{\mu\nu}(i))}}$$

Assume $\det M(U_\mu) \simeq 1$ i.e. fermions loops are removed from the action

The approximation is exact in the $m \to \infty$ and $N \to \infty$ limit ($g^2 N$ is fixed)
$\hookrightarrow$ the large N spectrum is quenched for $m \neq 0$

As N increases, unquenching effects are expected for smaller quark masses

Extracting the spectrum

Trial operators $\Phi_1(t), \dots, \Phi_n(t)$ with the quantum numbers of the state of interest

$$\begin{aligned}
C_{ij}(t) &= \langle 0| \left(\Phi_i(0)\right)^\dagger \Phi_j(t)|0\rangle \\
&= \langle 0| \left(\Phi_i(0)\right)^\dagger e^{-Ht}\Phi_j(0)e^{Ht}|0\rangle \\
&= \sum_n \langle 0| \left(\Phi_i(0)\right)^\dagger |n\rangle\langle n|e^{-Ht}\Phi_j(0)e^{Ht}|0\rangle \\
&= \sum_n e^{-\Delta E_n t}\langle 0| \left(\Phi_i(0)\right)^\dagger |n\rangle\langle n|\Phi_j(0)|0\rangle = \sum_n c^*_{in}c_{jn}e^{-\Delta E_n t}
\end{aligned}$$

Masses can be extracted from diagonal correlators

$$C_{ii} = \sum_n |c_{in}|^2 e^{-am_n t} \underset{t\to\infty}{\rightarrow} |c_{i1}|^2 e^{-am_1 t}$$

Variational calculations

Using the information of the whole correlation matrix allows us to better control the groundstate mass and to determine masses of excitations with a variational procedure

1. Find the eigenvector v that minimises
$$am_1(t_d) = -\frac{1}{t_d} \log \frac{v_i^* C_{ij}(t_d) v_j}{v_i^* C_{ij}(0) v_j}$$
for some t_d
2. Fit $v(t)$ with the law $Ae^{-m_1 t}$ to extract m_1
3. Find the complement to the space generated by $v(t)$
4. Repeat 1-3 to extract $m_2, \ldots, m_n$

Need a good overlap with the state of interest

Fermionic operators

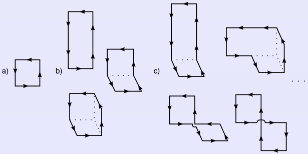
For isotriplet states (flavour index $\alpha \neq \beta$):

Particle	Bilinear	J^{PC}
a_0	$\bar{\psi}_\alpha \psi_\beta$	0^{++}
π	$\bar{\psi}_\alpha \gamma_5 \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_5 \psi_\beta$	0^{-+}
ρ	$\bar{\psi}_\alpha \gamma_i \psi_\beta$, $\bar{\psi}_\alpha \gamma_0 \gamma_i \psi_\beta$	1^{--}
a_1	$\bar{\psi}_\alpha \gamma_5 \gamma_i \psi_\beta$	1^{++}
b_1	$\bar{\psi}_\alpha \gamma_i \gamma_j \psi_\beta$	1^{+-}

Flavour singlet states more difficult to study

Glueball operators

- Using traced Wilson loops of various shapes, build operators $O_i(t)$ that are eigenstates of P and C and under the (discrete) group of rotations transform according to irreducible representations
- Examples of basic contours used



- Continuous spin reconstructed by looking the decomposition of the irreducible representations of the octahedral group in irreps of SO(3)

String tension

Confining potential: $V = \sigma R$

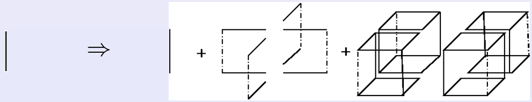
Polyakov loop $P_k(i) = \frac{1}{N} \mathrm{Tr} \prod_{j=0}^{L} U_k(i + j\hat{k})$

$$
\begin{aligned}
P(t) &= \sum_{\vec{n},k} P_k(\vec{n}, t) \\
C(t) &= \langle (P(0))^\dagger P(t) \rangle = \sum_j |c_j|^2 e^{-am_j t} \underset{t \to \infty}{\to} |c_l|^2 e^{-am_l t} \\
am_l &\simeq a^2 \sigma L - \frac{\pi(D-2)}{6L}
\end{aligned}
$$

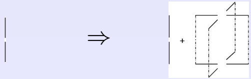
Building extended operators

Starting from links, we can built extended loop operators via

- Smearing



- Blocking



For mesons, extended operators can be built by smearing source and sink operators over a few lattice spacings with a weight function (e.g. Gaussian smearing, wall smearing etc.)

Outline

The 't Hooft's large-N limit

The Conjecture

In the limit $N \to \infty$ and $g \to 0$ with $\lambda = g^2 N$ fixed, physical quantities in SU(N) gauge theories can be expressed as functions of $1/N$ (if N_f fermions in the fundamental representations are present) or $1/N^2$ (in the Yang-Mills case), with a finite large-N limit and a convergent series expansion about that limit down to some $N = N^\star$

Relevance

- Explanation of observed QCD features (OZI rules, stability of particles, . . .)
- Potential for analytic calculations
- Connection with gauge-string dualities

Support

Large-N extrapolation of lattice results

[For a review, see B. Lucini and M. Panero, Phys. Rept. 526 (2013) 93]

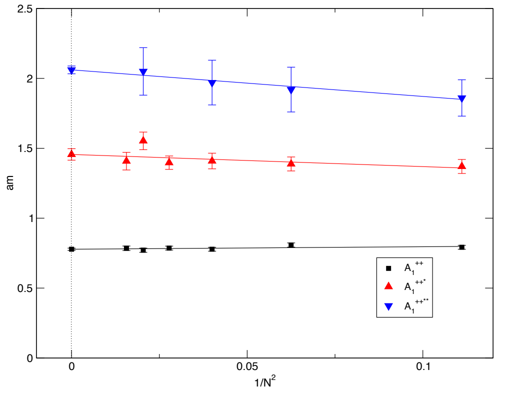
Large-N limit on the lattice

The lattice approach allows us to go beyond perturbative and diagrammatic arguments. In SU(N) YM, for a given observable

1. Continuum extrapolation
   - Determine its value at fixed a and N
   - Extrapolate to the continuum limit
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$
2. Fixed lattice spacing
   - Choose a in such a way that its value in physical units is common to the various N
   - Determine the value of the observable for that a at any N
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

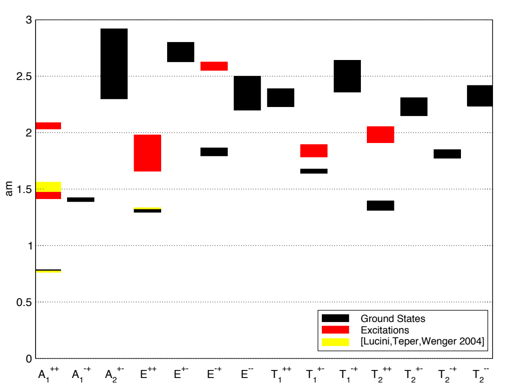
Study performed for various observables both at zero and finite temperature for $2 \leq N \leq 8$ (and $N = 17$!)

The A^{++} glueball channel



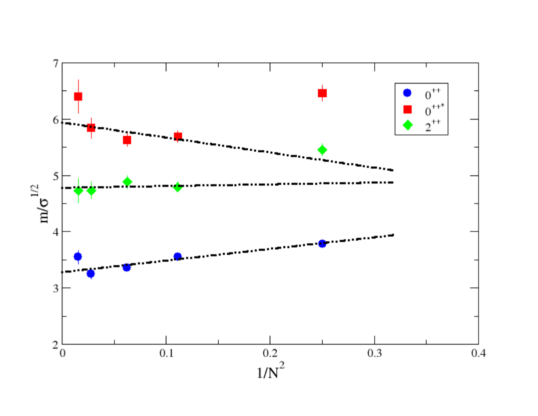
Lattice spacing fixed by requiring $aT_c = 1/6$
[B. Lucini, A. Rago and E. Rinaldi, JHEP 1008 (2010) 119]

The glueball spectrum at $aT_c = 1/6$



[B. Lucini, A. Rago and E. Rinaldi, JHEP 1008 (2010) 119]

Large-N extrapolation of glueball masses



[B. Lucini, M. Teper and U. Wenger, JHEP 0406 (2004) 012]

Glueball masses at $N = \infty$

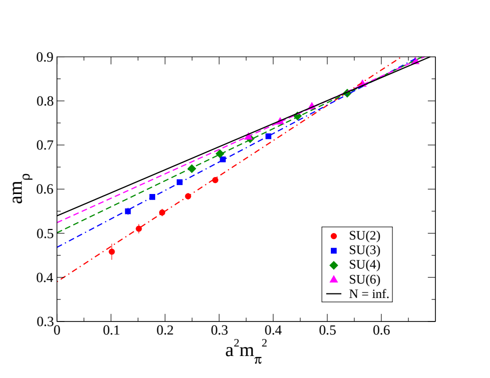
0^{++} $$\frac{m}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2}$$

0^{++*} $$\frac{m}{\sqrt{\sigma}} = 5.93(17) - \frac{2.7(2.0)}{N^2}$$

2^{++} $$\frac{m}{\sqrt{\sigma}} = 4.78(14) + \frac{0.3(1.7)}{N^2}$$

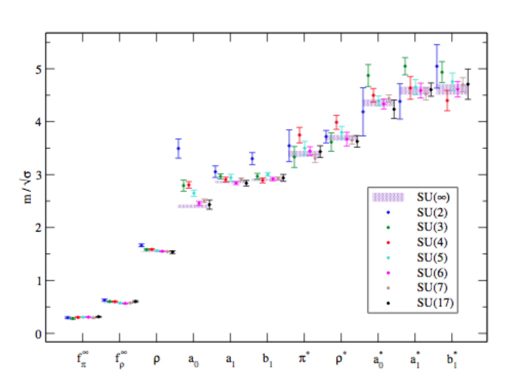
Accurate $N = \infty$ value, small $\mathcal{O}(1/N^2)$ correction

m_ρ vs. m_π^2 at $N = \infty$



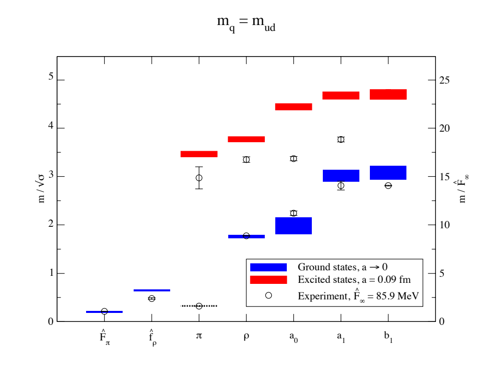
[L. Del Debbio, B. Lucini, A. Patella and C. Pica, JHEP 0803 (2008) 062]
[see also G. Bali and F. Bursa, JHEP 0809 (2008) 110]

The meson spectrum approaching $N = \infty$



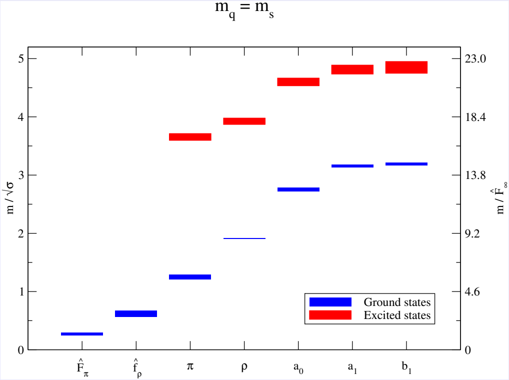
[G. Bali *et al*,, JHEP 1306 (2013) 071]

Comparison with QCD



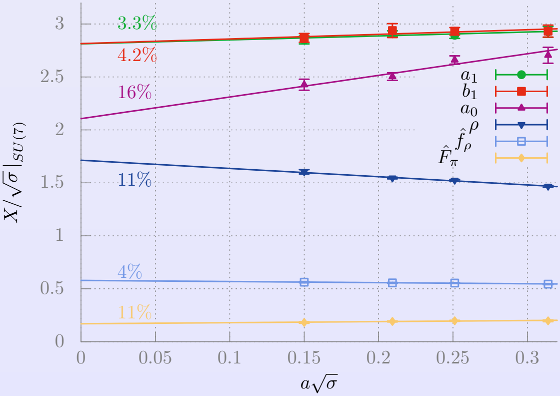
$\sqrt{\sigma}$ fixed from the condition $\hat{F}_\infty = 85.9\ \mathrm{MeV}$, m_{ud} from $m_\pi = 138\ \mathrm{MeV}$ [G. Bali *et al*,, JHEP 1306 (2013) 071]

The strange meson spectrum



m_s fixed from $m_\pi(m_s) = (m^2_{K^\pm} + m^2_{K^0} - m^2_{\pi^\pm})^{1/2} = 686.9\ \mathrm{MeV}$

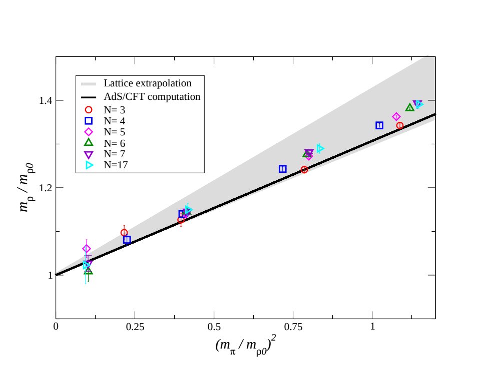
Continuum meson spectrum – SU(7)



What do we learn?

- Lowest-lying mesons broadly compatible with QCD, excitations off by 20% (however, excitations less controlled in our calculation)
- The calculated large-N masses $m_\rho = 753(14)\ \mathrm{MeV}$ and of the $m_\phi = 981(44)\ \mathrm{MeV}$ are remarkably close to their experimental values $m_\rho = 775$ and $m_\phi = 1019\ \mathrm{MeV}$
- Observed degeneracy of (ρ, a_0) and $(a_1, \pi^\star)$ (predicted by χPT)

Meson masses from gauge-string duality



Ads/CFT data from Erdmenger *et al.*, Eur.Phys.J. A35 (2008) 81-133 [arXiv:0711.4467]

The spectrum from the topological string

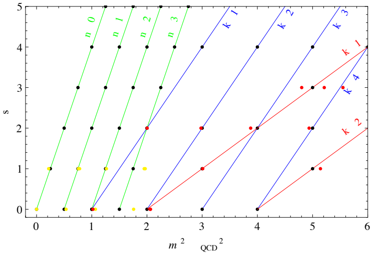


Figure 2. The glueball and meson spectrum of large-N massless QCD: The points in black, and the straight Regge trajectories labelled by the internal quantum numbers, k for glueballs and n for mesons, represent the spectrum implied by the laws Eqs. (1.1).

[M. Bochicchio, *Glueball and meson spectrum in large-N massless QCD*, arXiv:1308.2925 [hep-th]]

Outline

Casimir scaling of the string tension and of the scalar glueball mass

The conjecture

In a Yang-Mills theory, the ratio $m^2_{0^{++}}/\sigma$ is proportional to the ratio of the quadratic Casimir of the adjoint over that of the fundamental representation of the gauge group, with the proportionality constant η being **universal** (i.e. dependent only on the dimensionality):

$$\frac{m^2_{0^{++}}}{\sigma} = \eta(0^{++})\frac{C_2(A)}{C_2(F)}$$

Relevance

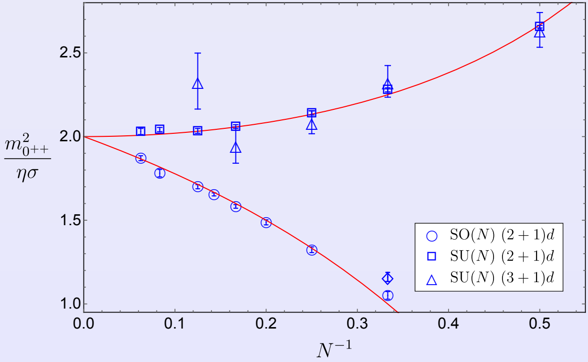
Mechanisms of colour confinement

Support

- Lattice data on glueball masses
- Semi-analytic arguments

[Hong, Lee, Lucini, Piai and Vadacchino, Phys. Lett. B775 (2017) 89, arXiv:1705.00286]

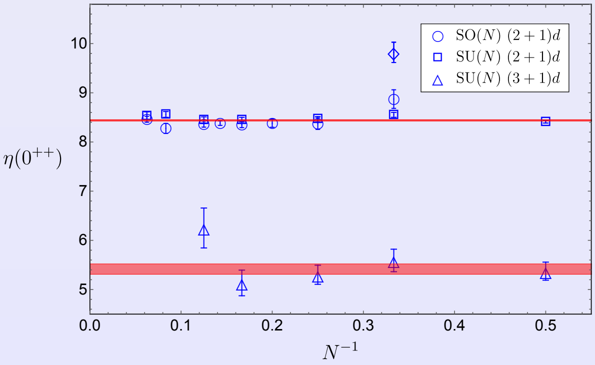
Testing with lattice results



SU(N) data in $d = 3 + 1$ from Lucini, Teper, Wenger, hep-lat/0404008
SU(N) data in $d = 2 + 1$ from Athenodorou, Lau, Teper, arXiv:1504.08126
SO(N) data in $d = 2 + 1$ from Lau, Teper, arXiv:1701.06941

(Also Sp(4) result in $d = 3 + 1$ from Bennett *et al.*, arXiv:1712.04220)

Universality of η

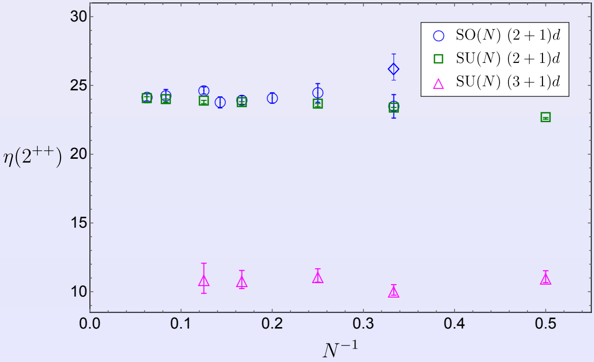


$$\eta(0^{++}) \equiv \frac{m^2_{0^{++}}}{\sigma} \cdot \frac{C_2(F)}{C_2(A)} = \begin{cases} 5.41(12)\,,\ \chi^2 \simeq 1 \text{ for } d = 3+1 \\ 8.440(14)(76)\,,\ \chi^2 \simeq 1.9 \text{ for } d = 2+1 \end{cases}$$

Some (semi-)analytic derivations

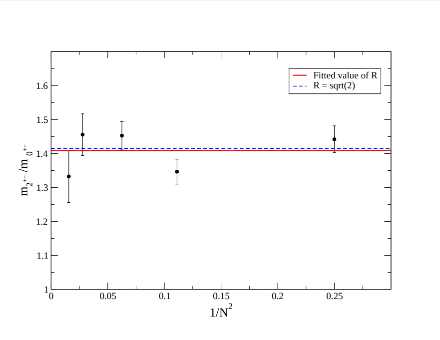
- Ground-state calculation in $d = 2 + 1$ (Leigh, Minik, Yelnikov, 2007)
- Constituent gluon model (Buisseret, Bicudo, . . .)
- Bethe-Salpeter equation (Hong *et al.*, 2017)
- Glueball as a dilaton (Hong *et al.*, 2017)
- Sum rules (Hong *et al.*, 2017)
- Coulomb gauge Hamiltonian (Greensite, private communication, 2017)

Casimir scaling of $m^2_{2^{++}}/\sigma$?



Data are inconclusive: bad constant ansatz for $d = 2+1$, plausible constant ansatz for $d = 3+1$

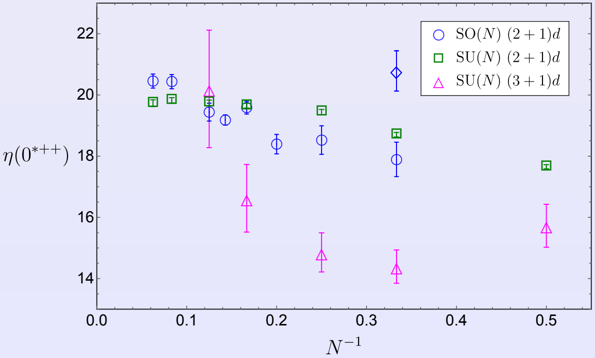
Universality of $m_{2^{++}}/m_{0^{++}}$?



Fit result: $m_{2^{++}}/m_{0^{++}} = 1.409(26)$, $\chi^2 \simeq 1.5$

In $d = 3+1$, is $m_{2^{++}}/m_{0^{++}} = \sqrt{2}$ independently of the gauge group?

(No) Casimir scaling of $m^2_{0^{++*}}/\sigma$



No clear sign of universality in either d=2+1 or d=3+1

Outline

Mass anomalous dimension and glueballs

The conjecture

In an asymptotically free SU(N) gauge theory with N_f fermions in the fundamental or a two index irreducible representation of the gauge group, the anomalous dimension of the fermion condensate is a monotonically increasing function of the glueball mass ratio $m_{2^{++}}/m_{0^{++}}$

Relevance

Studies of nearly-conformal and conformal gauge theories in relation to novel strongly interacting dynamics beyond the standard model

Support

- Gauge-string duality
- Lattice calculations

[A. Athenodorou *et al.*, JHEP 1606 (2016) 114, arXiv:1605.04258]

Perturbative IR fixed point

The running of the coupling in SU(N) gauge theories with N_f fermion flavours transforming in the representation R is determined by the β-function

$$\mu \frac{dg}{d\mu} = -b_0 g^3 - b_1 g^5 + \dots \ ,$$

with

$$b_0 = \frac{1}{(4\pi^2)} \left(\frac{11}{3} N - \frac{4}{3} T_R N_f \right) \ , \qquad b_1 = \frac{1}{(4\pi)^4} \left[\frac{34}{3} N^2 - \frac{20}{3} N T_R N_f - 4 \frac{N^2 - 1}{d_R} N_f \right]$$

Banks-Zaks (perturbative) fixed point (two-loops):

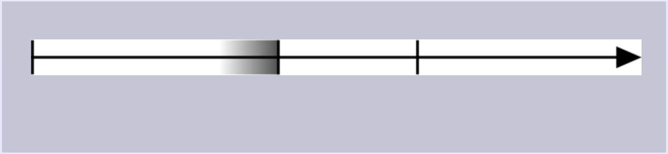
$$\left. \frac{dg}{d\mu} \right|_{2-L} = 0 \qquad \Rightarrow \qquad g^\star \simeq -\frac{b_0}{b_1} \ll 1$$

Starting from $g < g^\star$ in the ultraviolet, in the infrared $g \to g^\star$ (IR fixed point)

Phases of a gauge theory

At fixed N a critical number of flavours N_f^{cu} exists above which asymptotic freedom is lost

Banks and Zaks conjectured that an N_f^{lu} exists such that a non-trivial infrared fixed point appears for $N_f^{lu} \leq N_f \leq N_f^{cu}$ (conformal window)



At fixed fermion representation N_f^{lu} depends on the number of flavours

Near the BZ point naive scaling arguments can not be applied and walking can arise

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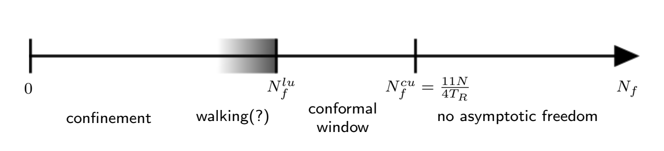

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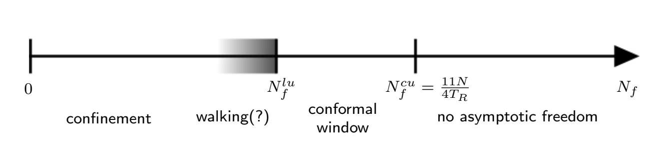

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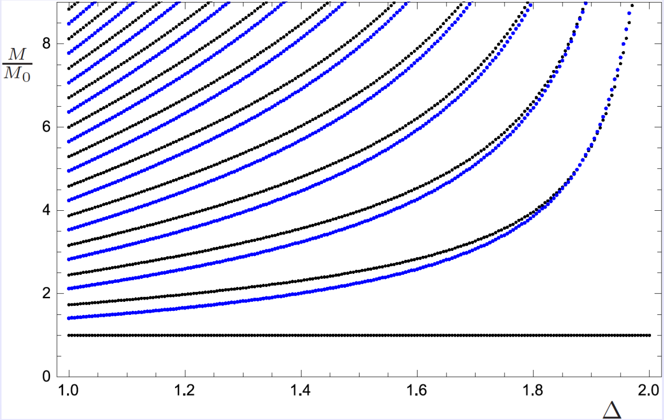
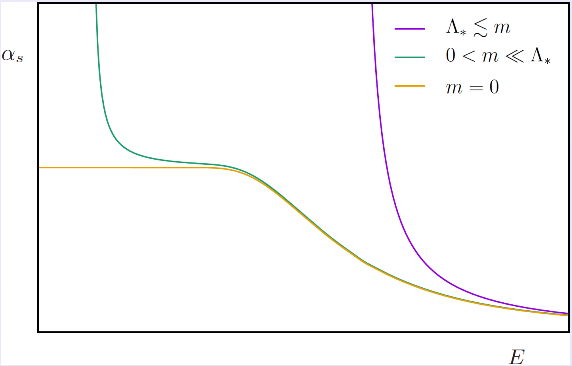
Glueball masses and anomalous dimension in the GPPZ model



Mass of the 2^{++} glueball monotonically increases with $\Delta = 1 + \gamma^\star$

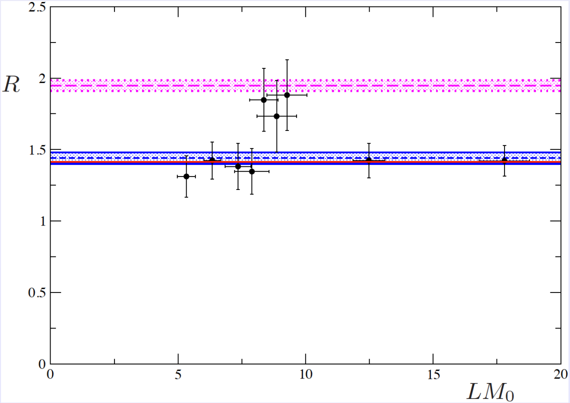
Near-conformality and lattice calculations



Strategy

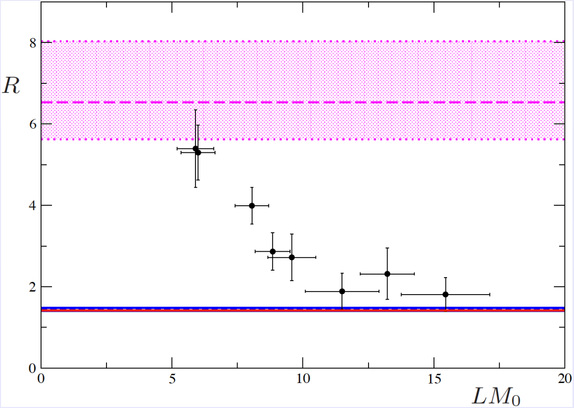
Compute $R = m_{2^{++}}/m_{0^{++}}$ as a function of the box size measured in units of $m_{0^{++}}$

Results for SU(2) with two adjoint flavours



Reference value $\gamma^{\star} = 0.371(20)$
[value from Del Debbio *et al.*, arXiv:1512.08242]

Results for SU(2) with one adjoint flavour



Reference value $\gamma^\star = 0.925(25)$
[value from Athenodorou *at al.*, arXiv:1412.5994]

Outline

Conclusions

Numerical results obtained in the framework of lattice gauge theories suggest that

- A large-N limit *à la* 't Hooft can be constructed in SU(N) gauge theories, with finite-N observables being close (in the sense of a perturbative expansion in $1/N$ or $1/N^2$) to the latter all the way down to the physical case of SU(3)
- In Yang-Mills gauge theories, the ratio of the scalar glueball mass squared over the string tension is proportional to a ratio of quadratic Casimirs, with the proportionality constant being universal (i.e. depending only on the dimension)
- The ratio of the tensor and scalar glueball is $\sqrt{2}$
- In theories with fermions in the fundamental or a two-index representation, the ratio of the tensor over the scalar glueball is a universal monotonically increasing function of the chiral condensate anomalous dimension

Future directions include

- to further explore numerically those conjectures
- to obtain more information from analytic arguments
- to understand the mechanisms that determine those universalities
- to identify potential violations

Relevant quadratic Casimir ratios

$$\frac{C_2(A)}{C_2(F)} = \begin{cases} \frac{2N^2}{N^2-1} \ \text{for} \ \mathrm{SU(N)} \\ \frac{2(N-2)}{N-1} \ \text{for} \ \mathrm{SO(N)} \\ \frac{4(N+1)}{2N+1} \ \text{for} \ \mathrm{Sp(2N)} \end{cases}$$