Loop Quantum Gravity on the Lattice: Examples and Surprises

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Loop Quantum Gravity (LQG) claims to be a quantization of GR. Is this true?

In other words: what is the **classical limit** of LQG?

Answering this question in general is very hard. For this, I will consider some examples, in particular **cosmology** and **black holes**.

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- semiclassical states in LQG
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- other examples
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GR – from continuum metric form to holonomy-flux lattice form

$$
\begin{array}{ccccc}
M & \longrightarrow & \Sigma & \longrightarrow & \gamma \\
g_{\mu\nu}(s,x) & \longrightarrow & (A^I_a(x), E^a_I(x)) & \longrightarrow & (h(e), P^I(e)) \\
G_{\mu\nu}(g, \partial g, \partial^2 g) & \longrightarrow & H(A,E) & \longrightarrow & H_{reg}(h,P)
\end{array}
$$

Note: γ is a collection of edges, $\{e_1, .., e_n\}$, and for each e we have $h(e) \in SU(2)$ and $P^I(e)\tau_I \in \mathfrak{su}(2)$.

LQG (on fixed γ):

- Hilbert space: $\mathcal{H}_\gamma = L_2(SU(2), d\mu)^{\otimes n}$
- state in $\mathcal{H}_\gamma$: $\Psi(g_1, .., g_n)$ with $g_e \in SU(2)$
- elementary operators (on each e): $\hat{h}(e)$ and $\hat{P}^I(e)$ given by

$$\hat{h}(e)\Psi(g_1, .., g_n) = g_e \Psi(g_1, .., g_n)$$

  and

$$\hat{P}^I(e)\Psi(g_1, .., g_n) = \frac{d}{d\epsilon}\Psi(g_1, .. e^{\epsilon\tau_I} g_e, .., g_n)\Big|_{\epsilon=0} = (R_e^I \Psi)(g_1, .., g_n)$$

Other classical functions $f(A, E)$ are promoted to operators by:

1. regularization: $f(A, E) \longrightarrow f_{reg}(h, P)$
2. quantization: $f_{reg}(h, P) \longrightarrow f_{reg}(\hat{h}, \hat{P})$

example: volume operator

Classical, continuum volume of Σ:

$$V = \int_\Sigma d^3x \sqrt{\det q(x)} = \int_\Sigma d^3x \sqrt{\epsilon_{abc}\epsilon_{IJK} E_I^a(x) E_J^b(x) E_K^c(x)}$$

where q_{ab} is the spatial part of $g_{\mu\nu}$.

After so-called "internal regularization", we get **[Ashtekar, Lewandowski 1997]**

$$V_{reg} \propto \sum_x \sqrt{|Q_x|}$$

where

$$Q_x = \sum_{e,e',e'' \in x} \epsilon(e, e', e'')\epsilon_{IJK} P^I(e) P^J(e') P^K(e'')$$

Quantization is now straightforward: replace each P with $\hat{P}$.

example: Hamiltonian operator

Given classical continuum GR Hamiltonian, $H(A, E)$, Thiemann regularization leads to $H_{reg}(h, P)$. Its quantization reads **[Thiemann 1998]**

$$\hat{H} = \sum_x \left[\hat{H}_E(x) - (1 + \beta^2)\hat{H}_L(x) \right]$$

where

$$\hat{H}_E(x) \propto \sum_{e,e',e'' \in x} \epsilon(e, e', e'')\mathrm{Tr}\left[\left(\hat{h}(\Box_{ee'}) - \hat{h}(\Box_{ee'})^\dagger \right) \hat{h}(e'')^\dagger [\hat{h}(e''), \hat{V}] \right]$$

and

$$\hat{H}_L(x) \propto \sum_{e,e',e'' \in x} \epsilon(e, e', e'')\mathrm{Tr}\left[\hat{K}(e)\hat{K}(e')\hat{h}(e'')^\dagger [\hat{h}(e''), \hat{V}] \right]$$

with

$$\hat{K}(e) \propto \hat{h}(e)^\dagger [\hat{h}(e), [\hat{H}_E(v), \hat{V}]]$$

dynamics

Once we have $\hat{H}$, quantum evolution on $\mathcal{H}_\gamma$ is given by

$$\Psi(s) = e^{-i\hat{H}s}\Psi(0)$$

Physically meaningful question: how does volume change in time?
Answer:

$$\langle\Psi(0)|e^{i\hat{H}s}\hat{V}e^{-i\hat{H}s}|\Psi(0)\rangle$$

However... $\hat{H}$ is too complicated to explicitely find its spectrum.

Not so surprising: knowing it would be the quantum equivalent of knowing all solutions to Einstein's equation! What to do?

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Fix γ. Consider a family of states in $\mathcal{H}_\gamma$:

$$\Psi(g_1, .., g_n) = \psi_{S_1}(g_1)..\psi_{S_n}(g_n)$$

where for each edge we have

$$\psi_{S_e}(g_e) := \frac{1}{N} a(g_e) e^{-S_e(g_e)/t}$$

and S_e has two properties:

- $\mathrm{Re}(S_e)$ has a single minimum, denoted u_e
- the Hessian of S_e at $g_e = u_e$ is non-degenerate

Then, it follows

$$\langle\Psi|\hat{h}(e)|\Psi\rangle = u_e + \mathcal{O}(t), \quad \langle\Psi|\hat{P}^I(e)|\Psi\rangle = \frac{1}{t}\left[(R^I \mathrm{Im}(S_e))(u_e) + \mathcal{O}(t)\right]$$

and relative dispersions are

$$\Delta\hat{h}/\langle\hat{h}\rangle = \mathcal{O}(t), \qquad \Delta\hat{P}^I/\langle\hat{P}^I\rangle = \mathcal{O}(t)$$

An example of such Ψ are the famous complexifier coherent states.

Thus, Ψ is labelled by $u_e \in SU(2)$ and $\xi_e := \tau_I R^I \mathrm{Im}(S_e) \in \mathfrak{su}(2)$:

$$\Psi_{(u_1,..,u_n;\xi_1,..,\xi_n)}(g_1,..,g_n) \text{ is peaked on } h(e) = u_e, P^I(e) = \xi_e^I$$

But $(h(e), P^I(e))$ is 1-to-1 to a **discrete** geometry on Σ.

Result: State $\Psi_{(u;\xi)}$ encodes a discrete 3-geometry.

Questions: How does this state evolve? Does it remain peaked?

To answer, we should compute $e^{-i\hat{H}s}\Psi_{(u,\xi)}$, but it is too hard!

Claim (not yet proven): The state we constructed satisfies

$$e^{-i\hat{H}s}\Psi_{(u;\xi)} = \Psi_{(u(s);\xi(s))} + \mathcal{O}(t)$$

where s-dependence of u_e and ξ_e is given by Hamilton's equations

$$\frac{d}{ds}u_e = \{u_e, H_{eff}(u,\xi)\}, \qquad \frac{d}{ds}\xi_e = \{\xi_e, H_{eff}(u,\xi)\}$$

with effective Hamiltonian given by

$$H_{eff}(u,\xi) := \langle \Psi_{(u;\xi)} | \hat{H} | \Psi_{(u;\xi)} \rangle$$

If the claim is true, then it means that for the leading order in semiclassicality parameter t **we only need to compute** H_{eff}!

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Metric on M: $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$.

- Ashtekar variables on Σ: $(A^I_a = c\delta^I_a, E^a_I = p\delta^a_I)$, where $p = a^2$ and c is conjugated
- choice of graph γ: cubic lattice adapted to coord's
- reading off the discrete data $(u_1, ..u_n; \xi_1, .., \xi_n)$: compute holonomies and fluxes of (A^I_a, E^a_I) along the edges of γ:

$$u_e \stackrel{!}{=} h(e_a) = e^{c\mu\tau_a/2}, \qquad \xi^I_e \stackrel{!}{=} P^I(e_a) = \delta^I_a \mu^2 p$$

- this data is sufficient to construct $\Psi_{(u_1,..,u_n;\xi_1,..,\xi_n)}$.

We call $\Psi_{(c,p)} := \Psi_{(u_1,..,u_n;\xi_1,..,\xi_n)}$ a **cosmological state** in LQG

[Liegener, AD 2017].

Computation of expectation value of $\hat{H}$ on cosmological states:

$$H_{eff}^{cosmo}(c,p) = \langle \Psi_{(c,p)} | \hat{H} | \Psi_{(c,p)} \rangle = = -\frac{3}{\kappa\beta^2}\sqrt{p}\frac{\sin^2(\mu c)}{\mu^2}\left[1-(1+\beta^2)\sin^2(\mu c)\right] + \mathcal{O}(t)$$

Compare with the classical Hamiltonian of cosmology:

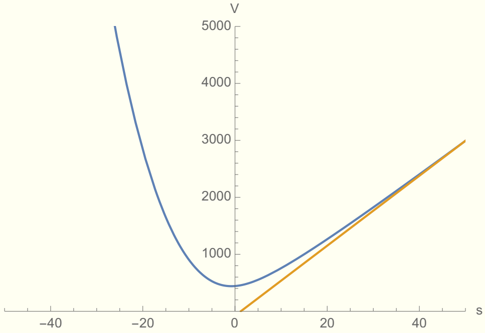
$$H_{class}^{cosmo}(c,p) = -\frac{3}{\kappa\beta^2}\sqrt{p}c^2$$

Remarks:

- $H_{eff}^{cosmo} \xrightarrow{\mu\to 0} H_{class}^{cosmo}$, so we recover the continuum limit.
- $H_{LQC}^{cosmo} \neq H_{eff}^{cosmo}$: the "holonomy correction" is not enough.

It is easy to (numerically) integrate Hamilton's equations.

Evolution of volume $V = p^{3/2}$:



(blue $= H_{eff}^{cosmo}$, gold $= H_{class}^{cosmo}$)

Remarks:

- Big Bang singularity replaced by a non-symmetric bounce
- late universe coincides with classical cosmology
- pre-bounce universe is an exponentially contracting spacetime
- the behavior in the contracting branch appears in classical comsmology in presence of a cosmological constant

$$\Lambda = \frac{3}{\mu^2(1+\beta^2)^2}$$

- natural choice of vacuum for perturbations **[Agullo 2018]**
- a modified LQC Hamiltonian operator can be found that reproduces this behavior **[Assanioussi, AD, Liegener, Pawlowski 2018]**

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example: Bianchi I

Metric on M: $g_{\mu\nu} = \mathrm{diag}(-1, a_1^2, a_2^2, a_3^2)$.

- Ashtekar variables on Σ: $(A_a^I = c_a \delta_a^I, E_I^a = p_a \delta_I^a)$
- choice of graph γ: same as for cosmology
- reading off the discrete data $(u_1, ..u_n; \xi_1, .., \xi_n)$:

$$u_e \overset{!}{=} h(e_a) = e^{c_a \mu \tau_a / 2}, \qquad \xi_e^I \overset{!}{=} P^I(e_a) = \delta_a^I \mu^2 p_a$$

- this data is sufficient to construct $\Psi_{(c_a, p_a)} := \Psi_{(u_1, .., u_n; \xi_1, .., \xi_n)}$.

Expectation value of $\hat{H}$ gives

$$\hat{H}_{eff}^{BI} = -\frac{1}{\kappa\beta^2}\sqrt{p_1 p_2 p_3}\frac{\sin(c_1\mu)\sin(c_2\mu)}{\mu^2}\left(1 - (1+\beta^2)\Gamma_{123}\right) + \text{cyclic}$$

where $\Gamma_{ijk} = 1 - \frac{\cos(c_i\mu) + \cos(c_k\mu)}{2}\frac{\cos(c_j\mu) + \cos(c_k\mu)}{2}$.

Observations: $\hat{H}_{eff}^{BI} \xrightarrow{\mu \to 0} H_{class}^{BI}$ but $\hat{H}_{eff}^{BI} \neq H_{LQC}^{BI}$.

example: the simplest black hole

In spherical coords:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) ds^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

We are interested in the BH interior, $r < 2M$: there, r is time, so

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) ds^2 + r^2 d\Omega^2$$

Change coords: $r \to T$ such that $dT^2 = \left(\frac{2M}{r} - 1\right)^{-1} dr^2$:

$$g_{\mu\nu}dx^\mu dx^\nu = -dT^2 + f(T)^2 ds^2 + g(T)^2 d\Omega^2$$

We consider a metric of this form with **general** $f(T)$ and $g(T)$.

example: the simplest black hole

In terms of Ashtekar variables A_a^I and E_I^a, we have

$$A_1^1 = -\beta a, \qquad A_2^2 = -\beta b, \qquad A_3^3 = -\beta b \sin\theta, \qquad A_3^1 = \cos\theta$$

and

$$E_1^1 = p_a \sin\theta, \qquad E_2^2 = \frac{p_b}{2}\sin\theta, \qquad E_3^3 = \frac{p_b}{2}$$

where $p_a = g^2$, $p_b = 2fg$ and a and b are conjugated to them.

Much less trivial than other examples, since:

- A not diagonal
- both A and E depend on the point (i.e., its coordinate θ)

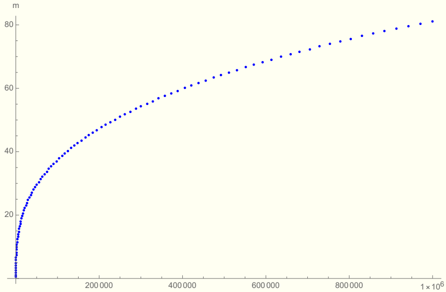
Moreover, we must change our choice of graph γ!

example: the simplest black hole

Numerical integration reveals BH $\rightarrow$ WH transition.

The boundary of the evolution is the WH horizon of mass

$$m(M) = M^{\frac{1}{3}}$$



Different from **[Ashtekar, Olmedo, Singh 2018]**, where a linear relation is found.

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summary

Results:

- Proposal for quantum gravity Hamiltonian $\hat{H}$ on the lattice γ
- Proposal for states $\Psi_{(u;\xi)} \in \mathcal{H}_\gamma$ representing discrete 3-geometries $(h(e), P^I(e)) = (u_e, \xi^I_e)$
- Theorem (to be shown): to leading order

$$e^{-i\hat{H}s}\Psi_{(u;\xi)} = \Psi_{(u(s);\xi(s))}$$

  with evolution of labels generated by

$$H_{eff}(u,\xi) = \langle \Psi_{(u;\xi)} | \hat{H} | \Psi_{(u;\xi)} \rangle$$

- Several examples:
  - Cosmology
  - Bianchi I
  - Schwartzschild-like black holes

summary

Open questions:

- What is role of lattice γ? Would $\gamma' \neq \gamma$ change the result?
- What is role of discreteness scale μ?
  If it is physical, then we have important implications:
  - Big Bang replaced by bounce
  - Schwartzschild singularity replaced by a transition to white hole

  On the other hand, the limit $\mu \to 0$ reproduces continuum GR.
- More examples to be studied:
  - $k \neq 0$ cosmologies
  - "real" black holes (spherical collapse)
  - gravitational waves
  - ...other ideas?

GRAZIE!