June 2018

1) Kinetic theory vs. hydrodynamics

Complementary: hydrodynamics $\longleftrightarrow$ macroscopic

$T(x), \mu_b(x), u^\mu(x)$

(derived quantities:

$e(x), n_b(x), p(x)$)

EoS: $p = p(e, n_b) = p(T, \mu_b)$

kinetic theory $\longleftrightarrow$ microscopic: $f(x,p)$

$f(x^\mu, p^\mu)$: classical kinetic theory: $\rho(E, \vec{p}) \sim \delta(E^2 - E_p^2)$

($g=0$)

$$= \frac{1}{2E_p}\left(\delta(E-E_p) + \delta(E+E_p)\right)$$

$$E_p = \sqrt{\vec{p}^{\,2} + m^2}$$

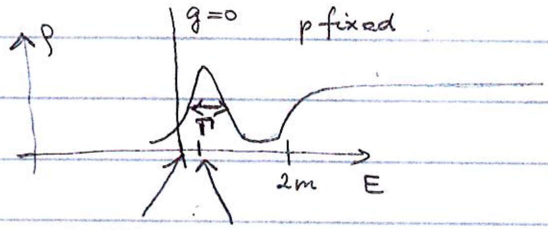
$\rightarrow$ on-shell particles, $f(\vec{x}, \vec{p}, t)$, $E = E_p$

quantum kinetic theory: ($g \neq 0$)

(i) weak coupling: $\rho(E, \vec{p}) \sim \dfrac{E^2 \Gamma^2(T)}{(E^2 - \vec{p}^{\,2} - m^2(T))^2 + E^2\Gamma^2}$

$$m^2(T) = m^2 + \#(gT)^2$$

$$\Gamma(T) \sim g^2 T \qquad \text{for } g \ll 1$$



$E_p = \sqrt{p^2+m^2}$ $\qquad \sqrt{p^2 + m^2 + \# g^2 T^2} = E_p(T)$

$f(x,p) \longrightarrow W(x^\mu, p^\nu)$ Wigner function

(ii) strong coupling $\longrightarrow$ no quasiparticles

structureless spectral function

no Boltzmann-like description

Boltzmann equation:

$$m \frac{df}{d\tau} = C \qquad \text{(collision term)}$$

$$\rightarrow \quad \underbrace{m\,\dot{x}^\mu}_{p^\mu} \frac{\partial f}{\partial x^\mu} + \underbrace{m\,\dot{p}^\mu}_{mF^\mu} \frac{\partial f}{\partial p^\mu} = C \qquad (\text{E\&M}: mF^\mu = q F^{\mu\nu} p_\nu \ \text{Lorentz force})$$

If no longrange forces $(F^{\mu\nu}=0)$

$$\Rightarrow \boxed{p^\mu \partial_\mu f(x,p) = C(x,p)}$$

If system is <u>dilute</u> $\left(f(x_1 x_2, p_1 p_2) \approx f(x, p_1)\, f(x_2, p_2)\right)$
and <u>weakly interacting</u> $\left(g \ll 1,\ \rho(E,\vec{p}) \approx \delta(E^2 - E_p^2(T))\right)$

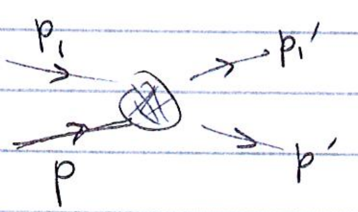
$\Rightarrow$ Boltzmann equation $\boxed{p \cdot \partial f = C[f]}$

$$C(x,p) = \frac{1}{2} \int \frac{d^3p_1}{2E_1} \frac{d^3p'}{2E'} \frac{d^3p_1'}{2E_1'} \delta^{(4)}(p + p_1 - p' - p_1')\, \sigma(s,\vartheta)$$

$$\times \Big[f(x,p') f(x,p_1') (1 \pm f(x,p))(1 \pm f(x,p_1))$$

$$- f(x,p) f(x,p_1) (1 \pm f(x,p'))(1 \pm f(x,p_1')) \Big]$$

where $s = (p+p_1)^2 = (p_1 + p_1')^2$ and



$$\cos\vartheta = \frac{(p-p_1)\cdot(p'-p_1')}{(p-p_1)^2}$$

(scattering angle in cm system

The collision term vanishes in two limits:

(1) free streaming ($g = 0 \Rightarrow \sigma = 0$, ideal gas)

$\Rightarrow$ solution $\quad f(\vec{x}, \vec{p}; t) = f\left(\vec{x} - \frac{\vec{p}}{E}(t - t_0), \vec{p}; t_0\right)$

(2) extremely strong coupling ($g \to \infty$, $\sigma \to \infty$, ideal fluid)

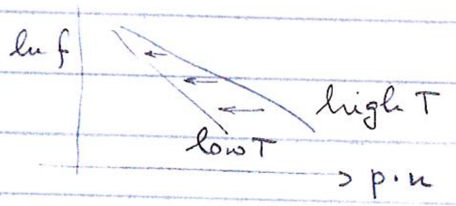
$$f(x,p) \xrightarrow{\text{collisions}} f_{eq}(x,p) = \frac{1}{e^{(p\cdot u(x) - \mu(x))/T(x)} + a}$$

$a = \begin{cases} 0 & \text{Boltzmann} \\ 1 & \text{Fermi} \\ -1 & \text{Bose} \end{cases}$

For the local equilibrium distribution $f_{eq}\left(\frac{p\cdot u(x)}{T(x)}, \frac{\mu(x)}{T(x)}\right)$

the gain and loss terms cancel in the collision term

If the system expands and accelerates, $T(x)$ decreases and $f_{eq}\left(\frac{p\cdot u(x)}{T(x)}, \frac{\mu(x)}{T(x)}\right)$ changes shape (steeper slope in a log-plot:



)

This requires particles to change their momenta (particles drift from higher to lower energies, on average)

Even for $g \to \infty$, this cannot happen instantaneously (quantum mechanics!)

$\rightarrow \quad f(x,p) = f_{eq}(x,p) + \delta f(x,p)$

For $g \to \infty$ or small expansion rates $\rightarrow$ δf small

For $g \to 0$ or large expansion rates $\rightarrow$ δf large

Will show: $\delta f = 0 \rightarrow$ ideal fluid dynamics
δf small $\rightarrow$ viscous fluid dynamics | δf large: hydrodynamic theory.

2) Ideal fluid dynamics

$$f(x,p) = f_{eq}(x,p) = \frac{1}{e^{p\cdot u(x)/T(x)} + a}$$ (neglect chem. potential mostly)

$u^\mu(x)$ = local fluid velocity $\longrightarrow u^\mu_{lrf} = (1, \vec{0}) \equiv \bar{u}^\mu$

$u^\mu u_\mu = 1.$

In l.r.f. $$f_{eq}(\bar{x},\bar{p}) = \frac{1}{e^{\bar{p}\cdot\bar{u}/T} + a} = \frac{1}{e^{\bar{E}/T}+a} = f_{eq}(\bar{E})$$

Moment equations:

Integrate Boltzmann equation
$$p^\mu \partial_\mu f(x,p) = C(x,p)$$

with integration measure $$dP \equiv \frac{1}{(2\pi\hbar)^3}\frac{d^3p}{E_p} = 2\theta(p^0)\delta(p^2-m^2(T))\frac{d^4}{(2\pi}$$

Write $$g^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu) + u^\mu u^\nu = \underbrace{u^\mu u^\nu}_{\text{timelike projector in lrf}} + \underbrace{\Delta^{\mu\nu}}_{\text{spacelike projector in l.r.f.}}$$

$$p^2 = p^\mu p_\mu = p_\mu u^\mu u^\nu p_\nu + p_\mu \Delta^{\mu\nu} p_\nu = (p\cdot u)^2 + p\cdot\Delta\cdot p$$
$$= \bar{E}^2 - \vec{\bar{p}}^2$$

$$d^4p = d^4\bar{p}\,, \quad \theta(p^0) = \theta(\bar{p}^0) = \theta(\bar{E})$$

$$\Rightarrow \int dP\, A(p)\, f_{eq}\left(\frac{p\cdot u}{T}\right) = \int \frac{d^4p}{(2\pi)^3} A(p)\, 2\theta(p^0)\,\delta(p^2-m^2) f_{eq}$$

$$= \int \frac{d^4\bar{p}}{(2\pi)^3} A(\bar{p})\, 2\theta(\bar{E})\, \underbrace{\delta(\bar{E}^2 - (\bar{p}^2+m^2))}_{\frac{1}{2E_p}\left(\delta(\bar{E}-E_p) + \delta(\bar{E}+E_p)\right)} f(\bar{E})$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3\bar{E}} f_{eq}(\bar{E})\, A(\bar{p})$$ $\Rightarrow$ in local equilibrium these moments can be easily worked out in lrf coords.

Define $\hat{I}^{\mu\nu\ldots\sigma}[f] = \int dP\, p^\mu p^\nu \ldots p^\sigma f(x,p)$

(i) 0^{th} moment:

$$\int dP\, p^\mu \partial_\mu f = \int dP\, C(x,p)$$

$$\Rightarrow \partial_\mu \underbrace{\int dP\, p^\mu f}_{N^\mu \text{ particle number current}} = \partial_\mu \int_{\vec{p}} v^\mu f = \int dP\, C$$

$\int_{\vec{p}} \equiv \int \frac{d^3p}{(2\pi)^3}$

$v^\mu = \frac{p^\mu}{E_p}$

Now net baryon number is conserved:

$$\partial_\mu (N^\mu - \bar{N}^\mu) = 0 \quad \Rightarrow \quad \int dP (C - \bar{C}) = 0$$

zeroth moment of collision kernel for quarks minus antiquarks vanishes (must hold for any approx. of C!)

(ii) 1^{st} moment

$$\int dP\, p^\lambda\, p^\mu \partial_\mu f = \int dP\, p^\lambda C$$

$$\partial_\mu \underbrace{\int dP\, p^\mu p^\lambda f(x,p)}_{T^{\mu\lambda}(x)} = \int dP\, p^\lambda C(x,p)$$

Since energy and momentum are conserved, we have

$$\partial_\mu (T^{\mu\nu}_q + T^{\mu\nu}_{\bar{q}} + T^{\mu\nu}_g) = 0 \quad \Rightarrow \quad \int dP\, p^\lambda (C_q + C_{\bar{q}} + C_g) = 0$$

first moment of sum of collision kernels vanishes (in any approximatio

(iii) Higher moments (needed in some derivations of viscous hydrodynam

$$\text{e.g. } \partial_\mu \underbrace{\int dP\, p^\mu p^\nu p^\lambda f}_{F^{\mu\nu\lambda}} = \int dP\, p^\nu p^\lambda C \equiv P^{\nu\lambda}$$

Since there are no other collision invariants, $P^{\nu\lambda} \neq 0$ in general.

But $F^{\mu\nu}{}_{\nu} = m^2 j^\nu$ and $(P_q - P_{\bar q})^\mu{}_\mu = 0$ since $p^2 = m^2$ and $\int dP (C - \bar C) = 0$

For ideal fluids $(f(x,p) \equiv f_{eq}(x,p))$ the moments have simple form:

$$N_B^\mu(x) = n_B(x)\, u^\mu(x) \quad \Big| \quad u \cdot N_B = n_B(x) = 3 \int_{\vec p} d^3\bar p \left(f_q(\bar E) - \bar f_q(\bar E) \right)$$

$$f_q = \frac{3_c \cdot 2_s \cdot N_f}{e^{(\bar E - \mu_q)/T} + 1} \qquad \bar f_q = \frac{3_c \cdot 2_s \cdot N_f}{e^{(\bar E + \mu_q)} + 1}$$

$$T^{\mu\nu}(x) = e(x)\, u^\mu(x)\, u^\nu(x) - p(x)\, \Delta^{\mu\nu}(x)$$

projection technique

$$e = u_\mu T^{\mu\nu} u_\nu = \int dP (u \cdot p)^2 f(u \cdot p) = \int d^3\bar p\, \bar E (f + \bar f + f_g)$$

$$f_g = \frac{8_c \cdot 2_s}{e^{\bar E / T} - 1}$$

$$p = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} = \int \frac{d^3 \bar p}{\bar E} \frac{\bar p^2}{3} (f + \bar f + f_g)$$

NB: $\boxed{T^{\mu\nu} u_\nu = e u^\mu}$ $\Rightarrow$ u^ν = timelike eigenvector of $T^{\mu\nu}$, with eigenvalue

Trace of $T^{\mu\nu}$: $\quad T^\mu_\mu(x) = e(x) - 3p(x)$

For a classical system of massless particles: $\boxed{T^\mu_\mu = 0}$

conformal symmetry!

$\Rightarrow$ $\boxed{p = \frac{e}{3}}$ for a conformally invariant system.

(CMB, QGP, early universe where $T \gg m_i$, etc

NB Strictly speaking, the ideas of an ideal fluid (requires $g \to \infty$, $\sigma \to \infty$) and of classical on-shell particles are mutually inconsistent. Also, an ideal fluid will always show deviations from conformal symmetry, since interactions break the scale invariance ($T^\mu_\mu \neq 0$, trace "anomaly")

But this only complicates the kinetic microscopic description — macroscopic hydrodynamics, with nonpert. EOS $p(e, n_B)$, remains valid.

3) Ideal fluid equations of motion

$\partial_\mu N_B^\mu = 0$	1 equ.	unknowns:	$n_B(x)$, $e(x)$, $p(x)$ } 3
$\partial_\mu T^{\mu\nu} = 0$	4 equs.		$u^\mu(x)$ } 3
$p = p(e, n_B)$	EoS (1 equ.)		
	6 equs.		6 unknowns ✓

(i) EoM in local rest frame (best for understanding the physics)

Write $\partial_\mu = u_\mu u^\nu \partial_\nu + \Delta_\mu{}^\nu \partial_\nu \equiv u_\mu D + \nabla_\mu$

D: time derivative in l.r.f.

∇_μ: spatial gradient in l.r.f.

Denote $Df \equiv \dot{f}$

(a) $\partial_\mu N_B^\mu = 0 = \partial_\mu (n_B u^\mu) = u^\mu \partial_\mu n_B + n_B \underbrace{\partial_\mu u^\mu}_{\theta\,=\,\text{local expansion rate}} = \dot{n}_B + n_B \theta$

note: $\partial_\mu u^\mu = \partial \cdot u = \nabla \cdot u = \nabla_\mu u^\mu$

since $\dot{u}_\mu$ is $\perp u^\mu$: $\dot{u}_\mu u^\mu = \frac{1}{2} D(\underbrace{u^\mu u_\mu}_{1}) = 0$

$$\Rightarrow \boxed{\dot{n}_B = -n_B \theta}$$ (similar for any other conserved density!)

n_B only changes because of expansion/contraction of the fluid

(b) $\partial_\mu T^{\mu\nu} = 0 = \partial_\mu \left[(e+p) u^\mu u^\nu - p g^{\mu\nu}\right] = u^\nu \underbrace{u^\mu \partial_\mu}_{D} (e+p) + (e+p)\theta u^\nu$

$+ (e+p) \underbrace{u^\mu \partial_\mu u^\nu}_{\dot{u}^\nu} - \partial^\nu p$

time-like component: project with u_ν:

(A) $D(e+p) + (e+p)\theta + (e+p) \underbrace{u_\nu D u^\nu}_{\frac{1}{2} D(u \cdot u) = 0} - Dp = 0 \Rightarrow \boxed{\dot{e} = -(e+p)\theta}$

energy density changes by expansion only, but faster than n_B due to work done by pressure

(B) Plug this back in

$$u^\nu(\dot{e}+\dot{p} + \underbrace{(e+p)\theta}) + (e+p)\dot{u}^\nu \overbrace{- u^\nu Dp - \nabla^\nu p}^{-\partial^\nu p} = 0$$

$$\Rightarrow \boxed{\dot{u}^\nu = \frac{\nabla^\nu p}{e+p}} \quad \left(\text{Newton's second law: } a = \frac{F}{m}\right)$$

pressure gradients are the driving force for hydrodynamic expansion

For an EOS of type $p = c_s^2 e$ $\quad \left(c_s^2 = \frac{\partial p}{\partial e}\right)$
this reduces to

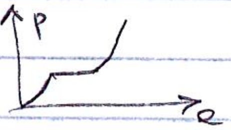
$$\boxed{\dot{u}^\nu = \frac{c_s^2}{1+c_s^2}\frac{\nabla^\nu e}{e}}$$

$\to$ scale invariance: magnitude of e doesn't matter.

acceleration driven by c_s^2 ("stiffness" of EOS)

- large c_s^2: fast acceleration, "stiff" EOS
- low c_s^2: slow acceleration, "soft" EOS

Special situation: 1st order phase transition $p(e) = \text{const.}$ in mixed phase



$$\Rightarrow \quad \frac{\partial p}{\partial e} = c_s^2 = 0 \quad \text{(sound cannot propagate)}$$

$\longrightarrow$ unaccelerated ("self-similar") expansion of mixed phase.

(ii) entropy conservation in ideal fluid dynamics (in the absence of shocks)

- Fundamental law of thermodynamics

$$Ts = p - \mu_B n_B + e$$

$$s = \frac{e+p}{T} - \frac{\mu_B}{T} n_B$$

entropy current: $S^\mu = s u^\mu = \underbrace{(e+p)\beta u^\mu}_{p\beta^\mu + \beta_\nu T^{\mu\nu}} - \alpha n_B u^\mu$ $\qquad \beta = \frac{1}{T}$

$\alpha = \frac{\mu_B}{T}$

$$\Rightarrow \boxed{S^\mu = p(\alpha,\beta)\beta^\mu - \alpha N_B^\mu + \beta_\nu T^{\mu\nu}} \quad \text{(fund. law. th. dyn.)} \qquad \beta^\mu = \frac{u^\mu}{T}$$

- gibbs-Duhem relation

$$dp = s\,dT + n_B\,d\mu$$

- 1^{st} law of thermodynamics

$$ds = \beta\,de - \alpha\,dn_B$$

$\Rightarrow$ entropy production rate

$$\partial_\mu S^\mu = \partial_\mu (s u^\mu) = \dot{s} + s\theta$$

1st law   fund. rel.

$$= \beta\dot{e} - \alpha\dot{n}_B + \left(\beta(e+p) - \alpha n_B\right)\theta = \beta\,(\underbrace{\dot{e} + (e+p)\theta}_{0}) - \alpha\,(\underbrace{\dot{n}_B + n_B\theta}_{0}) = 0$$

$\Rightarrow$ $\boxed{\partial \cdot S = 0}$ entropy is conserved in an ideal fluid

(iii) Ideal fluid equations in global reference frame:

write $u^\mu = \gamma(1, \vec{v})$ $\vec{v}$ = 3-velocity $\gamma = \frac{1}{\sqrt{1-v^2}}$

$N_B^\mu = n_B u^\mu = \gamma n_B (1, \vec{v}) \equiv R(1, \vec{v})$ $R = N_B^0$ lab frame density

$T^{00} = (e+p)u^0u^0 - p\,g^{00} = \gamma^2(e+p) - p \equiv \mathcal{E}$ lab frame energy dens.

$\vec{M} \equiv (T^{01}, T^{02}, T^{03}) = (e+p)u^0\vec{u} = \gamma^2(e+p)\vec{v}$ lab. frame mom. dens.

$$\partial \cdot N_B = 0 = \partial_t R + \vec{\nabla} \cdot (R\vec{v})$$
$$\partial_\mu T^{\mu 0} = 0 = \partial_t \mathcal{E} + \vec{\nabla} \cdot (\cancel{\gamma}(e+p)\vec{v})$$
$$\partial_\mu T^{\mu i} = 0 = \partial_t M^i + \vec{\nabla} \cdot (M^i \vec{v}) + \partial_i p$$

$\Rightarrow$ Generic form: (conservative form) $\boxed{\partial_t \mathcal{U} + \sum_{j=1}^{3} \partial_j F_j(\mathcal{U}) = 0}$ $(*)$

$$\mathcal{U} = R, \mathcal{E}, \vec{M}$$

Solution of equs. of the form$(*)$ require flux-corrected transport algorithm

We also need EOS $p(e, n_B)$.

To implement it must find e and $\vec{v}$ from $T^{\mu\nu}$:

$$n_B = R\sqrt{1-v^2} = \frac{N^0}{\gamma}$$

$$e = \mathcal{E} - \vec{M}\cdot\vec{v}$$

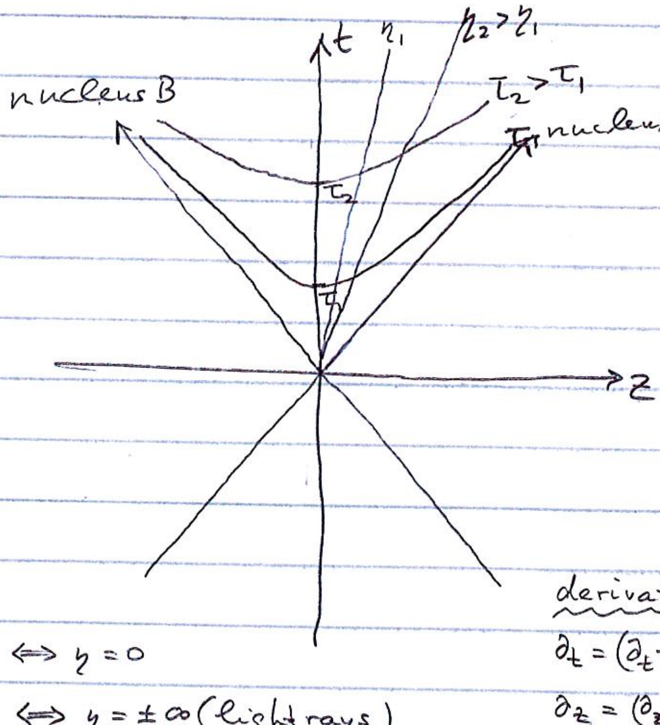
$\Big\}$ needs $\vec{v}$

Since $\vec{M} \parallel \vec{v}$, we only need to compute v and get $\vec{v}$ from $\vec{v} = v\frac{\vec{M}}{M}$

$$\vec{M} = \gamma^2 (e+p)\vec{v} = \mathcal{E}\vec{v} + p\vec{v} \Rightarrow M = (\mathcal{E}+p)v \Rightarrow \boxed{v = \frac{M}{\mathcal{E}+p(\underbrace{e,n_B}_{\text{from above}})}}$$

$$\Rightarrow \quad v = \frac{M}{\mathcal{E} + p\left(\mathcal{E}-Mv, R\sqrt{1-v^2}\right)} \Rightarrow \text{1-d zero search for } v.$$

4) A special solution: boost-invariant 1-dimensional expansion (Bjorken solution)



Milne coordinates

$$\left.\begin{aligned} t &= \tau\, \mathrm{ch}\,\eta \\ z &= \tau\, \mathrm{sh}\,\eta \end{aligned}\right\} \tau^2 = t^2 - z^2 \quad \text{"longitudinal proper time"}$$

$$\eta = \frac{1}{2}\ln\frac{t+z}{t-z} \quad \text{"space-time rapidity"}$$

$$= \frac{1}{2}\ln\frac{1+z/t}{1-z/t}$$

$z=0 \iff \eta = 0$

$z = \pm t \iff \eta = \pm\infty$ (light rays)

derivatives:

$$\partial_t = (\partial_t \tau)\partial_\tau + (\partial_t\eta)\partial_\eta = \mathrm{ch}\,\eta\, \partial_\tau - \frac{\mathrm{sh}\,\eta}{\tau}\partial_\eta$$

$$\partial_z = (\partial_z \tau)\partial_\tau + (\partial_z\eta)\partial_\eta = -\mathrm{sh}\,\eta\, \partial_\tau + \frac{\mathrm{ch}\,\eta}{\tau}\partial_\eta$$

Under boosts: $\Lambda^\mu{}_\nu(v_L) = \begin{pmatrix} \mathrm{ch}\,y_L & 0 & 0 & \mathrm{sh}\,y_L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{sh}\,y_L & 0 & 0 & \mathrm{ch}\,y_L \end{pmatrix}$ $\qquad y_L = \frac{1}{2}\ln\frac{1+v_L}{1-v_L}$, $\quad v_L = \tanh y_L$

$$\tau' = \tau, \quad \eta' = \eta - y_L \qquad \text{rapidities are additive!}$$

Matching momentum coordinates:

$E = m_\perp \mathrm{ch}\, y \qquad m_\perp^2 = E^2 - p_z^2 = m^2 + \vec{p}_\perp^{\,2}$

$p_z = m_\perp \mathrm{sh}\, y \qquad y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} = \frac{1}{2}\ln\frac{1+v_z}{1-v_z}$

Under boosts

$m_\perp' = m_\perp$

$y' = y - y_L$

Boost-invariance:

In phase-space:

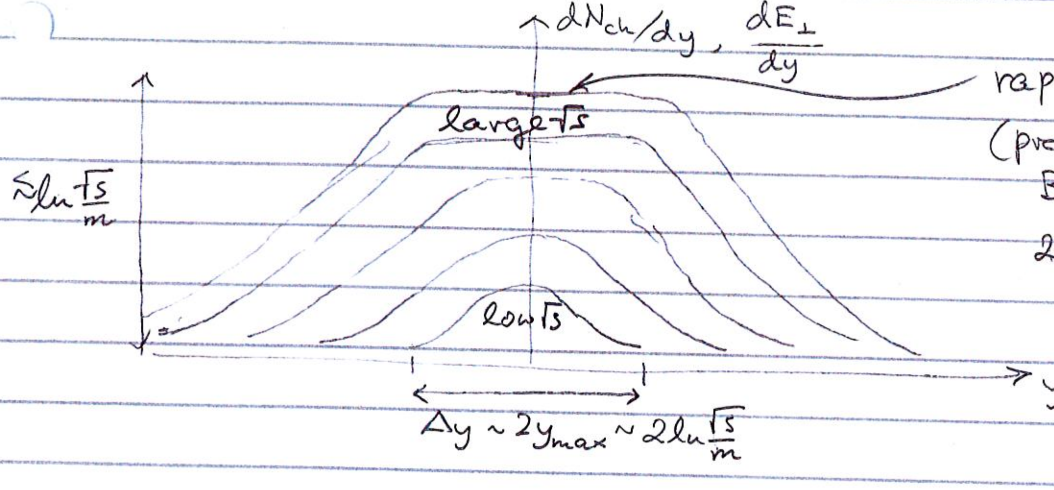
$$f(\vec{x},\vec{p},t) = f(\vec{x}_\perp, \eta; \vec{p}_\perp, y; t) = f(\vec{x}_\perp, \vec{p}_\perp, \underbrace{y-\eta}_{\text{boost-invariant combination}}; \tau)$$

In coordinate space:

$$e(x) = e(\vec{x}_\perp, \eta; \tau) = e(\vec{x}_\perp, \tau) \qquad \text{no } \eta\text{-dependence}$$

Why boost-invariance? Explain Bjorken's idea.

Phenomenological evidence:



rapidity plateau (predicted in 1983 by Bjorken, found in 2010 at LHC) ? not really

Solution of hydrodynamic equations with boost-invariant initial conditions:

$u_\mu(\vec{x}_\perp, \eta, \tau)$ cannot depend on $\eta \Rightarrow$ $\boxed{y_{flow} = \eta}$

$(y_{flow} - \eta = 0 \ \forall \tau)$

$$\boxed{u^\mu = \mathrm{ch}\, y_\perp (\mathrm{ch}\,\eta, v_x, v_y, \mathrm{sh}\,\eta)}$$ where $\mathrm{th}\, y_\perp(\tau, \vec{x}_\perp) = v_\perp(\tau, \vec{x}_\perp, \eta=0)$

general boost-invariant form

In Milne coordinates: $u^m \equiv (u^\tau, u^x, u^y, u^\eta) = \mathrm{ch}\, y_\perp (1, v_x, v_y, 0)$

$\Rightarrow u^\eta = 0 = \text{const},$

$\boxed{\dot{u}^\eta = 0}$ (since $\nabla_\eta p = 0$)

$\Rightarrow$ this reduces the dimensionality of the problem by 1.

If we ignore also transverse expansion
(nuclei = infinitely large, transversally homogeneous discs)

$v_x = v_y = 0$

$$\Rightarrow \quad \theta = \partial_\mu u^\mu = \frac{1}{\tau}, \qquad D = u^\mu \partial_\mu = \frac{d}{d\tau}$$

$$\Rightarrow \quad \frac{dn_B}{d\tau} = -\frac{n}{\tau} \quad \longrightarrow \quad \boxed{n_B(\tau) = n_B(\tau_0)\frac{\tau_0}{\tau}}$$

linear growth of volume

$$\frac{ds}{d\tau} = -\frac{s}{\tau} \quad \Rightarrow \quad \boxed{s(\tau) = s(\tau_0)\frac{\tau}{\tau_0}}$$

$$\frac{de}{d\tau} = -\frac{e+p}{\tau} = -(1+c_s^2)\frac{e}{\tau} \quad \Rightarrow \quad \boxed{e(\tau) = e(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}}$$

work done by long. pressure

For QGP $\quad e \sim p \sim T^4, \quad c_s^2 \approx \frac{1}{3}$
$\quad n \sim s \sim T^3 \quad \left(\frac{s}{n} \approx 4\right)$

$$\Rightarrow \quad \boxed{T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1/3}}$$

Bjorken solution

5) Viscous fluid dynamics

Deviations from local equilibrium:

(i) $f(x,p) = f_{eq}(x,p) + \delta f(x,p) \equiv f_{eq}\left[1+(1\pm f_{eq})\,\phi(x,p)\right]$

Expand δf in powers of gradients of the equilibrium parameters $T(x)$, $\mu_B(x)$, $u^\mu(x)$

To make the expansion unique, need to find optimal local equilibrium parameters for $f_{eq}(x,p)$: $T(x)$, $\mu_B(x)$, $u^\mu(x)$

→ Landau matching:

$$\delta n_B \equiv u_\mu \delta N_B^\mu = \int dP\,(u\cdot p)\left(\delta f(x,p) - \delta\bar{f}(x,p)\right) = 0 \qquad \Longrightarrow \frac{\mu_B}{T}(x)$$

$$\delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = \int dP\,(u\cdot p)^2\left(\delta f + \delta\bar{f} + \delta f_g\right) = 0 \qquad \Longrightarrow T(x)$$

fixing the local rest frame:

Landau frame: u^μ = time like eigenvector of $T^{\mu\nu}$: $T^{\mu\nu}u_\nu = e u^\nu$

$\Longleftrightarrow \boxed{u_\nu \delta T^{\mu\nu} = 0}$

Eckart frame: $u^\mu = \dfrac{N_B^\mu}{\sqrt{N_B\cdot N_B}} \Longleftrightarrow \boxed{\delta N_B^\mu = 0}$

We will use the Landau frame.

With $\delta f \neq 0$, we get from kinetic theory

$$N_B^\mu = \int dP\, p^\mu \left(f_{eq} - \bar{f}_{eq} + \delta f - \delta\bar{f}\right) = n_B u^\mu + V^\mu$$

$V^\mu = \Delta^{\mu\nu} N_{B\nu}$ = net baryon flow in l.r.f

$$T^{\mu\nu} = \int dP\, p^\mu p^\nu \left(\left(f_{eq} + \bar{f}_{eq} + f_{g,eq}\right) + \left(\delta f + \delta\bar{f} + \delta f_g\right)\right)$$
$$= e u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + \left(W^\mu u^\nu + W^\nu u^\mu\right)$$

Π = bulk viscous pressure; $\pi^{\mu\nu}$ = shear stress tensor; W^μ = momentum flow in l.r.f.

Using projection we get $\quad (p^\mu = p\cdot u\, u^\mu + \Delta^{\mu\nu} p_\nu \equiv \bar{E} u^\mu + \bar{p}^\mu$
$\bar{p}^\mu \bar{p}_\mu = -\bar{p}$

$$\underline{V^\mu} = \Delta^{\mu\nu} N_{B\nu} = \int dP\, \bar{p}^\mu (\delta f - \delta\bar{f}) \xrightarrow[lrf]{} \left(0, \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\vec{\bar{p}}}{\bar{E}} (\delta f - \delta\bar{f})\right)$$

$$p + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \Rightarrow \underline{\Pi} = -\frac{1}{3}\int dP\, p\cdot\Delta\cdot p\,(\delta f + \delta\bar{f} + \delta f_g) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{\bar{p}^2}{3\bar{E}} (\delta f + \delta\bar{f} + \delta f_g)$$

$$\underline{W^\mu} = \Delta^{\mu\nu} T_{\nu\alpha} u^\alpha = \int dP\, (p\cdot u)\, \Delta^{\mu\nu} p_\nu (\delta f + \delta\bar{f} + \delta f_g) \xrightarrow[lrf]{} \left(0, \int \frac{d^3\bar{p}}{(2\pi)^3} \vec{\bar{p}}\, (\delta f + \delta\bar{f} + \delta f_g)\right)$$

$$\underline{\pi^{\mu\nu}} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} \equiv \Big[\underbrace{\tfrac{1}{2}\big(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\big)}_{\text{transverse to } u^\mu, u^\nu} \underbrace{- \tfrac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}}_{\text{traceless}}\Big] T^{\alpha\beta}$$

$$= \int dP \left(\bar{p}^\mu \bar{p}^\nu - \frac{1}{3}\Delta^{\mu\nu}\bar{p}^2\right)(\delta f + \delta\bar{f} + \delta f_g) \xrightarrow[lrf]{} \int \frac{d^3\bar{p}}{(2\pi)^3} \left(\bar{p}^i \bar{p}^j - \frac{1}{3}\delta^{ij} \vec{\bar{p}}^2\right)(\delta f + \delta\bar{f} + \delta f_g)$$

for $i,j = 1,2,3$ (0 else)
(no time components in lrf)
traceless

$\underline{e, n_B \text{ and } p}$ still are the same integrals of f_{eq} as before — they receive no contributions from δf.

In the Landau frame $W^\mu = 0$ $\quad$ In this frame one writes $V^\mu = -\frac{n_B}{e+p} q^\mu$ with the "heat flow vector" q^μ

For a baryon-free ($n_B = 0$) system we expect $\delta f = \delta\bar{f}$ and thus $\underline{V^\mu = 0}$.

Henceforth consider this case ($n_B = V^\mu = 0$).

NB: This does not imply $\delta N^\mu = \int dP\, \bar{p}^\mu \delta f = \delta\bar{N}^\mu$, though.

So we expect in general $\delta N^\mu, \delta\bar{N}^\mu, \delta N_g^\mu \neq 0$.

Viscous fluid equations in comoving coordinates:

Using similar techniques as in ideal fluid case, we find the viscous fluid equations in the Landau frame

$$\boxed{\begin{aligned} \dot{e} &= -(e+p+\Pi)\theta + \pi_{\mu\nu}\sigma^{\mu\nu} \\ (e+p+\Pi)\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma \pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu \end{aligned}}$$

$\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta \equiv \nabla^{\langle\mu} u^{\nu\rangle}$
Velocity shear tensor

Still the same number of equations, but now we have 6 additional unknowns: $\underbrace{\Pi}_{1}, \underbrace{\pi^{\mu\nu}}_{10-4-1=5}$!

We need additional equations of motion for $\Pi, \pi^{\mu\nu}$.

Several approaches: (a) Chapman - Enskog: linearize C in terms of δf, and solve Boltzmann eq. for δf → integro-differential equation

(b) Grad 14-moment method: expand δf as $\sim \varepsilon(x) + \varepsilon_\mu(x) p^\mu + \varepsilon_{\mu\nu}(x) p^\mu p^\nu$ and use this expansion to solve the integro-differential equation

(c) Denicol - Molnar - Niemi - Rischke higher-order moment method → generalization of Grad method (PRD 85 (2012) 114047) (perhaps the most systematic approach)

(d) Israel - Stewart: purely macroscopic approach, using the second law of thermodynamics

(e) Navier - Stokes: macroscopic approach that ignores microscopic relaxation times (this causes serious problems in a relativistic thy.)

Here I will use the Israel - Stewart approach which also yields the Navier - Stokes approach in a specific limit.

Reference: W. Israel, Ann. Phys. 100 (1976) 310

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 349

Start from $\partial_\mu S^\mu \geq 0$ (2nd law of thermodynamics)

Remember the equilibrium identity

$$S^\mu_{eq} = p(\alpha, \beta)\beta^\mu - \alpha N^\mu_{B,eq} + \beta_\nu T^{\nu\mu}_{eq} \qquad \alpha = \frac{\mu_B}{T}, \ \beta^\mu = \beta u^\mu = \frac{u^\mu}{T}$$

In near-equilibrium we write

$$S^\mu = S^\mu_{eq} + \Phi^\mu = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_B + \beta_\nu T^{\mu\nu} + Q^\mu(\delta N^\mu_B, \delta T^{\mu\nu})$$

The terms $-\alpha\,\delta N^\mu_B + \beta_\nu \delta T^{\mu\nu}$ contribute linear terms in $q^\mu, \pi^{\mu\nu}, \Pi$; Q^μ contributes higher order terms

We can rewrite the Gibbs-Duhem relation from p. 9,

$$dp = s\,dT + n_B\,d\mu$$

(using thermodyn. identities in equilibrium) as

$$\partial_\mu\big(p(\alpha,\beta)\beta^\mu\big) = N^\mu_{B,eq}\,\partial_\mu\alpha - T^{\mu\nu}_{eq}\,\partial_\mu\beta_\nu$$

to find (using $\partial_\mu N^\mu_B = 0 = \partial_\mu T^{\mu\nu}$)

$$\boxed{\partial_\mu S^\mu = -\delta N^\mu_B\,\partial_\mu\alpha + \delta T^{\mu\nu}\,\partial_\mu\beta_\nu + \partial_\mu Q^\mu}$$

Using the decomposition of δN^μ_B and $\delta T^{\mu\nu}$ in terms of $q^\mu, \Pi, \pi^{\mu\nu}$, we can rewrite this as

$$\boxed{T\,\partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu}X_{\mu\nu} + T\,\partial_\mu Q^\mu}$$

where we introduced the thermodynamic forces

$$X \equiv -\theta = -\nabla\cdot u \qquad \text{scalar force}$$

$$X_\mu \equiv \frac{\nabla_\mu T}{T} - \dot{u}_\mu = -\frac{n_B T}{e+p}\nabla_\mu\left(\frac{\mu_B}{T}\right) \qquad \text{vector force}$$

$$X_{\mu\nu} \equiv \sigma_{\mu\nu} = \nabla_{\langle\mu}u_{\nu\rangle} \qquad \text{tensor force}$$

The second law of thermodynamics now requires

$$\boxed{\Pi X - q^\mu X_\mu + \pi^{\mu\nu}X_{\mu\nu} + \partial_\mu Q^\mu \geq 0}$$

(A) Relativistic Navier-Stokes theory

(See Landau-Lifshitz, Fluid Dynamics)

Ignore second-order terms, set $Q^\mu = 0$

$\Rightarrow$ 2^{nd} law is automatically satisfied if we demand

$$\boxed{\begin{array}{ll} \Pi = -\zeta\theta & \zeta = \text{bulk viscosity} \\ q^\nu = -\kappa \frac{nT^2}{e+P} \nabla^\nu\left(\frac{\mu_B}{T}\right) & \kappa = \text{heat conductivity} \\ \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} & \eta = \text{shear viscosity} \end{array}} \qquad (*)$$

such that $\partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q^\alpha q_\alpha}{2\kappa T} + \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0$

(remember: q^μ is spacelike!)

problem: the instantaneous response $(*)$ of δN_B^μ, $\delta T^{\mu\nu}$ to the thermodynamic forces is <u>acausal</u>!

$\Rightarrow$ superluminal signal propagation, unstable solutions of viscous hydro EOM $\lightning$

(B) Causal viscous relativistic fluid dynamics (Israel-Stewart theory)

keep Q^μ, parametrize it to second order in $\delta T^{\mu\nu}$

ignoring q^μ again, this is done as follows

$$\boxed{Q^\mu = -\left(\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda}\pi^{\nu\lambda}\right)\frac{u^\mu}{2T}}$$

Then

$$T\partial \cdot S = \Pi\left[-\theta - \beta_0\dot{\Pi} - \Pi T \partial_\mu\left(\frac{\beta_0 u^\mu}{2T}\right)\right] + \pi^{\alpha\beta}\left[\sigma_{\alpha\beta} - \beta_2\dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta} T\partial_\mu\left(\frac{\beta_2 \ldots}{2 \ldots}\right.\right.$$

This is positive definite if we set the expressions in [...] proportional to Π, $\pi_{\alpha\beta}$ such that $\partial \cdot S$ looks like in NS theory.

This gives

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T \partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right]$$

$$\Delta^{\alpha\beta}_{\mu\nu}\dot{\pi}^{\mu\nu} = -\frac{1}{\tau_\pi}\left[\pi^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} + \pi^{\alpha\beta}\eta T \partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)\right]$$

$\Rightarrow$ dynamical relaxation equations for Π, $\pi^{\mu\nu}$ with relaxation times $\tau_\Pi = \zeta\beta_0$ and $\tau_\pi = 2\eta\beta_2$

$\Rightarrow$ Causality restored!

Using $\gamma_\Pi = \zeta T \partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)$, $\gamma_\pi = \eta T \partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)$

this can be rewritten as

$$\boxed{\begin{aligned} &\dot{\Pi} = -\frac{1}{\tau'_\Pi}\left[\Pi + \zeta'\theta\right] \\ &\Delta^{\mu\nu}_{\alpha\beta}\dot{\pi}^{\alpha\beta} = -\frac{1}{\tau'_\pi}\left[\pi^{\mu\nu} - 2\eta'\sigma^{\mu\nu}\right] \\ &\text{where } \tau'_\Pi = \frac{\tau_\Pi}{1+\gamma_\Pi} \qquad \tau'_\pi = \frac{\tau_\pi}{1+\gamma_\pi} \\ &\qquad\quad \zeta' = \frac{\zeta}{1+\gamma_\Pi} \qquad \eta' = \frac{\eta}{1+\gamma_\pi} \end{aligned}} \qquad (\star)$$

$\rightarrow$ fast expansion (γ_Π, γ_π large) effective reduces the viscosities and relaxation times

$\star$ must be solved together with the viscous hydro equations.

For a conformal system $\boxed{\gamma_\Pi = \gamma_\pi = \frac{4}{3}\theta\tau_\pi}$

For a discussion of anisotropic hydrodynamics please refer to

M. McNelis, D.Bazow, U. Heinz, Phys. Rev. C 97, 054912 (2018)

"(3+1)-dimensional anisotropic fluid dynamics with a lattice QCD equation of state"