# Initial stages of heavy ion collisions 

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## 1 Literature

A large part of these lectures are originally based on selected material from courses lectured at the University of Jyväskylä: "High energy scattering in QCD" lectured in the spring of 2011, http: / / users. jyu.fi/~tulappi/fysh560kl11/ (original handwritten notes for the blackboard lectures are partly
in Finnish.) and later a part of the course "QCD" lectured in the spring of 2014, http://users.jyu. fi/~tulappi/fysh555kl14/ (slides and handwritten lecture notes in English).

Below we have a list of literature on the subject. Many of these review articles concentrate more on the current state of the phenomenology in the field, and less on the theoretical foundations that will be te focus of these lectures.

Recent reviews:
[1] F. Gelis, "Initial state and thermalization in the Color Glass Condensate framework," Int. J. Mod. Phys. E 24 (2015) no.10, 1530008, [arXiv:1508. 07974 [hep-ph]]
[2] J. L. Albacete and C. Marquet, "Gluon saturation and initial conditions for relativistic heavy ion collisions," Prog. Part. Nucl. Phys. 76 (2014) 1 [arXiv:1401.4866 [hep-ph]]
Somewhat less recent reviews:
[3] E. Iancu and R. Venugopalan, "The Color glass condensate and high-energy scattering in QCD," In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 249-3363 [hep-ph/0303204]
[4] F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, "The Color Glass Condensate," Ann. Rev. Nucl. Part. Sci. 60 (2010) 463 [arXiv:1002. 0333 [hep-ph] ]
[5] T. Lappi, "Small x physics and RHIC data," Int. J. Mod. Phys. E 20 (2011) no.1, 1 [arXiv:1003.1852 [hep-ph] ]
[6] H. Weigert, "Evolution at small $\mathrm{x}(\mathrm{bj})$ : The Color glass condensate," Prog. Part. Nucl. Phys. 55 (2005) 461 [hep-ph/0501087]
[7] J. Jalilian-Marian and Y. V. Kovchegov, "Saturation physics and deuteron-Gold collisions at RHIC," Prog. Part. Nucl. Phys. 56 (2006) 104 [hep-ph/0505052]
[8] F. Gelis, T. Lappi and R. Venugopalan, "High energy scattering in Quantum Chromodynamics," Int. J. Mod. Phys. E 16 (2007) 2595 [arXiv: 0708.0047 [hep-ph] ]

Very good books, focused more on small- $x$ evolution, DIS and other dilute-dense processes and less on heavy ion phenomenology
[9] Y. V. Kovchegov and E. Levin, "Quantum chromodynamics at high energy," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 33 (2012).
[10] V. Barone and E. Predazzi, "High-Energy Particle Diffraction," Springer 2002

## 2 Plan of lectures

We will start from the 18th century physics of wave mechanics to understand scattering in a very old fashioned way, and end up in the weak coupling picture of isotropization that matches this picture to the hydrodynamical modeling of heavy ion collisions that Uli will discuss in his lectures.

- Monday: diffraction of light, impact parameter, eikonal approximation, eikonal propagation in target color field, Secs 3, 4
- Tuesday: DIS in dipole picture, BK equation, Secs. 5, 6
- Wednesday: Gluon saturation, MV model, Sec. 7
- Thursday: The glasma initial state of a heavy ion collision, Sec. 8
- Friday: thermalization in the bottom-up scenario Sec. 9


## 3 Scattering of scalar light off a plane

The discussion here follows very closely the one in Barone \& Predazzi, sec 2.1.

Aim We start familiarizing ourselves with high energy scattering in an "old-fashioned" way by looking at the simplest possible relativistic problem: the scattering of light off a planar (semi)absorbing target. This will help us illustrate in a simple context several concepts that are essential for understanding the scattering of hadrons and nuclei at high energies, such as:

- the impact parameter
- the optical theorem
- elastic and total cross sections
- the scattering amplitude, its phase and unitarity constraints


### 3.1 Scattering from a hole

Before discussig scattering off a disk, we start with the opposite situation of scattering off a hole in an infinite opaque plane. The experimental situation at first is the following: we have an absorbing sheet on the $x y$ plane (surface $\Sigma$ ), with a hole (denoted $\Sigma_{0}$ ) in the middle. Upon this whole we have an incoming ray of light with momentum (wave vector) $\mathbf{k}$ along the $z$-direction, which we measure behind the hole at $z \rightarrow \infty$, keeping track of the pattern of light far away from the whole as a function of the angle $\theta$ with respect to the $z$-axis.


We look at "scalar light" (forgetting the polarization/spin) for now, so our ray of light obeys a relativistic Klein-Gordon wave equation. Since the disk is static (does not depend on time), the time dependence of the wave function can be described by a single frequency ${ }^{1} \omega=k$, and we can factorize the solution of the wave equation as

$$
\begin{equation*}
\square \Phi(x) \equiv \partial_{\mu} \partial^{\mu} \Phi(x)=0, \quad \Phi(t, \mathbf{x})=e^{-i k t} \varphi(\mathbf{x}) \Longrightarrow\left(k^{2}+\nabla^{2}\right) \varphi(\mathbf{x})=0, \quad k \equiv|\mathbf{k}| \tag{1}
\end{equation*}
$$

We now want to find the scattered wave after it has gone through the hole, $\varphi(\mathbf{x}) \mid z>0$. The way we will do this is physically known as the Huygens principle: every point on $\Sigma_{0}$ is itself a source of spherical light waves. In terms of mathematics we will do this using a Green function and Green's theorem:

## Green function $G$ :

$$
\begin{align*}
\left(k^{2}+\nabla_{\mathbf{x}}^{2}\right) G(\mathbf{x}, \mathbf{y}) & =-\delta^{(3)}(\mathbf{x}-\mathbf{y})  \tag{2}\\
\Longrightarrow G(\mathbf{x}, \mathbf{y}) & =\frac{e^{i k|\mathbf{x}-\mathbf{y}|}}{4 \pi|\mathbf{x}-\mathbf{y}|} \tag{3}
\end{align*}
$$

## Green's theorem

$$
\begin{equation*}
\varphi(\mathbf{x})=\int_{\mathbf{y} \in \Sigma_{0}} \mathrm{~d} \mathbf{S} \cdot\left[\left(\nabla_{\mathbf{y}} G(\mathbf{x}-\mathbf{y})\right) \varphi(\mathbf{y})-G(\mathbf{x}-\mathbf{y}) \nabla_{\mathbf{y}} \varphi(\mathbf{y})\right] \tag{4}
\end{equation*}
$$

[^0]The Green function (3) can be derived by taking a Fourier-transform of equation (2). This is a good exercise to go through and is presented below in more detail. An important thing to recall is that the solution to the second order differential equation (2) is not unique, because from one solution one can always obtain another solution by adding to it a solution of the homogenous equation

$$
\begin{equation*}
\left(k^{2}+\nabla_{\mathbf{x}}^{2}\right) h(\mathbf{x}, \mathbf{y})=0 \quad \Longrightarrow \quad\left(k^{2}+\nabla_{\mathbf{x}}^{2}\right)(G(\mathbf{x}, \mathbf{y})+h(\mathbf{x}, \mathbf{y}))=-\delta^{(3)}(\mathbf{x}-\mathbf{y}) . \tag{5}
\end{equation*}
$$

The solution becomes unique when one supplements it with a boundary condition. Here we have chosed a boundary condition that corresponds to the physical situation: we want to have an outgoing plane wave $e^{i \mathbf{k} \cdot \mathbf{x}}$ at large $\mathbf{x}$.

The Green's theorem is a way to somewhat formally construct a solution to the wave equation (1) that satisfies a desired boundary condition in the hole $\Sigma_{0}$; the first term guarantees that $\varphi(\mathbf{x})$ has the correct value at $\Sigma_{0}$, and the second term that the derivative with respect to $z$ also does so. Recall that we know the solution at $z<0$, it is just the incoming plane wave; thus we also know the solution on $\Sigma_{0}$.

## Digression: Green function

To derive the Green function (3) we represent it as a Fourier-transform. Note that since the r.h.s. of equation (2) only depends on the difference $\mathbf{x}-\mathbf{y}$, the solution must also be a function of just the difference.

$$
\begin{equation*}
G(\mathbf{x}, \mathbf{y})=\int \frac{\mathrm{d}^{3} \mathbf{p}}{(2 \pi)^{3}} e^{i \mathbf{p} \cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{p}) \tag{6}
\end{equation*}
$$

We now insert this into the defining equation (2) to get

$$
\begin{align*}
\left(k^{2}+\nabla_{\mathbf{x}}^{2}\right) G(\mathbf{x}, \mathbf{y}) & =\int \frac{\mathrm{d}^{3} \mathbf{p}}{(2 \pi)^{3}} e^{i \mathbf{p} \cdot(\mathbf{x}-\mathbf{y})}\left(k^{2}-\mathbf{p}^{2}\right) G(\mathbf{p})  \tag{7}\\
& =-\delta^{(3)}(\mathbf{x}-\mathbf{y})=-\int \frac{\mathrm{d}^{3} \mathbf{p}}{(2 \pi)^{3}} e^{i \mathbf{p} \cdot(\mathbf{x}-\mathbf{y})} \tag{8}
\end{align*}
$$

We know that Fourier-transforms are invertible, so we can deduce that

$$
\begin{equation*}
\left(\mathbf{p}^{2}-k^{2}\right) G(\mathbf{p})=1 \tag{9}
\end{equation*}
$$

Now we should divide both sides of this equation with $\left(\mathbf{p}^{2}-k^{2}\right)$ to solve for $G(\mathbf{p})$. This is fine except at the point $k^{2}=\mathbf{p}^{2}$ : where we have to regularize the expression somehow. In fact there are several ways of doing this, which correspond to several different possible boundary conditions for the original equation. We will here replace $k^{2}$ by $(k+i \varepsilon)^{2}$ to get the correct result (outgoing wave), other possible options would be $(k-i \varepsilon)^{2}$ or $k^{2} \pm i \varepsilon$, which would correspond to an incoming wave, or a linear combination between incoming and outgoing ones. We now get

$$
\begin{equation*}
G(\mathbf{p})=\frac{1}{\mathbf{p}^{2}-(k+i \varepsilon)^{2}}, \tag{10}
\end{equation*}
$$

and can invert the Fourier-transform as

$$
\begin{align*}
G(\mathbf{x}, \mathbf{y}) & =\int \frac{\mathrm{d}^{3} \mathbf{p}}{(2 \pi)^{3}} \frac{e^{i \mathbf{p} \cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^{2}-(k+i \varepsilon)^{2}}  \tag{11}\\
& =\frac{1}{(2 \pi)^{2}} \int_{0}^{\infty} \mathrm{d} p p^{2} \int_{-1}^{1} \mathrm{~d} \cos \theta \frac{e^{i p|\mathbf{x}-\mathbf{y}| \cos \theta}}{\mathbf{p}^{2}-(k+i \varepsilon)^{2}}  \tag{12}\\
& =\frac{1}{(2 \pi)^{2}} \int_{0}^{\infty} \mathrm{d} p p^{2} \frac{1}{i p|\mathbf{x}-\mathbf{y}|} \frac{e^{i p|\mathbf{x}-\mathbf{y}|}-e^{-i p|\mathbf{x}-\mathbf{y}|}}{\mathbf{p}^{2}-(k+i \varepsilon)^{2}}  \tag{13}\\
& =\frac{-i}{(2 \pi)^{2}|\mathbf{x}-\mathbf{y}|} \int_{-\infty}^{\infty} \mathrm{d} p p \frac{e^{i p|\mathbf{x}-\mathbf{y}|}}{(p+k+i \varepsilon)(p-k-i \varepsilon)} \tag{14}
\end{align*}
$$

This integral is performed by the theorem of residues, closing the contour in the upper complex p-plane, where one picks up the pole at $p=k+i \varepsilon$, with residue $e^{i k|\mathbf{x}-\mathbf{y}|} / 2$. Together with the factor $2 \pi i$ from the residue theorem this gives the result

$$
\begin{equation*}
G(\mathbf{x}, \mathbf{y})=\frac{e^{i k|\mathbf{x}-\mathbf{y}|}}{4 \pi|\mathbf{x}-\mathbf{y}|} \tag{15}
\end{equation*}
$$

## Digression: Green's theorem

See e.g. Jackson, Classical Electrodynamics, sec 10.5.


Let us look at the volume $V$, defined as the halph-sphere with very large radius, on the positive side of the $z$-axis, i.e. $z>0,\left|x^{2}+y^{2}+z^{2}\right|<R^{2}, R \rightarrow \infty$. Recalling the definition of the Green function (2), we can write the value of the wave function $\varphi(\mathrm{x})$ within this sphere as

$$
\begin{align*}
\left.\varphi(\mathbf{x})\right|_{\mathbf{x} \in V} & =-\int_{V} \mathrm{~d}^{3} \mathbf{y} \overbrace{\left[\left(\nabla_{\mathbf{y}}^{2}+k^{2}\right) G(\mathbf{x}, \mathbf{y})\right]}^{-\delta^{(3)}(\mathbf{x}-\mathbf{y})} \varphi(\mathbf{y})  \tag{16}\\
& =-\int_{V} \mathrm{~d}^{3} \mathbf{y} \nabla_{\mathbf{y}} \cdot\left[\left(\nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})\right) \varphi(\mathbf{y})-G(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} \varphi(\mathbf{y})\right] \tag{17}
\end{align*}
$$

where we used the equation of motion $\left(\nabla_{\mathbf{y}}^{2}+k^{2}\right) \varphi(\mathbf{y})=0$. Now we can transform the volume integral into a surface integral, neglecting the halph-sphere at infinity where the wave function (and Green function) vanish. Denoting by $\mathrm{d} \vec{\Sigma}$ the directed surface element pointing in the positive $z$-direction (towards the inside of the volume $V$, i.e. opposite to the conventional direction of a surface element) we now have

$$
\begin{equation*}
\left.\varphi(\mathbf{x})\right|_{\mathbf{x} \in V} \int_{\Sigma} \mathrm{d} \vec{\Sigma} \cdot\left[\left(\nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})\right) \varphi(\mathbf{y})-G(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} \varphi(\mathbf{y})\right] \tag{18}
\end{equation*}
$$

In order to apply Green's theorem to our scattering problem, we need to calculate the required quantities on the boundary $\Sigma$, which we will parametrize now with the 2 -dimensional coordinate known as the impact parameter $\mathbf{b}_{\perp}$.

$$
\begin{align*}
\left.\partial_{y^{3}} \varphi(\mathbf{y})\right|_{\mathbf{y}=\left(\mathbf{b}_{\perp}, 0\right)} & =i k \varphi_{0}  \tag{19}\\
\left.\partial_{y^{3}} \frac{e^{i k|\mathbf{x}-\mathbf{y}|}}{4 \pi|\mathbf{x}-\mathbf{y}|}\right|_{\mathbf{y}=\left(\mathbf{b}_{\perp}, 0\right)} & =-\varphi_{0} \cos \theta\left(i k-\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) \tag{20}
\end{align*}
$$

where $\theta$ is the angle between the $z$-axis and the vector $\mathbf{x}-\mathbf{y}$ connecting the point $\mathbf{y}$ on the $z=0$-plane $\Sigma$ that is the target of our scattering, and the far away location $\mathbf{x}$ where we are observing the outcoming wave. here we have used

$$
\begin{align*}
& \partial_{y^{3}}|\mathbf{y}-\mathbf{x}|=\partial_{y^{3}} \sqrt{\left(x^{1}-y^{1}\right)^{2}+\left(x^{2}-y^{2}\right)^{2}+\left(x^{3}-y^{3}\right)^{2}} \\
&=\frac{2\left(y^{3}-x^{3}\right)}{2 \sqrt{\left(x^{1}-y^{1}\right)^{2}+\left(x^{2}-y^{2}\right)^{2}+\left(x^{3}-y^{3}\right)^{2}}}=-\cos (\theta) \tag{21}
\end{align*}
$$

with

$$
\begin{equation*}
\cos (\theta)=\frac{\left(x^{3}-y^{3}\right)}{|\mathbf{y}-\mathbf{x}|} \tag{22}
\end{equation*}
$$

$$
\varphi(\mathbf{x})=\varphi \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}\left[-\cos (\theta)\left(i k-\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)-i k\right] \frac{e^{i k|\mathbf{x}-\mathbf{y}|}}{4 \pi|\mathbf{x}-\mathbf{y}|}
$$

This is an exact result. Now we are going to make certain approximations that limit our interest to a detector that is far from the scattering compared to the wavelength of the wave $|\mathbf{y}-\mathbf{x}| \gg 1 / k$, and to scattering at small angles: $\theta \ll 1$

- $|\mathbf{y}-\mathbf{x}| \gg 1 / k$
- $\theta \ll 1 \Longrightarrow \cos \theta \approx 1$

$$
\begin{gather*}
k|\mathbf{y}-\mathbf{x}|=k \sqrt{\left(x^{3}\right)^{2}+\left(\mathbf{x}_{\perp}-\mathbf{b}_{\perp}\right)^{2}} \approx k r\left(1-\frac{\mathbf{x}_{\perp} \cdot \mathbf{b}_{\perp}}{r^{2}}\right) \approx k r-\mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp} \quad r \equiv|\mathbf{x}|  \tag{24}\\
\mathbf{q}=\mathbf{k}^{\prime}-\mathbf{k}, \frac{\mathbf{q}_{\perp}}{k}=\frac{\mathbf{x}_{\perp}}{r} \sim \theta  \tag{25}\\
\varphi(\mathbf{x})=-i k \varphi_{0} \frac{e^{i k r}}{2 \pi r} \int_{\Sigma} \mathrm{d}^{2} \mathbf{b}_{\perp} \Gamma\left(\mathbf{b}_{\perp}\right) e^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}}  \tag{26}\\
\Gamma\left(\mathbf{b}_{\perp}\right)=\left\{\begin{array}{l}
1, \mathbf{b}_{\perp} \in \Sigma_{0} \\
0, \mathbf{b}_{\perp} \notin \Sigma_{0}
\end{array}\right. \tag{27}
\end{gather*}
$$

To be specific: here $\mathbf{k}^{\prime}$ is defined as a vector of absolute value $k$, pointing in the direction of $\mathbf{x}$, i.e. from the origin of our coordinate system (which is somewhere inside the target hole) to our detector.

## Digression: Energy conservation check

We made several small angle approximations in this process. It can be useful to check to what extent energy is still conserved in our scattering. We can do this by comparing the radiation power (= intensity times area) incoming on the hole, and the power going out. The incoming power is (up to a factor of $k$ ) the intensity of the incoming wave $\left|\varphi_{0}\right|^{2}$ integrated over the area of the hole

$$
\begin{equation*}
P_{\text {in }}=\left|\varphi_{0}\right|^{2} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp} . \tag{28}
\end{equation*}
$$

The outgoing power is obtained by integrating, at a distance $r=R$ from the target, the intensity $|\varphi(\mathbf{x})|^{2}$ over the area, where the area element is the solid angle times the squared radius: $\mathrm{d} S=R^{2} \mathrm{~d} \Omega=R^{2} \mathrm{~d}^{2} \mathbf{q}_{\perp} / k^{2}$

$$
\begin{align*}
P_{\text {out }} & =R^{2} \int \frac{\mathrm{~d}^{2} \mathbf{q}_{\perp}}{k^{2}}|\varphi(\mathbf{x})|^{2}  \tag{29}\\
& =R^{2}\left|\varphi_{0}\right|^{2} \int \frac{\mathrm{~d}^{2} \mathbf{q}_{\perp}}{k^{2}} \frac{1}{(2 \pi r)^{2}} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}^{\prime} \Gamma\left(\mathbf{b}_{\perp}\right) \Gamma\left(\mathbf{b}_{\perp}^{\prime}\right) e^{-i \mathbf{q}_{\perp} \cdot\left(\mathbf{b}_{\perp}-\mathbf{b}_{\perp}^{\prime}\right)} \tag{30}
\end{align*}
$$

Integrating first over $\mathbf{q}_{\perp}$ gives a $\delta$-function in $\mathbf{b}_{\perp}-\mathbf{b}_{\perp}^{\prime}$, which can used to do the $\mathbf{b}_{\perp}^{\prime}$-integral, resulting in $P_{\text {out }}=P_{\text {in }}$. This is of course not exact, because the identification $\mathrm{d} \Omega=\mathrm{d}^{2} \mathbf{q}_{\perp} / k^{2}$ is only valid for small angles.

### 3.2 Scattering off disk

So far we have been looking at light going through a hole in an opaque plane. Let us now swith to a situation that is closer to the case of high enery particle, and replace the hole by an opaque disk of the same shape $\Sigma_{0}$. Because the wave equation is linear, the waves scattered off the hole and the disk should add up to the original wave. We can therefore obtain the wave scattered from the disk by simply subtracting from the original incoming wave the wave scattered off a hole, i.e. replacing our profile function $\Gamma\left(\mathbf{b}_{\perp}\right)$ by $1-\Gamma\left(\mathbf{b}_{\perp}\right)$

Babinet: hole + disk = original wave
$\Longrightarrow$ Scattering off disk $\Sigma_{0}$

$$
\begin{equation*}
\varphi(\mathbf{x})=-i k \varphi_{0} \frac{e^{i k r}}{2 \pi r} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}\left(1-\Gamma\left(\mathbf{b}_{\perp}\right)\right) e^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}} \approx \varphi(\mathbf{x})+\varphi_{0} f(\mathbf{q}) \frac{e^{i k r}}{r} \tag{31}
\end{equation*}
$$

$f(\mathbf{q})=$ QM scattering amplitude, cross section:

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d}^{2} \mathbf{q}_{\perp}}=\frac{1}{k^{2}} \frac{\mathrm{~d} \sigma}{\mathrm{~d} \omega}=\frac{1}{k^{2}}|f(\mathbf{q})|^{2}=\left|\frac{i}{2 \pi} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp} \Gamma\left(\mathbf{b}_{\perp}\right) e^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}}\right|^{2} \tag{32}
\end{equation*}
$$

Note that because we have written the outgoing wave as a spherical one, it is a bit tricky to directly mathematically reconstruct the original wave by taking just the " 1 " in the $1-\Gamma$. The physical interpretation should however be clear: the 1 corresponds to no target, and the $\Gamma$ to the opaqure disk.

He wwe constantly pass between the solid angle $\mathrm{d} \Omega$ and the momentum transfer $\mathrm{d} \Omega \approx \mathrm{d}^{2} \mathbf{q}_{\perp} / k^{2}$. For an azimuthally symmetric target we can write this in terms of the scalar momentum transfer (the usual Mandelstam variable $t$ ) using $\mathrm{d}^{2} \mathbf{q}_{\perp}=2 \pi \mathrm{dq}_{\perp}^{2}=-2 \pi \mathrm{~d} t$.

Note that the quantum mechanical scattering amplitude $f$ in this case is imaginary. This corresponds, in the usual conventions, to a purely absorptive potential, as is our target of a fully opaque disk. Mathematically it follows from the derivatives acting on the plane wave and the Green function in the Green's theorem formula. It is, however, easy to generalize this to a scattering amplitude that has both absorptive and nonabsorbtive parts by just assuming that the function $\Gamma$ can be take arbitrary complex values. In this case the expression (32) must be interpreted as elastic scattering, since what comes out is the same ray of light that went in, just bent in a different direction. We can also calculate an absorptive cross section by taking the difference between incoming intensity (the 1 ) and the outgoing one (the $1-\Gamma$ ) and integrating over the momentum $\mathbf{q}_{\perp}$ (which now is not directly an obsevable quantity, but is rather interpreted as the momentum that the outgoing wave would have had had it not been absorbed). The sum of the elastic and absorptive cross sections is the total cross section.

Generalization: $\Gamma\left(\mathbf{b}_{\perp}\right) \in \mathbb{C}$

- Elastic

$$
\begin{equation*}
\frac{\mathrm{d} \sigma_{\mathrm{el}}}{\mathrm{~d}^{2} \mathbf{q}_{\perp}}=\left.\left.\left|\frac{i}{2 \pi} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}\right| \Gamma\left(\mathbf{b}_{\perp}\right)\right|^{2} e^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}}\right|^{2} \tag{33}
\end{equation*}
$$

- Absorptive

$$
\begin{align*}
\frac{\mathrm{d} \sigma_{\mathrm{abs}}}{\mathrm{~d}^{2} \mathbf{q}_{\perp}}=\left|\frac{i}{2 \pi} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp} 1 \mathrm{e}^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}}\right|^{2} & -\left|\frac{i}{2 \pi} \int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}\right| 1-\left.\left.\Gamma\left(\mathbf{b}_{\perp}\right)\right|^{2} e^{-i \mathbf{q}_{\perp} \cdot \mathbf{b}_{\perp}}\right|^{2} \\
& \Longrightarrow \sigma_{\mathrm{abs}}=\int_{\Sigma_{0}} \mathrm{~d}^{2} \mathbf{b}_{\perp}\left[2 \Re\left(\Gamma\left(\mathbf{b}_{\perp}\right)\right)-\left|\Gamma\left(\mathbf{b}_{\perp}\right)\right|^{2}\right] \tag{34}
\end{align*}
$$

- Total

$$
\begin{equation*}
\sigma_{\mathrm{tot}}=\sigma_{\mathrm{el}}+\sigma_{\mathrm{abs}}=\sigma_{\mathrm{abs}}=2 \int_{\Sigma} \mathrm{d}^{2} \mathbf{b}_{\perp}\left[\Re \Gamma\left(\mathbf{b}_{\perp}\right)\right]=\frac{4 \pi}{k} \Im(f(\mathbf{q}=0)) \tag{35}
\end{equation*}
$$

$\Longrightarrow$ optical theorem

- Black disk $=\Gamma=1 \Longrightarrow \sigma_{\mathrm{el}}=\sigma_{\mathrm{abs}}$

This is a concrete and explicit demonstration of the optical theorem, which of course is constantly used in scattering theory. In fact it is so frequent that one often does not explicitly mention it when using it. Note that there is a difference of $i$ between our opacity function $\Gamma$ and the scattering amplitude $f$, i.e. when $f$ is
purely imaginary, $\Gamma$ is purely real.
Note that contrary to the usual relativistic field theory expressions, we have not derived these in terms of Lorentz-invariant matrix elements and flux factors; the flux factor is absorbed into the normalization of $\Gamma$, see [9]. In stead we have a very good physical interpretation of what the cross section means: it is a quantity with the dimensions of area that results from integrating over the surface of the target the dimensionless probability that the probe will scatter if it hits the specific impact parameter $\mathbf{b}_{\perp}$. This wording might sound classical, but the importance of interference and the complex phase of $\Gamma$ should serve as reminders that these results reflect the wave nature of quantum particles.

## 4 Eikonal scattering

### 4.1 Light cone coordinates

We shall be interested in these lectures at particles traveling practically at the speed of light, at very high energy. In this case it is convenient to think of the scattering problem in a coordinate system well adapted to things traveling at the speed of light in the $z$ - or 3-direction.

Light-cone coordinates

$$
\begin{align*}
a^{ \pm} & \equiv \frac{1}{\sqrt{2}}\left(a^{0} \pm a^{3}\right)  \tag{36}\\
a \cdot b & =a^{\mu} b_{\mu}=a^{+} b^{-}+a^{-} b^{+}-\mathbf{a}_{\perp} \cdot \mathbf{b}_{\perp}  \tag{37}\\
& \Longrightarrow a_{+}=a^{-} \quad a_{-}=a^{+} \tag{38}
\end{align*}
$$

## Note $\partial_{+} \equiv \frac{\partial}{\partial x^{+}}$

In particular for a particle with mass $m$ the on mass shell 4-momentum is $m^{2}=p^{2}=2 p^{+} p^{-}-\mathbf{p}_{\perp}{ }^{2}$.
Using light cone coordinates is just a change of variables and does not affect the physics in any way. However, the coordinate system is particularly useful in the context of light cone quantization [11, 9]. In ordinary ("instant form") one quantizes the theory by imposing equal time ( $t=t^{\prime}$ ) commutation relations between Heisenberg theory time-dependent operators, and studies the evolution of the system in time $t$ generated by a Hamiltonian $H$, which is the 0 -component of a Lorentz 4 -vector. In light-cone (or light front) quantization one treats instead the coordinate $x^{+}$as the time coordinate (light cone time), and imposes canonical commutation relations at equal $x^{+}$. The evolution of the Heisenberg picture operators is then generated by the ligh-cone Hamiltonian $\hat{P}^{-}$, which is the --component of a Lorentz 4 -vector.

### 4.2 Eikonal approximation

Scattering at high energy becomes much simpler with the eikonal approximation. Many textbooks, (e.g. Barone \& Predazzi (Sec 2.3) start by describing the eikonal approximation for nonrelativistic scattering off a classical potential). A more formal field theoretic derivation for gauge theory (e.g. [8]) follows the important work of Bjorken et al [12] by starting from the operator definition of the scattering amplitude and looking at its properties when the incoming state is boosted to a very high energy. We will here follow an approach that is between these in the sense that we look just at solutions of wave equation, but inspect a situation that is

- relativistic and
- involves a vector potential (for the Abelian theory at first)

Klein-Gordon eq. in external vector potential:

$$
\begin{equation*}
\left[\left(i \partial_{\mu}+e A_{\mu}(x)\right)\left(i \partial^{\mu}+e A^{\mu}(x)\right)-m^{2}\right] \phi(x)=0 \tag{39}
\end{equation*}
$$

We are now interested in the scattering problem where the boundary condition is that of a high energy particle approaching the target (represented by the gauge potential $A_{\mu}$ ). Thus the boundary condition for the equation is a plane wave $e^{-i k \cdot x}$ at $x^{+} \rightarrow-\infty$. The assumption that we are interested in high energy means that $k^{+}$is much larger than any other momentum scale in the problem, in particular any gradients of the potential $A_{\mu}$. It is natural to make this assumption about the momentum scales explicit and to separate
out the rapidly oscillating behavior of the wave function from the smoother scales by using an ansatz where it is explicitly factorized. We can also assume that we are in Lorentz gauge $\partial_{\mu} A^{\mu}=0$.

$$
\begin{equation*}
\phi(x) \rightarrow e^{-i k \cdot x}, \quad x^{+} \rightarrow-\infty \tag{40}
\end{equation*}
$$

Ansatz

$$
\begin{align*}
\phi(x) & =e^{-i k \cdot x} \varphi(x)  \tag{41}\\
k^{+} & \gg \frac{\partial_{\mu} \varphi(x)}{\varphi(x)}  \tag{42}\\
\partial_{\mu} A^{\mu} & =0 \tag{43}
\end{align*}
$$

Inserting the ansatz (41) into the original equation (39) leads to

$$
\begin{equation*}
e^{-i k \cdot x}[\overbrace{k^{2}-m^{2}}^{=0}+2 e k^{\mu} A_{\mu}+2 i k^{\mu} \partial_{\mu}+2 i e A^{\mu} \partial_{\mu}+e^{2} A_{\mu} A^{\mu}] \varphi(x)=0 \tag{44}
\end{equation*}
$$

Now comes the crux of the eikonal approximation. We are assuming that $k^{+}$is a large variable, thus at leading order we only need to keep the terms proportional to $k^{+}$in equation (44). These only appear in the terms that are underlined in the equation. Thus, and recalling that now the initial condition is $\varphi(x) \rightarrow$ $1, x^{+} \rightarrow-\infty$, and factoring out the $2 i k^{\mu}$ we end up with

$$
\begin{equation*}
\left[\partial_{+}-i e A^{-}(x)\right] \varphi(x)=0 \Longrightarrow \varphi(x)=\exp \left\{i e \int_{-\infty}^{x^{+}} \mathrm{d} y^{+} A^{-}\left(y^{+}, x^{-}, \mathbf{x}_{\perp}\right)\right\} \tag{45}
\end{equation*}
$$

In the far future after the target, $x^{+} \rightarrow \infty$, the outgoing wave is written in terms of the incoming one in terms of the

Eikonal phase

$$
\begin{gather*}
\chi\left(\mathbf{x}_{\perp}\right)=e \int_{-\infty}^{\infty} \mathrm{d} y^{+} A^{-}\left(y^{+}, x^{-}, \mathbf{x}_{\perp}\right)  \tag{46}\\
\left.\phi(x)\right|_{x^{+} \rightarrow \infty} \approx e^{-i k \cdot x}\left[1-\left(1-e^{i \chi\left(\mathbf{x}_{\perp}\right)}\right)\right] \tag{47}
\end{gather*}
$$

At this point we will just state, without trying to go through a formal derivation, that one minus the exponential of the eikonal phase is analogous to the opacity/transparency of the optical problem, and that the elastic cross section is given (see equation (33)) by

$$
\begin{equation*}
1-e^{i \chi\left(\mathbf{x}_{\perp}\right)} \Longleftrightarrow \Gamma\left(\mathbf{x}_{\perp}\right) \Longrightarrow \frac{\mathrm{d} \sigma_{\mathrm{el} .}}{\mathrm{d}^{2} \mathbf{q}_{\perp}}=\left|\frac{-i}{2 \pi} \int \mathrm{~d}^{2} \mathbf{x}_{\perp} e^{-i \mathbf{q}_{\perp} \cdot \mathbf{x}_{\perp}}\left(1-e^{i \chi\left(\mathbf{x}_{\perp}\right)}\right)\right|^{2} \tag{48}
\end{equation*}
$$

This was a sketchy derivation. Some points of interest:

- We haven't specified anything about $x^{-}$: we will come back to this later
- The outgoing wave does not appear to expand spherically as was the case for our optical problem. This is mainly an artefact of the coordinate system; and a result of the eikonal approximation that assumes small angle scattering.
- If we do expand to leading order in the eikonal phase $\chi$, we see that $\Gamma$ is imaginary, i.e. the scattering amplitude real. But when expanded to second order, the scattering amplitude develops an imaginary phase. The eikonal approximation builds in a relation between the real and complex parts of the scattering amplitude, i.e. between elastic and inelastic (absorptive) scattering, that is consistent with unitarity of the scattering $S$-matrix.


### 4.3 Eikonal scattering off target of glue

So far we have only been discussing Abelian theory. Let us now move to QCD in the high energy limit. We will, however still stay within the eikonal picture of scattering. In particular, we will assume that QCD
scattering is dominated by scattering off the gluon field of the target. This is in fact not just an assumption, but a consequence of the spin of the gluons: the coupling strength of a high energy particle to the gluon field is proportional to the momentum of the particle. The coupling to the fermion field of the target, on the other hand, is proportional to the square root of the high energy, because the fermion field has spin half.

We also know from experiment, and can show in perturbative QCD (BFKL evolution), that the number of gluons in the proton or nucleus grows at higher energy (i.e. smaller $x$ ). We may therefore assume that there are many of these gluons, in fact so many that eventually at some point the gluon field becomes nonperturbatively strong.

## Color Glass Condensate (CGC)

Target has many gluons: $A_{\mu} \gg 1 / g \Longrightarrow$ described as classical gluon field


To see how a colored particle interacts with such a target color field, we can follow the same procedure as previously, by solving the Dirac equation of a colored fermion in this field. Since the argument is essentially the same as previously in the abelian theory, we will skip the details here. The only difference with respect to eq. (45) is that now the field is a matrix in color space, and does not commute with the field at a different coorinate $x^{+}$. Thus, when multiplying or exponentiating matrices, one naturally ends up with a path ordered exponential.

Quark in color field, Dirac:

$$
\begin{equation*}
(i \not \partial-g \not A) \psi(x)=0 \tag{49}
\end{equation*}
$$

Ansatz $\psi(x)=V(x) e^{-i p \cdot x} u(p)+$ eikonal approx.

$$
\begin{equation*}
\partial_{+} V\left(x^{+}, x^{-}, \mathbf{x}_{\perp}\right)=-i g A^{-}\left(x^{+}, x^{-}, \mathbf{x}_{\perp}\right) V\left(x^{+}, x^{-}, \mathbf{x}_{\perp}\right) \tag{50}
\end{equation*}
$$

## Wilson line = path-ordered exponential

$$
\begin{gather*}
V\left(x^{+}, x^{-}, \mathbf{x}_{\perp}\right)=\mathbb{P} \exp \left\{-i g \int^{x^{+}} \mathrm{d} y^{+} A^{-}\left(y^{+}, x^{-}, \mathbf{x}_{\perp}\right)\right\}  \tag{51}\\
\mathbb{P}\left[A\left(y^{+}\right) A\left(x^{+}\right)\right] \equiv \theta\left(y^{+}-x^{+}\right) A\left(y^{+}\right) A\left(x^{+}\right)+\theta\left(x^{+}-y^{+}\right) A\left(x^{+}\right) A\left(y^{+}\right) \tag{52}
\end{gather*}
$$

These path ordered exponentials are the same quantities that one uses to formulate lattice field theory (the link matrices). The difference is that now, we are following a light-like path, whereas in lattice field theory the links are in the spatial or imaginary time (also $\sim$ spatial) direction. This is because we are interested in the propagation of a very energetic particle moving at the speed of light in stead of a static equilibrium system.

## Recap

- We used an eikonal approximation: assume that longitudinal momentum is larger than other momenta involved
- Our result is not perturbative, i.e. it sums a series of different powers of the external field $A_{\mu}$
- Only one component of the vector field $A^{-}$matters. This is related to the high energy approximation: the current of charged particles with a large momentum $k^{+}$has a $J^{+}$-component that is larger than all others, and this component of the current only couples to the $A^{-}$-component of the gauge field. Incidentally, this feature is independent of the spin of the probe, and is thus the same also if we replace our incoming charged scalar particles by fermions or charged vector bosons. On the other hand, this feature does very much depend on the fact that we are scattering off a spin- 1 field (a vector field), it
is this feature that causes the strength of the interaction to be proportional to the momentum of the incoming particle. Theh graviton field would couplt to the momentum squared, whereas a scalar field does not couple to the momentum. The same feature can easily be seen by calculating some tree level Feynman diagrams for such scattering [exercise]. In terms of Feynman diagrams one can state the eikonal approximation in terms of the "eikonal vertex"


## Digression: Probe and target light cone gauges

If we wanted to count the photons of the target in a rigorous well-defined quantum mechanical way, we would want to go to the light-cone gauge of the leftmoving target, $A^{-}=0$. Then our interaction would look very different from what we have now. Thus, even the perturbative limit (power series expansion in $A_{\mu}$ ) of our result is a perturbation series in the interaction with a Coulomb field of the target, not really with a specific number of actual radiation quanta (photons) in it.

## 5 DIS in dipole picture

### 5.1 DIS introduction

Before understanding what happens when two nuclei collide at high energy, we have to know what the nuclei are made of. We know that they consist of quarks and gluons, but how many of these are there? We know that the nuclei are nonperturbative objects, so we cannot understand them from first principles at weak coupling alone. We can, however, measure the content of the hadron experimentally, by studying it with a simple probe: an electron. This experimental situation is known as deep inelastic scattering.

## DIS kinematics



$$
\begin{align*}
s & =(k+P)^{2}  \tag{53}\\
q & =k-k^{\prime} q^{2} \equiv-Q^{2}  \tag{54}\\
W^{2} & =(P+q)^{2}  \tag{55}\\
x & =\frac{Q^{2}}{2 P \cdot q}=\frac{Q^{2}}{W^{2}+Q^{2}-m_{N}^{2}} \tag{56}
\end{align*}
$$

## High energy limit

$$
\begin{equation*}
x \rightarrow 0 \quad W^{2} \rightarrow \infty \tag{57}
\end{equation*}
$$

In addition to the variables above, one also conventionally defines the following Lorentz-invariant variables

$$
\begin{align*}
y & =\frac{2 P \cdot q}{2 P \cdot k}=\frac{W^{2}+Q^{2}-m_{N}^{2}}{s m_{N}^{2}}  \tag{58}\\
\nu & =P \cdot q / m_{N}=\frac{W^{2}+Q^{2}-m_{N}^{2}}{2 m_{N}} \tag{59}
\end{align*}
$$

Before moving on let us discuss a little the physical interpretatation of the kinematical variables.

- The total $e^{-} p$ center of mass energy $s$ is of course important for an experimentalist. For the QCD part of the scattering process it is, completely irrelevant; its only practical consequence is that kinematically it sets the upper limit for the quantity we are really interested in, which is
- $W^{2}$, the c.m.s. energy of the $\gamma^{*} p$-system. This is the collision energy that is relevant for the QCD part of the scattering, and the high $W^{2}$ is what defines the high energy limit that we are interested in. The collision energy $W^{2}$ is practically the same as
- $\nu$, which is the energy of the $\gamma^{*}$, i.e. the energy lost by the electron, in the target rest frame. This is important historically when most experiments are done at fixed targets, but $W^{2}$ is nicer for a theorist since it is Lorentz-invariant.
- The "inelasticity" $y$ is really related to the electron-photon vertex kinematics. It determines how much of the electron energy is taken by the photon, i.e. the relation between the uninteresting variable $s$ and the interesting one $W^{2}$, and is thus uninteresting for us here. ${ }^{2}$
- $Q^{2}$ should be thought of as the spatial resolution scale, it tells us what is the wavelength of the spacelike virtual photon and therefore the resolution scale at which we can probe the contents of the target. Unless we have heavy quarks, $Q^{2}$ has to be large enough to justify weak coupling. As we will see later, it is useful to think of $Q^{2}$ as a transverse momentum scale, and $W^{2}$ as a longitudinal momentum, or energy scale.
- Finally there is $x$; which for the purposes of these lectures is mostly just a way of parametrizing $W^{2}$. However, it is conventionally thought of as a momentum fraction of the struck parton, and we will first take a short detour to see why this is the case.

Usually the textbook description of DIS is done in the infinite momentum frame, where the hadron is moving fast. One then makes the parton model assumption that the hadron consists of partons, quarks and gluons, whose momenta are parallel to that of the hadron (since the hadron momentum is very high, the transverse momenta of the partons with respect to the large longitudinal momenta are small). The physical picture of a DIS process is then that the virtual photon strikes one of these parton (at LO it has to be a quark, since photons couple to gluons only through a higher order process with intermediate quarks).

Let us now assume that

1. the 4 -momentum of the quark before it hits the photon is parallel to the hadron and carries a fraction $\alpha$ of the momentum of the hadron $P$,
2. the quark after it is struck by the photon is an asymptotic final state particle and thus on shell (this is certainly always the case in inclusive cross sections, because there one is summing over all possible final states, and this can be done using any complete set of states, including of course the free particle states)
3. we can neglect the mass of the hadron (we always do in these lectures whenever we can).

This allows us to show that actually the momentum fraction $\alpha$ is with these assumptions is actually the same as the kinematical variable $x$ :

## Infinite momentum frame IMF



[^1]$$
0=(q+\alpha P)=-Q^{2}+2 \alpha P \cdot q+\overbrace{\alpha^{2} m_{N}^{2}}^{\approx 0} \Longrightarrow \alpha=\frac{Q^{2}}{2 P \cdot q}=x
$$
$$
x=\text { momentum fraction }
$$

The identification of the kinematical variable $x$ (or Bjorken $x$, i.e. $x_{B j}$ ), which is a well defined Lorentzinvariant constructed from the 4-momenta of the incoming and outgoing electron and the incoming proton, with the momentum fraction of the quark that is struck is so common, that the two concepts are often used completely interchangeably. It is good to remember, however, that $x$ and the momentum fraction are conceptually separate things and identifying them requires a certain physical picture, that of the quarks as completely collinear to the hadron moving at infinite velocity.

### 5.2 Dipole picture

Now let us move to a different Lorentz frame, i.e. the target (hadron/nucleus) rest frame. Crosos sections are Lorentz-invariant, but the physical picture of the scattering is not. In fact, in this frame, and at small $x$, we think of the $\gamma^{*}$ as a kind of a hadron, which itself consists of quarks and gluons. DIS (and even $\gamma \gamma$-scattering) are now hadronic processes. In particular one would expect the high energy behavior of the cross sections to be similar to hadronic scattering, just scaled by a factor of $1 / 137$ (or $(1 / 137)^{2}$ ) because at some point $\gamma p$ (and respectively $\gamma \gamma$ ) involve one (or two) additional electromagnetic vertices. In fact I want to convince you that the $\gamma^{*}$ with a largish $Q^{2}$ is the theorist's favorite hadron it is a QCD bound state; but unlike the proton, its partonic structure can be fully understood in perturbation theory!


Figure 1: $p p, \gamma p$, and $\gamma \gamma$ total cross sections vs $\sqrt{s}$
Let us now look at the kinematics of DIS in the target rest frame
Target Rest Frame TRF
$x^{ \pm}=\frac{1}{\sqrt{2}}(t \pm z)$

$$
\begin{align*}
P^{\mu} & =(\stackrel{0}{m}, \stackrel{\perp}{0}, \stackrel{z}{0}) \Longrightarrow(m / \sqrt{2}, m / \sqrt{2}, \stackrel{\perp}{0})  \tag{61}\\
q^{\mu} & \approx\left(W^{2} /(2 m), \stackrel{\perp}{0}, \sqrt{\left(W^{2} / 2 m\right)^{2}+Q^{2}}\right) \Longrightarrow\left(q^{+},-Q^{2} /\left(2 q^{+}\right), \stackrel{\perp}{0}\right) \tag{62}
\end{align*}
$$

High energy: $q^{+} \approx W^{2} /(\sqrt{2} m) \rightarrow \infty$
$\gamma^{*}$ wave $e^{-i\left(q^{+} x^{-}+q^{-} x^{+}\right)}$


- Accurate $x^{-}$
- Target "shockwave" in $x^{+}$

In particular, since the light cone time $x^{+}$evolution of the high energy $\gamma^{*}$ happens with the very light cone energy $q^{-}$, the $\gamma^{*}$ cannot change into a hadronic final state inside the proton. In stead the scattering is dominated by contributions where the $\gamma^{*}$ fluctuates into hadrons before the collision. This hadronic component of the $\gamma^{*}$ state then scatters off the target. This physical picture of high energy scattering is not specific to DIS, but common to all high energy scattering in gauge theory; this was first understood by Bjorken, Kogut and Soper [12] who look at various QED processes.

In the case of DIS, recall that we are now assuming that the target is described by a gluon field. The photon does not couple to this field directly. Instead, it has to fluctuate into a hadronic state before the interaction. The simplest (i.e. lowest order in perturbation theory) such hadronic state is a quark-antiquark dipole. This leads to the so called

## Dipole picture of DIS



High energy: factorize

$$
\begin{equation*}
\sigma_{T, L}^{\gamma^{*} p}=\int \mathrm{d}^{2} \mathbf{r}_{\perp} \mathrm{d} z\left|\psi^{\gamma^{*} \rightarrow q \bar{q}}\left(\mathbf{r}_{\perp}, z\right)_{T, L}\right|^{2} 2 \mathcal{N} \tag{63}
\end{equation*}
$$

- "light cone wave function" $\psi^{\gamma^{*} \rightarrow q \bar{q}}\left(\mathbf{r}_{\perp}, z\right)_{T, L}$
- "dipole cross section" $2 \mathcal{N}$

The justification for this factorized formula is given by the high energy kinematics: since the scattering is instantaneous (in $x^{+}$), the fluctuation of the photon into a quark-antiquark pair must happen before the target (in the target rest frame).

Note that we are here using the optical theorem: the $\gamma^{*} p$-cross section is calculated as twice the imaginary part of the elastic forward $\gamma^{*} p$-scattering amplitude. Correspondingly the dipole-target cross section is twice the dipole-target elastic scattering amplitude.

## Digression: Note on 'S-matrix" and optical theorem

In QFT in general, the scattering amplitude is defined in terms of the $S$-matrix, from which one separates the trivial identity and an imaginary unit to define the scattering matrix or $T$-matrix as

$$
\begin{equation*}
S_{f i}=\langle f| \hat{S}|i\rangle=1+i T_{f i} \tag{64}
\end{equation*}
$$

The usual way to state the optical theorem is

$$
\begin{equation*}
\sigma_{\text {tot }}=2 \operatorname{Im} T_{i i} \tag{65}
\end{equation*}
$$

Now we are assuming (as is the case in lowest order in QCD perturbation theory, and also to a good approximation experimentally) that at high energy the scattering matrix is purely imaginary. Thus our (forward) scattering amplitude $\mathcal{N}$ is related to the $T$-matrix as

$$
\begin{equation*}
\mathcal{N} \equiv \operatorname{Im} T_{i i} \tag{66}
\end{equation*}
$$

and the diagonal element of the $S$-matrix is given by

$$
\begin{equation*}
S_{i i}=\delta_{i i}-\mathcal{N}+\mathrm{imag}, \tag{67}
\end{equation*}
$$

where we are all the time neglecting the imaginary part of the diagonal elements of the $S$-matrix, i.e. the real part of the diagonal element of the $T$-matrix.

In the infinite momentum frame, in the case scattering off a dilute gluon field that can be pictured as one gluon exchange, the same process would look like this:

Same process in IMF:


In the usual parton model/IMF thinking this process is formally higher order in $\alpha_{\mathrm{s}}$. It can however be argued to dominate at small $x$ since the gluon distribution $x g\left(x, Q^{2}\right)$ is so large compared to the quark distribution. However, this kind of a picture cannot describe valence quarks, or large $x$ physics more generally.

Without any more detailed calculation we will now just state that the information of the target color field in the CGC picture is built from two Wilson lines, one for the quark and another one, hermitian conjugate, for the gluon. In the eikonal approximation the quark and antiquark stay at the same coordinate when passing through the target color field. Thus the dipole cross section is needed in coordinate space, it depends ont he transverse coordinate separation of the quark and the antiquark. In general the Wilson lines in (51) depend also on the longitudinal coordinate $x^{-}$(they do not depend on $x^{+}$since this is a quantity that is integrated over). However, here we again refer to the timescale argument above: since the probe has a high $q^{+}$, it has a very high resolution in $x^{-}$. In the strict high energy limit this means that we are probing the degrees of freedom in the target at a fixed, instantaneous $x^{-}$, which we can take to be 0 here. Thus in practice we do not worry about an $x^{-}$-dependence in the Wilson line, and in stead consider it as a function of only the transverse coordinate.

## Dipole amplitude in the CGC

$$
\begin{equation*}
\mathcal{N}_{q \bar{q}}=1-\frac{1}{N_{\mathrm{c}}} \operatorname{Tr} V\left(\mathbf{x}_{\perp}\right) V^{\dagger}\left(\mathbf{y}_{\perp}\right) \tag{68}
\end{equation*}
$$

$V\left(\mathbf{x}_{\perp}\right)=$ familiar Wilson line (51)

### 5.3 Virtual photon light cone wave function

In the interest of time, we will probably not want to go through the calculation of the wave function $\psi^{\gamma^{*} \rightarrow q \bar{q}}\left(\mathbf{r}_{\perp}, z\right)$ in the lecture. The most important thing to remember is that this is a purely QED process, and therefore very well understood.

The concept of a light cone wave function makes sense in the framework of Light Cone Perturbation Theory (LCPT). In LCPT one starts fron the standard Lagrangian of the theory, and quantizes it by postulating canonical commutation relations at equal light cone time $x^{+}$among interaction picture (i.e. time-dependent) field operators. One then works in "old fashioned" quantum mechanical perturbation theory (in stead of the "modern" covariant theory of Feyman et al). This is a framework that is used to understand the partonic structure of hadronic bound (or more generally composite) states as they are measured in high energy scattering processes. In the case of the dipole picture of DIS the "bound state" that we are quantizing is
the virtual photon. We will not go into any details here, for a review see $[11,9]$ and for the notations and technical details of the calculation done here (and its extension to NLO) see [13].

The basic idea of the LCPT calculations here is that in perturbation theory we know the free particle Fock states, for a photon $\left|\gamma^{*}\right\rangle_{0}, \quad|q \bar{q}\rangle_{0}, \quad|q \bar{q} g\rangle_{0}$ etc. Physical particles are eigenstates of the interacting theory; they can always be written as linear superpositions of the free particle states

$$
\begin{equation*}
\left|\gamma^{*}\right\rangle=(1+\ldots)\left|\gamma^{*}\right\rangle_{0}+\psi^{\gamma^{*} \rightarrow q \bar{q}} \otimes|q \bar{q}\rangle_{0}+\psi^{\gamma^{*} \rightarrow q \bar{q} g} \otimes|q \bar{q} g\rangle_{0}+\ldots \tag{69}
\end{equation*}
$$

It is the coefficient functions of this expansion that are called light cone wave functions. In quantum mechanical perturbation theory the first perturbative correction to the ground state $|0\rangle$ wavefunction is

$$
\begin{equation*}
|0\rangle \Longrightarrow|0\rangle+\sum_{n} \frac{\langle n| \hat{V}|0\rangle}{E_{0}-E_{n}}|n\rangle, \tag{70}
\end{equation*}
$$

LCPT: $E \Longrightarrow k^{-}$
where $V$ is the interaction part of the Hamiltonian.Here the energy denominator $1 / \Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to $n$; it suppresses short-lived fluctuations whose energy differs by a large amount from the unperturbed state. The LCPT "energy" is the operator generating translations in light cone time $x^{+}$, i.e. $k^{-}$. The energy denominator corresponds to the propagator of covariant theory, integrated over the $k^{-}$momentum using the pole. The matrix elements $\langle n| \hat{V}|0\rangle$ are vertices in Feynman rules: the interaction terms in field theory are operators that change the number of particles.

Now let us briefly review what goes into the calculation of the virtual photon to quark-antiquark wavefunction. For this we need to evaluate the energy denominator and the vertex in the following diagram:


The matrix element is

$$
\begin{equation*}
e \bar{u}_{s}(p) \not \varliminf_{\lambda} v_{s^{\prime}}\left(p^{\prime}\right) \quad ; \quad s, s^{\prime}= \pm \frac{1}{2} ; \quad \lambda=0=L, \quad \lambda= \pm 1=T \tag{71}
\end{equation*}
$$

The transversely polarized virtual photon polarization vector in the LC gauge is given by

$$
\begin{equation*}
\varepsilon_{\lambda= \pm}^{\mu}(q)=\left(0, \frac{\mathbf{q}_{\perp} \cdot \varepsilon_{\lambda}}{q^{+}}, \varepsilon_{\lambda}\right)=\left(0,0, \varepsilon_{\lambda}\right) \tag{72}
\end{equation*}
$$

where the circularly polarized transverse polarization vectors can be taken as

$$
\begin{equation*}
\varepsilon_{ \pm}=\frac{1}{\sqrt{2}}\binom{\mp 1}{-i} \tag{73}
\end{equation*}
$$

Using Eq. (72) and explicitly evaluating the matrix elements between the Dirac spinors (these are independent of the representation of the $\gamma$-matrices, the needed elements can be found e.g. in [14]) we get

$$
\begin{equation*}
\bar{u}_{s}(p) \not \ddagger_{\lambda= \pm 1} v_{s^{\prime}}\left(p^{\prime}\right)=\delta_{s,-s^{\prime}} \frac{2}{\sqrt{z(1-z)}}\left(z \delta_{\lambda, 2 s}-(1-z) \delta_{\lambda,-2 s}\right) \varepsilon_{\lambda} \cdot \mathbf{m}_{\perp}+\delta_{s, s^{\prime}} \delta_{\lambda, 2 s} \frac{\sqrt{2} m}{\sqrt{z(1-z)}} \tag{74}
\end{equation*}
$$

Here we clearly have the separated quark helicity conserving vertex which is independent of the mass, and the helicity-flip matrix element that is proportional to the mass.

A longitudinally polarized virtual photon is strictly speaking not a part of the Fock space of the theory. In stead it is part of an additional "instantaneous" vertex that arises from explicitly solving non-dynamical constraints in light cone quantization. However, for the purposes of this calculation we can in practice define a longitudinal polarization vector for a virtual photon in the $\epsilon^{+}=0$ light cone gauge

$$
\begin{equation*}
\varepsilon_{\lambda=0}^{\mu}(q)=\left(0, \frac{\sqrt{Q^{2}-\mathbf{q}_{\perp}^{2}}}{q^{+}}, \frac{\mathbf{q}_{\perp}}{\sqrt{Q^{2}-\mathbf{q}_{\perp}^{2}}}\right) \underset{\mathbf{q}_{\perp}=0}{=}\left(0, \frac{Q}{q^{+}}, 0\right), \tag{75}
\end{equation*}
$$

where we chose $\mathbf{q}_{\perp}=0$. Using this we get

$$
\begin{equation*}
\bar{u}_{s}(p) \not \oint_{\lambda=0} v_{s^{\prime}}\left(p^{\prime}\right)=-2 Q \sqrt{z(1-z)} \delta_{s,-s^{\prime}} \tag{76}
\end{equation*}
$$

The energy denominator $\left(q^{-}-p^{-}-p^{\prime-}\right)^{-1}$ is:

$$
\begin{equation*}
=-\left(\frac{Q^{2}}{2 q^{+}}+\frac{\mathbf{p}_{\perp}{ }^{2}+m^{2}}{2 z q^{+}}+\frac{\left(\mathbf{p}_{\perp}-\mathbf{q}_{\perp}\right)^{2}+m^{2}}{2(1-z) q^{+}}\right)=\underbrace{\frac{-2 q^{+} z(1-z)}{Q^{2} z(1-z)+m^{2}+\mathbf{m}_{\perp}{ }^{2}} . . . . . ~}_{\equiv \varepsilon^{2}} \tag{77}
\end{equation*}
$$

Note that both the energy denominator and the vertex do not depend on $\mathbf{p}_{\perp}, \mathbf{p}_{\perp}^{\prime}, \mathbf{q}_{\perp}=\mathbf{p}_{\perp}+\mathbf{p}_{\perp}^{\prime}$ separately, but only though the combination $\mathbf{m}_{\perp} \equiv \mathbf{p}_{\perp}-z \mathbf{q}_{\perp}$. From now on we will take $\mathbf{q}_{\perp}=0$ so that $\mathbf{m}_{\perp}=\mathbf{k}_{\perp}$.

The eikonal scattering with the target happens such that the transverse coordinates of the quark and antiquark stay constant. Thus we need the Fourier transforms of the wave functions from transverse momenta to transverse coordinates

$$
\begin{equation*}
\psi^{\gamma^{*} \rightarrow q \bar{q}}\left(\mathbf{x}_{\perp}, \mathbf{y}_{\perp}\right)=\int \frac{\mathrm{d}^{2} \mathbf{k}_{\perp}}{(2 \pi)^{2}} \int \frac{\mathrm{~d}^{2} \mathbf{k}_{\perp}^{\prime}}{(2 \pi)^{2}}(2 \pi)^{2} \delta^{(2)}\left(\mathbf{q}_{\perp}-\mathbf{k}_{\perp}-\mathbf{k}_{\perp}^{\prime}\right) \psi^{\gamma^{*} \rightarrow q \bar{q}}\left(\mathbf{k}_{\perp}, \mathbf{k}_{\perp}^{\prime}, z\right) e^{i \mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} e^{i \mathbf{p}_{\perp}^{\prime} \cdot \mathbf{y}_{\perp}} \tag{78}
\end{equation*}
$$

The Fourier-transforms of polynomials in $\mathbf{k}_{\perp}$ divided by the energy denominator (77) are modified Bessel functions (exercise).

Summing over spins and including some additional phase space factors related to the correct normalization of the cross section (see Sec. III of [13]) the squares of the wave functions needed in (63) are:

$$
\begin{gather*}
\left|\psi_{T}^{\gamma^{*} \rightarrow q \bar{q}}\right|^{2}=\frac{\alpha_{\text {e.m. }}}{2 \pi^{2}} N_{\mathrm{c}} e_{f}\left(\left[z^{2}+(1-z)^{2}\right] K_{1}^{2}(\varepsilon r)+m_{f}^{2} K_{0}^{2}(\varepsilon r)\right)  \tag{79}\\
\left|\psi_{L}^{\gamma^{*} \rightarrow q \bar{q}}\right|^{2}=\frac{\alpha_{\text {e.m. }}}{2 \pi^{2}} N_{\mathrm{c}} e_{f} 4 Q^{2} z^{2}(1-z)^{2} K_{0}^{2}(\varepsilon r)  \tag{80}\\
\varepsilon^{2}=Q^{2} z(1-z)+m^{2} \tag{81}
\end{gather*}
$$

with $r=\mathbf{x}_{\perp}-\mathbf{y}_{\perp}$.
To summarize, we have understood the DIS process as a virtual photon $\gamma^{*}$ scattering on target at rest. In the high energy limit this process factorizes into two parts:

- $\gamma^{*}$ fluctuates into $q \bar{q}$ : this is a well understood QED process
- The $q \bar{q}$ scatters off the color field of the target: this is the part that tells us about the color field of the target.
Some side remarks about this process:
- The modified Bessel function $K_{n}$ decreases exponentially in its argument. This means that the typical dipole transverse size contributing to the scattering process is $r \sim 1 / Q$. This gives a concrete meaning to our earlier statement that $Q$ should be thought of as a transverse momentum or resolution scale in the scattering.
- We used the optical theorem expressing the total cross section as $2 \mathcal{N}$, where $\mathcal{N}$ is, in our notation, the imaginary part of the scattering amplitude. But the quantity that we really are using to describe the target is the elastic scattering amplitude. Thus we get a direct connection between inclusive scattering (total cross section) and exclusive scattering, where the target indeed stays intact. In DIS the exclusive scattering process is referred to as Diffractive DIS, and it can be independently measured. In the parton picture inclusive and exclusive DIS are different processes described by independent, separate parton distributions. But in the dipole picture one calculates the cross sections for both from the same scattering amplitude, i.e. Wilson line correlator. This enables one to make nontrivial predictions for DIS cross sections that are not possible in the parton model.
- Another way to state the content of the eikonal approximation is that we are assuming that fixed-size dipoles are the basis of states that diagonalizes the imaginary part of the $T$-matrix. In general, the fact that in the high energy/eikonal approximation particles fly through target at fixed $\mathbf{x}_{\perp}$ does not imply zero momentum transfer! Rather it should be understood that even if they do get a momentum transfer, their energy is so high (and/or the longitudinal extent of the target so short, these statements are Lorentz-transforms of each other) that the transverse momentum transfer is not enough to modify the transverse position of the particle during the time that it spends inside the target.


## 6 BK equation

### 6.1 Soft gluon radiation

As discussed above, the dipole picture of DIS represents just the leading order in a QCD perturbationtheory expansion. The next correction comes in, when one adds a gluon, either to make a loop correction, or to interact with the target. We shall here consider the kinematical limit of these processes where the gluon is very soft because, as we will see, it contributes to the cross section as a large logarithm of the energy or $x$. When we want to study what happens at very small $x$, we have to resum these logarithms of $\alpha_{\mathrm{s}} \ln 1 / x$. This resummation is performed using the Balitsky-Kovchegov (BK) equation. Our purpose here is to derive this equation. In order to do this it is enough to consider the radiative correction, where the dipole emits a gluon, and this gluon then interacts with the target nucleus. The loop corrections can, in the soft gluon limit but not generally, be deduced indirectly from a unitarity argument.

NLO: radiate soft gluon $z \ll 1$


Light cone wavefunction

$$
\begin{equation*}
\psi^{q \rightarrow q g}\left(z, \mathbf{k}_{\perp}\right)=\frac{1}{\frac{\mathbf{p}_{\perp}{ }^{2}}{2 p^{+}}-\frac{\mathbf{k}_{\perp}^{2}}{2 k^{+}}-\frac{\mathbf{p}_{\perp}^{\prime} \perp^{\prime}}{2 p^{\prime+}}} \bar{u}_{s^{\prime}}\left(p^{\prime}\right)(-g) t_{j i}^{a} q^{*}(k) u_{s}(p) \tag{82}
\end{equation*}
$$

Similarly as for the $\gamma^{*} \rightarrow q \bar{q}$ wavefunction in Sec. 5, we need the matrix element, which can be calculated in any representation of the $\gamma$-matrices. The full matrix element (now for massless quarks for simplicity) is

$$
\begin{equation*}
\bar{u}_{s^{\prime}}\left(p^{\prime}\right)(-g) t_{j i}^{a} \xi^{*}(k) u_{s}(p)=\frac{-2 g t_{j i}^{a}}{z \sqrt{1-z}}\left(\delta_{\lambda, 2 s}+(1-z) \delta_{\lambda,-2 s}\right) \mathbf{q}_{\perp} \cdot \varepsilon_{\lambda}^{*}, \quad \mathbf{q}_{\perp}=\mathbf{k}_{\perp}-z \mathbf{p}_{\perp} \tag{83}
\end{equation*}
$$

We will here be interested only in the high energy limit where the quark longitudinal momentum $p^{+}$is large, so that there is a lot of phase space for emission of gluons with a much smaller $k^{+}$. We will therefore assume that the emitted gluon is sozt, and take $z \rightarrow 0$. In this limit the emitted gluon light cone energy $k^{-}$is large and dominates the energy denominator in (82). Also the emission vertex simplifies, in particular we see that it becomes the same for both of the gluon polarization states $\lambda= \pm 2 s$. This is very general feature at high energy limit and related to what we have already discussed in the context of the eikonal approximation: the interaction of the gauge soft gauge field with the hard particles cares only about one vector: that of the hard particle, and not e.g. the spin states of the quark or gluon. In the soft limit also the relative momentum $\mathbf{q}_{\perp}=\mathbf{k}_{\perp}-z \mathbf{p}_{\perp} \rightarrow \mathbf{k}_{\perp}$

In soft limit $z \rightarrow 0$ :

$$
\begin{equation*}
\psi^{q \rightarrow q g}\left(k^{+}, \mathbf{k}_{\perp}\right) \approx \frac{-2 z p^{+}}{\mathbf{k}_{\perp}^{2}} \frac{-2 g t_{j i}^{a}}{z} \mathbf{k}_{\perp} \cdot \varepsilon_{\lambda}^{*}=\frac{4 g t_{j i}^{a} p^{+}}{\mathbf{k}_{\perp}^{2}} \mathbf{k}_{\perp} \cdot \varepsilon_{\lambda}^{*} \tag{84}
\end{equation*}
$$

From this one can calculate (somewhat sketchily) a "probability to emit a gluon" by squaring the wavefunction and integrating over a Lorentz-invariant phase space for gluon emission, which exhibits the typical gauge theory logarithmic divergences fr both soft $z \rightarrow 0$ and collinear $\mathbf{k}_{\perp} \rightarrow 0$ gluon emission.

$$
\begin{equation*}
\mathrm{d} P_{q \rightarrow q g}=\left|\psi^{q \rightarrow q g}\left(k^{+}, \mathbf{k}_{\perp}\right)\right|^{2} \frac{\mathrm{~d} k^{+} \mathrm{d}^{2} \mathbf{k}_{\perp}}{2 k^{+}(2 \pi)^{3}} \sim \frac{\mathrm{~d} z}{z} \frac{\mathrm{~d}^{2} \mathbf{k}_{\perp}}{\mathbf{k}_{\perp}^{2}} \quad\left(\sum_{\lambda= \pm 1} \varepsilon_{i} \varepsilon_{j}^{*}=\delta_{i j}\right) \tag{85}
\end{equation*}
$$

soft $\frac{d z}{z}$
collinear $\frac{\mathrm{d}^{2} \mathbf{k}_{\perp}}{\mathbf{k}_{\perp}^{2}}$
Out of these divergences the collinear one, corresponding to the emission of very large (transverse) wavelength gluons will cancel when we consider the emission from a color neutral dipole, beucase the emissions from the quark and the gluon must destructively interfere in this limit.

The soft divergence, on the other had, does not cancel. In stead it is something that we must absorb into a redefinition of what exactly we mean by the target. To visualize this it is conventional to draw diagrams where the vertical axis corresponds to rapidity. At the top is the probe photon with a large $q^{+}$and at the bottom the target, with the smallest $p^{+}$, i.e. the largest $p^{-}$, with the small mometum fraction $x \sim p^{+} / q^{+}$. The large logarithmic contributions come from integrating over the longitudinal momentum of the emitted gluons over the whole rapidity interval. Since the phase space integral is logarithmic, each gluon contributes a factor $\sim \alpha_{\mathrm{s}} \ln 1 / x$, which we are assuming to be of order one. Therefore we must resum the emission of arbitrary numbers of these gluons.

### 6.2 Idea of a renormalization group equation

This resummation is done by using a renormalization group equation, in this case the BK equation. To start with, we have to introduce some separation scale in rapidity $y \sim \ln k^{+}$. Now whatever is at a higher $k^{+}$will be called a part of the probe photon, and all that is at a lower $p^{+}$we decide to call a part of the target. The separation scale now appears as a cutoff in the integral over the longitudinal momentum fraction $z$. Now the smart thing to do is to choose in the end the separation scale so close to the rapidity of the probe $q^{+}$that the integral $\sim \alpha_{\mathrm{s}} \int_{z_{\text {cuturf }}} \mathrm{d} z / z$ is small. Then we do not have a large logarithm left, but this comes at the expense of our parametrization of target depending on this cutoff. The advantage is that now we can calculate how exactly the target depends on this cutoff: this dependence is precisely the RGE equation that tells us how the cross section depends on the energy.


If we now look at an individual gluon (at $y$ ) the cross section has to be the same whether this gluon is a part of the probe $\gamma^{*}$ Fock state, or a part of the target $p$. The equality of the cross section gives us an equation, which is the $B K$ evolution equation.

$$
\begin{align*}
\sigma^{\gamma^{*} p}=\overbrace{\left|\psi^{\gamma^{*} \rightarrow q \bar{q}}\right|_{y}^{2} \otimes 2 \mathcal{N}_{y}^{q \bar{q} p}+\left|\psi^{\gamma^{*} \rightarrow q \bar{q} g}\right|_{y}^{2} \otimes 2 \mathcal{N}_{y}^{q \bar{q} g p}+\ldots}^{\text {gluons up to y part of proton }} \\
=\underbrace{\left|\psi^{\gamma^{*} \rightarrow q \bar{q}}\right|_{y+\Delta y}^{2} \otimes 2 \mathcal{N}_{y+\Delta y}^{q \bar{q} p}+\left|\psi^{\gamma^{*} \rightarrow q \bar{q} g}\right|_{y+\Delta y}^{2} \otimes 2 \mathcal{N}_{y+\Delta y}^{q \bar{q} g p}+\ldots}_{\text {gluons up to } y+\Delta y \text { part of proton }} \tag{86}
\end{align*}
$$

### 6.3 The Balitsky-Kovchegov equation: a short derivation

Let us now put this idea into practice. To derive the BK equation one needs to

- Calculate $\psi^{\gamma^{*} \rightarrow q \bar{q} g}(z)$
- Take soft gluon limit $z \rightarrow 0$; these we have already done
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q \bar{q}$ cross section

It turns out, as you might already expect, that the calculation is more naturally done in coordinate space. We therefore need to Fourier-transform the gluon emission wave function, which becomes:

$$
\begin{equation*}
\psi^{q \rightarrow q g}\left(k^{+}, \mathbf{r}_{\perp}\right)=\int \frac{\mathrm{d}^{2} \mathbf{k}_{\perp}}{(2 \pi)^{2}} e^{i \mathbf{k}_{\perp} \cdot \mathbf{r}_{\perp}} \psi^{q \rightarrow q g}\left(k^{+}, \mathbf{k}_{\perp}\right)=-i 2 p^{+} \frac{2 g t_{j i}^{a}}{2 \pi} \frac{\varepsilon \cdot \mathbf{r}_{\perp}}{\mathbf{r}_{\perp}{ }^{2}} \delta_{s, s^{\prime}} \tag{87}
\end{equation*}
$$

Now we need to coherently sum the emissions from the quark and the antiquark, where for the latter there is a relative minus sign in the emission vertex.


$$
\begin{equation*}
\mathbf{r}_{\perp}=\mathbf{x}_{\perp}-\mathbf{y}_{\perp} \quad \mathbf{r}_{\perp}^{\prime}=\mathbf{x}_{\perp}-\mathbf{z}_{\perp} \quad \mathbf{z}_{\perp}-\mathbf{y}_{\perp}=\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime} \tag{88}
\end{equation*}
$$

We are being sloppy with the exact factors from the phase space. However, up to overall constants we can now write the Fock state wavefunction of the virtual photon as

$$
\begin{align*}
\left|\gamma^{*}\right\rangle_{\mathrm{int}}=\left|\gamma^{*}\right\rangle+\int_{z, \mathbf{r}_{\perp}} C\left(\mathbf{r}_{\perp}\right) \psi^{\gamma^{*} \rightarrow q \bar{q}}\left(z, \mathbf{r}_{\perp}\right) \mid q_{i}\left(\mathbf{x}_{\perp}, z\right) & \left.\bar{q}_{i}\left(\mathbf{y}_{\perp}, 1-z\right)\right\rangle \\
+\int_{z, \mathbf{r}_{\perp}, \mathbf{r}_{\perp}^{\prime}} \psi^{\gamma^{*} \rightarrow q \bar{q}}\left(z, \mathbf{r}_{\perp}\right) \int \frac{\mathrm{d} z^{\prime}}{4 \pi z^{\prime}} \frac{-i 2 g}{2 \pi} t_{j i}^{a} & {\left[\frac{\left(\mathbf{x}_{\perp}-\mathbf{z}_{\perp}\right) \cdot \varepsilon}{\left(\mathbf{x}_{\perp}-\mathbf{z}_{\perp}\right)^{2}}-\frac{\left(\mathbf{y}_{\perp}-\mathbf{z}_{\perp}\right) \cdot \varepsilon}{\left(\mathbf{y}_{\perp}-\mathbf{z}_{\perp}\right)^{2}}\right] } \\
& \times\left|q_{i}\left(\mathbf{x}_{\perp}, z\right) \bar{q}_{j}\left(\mathbf{y}_{\perp}, 1-z\right) g_{a}\left(\mathbf{z}_{\perp}, z^{\prime}\right)\right\rangle, \tag{89}
\end{align*}
$$

To be a bit more precise let us write the exact expressions for the longitudinal phase space integrals here. These require a bit of care to be done correctly, because the Lorentz-invariant phase space element for the integrations is

$$
\begin{equation*}
\frac{\mathrm{d} k^{+}}{4 \pi k^{+}} \tag{90}
\end{equation*}
$$

but the factor $2 k^{+}$is not there in the momentum conservation delta function: each vertex yields a

$$
\begin{equation*}
2 \pi \delta\left(\sum p^{+}\right) \tag{91}
\end{equation*}
$$

For the quark-antiquark dipole the phase exact expression for the phase space integral is

$$
\begin{equation*}
" \int_{z} " \equiv \int \frac{\mathrm{~d} p^{+}}{4 \pi p^{+}} \frac{\mathrm{d} p^{++}}{4 \pi p^{\prime+}}(2 \pi) \delta\left(p^{+}+p^{+}-q^{+}\right)=\frac{1}{2 q^{+}} \int \frac{\mathrm{d} z}{4 \pi z(1-z)} . \tag{92}
\end{equation*}
$$

For the quark-antiquark-gluon state we have in (89) already assumed that the gluon is soft $k^{+}=z^{\prime} q^{+} \ll$ $p^{+}, p^{+}, q^{+}$. This has enabled us to factor out the kinematics of the leading order vetex; this is possible only in the soft limit. For the full kinematics the whole expressions are a bit more complicated [13]. For the term where the gluon is emitted from the quark, the phase space integral (denoting $p^{\prime \prime}$ the momentum of the quark before emitting the gluon) is

$$
\begin{array}{r}
" \int_{z, z^{\prime}} " \equiv \int \frac{\mathrm{~d} p^{+}}{4 \pi p^{+}} \frac{\mathrm{d} p^{+}}{4 \pi p^{\prime+}} \frac{\mathrm{d} p^{\prime \prime+}}{4 \pi p^{\prime \prime+}} \frac{\mathrm{d} k^{+}}{4 \pi k^{+}}(2 \pi) \delta\left(p^{\prime \prime+}+p^{\prime+}-q^{+}\right)(2 \pi) \delta\left(p^{+}+k^{+}-p^{\prime \prime+}\right) \\
=\frac{1}{\left(q^{+}\right)^{2}} \int \frac{\mathrm{~d} z}{4 \pi} \frac{\mathrm{~d} z^{\prime}}{4 \pi} \frac{1}{z} \frac{1}{1-z-z^{\prime}} \frac{1}{z+z^{\prime}} \frac{1}{z^{\prime}} \approx \frac{1}{q^{+}} \int \frac{\mathrm{d} z}{8 \pi z(1-z)} \frac{1}{z q^{+}} \int \frac{\mathrm{d} z^{\prime}}{8 \pi z^{\prime}} \\
=" \int_{z}^{"} \times \frac{1}{2 p^{+}} \int \frac{\mathrm{d} z^{\prime}}{4 \pi z^{\prime}}, \tag{93}
\end{array}
$$

where we separated out an explicit $1 / 2 p^{+}$to recover the normal Lorentz-invariant phase space $\mathrm{d} z^{\prime} /\left(4 \pi z^{\prime}\right)$. Likewise, when the gluon is emitted from the antiquark we get, using this time $p^{\prime \prime}$ for the momentum of the
antiquark before the emission

$$
\begin{align*}
" \int_{z, z^{\prime}} " & \equiv \int \frac{\mathrm{~d} p^{+}}{4 \pi p^{+}} \frac{\mathrm{d} p^{+}}{4 \pi p^{+}} \frac{\mathrm{d} p^{\prime \prime+}}{4 \pi p^{\prime \prime+}} \frac{\mathrm{d} k^{+}}{4 \pi k^{+}}(2 \pi) \delta\left(p^{+}+p^{\prime \prime+}-q^{+}\right)(2 \pi) \delta\left(p^{++}+k^{+}-p^{\prime \prime+}\right) \\
& =\frac{1}{\left(q^{+}\right)^{2}} \int \frac{\mathrm{~d} z}{4 \pi} \frac{\mathrm{~d} z^{\prime}}{4 \pi} \frac{1}{z} \frac{1}{1-z-z^{\prime}} \frac{1}{1-z} \frac{1}{z^{\prime}} \approx \frac{1}{q^{+}} \int \frac{\mathrm{d} z}{8 \pi z(1-z)} \frac{1}{(1-z) q^{+}} \int \frac{\mathrm{d} z^{\prime}}{8 \pi z^{\prime}} . \\
& =" \int_{z}^{"} \times \frac{1}{2 p^{\prime+}} \int \frac{\mathrm{d} z^{\prime}}{4 \pi z^{\prime}}, \tag{94}
\end{align*}
$$

Here the explicit factors $\frac{1}{2 p^{+}}, \frac{1}{2 p^{+}}$in Eqs. (94) and (93) cancel against the $2 p^{+}$in Eq. (87) and the corresponding $2 p^{\prime+}$ in the emission wavefunction from the antiquark, and therefore do not appear in Eq. (89).

It is important to note that the $\int_{z^{\prime}}$-integral is divergent because of the phase space $\int_{z^{\prime}} \sim \int \mathrm{d} z^{\prime} / z^{\prime}$. We have also added a coefficient $C\left(\mathbf{r}_{\perp}\right)$ in front of the leading order term, because the normalization of this term needs to be adjusted in order to keep the wavefunction normalized. We can determine this coefficient fron the requirement that the normalization of the wave function with the extra gluon, i.e. square of the dipole term with the normalization constant plus the square of the $q \bar{q} g$ term, are the same as the square of the dipole term at lowest order. Note that here squaring involves also one sum over the colors of the quarks, which yields a factor $N_{\mathrm{c}}$ for the dipoles. We get the condition

$$
\begin{align*}
N_{\mathrm{c}}\left|C\left(\mathbf{r}_{\perp}\right)\right|^{2} & =N_{\mathrm{c}}-\frac{(2 g)^{2}}{(2 \pi)^{2}} \frac{1}{4 \pi} t_{i j}^{a} t_{j i}^{a} \int \frac{\mathrm{~d} z^{\prime}}{z^{\prime}} \int \mathrm{d}^{2} \mathbf{r}_{\perp}^{\prime} \sum_{\lambda= \pm 1}\left|\frac{\left(\mathbf{x}_{\perp}-\mathbf{z}_{\perp}\right) \cdot \varepsilon_{\lambda}}{\left(\mathbf{x}_{\perp}-\mathbf{z}_{\perp}\right)^{2}}-\frac{\left(\mathbf{y}_{\perp}-\mathbf{z}_{\perp}\right) \cdot \varepsilon_{\lambda}}{\left(\mathbf{y}_{\perp}-\mathbf{z}_{\perp}\right)^{2}}\right|^{2} \\
& =N_{\mathrm{c}}-\frac{\alpha_{\mathrm{s}}}{\pi^{2}} \frac{N_{\mathrm{c}}{ }^{2}-1}{2} \Delta y \int \mathrm{~d}^{2} \mathbf{r}_{\perp}^{\prime} \frac{\mathbf{r}_{\perp}{ }^{2}}{\mathbf{r}_{\perp}^{\prime}\left(\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}\right)^{2}} \quad \sum_{\lambda= \pm 1} \varepsilon_{i}^{(\lambda)} \varepsilon_{j}^{(\lambda) *}=\delta_{i j} \tag{96}
\end{align*}
$$

Putting these together, and remembering that we now redefine our scattering amplitude so that the scattering off the dipole and the dipole-gluon system from the old target becomes the same as the scattering of just a dipole off a new, renormalized, target.

$$
\begin{equation*}
\mathcal{N}_{q \bar{q}}^{y+\Delta y}=\mathcal{N}_{q \bar{q}}^{y}+\frac{\alpha_{s}}{\pi^{2}} \frac{N_{c}{ }^{2}-1}{2 N_{c}} \int_{y}^{y+\Delta y} \mathrm{~d} \ln 1 / z^{\prime} \int \mathrm{d}^{2} \mathbf{r}_{\perp}^{\prime} \frac{\mathbf{r}_{\perp}{ }^{2}}{\mathbf{r}_{\perp}^{\prime}\left(\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}\right)^{2}}\left[\mathcal{N}_{q \bar{q} \bar{g}}^{\ln 1 / z^{\prime}}-\mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\right] \tag{97}
\end{equation*}
$$



What this equation now tells us is that the dipole scattering on new target $\mathcal{N}_{q \bar{q}}^{y+\Delta y}$ consists of

- A dipole scattering off original target $\mathcal{N}_{q \bar{q}}^{y}$
- A contribution where the dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off the target and
- A correction to the normalization of original dipole (There are now less dipoles in $\gamma^{*}$ )

Now we are almost there. We are looking for an evolution equation for the scattering amplitude $\mathcal{N}_{q \bar{q}}$ : but enocuntered new quantity $\mathcal{N}_{q \bar{q} q}$, which needs to be related to $\mathcal{N}_{q \bar{q}}$. Let us do this in the large $N_{\mathrm{c}}$ approximation. At large $N_{\mathrm{c}}$ a gluon is approximately the same as a quark-antiquark pair (note: not a color neutral dipole, in fact a general quark-antiquark pair is $1 / N_{c}{ }^{2}$ parts dipole and $\left(N_{c}{ }^{2}-1\right) / N_{c}{ }^{2}$ parts octet, i.e. gluon, and at large $N_{\mathrm{c}}$ we can forget the color neutral dipole). So when we have a quantum state $\left|q\left(\mathbf{x}_{\perp}\right) \bar{q}\left(\mathbf{y}_{\perp}\right) g\left(\mathbf{z}_{\perp}\right)\right\rangle$, we can approximate it by $\left|q\left(\mathbf{x}_{\perp}\right) \bar{q}\left(\mathbf{z}_{\perp}\right) q\left(\mathbf{z}_{\perp}\right) \bar{q}\left(\mathbf{y}_{\perp}\right)\right\rangle$

- $N_{\mathrm{c}} \rightarrow \infty \Longrightarrow g \approx q \bar{q}$ (not dipole!)
- $N_{c}{ }^{2}-1$ gluon colors $\approx N_{c}{ }^{2} q \bar{q}$ colors.


Now, instead of $\mathcal{N}_{q \bar{q} g}$, we need $\mathcal{N}_{q \bar{q} q \bar{q}}$; the amplitude for the simultaneous scattering of two dipoles. We can obtain it with the following argument:

- $\mathcal{N}$ is a really a scattering probability;
- $S=1-\mathcal{N}$ is a probability not to scatter

For two dipoles no scattering means that neither dipole scatters, from which we deduce

$$
\begin{equation*}
S_{q \bar{q} q \bar{q}}=S_{q \bar{q}} S_{q \bar{q}} \tag{98}
\end{equation*}
$$

Thus the scattering probability for a two-gluon system is

$$
\begin{equation*}
\mathcal{N}_{q \bar{q} q \bar{q}}=1-S_{q \bar{q} q \bar{q}}=1-\left(1-\mathcal{N}_{q \bar{q}}\right)\left(1-\mathcal{N}_{q \bar{q}}\right) \tag{99}
\end{equation*}
$$

In order to justify this argument we need to make an additional mean field approximation, namely that the scattering probabilities of the two dipoles are independent of each other, which enables the factorized form (98). In fact this mean field approximation can be argued to hold in the large $N_{\mathrm{c}}$ limit, which we are already assuming here.

With this argument we end up with the approximation

$$
\begin{equation*}
\mathcal{N}\left(q\left(\mathbf{x}_{\perp}\right) \bar{q}\left(\mathbf{y}_{\perp}\right) g\left(\mathbf{z}_{\perp}\right)\right) \approx \mathcal{N}\left(q\left(\mathbf{x}_{\perp}\right) \bar{q}\left(\mathbf{z}_{\perp}\right)\right)+\mathcal{N}\left(q\left(\mathbf{z}_{\perp}\right) \bar{q}\left(\mathbf{y}_{\perp}\right)\right)-\mathcal{N}\left(q\left(\mathbf{x}_{\perp}\right) \bar{q}\left(\mathbf{z}_{\perp}\right)\right) \mathcal{N}\left(q\left(\mathbf{z}_{\perp}\right) \bar{q}\left(\mathbf{y}_{\perp}\right)\right) . \tag{100}
\end{equation*}
$$

The argument leading to this can also be stated in another way: in order to calculate the probability for a scattering to happen, one must add the probabilities that one of the two dipoles scatters, and then subtract the case that both dipoles scatter, since this would otherwise get counted twice.

$$
\begin{align*}
& \mathcal{N}_{q \bar{q}}^{y+\Delta y}=\mathcal{N}_{q \bar{q}}^{y}+\frac{\alpha_{\mathrm{s}}}{\pi^{2}} \frac{N_{\mathrm{c}}^{2}-1}{2 N_{\mathrm{c}}} \int_{y}^{y+\Delta y} \mathrm{~d} \ln 1 / z^{\prime} \int \mathrm{d}^{2} \mathbf{z}_{\perp} \frac{\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)^{2}}{\left(\mathbf{x}_{\perp}-\mathbf{z}_{\perp}\right)^{2}\left(\mathbf{z}_{\perp}-\mathbf{y}_{\perp}\right)^{2}} \\
& \left.\quad \times\left[\mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\left(\mathbf{x}_{\perp}, \mathbf{z}_{\perp}\right)+\mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\left(\mathbf{z}_{\perp}, \mathbf{y}_{\perp}\right)-\mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\left(\mathbf{x}_{\perp}, \mathbf{z}_{\perp}\right) \mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\left(\mathbf{z}_{\perp}, \mathbf{y}_{\perp}\right)-\mathcal{N}_{q \bar{q}}^{\ln 1 / z^{\prime}}\left(\mathbf{x}_{\perp}, \mathbf{y}_{\perp}\right)\right)\right] \tag{101}
\end{align*}
$$

## Differentially in $\Delta y$

$$
\begin{equation*}
\partial_{y} \mathcal{N}\left(\mathbf{r}_{\perp}\right)=\frac{\alpha_{\mathrm{s}} N_{\mathrm{c}}}{2 \pi^{2}} \int \mathrm{~d}^{2} \mathbf{r}_{\perp}^{\prime} \frac{\mathbf{r}_{\perp}{ }^{2}}{\mathbf{r}_{\perp}^{\prime 2}\left(\mathbf{r}_{\perp}^{\prime}-\mathbf{r}_{\perp}\right)^{2}}\left[\mathcal{N}\left(\mathbf{r}_{\perp}^{\prime}\right)+\mathcal{N}\left(\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}\right)-\mathcal{N}\left(\mathbf{r}_{\perp}^{\prime}\right) \mathcal{N}\left(\mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}\right)-\mathcal{N}\left(\mathbf{r}_{\perp}\right)\right] \tag{102}
\end{equation*}
$$

This is the BK equation (1995), that I personally like to call the holy grail of small $x$ physics. It is the basic tool for a lot of practical phenomenology.

- Given initial condition $\mathcal{N}\left(\mathbf{r}_{\perp}\right)$ at $y=y_{0}$ the equation predicts the scattering amplitude at larger $y=$ smaller $x=$ higher $\sqrt{s}$.
- If one drops the nonlinear term one gets the BFKL (Balitsky-Fadin-Kuraev-Lipatov) equation, which was derived by summing normal Feynman diagrams for the scattering, already a longer time ago.
- The divergences of the kernel at $\mathbf{r}_{\perp}^{\prime} \rightarrow 0$ and $\mathbf{r}_{\perp}^{\prime} \rightarrow \mathbf{r}_{\perp}$ are regulated because $\mathcal{N}(0)=0$. This feature is referred to as "color transparency": a dipole is a color neutral object as a whole, and if its size is zero it should not scatter at all.
- The equation enforces the black disk limit (unitarity) $\mathcal{N}<1$
- For practical work the coupling $\alpha_{\mathrm{s}}$ should not be fixed, but depend on a distance scale, which can be some combination of $\mathbf{r}_{\perp}, \mathbf{r}_{\perp}^{\prime}, \mathbf{r}_{\perp}-\mathbf{r}_{\perp}^{\prime}$. What, in the end, is both theoretically and practically the correct way to do this is still a matter of discussion in the field.

The equation can be solved numerically, let us look at a few features of the solution

- Small dipoles $r \lesssim 1 / Q_{\mathrm{s}}$ scatter very little, color transparency is satisfied.
- Large dipoles $r \gtrsim 1 / Q_{\mathrm{S}}$ scatter with probability almost one, but not more. This is gluon saturation in action, although in this calculation gluon saturation ndoes not appear as some mystical complicated phenomenon, but just the obvious fact that the scattering can only happen with a probability of at most one.
- When going towards smaller $x$, the "front" of the solution moves towards the left. This means that the sensity of gluons in the target increases, justifying a posteriori our assumption that at smaller $x$ the classical field picture of the target gets better and better.


(Actually cheating, this particular plot from [15] is a solution of JIMWLK, which generalizes BK)
To set this in context remember that for the DIS cross sections $F_{2}, F_{L}$ this solution of the BK equation has to be convoluted with the (known) $\gamma^{*}$ wavefunction.

$$
\begin{equation*}
\sigma_{T, L}^{\gamma^{*} p}=\int \mathrm{d}^{2} \mathbf{b}_{\perp} \mathrm{d}^{2} \mathbf{r}_{\perp} \mathrm{d} z\left|\psi^{\gamma^{*} \rightarrow q \bar{q}}(r, z)_{T, L}\right|^{2} 2 \mathcal{N}\left(\mathbf{r}_{\perp}, \mathbf{b}_{\perp}, x\right) \tag{103}
\end{equation*}
$$

Fits HERA data ( $x<0.01 Q^{2}$ moderate) with this formula work extremely well, althugh this depends on a specific way to model the $b$-dependence of the dipole amplitude.

The JIMWLK equation is a generalization of the BK equation, in the sense of giving uo the mean field approximation that we had to use. In fact, even without this approximation and at finite $N_{\mathrm{c}}$ it turns out, after some color algebra magic, that the dipole amplitude satisfies an equation that has the form (102). However, the amplitudes $\mathcal{N}$ are replaced by expectation values of the amplitudes in some distribution. In general, the expectation value of the product of amplitudes $\langle\mathcal{N N}\rangle$ cannot be expressed in terms of the expectation value of the amplitude $\langle\mathcal{N}\rangle$. Thus the resulting equation does not close. One can then derive an additional evolution equation for the $x$-dependence of $\langle\mathcal{N} \mathcal{N}\rangle$, but this will involve a new quantity $\langle\mathcal{N} \mathcal{N} \mathcal{N}\rangle$, and so on ad infinitum. This set of evolution equations is known as the Balitsky hierarchy. It is equivalent to the JIMWLK equation, which is written as an evolution equation for the probability distribution of the Wilson lines from which the amplitudes are constructed. Knowing the probability distribution is equivalent to knowing all possible expectation values.

## 7 Gluon saturation \& CGC

### 7.1 Spacetime structure in 2 gauges

Until now we have been describing the high energy nucleus as a classical field with one Lorentz-component $A^{-}$, which we then developed into a path-ordered exponential or Wilson line, which is the eikonal scattering amplitude for a high energy probe passing through the nucleus. What does this picture imply in terms of the partonic content of the nucleus, i.e. the gluon distribution?

Let us now reflect the $z$-axis and consider a nucleus (in stead of the probe) moving in the + -direction. This means that the color field of the nucleus has a large $A^{+}$component. The physical picture of "gluons as partons" requires two things

- Infinite momentum frame: we have to boost to a frame where the nucleus is moving fast, so that the partons are collinear (at least approximately) to the nucleus.
- Light cone gauge: in order to have a partonic interpretation we also have to gauge transform to light cone gauge $A^{+}=0$.

Why light cone gauge? ...
But let us start with the "classical" part. The basic idea of the CGC is that we separate the microscopic degrees of freedom inside a high energy nucleus into two things:

CGC separation of scales:

- small $x$ : classical field
- large $x$ : color charge

For a first simplistic picture one can think of the color charge as valence quarks, which then radiate gluons, which are represented by the classical field. A classical field radiated by a classical current follows the classical equation of motion. For Yang-Mills theory this is given by

$$
\begin{equation*}
\left[D_{\mu}, F^{\mu \nu}\right]=J^{\nu} \tag{104}
\end{equation*}
$$

This is written in terms of some yet unspecified color current $J^{\mu}$. The commutator in (104) refers to the fact that the gauge potential $A_{\mu}$, field strength $F_{\mu \nu}$ and color current $J^{\mu}$ are matrices in color space. Incidentally, note that sice the field strength is a commutator of covariant derivatives, $F_{\mu \nu} \sim\left[D_{\mu}, D_{\nu}\right]$, one can show using the Jacobi identity that the color current must be covariantly conserved,

$$
\begin{equation*}
\left[D_{\nu}, J^{\nu}\right]=0 \tag{105}
\end{equation*}
$$

in stead of the usual abelian charge conservation $\partial_{\mu} J^{\mu}=0$. This means that the conservation of color charge actually requires $J^{\mu}$ to rotate in color space.

If we have only one important component of the gauge field, $A^{+}$, the equation of motion (104) simplifies to

$$
\begin{equation*}
\nabla_{\perp}^{2} A^{+}=-J^{+} \tag{106}
\end{equation*}
$$

This looks very nice nice, the big +-component of the field corresponds to a color current in the + direction, precisely what we would expect in the physical situation of a color charged object (which the nucleus is, if you only look at it within a region of size $\ll 1 / \Lambda_{\mathrm{QCD}}$ in the transverse plane).


Now we have to somehow implement the statement that the charge is supposed to correspond to large $x$ and the fields to small $x$ in a spacetime picture. The consequence of this is that we can treat the color current
$J^{+}$as a very thin sheet in $x^{-}$. Sometimes one quotes a somewhat naive justification for this approximation in terms of a Lorenz-contraction of the nucleus from its rest frame thickness $2 R_{A}$ to $\Delta z \sim 2 R_{A} m_{A} / \sqrt{s}$ by the boost from the rest frame energy $m_{A}$ to the large energy $\sqrt{s}$. However, the more proper argument for this assumption is the combination of the uncertainty principle with the assumption about a scale separation. The current represents large $x$ degrees of freedom, which for a fast nucleus have a large $p^{+}$momentum. The classical field, on the other hand, corresponds to degrees of freedom with smaller $p^{+}$. As a consequence of this the charges are better localized $x^{-}$than the field, so the field (which is what we are interested in) sees the charges as localized in $x^{-}$.

Charges are thin sheet in $x^{-}$

## 1. Naive explanation: Lorentz-contraction

2. Real explanation: Heisenberg $\Delta x^{-} \sim 1 / p^{+}$, charges have large $p^{+}$

A similar argument must be made for the dependence of the current on the light cone time coordinate $x^{+}$. A simplistic argument states that for a fast-moving nucleus time is Lorentz-dilated so that all the internal dynamics happens very slowly in $x^{+}$. A more detailed statement of the same effect is that any probe of the small $x$ (small $p^{+}$) degrees of freedom will have a larger $p^{-} \sim p_{T}^{2} / p^{+}$than the light cone energies $p^{-}$of the color charges. Since the color charges have a small $p^{-}$, they evolve more slowly in $x^{+}$than the fields, or an external probe of the system operating at the timescale of the small- $x$ fields. Therefore from the point of view of the field, or of this external probe, the charges appear as slowly dependent of $x^{+}$. The extreme limit of this argument is that we can consider the charge density to be independent of the light cone time $x^{+}$, i.e. static.

## Digression: Glass

A system that is dime dependent, but whose time dependence is unnaturally slow compared to the timescales that one is probing it with, is known as a glass in statistical physics. An actual glass is a "glass" in the statistical physics sense, because glasses are non-chrystalline like liquids, but nevertheless do not flow. In a loose (and not technically correct) sense one can describe a glass as a liquid that flows so slowly that on the timescales of interest it behaves like a solid.

Taking these energy- and momentum scale arguments to the extreme now arrive at the desired form for the color current as a $\delta$-function in $x^{-}$, independent of $x^{+}$, which leaves us with a purely 2-dimensional transverse color charge density. The notation $\delta\left(x^{-}\right)$should be thought of as a somewhet formal limit, we will still need to maintain the concept of path ordering (now in $x^{-}$because we switched the direction of motion of the nucleus): so we should stil lthink of the nucleus as slightly elongated in $x^{-}$, byt very thin compared to any other $x^{-}$-scale in the scattering problem.

Current static, independent of $x^{+} \Longrightarrow$ glass

## 1. Time is dilated for the nucleus

2. Charges: small $p^{-}$: slow dependence on $x^{+}$

$$
\begin{align*}
x^{-} & \uparrow t \\
J^{+}\left(x^{-}, \mathbf{x}_{\perp}\right) & \approx \delta\left(A^{-}\right) \rho\left(\mathbf{x}_{\perp}\right)  \tag{107}\\
A^{+}\left(x^{-}, \mathbf{x}_{\perp}\right) & \approx-\delta\left(x^{-}\right) \frac{1}{\nabla_{\perp}^{2}} \rho\left(\mathbf{x}_{\perp}\right)  \tag{108}\\
F^{+i} & =\partial_{i} A^{+} \tag{109}
\end{align*}
$$

Note that the only nonzero component of the field strength tensor $F^{\mu \nu}$ is $F^{+i}$, which corresponds to $F^{0 i}$ and $F^{z i}$, i.e. chromoelectric and - magnetic fields perpendicular to the $z$-axis. This is the same as the

Weizsäcker-Williams field of an electric charge boosted to the speed of light, which can be thought of as an "equivalent photon" cloud accompanying the photon.

As stated before, in order to have a partonic interpretation we have to transform our field to the light cone gauge $A^{+}=0$. In addition to satisfying the requirement of a formal partonic interpretation in terms of a light cone quantization of the degrees of freedom in the nucleus, this gauge transformation will be required to calculate what happens in the collision of our nucleus with another one. We will come back to this in Sec. 8. But already at this stage it is useful to note that the other colliding nucleus will have a color current $j^{-}$, and if our color field has a $A^{+}$-component it will cause the color current of the other nucleus to precess in color space. But if we transform our field to $A^{+}=0$-gauge, the current of the other nucleus can stay unaffected by it, leaving only the fields to interact. This is easier to deal with, in fact a major reason for using the classical field approximation in the first place is that classical fields are easier to deal with. So let us now make this gauge transformation to $A^{+}=0$. We will keep our static approximation, nothing depends on $x^{+}$, but maintain a general $x^{-}$-dependence remembering that the support of our color current in $x^{-}$is very narrow. Note that since there is no dependence on $x^{+}$, the gauge transform will not generate an $A^{-}$-component. This should not be seen as a gauge choice, but as a result of the high energy kinematics.

Gauge transform:

$$
\begin{align*}
A^{+} & \Rightarrow V^{\dagger}\left(\mathbf{x}_{\perp}, x^{-}\right) A^{+} V\left(\mathbf{x}_{\perp}, x^{-}\right)-\frac{i}{g} V^{\dagger}\left(\mathbf{x}_{\perp}, x^{-}\right) \partial_{-} V\left(\mathbf{x}_{\perp}, x^{-}\right)=0  \tag{110}\\
A^{-} & \Rightarrow-\frac{i}{g} V^{\dagger}\left(\mathbf{x}_{\perp}, x^{-}\right) \partial_{+} V\left(\mathbf{x}_{\perp}, x^{-}\right)=0, \text { still }  \tag{111}\\
A^{i} & \Rightarrow \frac{i}{g} V^{\dagger}\left(\mathbf{x}_{\perp}, x^{-}\right) \partial_{i} V\left(\mathbf{x}_{\perp}, x^{-}\right) \quad \text { transverse pure gauge } \tag{112}
\end{align*}
$$

(110) solved by

$$
V\left(\mathbf{x}_{\perp}, x^{-}\right)=\mathbb{P} \exp \left[-i g \int_{-\infty}^{x^{-}} \mathrm{d} y^{-} A^{+}\right]
$$

$\Longrightarrow$ Wilson line!
Now since for $A^{+}\left(y^{-}\right)$is very localized around $y^{-}=0$, the Wilson line $V$ jumps very rapidly from unity (independently of the transverse coordinate) to a nontrivial $\mathbf{x}_{\perp}$-dependent value around $y^{-}=0$. Thus its derivative, which gives $A^{i}$, has a $\sim \theta\left(x^{-}\right)$-function like discontinuity around $x^{-}=0$. This (unlike the localized $A^{+}$-field) now corresponds to the physical picture of how the color field of small $x$ (and therefore small $p^{+}$) gluons should look like - delocalized in $x^{-}$.

$$
A^{i} \sim \theta\left(x^{-}\right)
$$



### 7.2 McLerran-Venugopalan model

In order to be able to actually calculate things we will now need a concrete model for what the charge density $\rho$ in Eqs. (107), (108) could look like. Such a model is provided by the Mclerran-Venugopalan (MV) model [16, 17, 18].

Let us briefly review the physics argument behind this model. The situation one is thinking about is that of small enough $x$ so that the CGC spacetime picture discussed above is valied; but not necessarily asymptotically small. In stead, we are working in the limit where the nuclear mass number $A$ is very large. This implies that at a fixed coordinate in the transverse plane there is an parametrically large number $\sim A^{1 / 3}$ of valence quark-like colored degrees of freedom. Since these color charges come from separate nucleons, they should be uncorrelated with each other in the longitudinal direction. Also in the transverse plane, at two different coordinates, the total color charge commes from a sum of at only partially overlapping color charges in the different nucleons, and is therefore different. Thus one can also assume the color charges at
different transverse coordinates to be uncorrelated. Finally, since the color charge at one coordinate is a sum of many uncorrelated color charges, the central limit theorem states that its distribution should be Gaussian.

MV model: charge density $\rho\left(x_{\perp}\right)$ is

- stochastic, Gaussian random
- local in $\times \AA$ - and $x_{\perp}$

$$
\begin{equation*}
\left\langle\rho^{a}\left(\mathbf{x}_{\perp}, x^{-}\right) \rho^{b}\left(\mathbf{y}_{\perp}, y^{-}\right)\right\rangle=g^{2} \delta^{a b} \mu^{2}\left(x^{-}\right) \delta\left(x^{-}-y^{-}\right) \delta^{(2)}\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right) \tag{113}
\end{equation*}
$$

The importance of this model is that it is based on a very general physical argument, but is nevertheless is very explicit, enabling one to calculate a large variety of things in various approximations. It is also simple in terms of parameters, since as we will see physical observables will end up depending on only one new dimensionful parameter, which is essentially the saturation scale

$$
\begin{equation*}
Q_{\mathrm{s}}^{2} \sim g^{4} \int_{-\infty}^{\infty} \mathrm{d} x^{-} \mu^{2}\left(x^{-}\right) \tag{114}
\end{equation*}
$$

For further developments it is important to note that one is making two independent approximations that need to be considered separately.

- We assumed that the charge density correlator is proportional to $\delta^{(2)}\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$. This, as we will see, leads to a very specific dependence of the unintegrated gluon distribution $\varphi^{\mathrm{wW}}\left(\mathbf{k}_{\perp}\right)$ on $k_{T}$ at high momentum. The resulting distribution is roughly consistent with a DGLAP-like behavior of the gluon distribution, but not for example with the solution of the BK equation at very small-x. This assumption is therefore usually thought of as appropriate for moderately small $x$ and large $A$, but not parametrically small $x$.
- The other assumptions; namely that the charges at different $x^{-}$are uncorrelated, and that the distribution is Gaussian (which implies a specific relation between the higher 4-, 6- etc. correlators and the 2-point function (113)) are much more general. These assumptions can be used to to calculate relations between different Wilson line correlators, and in the cases where this has been checked (see e.g. [19]) this assumption has been found to be consistent with JIMWLK evolution. What this implies in practice is that one can use the assumption of locality in $x^{-}$and Gaussian correlators to calculate multiparticle correlations, and combine this with a transverse coordinate structure that is not given by $\delta^{(2)}\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$, but taken e.g. from the solution of the BK equation. This procedure or approximation is referred to in the literature as the "nonlinear Gaussian" [19], "nonlocal Gaussian" or "Gaussian truncation" [6] approach.


### 7.3 Dipole cross section

In order to get a more physical picture of what the MV model means, let us now use it to calculate the two unintegrated gluon distributions that we have discussed, the dipole distribution (dipole cross section of Secs. 5 and 6.

Dipole

$$
\begin{gather*}
S\left(r=\left|\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right|\right) \equiv \frac{1}{N_{\mathrm{c}}}\left\langle\operatorname{Tr} V\left(\mathbf{x}_{\perp}\right) V^{\dagger}\left(\mathbf{y}_{\perp}\right)\right\rangle \quad V\left(\mathbf{x}_{\perp}, x^{-}\right)=\mathbb{P} \exp \left[-i g \int_{-\infty}^{x^{-}} \mathrm{d} y^{-} A^{+}\right]  \tag{115}\\
\left\langle A_{a}^{+}\left(\mathbf{x}_{\perp}, x^{-}\right) A_{b}^{+}\left(\mathbf{y}_{\perp}, y^{-}\right)\right\rangle=g^{2} \delta^{a b} \mu^{2}\left(x^{-}\right) \delta\left(x^{-}-y^{-}\right) L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right) \tag{116}
\end{gather*}
$$

In order to make sense of the path ordered exponential in practice, it is not very convenient it is not in practice very convenient to use the path-ordered product definition (51). In stead, we will discretize the integral over $x^{-}$into many infinitesimally thin pieces of width $\Delta^{-} \rightarrow 0$. Now a path ordered exponential of an integral turns into a path ordered exponential of a sum, which is nicely interpreted as a path ordered product of exponentials of individual terms in the sum. Notice that if the path ordering in the Wilson line is from right to left, in the hermitian conjugate the corresponding ordering is from left to right. Thus in the trace (115) the elements at the lower limit in $x^{-}$are next to each other.

## Discretize

$$
\begin{array}{r}
x^{-}=n \Delta^{-} \Longrightarrow\left\langle A_{m a}^{+}\left(\mathbf{x}_{\perp}\right) A_{n b}^{+}\left(\mathbf{y}_{\perp}\right)\right\rangle=g^{2} \delta^{a b} \mu_{n}^{2} \frac{1}{\Delta^{-}} \delta_{m n} L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right) \\
S(r)=\lim _{N \rightarrow \infty} \frac{1}{N_{\mathrm{c}}} \operatorname{Tr}\left\langle e^{-i g \Delta^{-} A_{N}^{+}\left(\mathbf{x}_{\perp}\right)} \cdots e^{-i g \Delta^{-} A_{n}^{+}\left(\mathbf{x}_{\perp}\right)} \cdots e^{-i g \Delta^{-} A_{-N}^{+}\left(\mathbf{x}_{\perp}\right)}\right. \\
 \tag{118}\\
\left.\times e^{i g \Delta^{-} A_{-N}^{+}\left(\mathbf{y}_{\perp}\right)} \cdots e^{i g \Delta^{-} A_{n}^{+}\left(\mathbf{u}_{\perp}\right)} \cdots e^{i g \Delta^{-} A_{-N}^{+}\left(\mathbf{y}_{\perp}\right)}\right\rangle
\end{array}
$$

Now according to the MV model assumption the color fields at different steps in $x^{-}$are indpendent of each other. This means that the expectation value of the product is the product of expectation values, and we can start calculating the expectation values one rapidity step at the time, starting with the innermost $n=-N$. Now we also use the fact that $\Delta^{-}$is small and expand

$$
\begin{align*}
\left\langle e^{-i g \Delta^{-} A_{n}^{+}\left(\mathbf{x}_{\perp}\right)} e^{i g \Delta^{-} A_{n}^{+}\left(\mathbf{y}_{\perp}\right)}\right\rangle \approx & \left\langle 1-i g \Delta^{-} A_{n}^{+}\left(\mathbf{x}_{\perp}\right)+i g \Delta^{-} A_{n}^{+}\left(\mathbf{y}_{\perp}\right)+g^{2}\left(\Delta^{-}\right)^{2} A_{n}^{+}\left(\mathbf{x}_{\perp}\right) A_{n}^{+}\left(\mathbf{y}_{\perp}\right)\right. \\
& \left.\left.-\frac{1}{2} g^{2}\left(\Delta^{-}\right)^{2}\left(\left(A_{n}^{+}\left(\mathbf{x}_{\perp}\right)\right)^{2}+\left(A_{n}^{+}\left(\mathbf{y}_{\perp}\right)\right)^{2}\right)\right\rangle+\mathcal{O}\left(\Delta^{-}\right)^{3 / 2}\right) \tag{119}
\end{align*}
$$

The expectation value of a single $A^{+}$vanishes and we use (117) for the two point function:

$$
\begin{align*}
&\left\langle e^{-i g \Delta^{-} A_{n}^{+}\left(\mathbf{x}_{\perp}\right)} e^{i g \Delta^{-} A_{n}^{+}\left(\mathbf{y}_{\perp}\right)}\right\rangle \\
& \approx 1+g^{4}\left(\Delta^{-}\right)^{2} \frac{\mu_{n}^{2}}{\Delta^{-}}\left[L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)-\right.\left.\frac{1}{2} L\left(\mathbf{x}_{\perp}-\mathbf{x}_{\perp}\right)-\frac{1}{2} L\left(\mathbf{y}_{\perp}-\mathbf{y}_{\perp}\right)\right] \overbrace{t^{a} t^{a}}^{=C_{\mathrm{F}} \mathbb{I}_{N_{\mathrm{C}} \times N_{\mathrm{c}}}} \\
& \approx e^{g^{4} \Delta^{-} C_{\mathrm{F}} \mu_{n}^{2}\left[L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)-\frac{1}{2} L\left(\mathbf{x}_{\perp}-\mathbf{x}_{\perp}\right)-\frac{1}{2} L\left(\mathbf{y}_{\perp}-\mathbf{y}_{\perp}\right)\right]} \tag{120}
\end{align*}
$$

Now that this expectation value is just a number times the identity matrix, it commutes with all the other infinitesimal Wilson line factors in the infinite product (118). We can therefore pull it out and repeat the same procedure for the next factor (next $n$ ). Doing this one by one we finally get just a product of terms like (120), each of them proportional to the identity matrix. The overall trace now just gives a factor $N_{\mathrm{c}}$ that cancels with the normalization. We can now return from the discrete to the continuous notation as

$$
\begin{align*}
S(r)= & \lim _{N \rightarrow \infty} \overbrace{\frac{1}{N_{c}} \operatorname{Tr}}^{\operatorname{Tr}_{n=-N}} e^{g^{4} \Delta^{-} C_{\mathrm{F}} \mu_{n}^{2}\left[L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)-L(0)\right]} \\
& =\exp \{g^{4} C_{\mathrm{F}} \int_{-\infty}^{\infty} \mathrm{d} x^{-} \mu^{2}\left(x^{-}\right) \overbrace{\left[L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)-L(0)\right]}^{\equiv-\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)}\} \tag{121}
\end{align*}
$$

This is now practically our final result. Note that the only thing we have used so far from the MV model is the more general of the two approximations involved: the independence of different $x^{-}$color charges and the Gaussian expectation value. We have not yet actually used the assumption that the correlator (113) is proportional to $\delta^{(2)}\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$ at all. Therefore the result (121) can be used independently of the MV model assumption. In particular, we can take a solution of the BK equation for $S(r)$, and consider (121) as the definition of the correlator $\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right) \equiv-L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)+L(0)$.

But in order to see what the more restrictive local charge correlation assumption gives us, let us now calculate $\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$ in the actual model, recalling that physically we expect this to be a good approximation in the limit of smallish $x$ and very large $A$. In order to get the $A^{+}$correlator from the MV model assumption for the correlator of $\rho$ 's we need to solve the equation (108). This is done by Fourier-transforminng; we also
need to introduce an additional IR cutoff to deal with the very divergent integrals in intermediate stages of our calculation. For now we will suppress the argument $x^{-}$for brevity, the following steps happen locally in $x^{-}$.

$$
\begin{gather*}
A^{+}\left(k_{T}\right)=\frac{1}{\mathbf{k}_{\perp}^{2}} \rho\left(\mathbf{x}_{\perp}\right)  \tag{122}\\
\begin{aligned}
\left\langle\rho^{a}\left(\mathbf{k}_{\perp}\right) \rho^{b}\left(\mathbf{p}_{\perp}\right)\right\rangle=g^{2} \delta^{a b} \mu^{2} \int \mathrm{~d}^{2} \mathbf{x}_{\perp} \mathrm{d}^{2} \mathbf{y}_{\perp} e^{i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}+i \mathbf{p}_{\perp} \cdot \mathbf{y}_{\perp} \delta^{(2)}\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)} \\
=g^{2} \delta^{a b} \mu^{2}(2 \pi)^{2} \delta^{(2)}\left(\mathbf{k}_{\perp}+\mathbf{p}_{\perp}\right)
\end{aligned} \\
\begin{array}{r}
\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)=\int \frac{\mathrm{d}^{2} \mathbf{k}_{\perp}}{(2 \pi)^{2}} \frac{\mathrm{~d}^{2} \mathbf{p}_{\perp}}{(2 \pi)^{2}}(2 \pi)^{2} \frac{\delta^{(2)}\left(\mathbf{k}_{\perp}+\mathbf{p}_{\perp}\right)}{p_{T^{2} k_{T}{ }^{2}}}\left[1-e^{\left.-i \mathbf{k}_{\perp} \cdot \mathbf{x}_{\perp}-i \mathbf{p}_{\perp} \cdot \mathbf{y}_{\perp}\right]}\right. \\
\\
=\frac{1}{2 \pi} \int \mathrm{~d} k \frac{1-J_{0}(k r)}{k^{3}} \approx \frac{1}{8 \pi} r^{2} \ln \overbrace{\frac{1}{r \Lambda}}^{>1}
\end{array} \tag{123}
\end{gather*}
$$

This integral is still IR divergent at $k=0$, but only logarithmically so. Note that the leading power law divergence cancels in the combination $\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right) \equiv-L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)+L(0)$. This is true generally: expectation values of nonsinglet operators are very very sick because they depend on $L\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$ alone. They diverge (or vanish) as an exponential function of a power of the IR cutoff; they do not describe meaningful physical quantities. Expectation values of singlet operators, such as the dipole that we are calculating here, depend always on the combination $\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)$ which, in the MV model, is only logarithmically divergent. One can always argue away the logarithm for coordinate space quantities like $S(r)$, but it does affect the analytical properties and therefore the momentum space behavior, so it is better to keep it for now. Note that since $\Lambda$ in Eq. (124) is an IR cutoff in the $k$-integral, we should assume that it is always small compared to any coordinate value that we are interested in: $r \Lambda \ll 1$. The full $\Gamma(r)$ is, however finite, becaus of the power of $r$ multiplying the logarithm.

There are many ways to regulate the remainig IR divergence, which lead to different values for the remaining finite term. The leading log, however, is universal. Our result is for the dipole cross section is now:

$$
\begin{gather*}
S(r) \equiv \frac{1}{N_{\mathrm{c}}}\left\langle\operatorname{Tr} V\left(\mathbf{x}_{\perp}\right) V^{\dagger}\left(\mathbf{y}_{\perp}\right)\right\rangle=\exp \left\{-\frac{g^{4} C_{\mathrm{F}}}{8 \pi}\left[\int_{-\infty}^{\infty} \mathrm{d} x^{-} \mu^{2}\left(x^{-}\right)\right] r^{2} \ln \frac{1}{r \Lambda}\right\}  \tag{125}\\
Q_{\mathrm{s}}^{2} \sim \frac{g^{4} C_{\mathrm{F}}}{4 \pi}\left[\int_{-\infty}^{\infty} \mathrm{d} x^{-} \mu^{2}\left(x^{-}\right)\right] \tag{126}
\end{gather*}
$$

This provides a concrete link between the MV model variable $\mu^{2}$ (as already promised, this only enters as the integral over $x^{-}$) and the dipole cross section, i.e. the DIS cross section. We can now make a fit to DIS data, use that to extract a value for $\mu^{2}$, and apply this to calculating heavy ion collisions: this is indeed what is done in the very successful IPglasma model for the initial stage of a heavy ion collision [20]. Note also that the dipole cross section interpolates between 1 at $r=0$ and 0 at $r \rightarrow \infty$ as it should, satisfying the expectations for a $\mathrm{SU}(3)$ matrix correlator. We can identify the saturation scale $Q_{\mathrm{s}}$ from the characteristic scale in this $r$-dependence.

In momentum space, the Fourier-transform of the dipole expectation value, multiplied by $k_{T}{ }^{2}$, is known as the dipole gluon distribution

$$
\begin{equation*}
\varphi_{\text {dip. }}\left(k_{T}\right) \sim k_{T}^{2} \int \mathrm{~d}^{2} \mathbf{r}_{\perp} e^{i \mathbf{r}_{\perp} \cdot \mathbf{k}_{\perp}} S(r) \tag{127}
\end{equation*}
$$

Thanks to the logarithm of $r$ in the MV model expression, it behaves [Exercise!] like $1 / k_{T}{ }^{2}$ at large $k_{T}$. At small $k_{T}$, the dipole distribution behaves as $\sim k_{T}^{2}$ (this is easy to see, since the integral of $S(r)$ over $\mathbf{r}_{\perp}$ is finite, therefore $S\left(k_{T}=0\right)$ is a constant and $\varphi_{\text {dip. }}\left(k_{T}\right) \sim k_{T}{ }^{2} S(k)$ ).

### 7.4 Weizsäcker-Williams distribution

If one wants to characterize the number distribution of gluons in the nucleus, one nees to light cone quantize the gluon field. Note that now we are light cone quantizing the target nucleus, not the probe. We cannot light cone quantize both at the same time because we cannot simultaneously fix the gauge to be noth $A^{+}=0$ and $A^{-}=0$. This gluon number density corresponds to what is know Weizsäcker-Williams gluon distribution. The full calculation in the MV model was first done in [21], following a procedure very similar to the above. Repeating this would take up too much time in this lecture, so we will just quote the result below. But before that it is useful to look at the dilute limit of the calculation. This can be derived by assuming that $\rho \sim g^{2} \mu$ is small, in which case one can expand the Wilson lines to lowest order. Since in the MV model everything is expressed in terms of $A^{+} \sim \rho / \nabla_{\perp}^{2}$, the dilute limit of small $g^{2} \mu$ is at the same time the limit of large transverse momenta.

$$
\begin{equation*}
\frac{\mathrm{d} N}{\mathrm{~d}^{2} \mathbf{k}_{\perp} \mathrm{d} y}=\varphi^{\mathrm{ww}}\left(\mathbf{k}_{\perp}\right) \sim\left\langle A_{\mathrm{a}}^{i}\left(\mathbf{k}_{\perp}\right) A_{\mathrm{a}}^{i}\left(-\mathbf{k}_{\perp}\right)\right\rangle \tag{128}
\end{equation*}
$$

High $k_{T}$ - small $\mu$

$$
\begin{equation*}
A_{a}^{i}\left(\mathbf{k}_{\perp}\right)=\frac{k^{i}}{k_{T}{ }^{2}} \rho^{a}\left(\mathbf{k}_{\perp}\right) \Longrightarrow\left\langle A_{a}^{i}\left(\mathbf{k}_{\perp}\right) A_{a}^{i}\left(-\mathbf{k}_{\perp}\right)\right\rangle=\overbrace{(2 \pi)^{2} \delta^{(2)}\left(\mathbf{k}_{\perp}=0\right)}^{\pi R_{\perp}^{2}}\left(N_{\mathrm{c}}{ }^{2}-1\right) \frac{g^{2} \mu^{2}}{\mathbf{k}_{\perp}^{2}} \tag{129}
\end{equation*}
$$

Here we used the momentum space correlator (123) and the fact that in the dilute limit the light cone gauge transverse gauge field in (112) reduces to just a derivative of covariant gauge $A^{+}$field. This gluon distibution reduced in a very natural way to the DGLAP-evolved integrated gluon distribution, for which the definition in terms of the quantized light cone gauge gluonic field operators leads to

$$
\begin{equation*}
x G\left(x, Q^{2}\right)=\int^{Q^{2}} \mathrm{~d}^{2} \mathbf{k}_{\perp} \frac{\mathrm{d} N}{\mathrm{~d}^{2} \mathbf{k}_{\perp} \mathrm{d} y} \sim \pi R_{A}^{2}\left(N_{\mathrm{c}}^{2}-1\right)\left(g^{2} \mu\right)^{2} \ln Q^{2}, \tag{130}
\end{equation*}
$$

i.e. a function that grows logarithmically with $Q^{2}$. This growth can be identified with the growth in the number of gluons in DGLAP evolution when increasing the resolutions scale $Q^{2}$.

The result from [21] (see e.g. [22] for a mathematically clearer but more formal derivation) for the Weizsäcker-Williams gluon distribution is

$$
\begin{equation*}
A^{i}=\frac{i}{g} V\left(\mathbf{x}_{\perp}\right) \partial_{i} V\left(\mathbf{x}_{\perp}\right) \tag{131}
\end{equation*}
$$

$\Longrightarrow$
$\left\langle A_{a}^{i}\left(\mathbf{x}_{\perp}\right) A_{b}^{i}\left(\mathbf{y}_{\perp}\right)\right\rangle=\delta^{a b} \frac{2 C_{\mathrm{F}}}{g^{2}\left(N_{c}^{2}-1\right)} \frac{\nabla_{\perp}^{2} \Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)}{\Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)}\left(\exp \left\{-g^{4} N_{\mathrm{c}} \int_{-\infty}^{\infty} \mathrm{d} x^{-} \mu^{2}\left(x^{-}\right) \Gamma\left(\mathbf{x}_{\perp}-\mathbf{y}_{\perp}\right)\right\}-1\right)$
$\Longrightarrow$

$$
\varphi_{W W}\left(k_{T}\right) \sim\left(\begin{array}{ll}
\frac{1}{\alpha_{s}} \ln k_{T}, & k_{T} \ll Q_{s}^{2}  \tag{133}\\
\frac{1}{\alpha_{s}} \frac{Q_{s}^{2}}{k_{T}^{2}}, & k_{T} \gg Q_{s}^{2} .
\end{array}\right.
$$



Here we have identified the saturation scale as $Q_{\mathrm{s}} \sim g^{2} \mu$. Some points to note about the expression (133):

- The factor in the exponent is an adjoint representation Casimir $C_{\mathrm{A}}$, compared to the fundamental Casimir $C_{\mathrm{F}}=\left(N_{\mathrm{c}}^{2}-1\right) /\left(2 N_{\mathrm{c}}\right)$ in the fundamental representation dipole (125). This is what one would expect for gluons.
- Both the dipole gluon distribution (127) and the WW distribution (133) have (when properly normalized) the same large $k_{T}$ limit. However, they look quite different at small $k_{T}$. Both exhibit saturation: the behavior changes at $k_{T} \lesssim Q_{\mathrm{S}}$; the dipole distribution changes to $\sim k_{T}{ }^{2}$ and the WW distribution to $\sim \ln k_{T}$. You can deduce the latter property by noting that due to the 1 inside the bracket in (131), the integral over $r$ of the coordinate space distribution diverges logarithmically; this would be the $k_{T}=0$ value of the momentum space distribution.
- In a strict operatorial sense the dipole and WW distributions are independent quantities, some cross sections depend on one, some on the other. Without evoking a model such as MV one has t measure both of them separately experimentally. The MV model (or in general the nonlinear Gaussian assumption) provides an additional physical model that allows one to relate them to each other.


## 8 Glasma

Now let us finally move to the collision of two high energy nuclei. We will represent both in a symmetrical way, by their own classical color currents and radiated fields, with nucleus (1) moving in the positive $z$ direction and nucleus (2) in the negative one.

$$
\begin{equation*}
J^{\mu}=\underbrace{\delta^{\mu+} \rho_{(1)}\left(\mathbf{x}_{\perp}\right) \delta\left(x^{-}\right)}_{A^{i} \sim \theta\left(x^{-}\right)}+\underbrace{\delta^{\mu-} \rho_{(2)}\left(\mathbf{x}_{\perp}\right) \delta\left(x^{+}\right)}_{A^{i} \sim \theta\left(x^{+}\right)} \tag{134}
\end{equation*}
$$

For each of these currents separately, we already know the solution of the field in the light cone gauge of each of the currents $A^{ \pm}=0$, and these solutions have the nice property that also the other longitudinal component vanishes: $A^{\mp}=0$. I.e. the light cone field of nucleus (1) also "accidentally" satisfies the ligh cone gauge condition of the other one. Thus in some region of spacetime, which is causally connected to only one of the nuclei (we chose the boundary conditions for the Yang-Mills equation to be causal, so that one gets a color field only in the region after the passage of the nucleus).


Let us write the expressions for these "transverse pure gauge" fields without the $\theta$-functions, which we will deal with separately.

$$
\begin{gather*}
A_{(1,2)}^{i}=\frac{i}{g} V_{(1,2)}\left(\mathbf{x}_{\perp}\right) \partial_{i} V_{(1,2)}^{\dagger}\left(\mathbf{x}_{\perp}\right)  \tag{135}\\
V_{(1,2)}\left(\mathbf{x}_{\perp}\right)=P e^{i g \int \mathrm{~d} x^{-} \frac{\rho\left(\mathbf{x}_{\perp}, x^{-}\right)}{\nabla_{\perp}^{2}}} \tag{136}
\end{gather*}
$$

These are valid solutions to the equations of motion in the regions (1) and (2) in the figure above, which are causally connected to only one of the two nuclei. Let us now derive an initial condition for the field inside the future light cone, which is causally connected to both. Inside the region (3) there is no current (since the color currents are proportional to delta functions)

Region (3):

- Inside: $\left[D_{\mu}, F^{\mu \nu}\right]=0$; no source
- Need initial conditions at $\tau=0$
- $\sqrt{s} \rightarrow \infty$ : should be boost invariant: $A_{\mu}\left(\tau, \mathbf{x}_{\perp}\right)$


## Gauge choice:

- Avoid color precession: $A^{-}=0$ at $x^{-}=0$ where $J^{+}$lives, + vice versa
- Boost invariant: $\tau, \eta$-components
$\Longrightarrow$ Fock-Schwinger $A_{\tau}=\left(x^{+} A^{-}+x^{-} A^{-}\right) / \tau=0$
The Fock-Schwinger or temporal gauge condition has the very nice property that on the $x^{+}$-axis, i.e. $x^{-}=0$, it reduces to $x^{+} A^{-}=0$, i.e. $A^{-}=0$. Thus the current that lives on this light cone is not only covariantly conserved $\left[D_{\mu}, J^{\mu}\right]=\partial_{+}+i g\left[A^{-}, J^{+}\right]=0$, but in fact does not color precess and is constant $\partial_{+} J^{+}=0$. This significantly simplifies our following calculation. The other advantage of the $A_{\tau}=0-$ condition is that in order to solve the equations of motion numerically in a Hamiltonian formalism, which is the standard thing to do, one needs to choose a temporal gauge condition. There are a few calculations of gluon production inside the forward light cone in another gauge, the covariant gauge, but this is done only in the dilute limit, not including any final state interactions [23, 24]. I am not aware of a numerical implementation of the full equations in another gauge.

Now, as discussed above, we need an initial condition for the field inside the future light cone. For the case of one nucleus we could have two components equal to zero, one as a gauge choice and the other one as a result of our current being independent of the light cone time $x^{+}$. In the case of two nuclei there is a dependence on both $x^{-}$and $x^{+}$in the problem, and we cannot have this "accidental" vanishing of one component of the gauge field any more. So we can only make one component of the gauge field vanish with a gauge choice, but our solution inside the future light cone will have nonzero values for the other three. With $A_{\tau}=0$, this third component is the longitudinal field orthogonal to it, namely $A_{\eta}$ or $A^{\eta}=-A_{\eta} / \tau^{2}$. In the following we will use the latter, because it is larger at $\tau=0$.

Since the currents exist only on the light cones, the field can have discontinuities across the light cone. We can write [25] an ansatz for the solution in terms of $\theta$ functions. Inserting this into the equations of motion will yield $\delta$-functions arising from derivatives of the $\theta$-functions. We can determine the initial conditions for the field at $\tau=0^{+}$by requiring that the coefficients of these $\delta$-functions vanish. This is enough to determine the initial condition, we do not need the terms that do not have a $\delta\left(x^{ \pm}\right)$.

$$
\begin{align*}
\text { Ansatz: } A_{i} & =\overbrace{A_{i}^{(1)}\left(\mathbf{x}_{\perp}\right) \theta\left(-x^{+}\right) \theta\left(x^{-}\right)+A_{i}^{(2)}\left(\mathbf{x}_{\perp}\right) \theta\left(x^{+}\right) \theta\left(-x^{-}\right)}^{\text {known }}+A_{i}^{(3)}\left(\mathbf{x}_{\perp}, \tau\right) \theta\left(x^{+}\right) \theta\left(x^{-}\right)(1 \\
A^{ \pm} & = \pm \theta\left(x^{+}\right) \theta\left(x^{-}\right) x^{ \pm} A^{\eta}\left(\mathbf{x}_{\perp}, \tau\right) \tag{138}
\end{align*}
$$

$$
\begin{equation*}
\left[D_{\mu}, F^{\mu \nu}\right]=J^{\nu} \quad \text { match } \quad \delta\left(x^{ \pm}\right) \tag{139}
\end{equation*}
$$

Coefficient of $\delta\left(x^{-}\right) \delta\left(x^{+}\right)$:

$$
\begin{equation*}
J^{i}=0=\partial_{-} \partial_{+} A^{i}+\cdots=\delta\left(x^{-}\right) \delta\left(x^{+}\right)\left(-A_{i}^{(1)}\left(\mathbf{x}_{\perp}\right)-A_{i}^{(2)}\left(\mathbf{x}_{\perp}\right)+A_{i}^{(3)}\left(\mathbf{x}_{\perp}\right)\right) \tag{140}
\end{equation*}
$$

Note: $\partial_{+} \partial_{-} A^{+} \sim \delta\left(x^{-}\right) \delta\left(x^{+}\right) x^{-}=0$

## Coefficient of $\delta\left(x^{-}\right) \theta\left(x^{+}\right)$in $J^{+}$-component

$$
\begin{align*}
\partial_{-} F^{-+} & =\left[\partial_{-}\left(-\partial_{-}\right)\left(-x^{-} \theta\left(x^{+}\right) \theta\left(x^{-}\right)\right)+\partial_{-} \partial_{+}\left(x^{+} \theta\left(x^{+}\right) \theta\left(x^{-}\right)\right)\right] A^{\eta}\left(\mathbf{x}_{\perp}, \tau\right)  \tag{141}\\
& =2 \delta\left(x^{-}\right) \theta\left(x^{+}\right) A^{\eta}\left(\mathbf{x}_{\perp}, \tau\right)+\ldots  \tag{142}\\
\partial_{i}\left(-\partial_{-} A^{i}\right) & =\delta\left(x^{-}\right) \theta\left(x^{+}\right) \partial_{i}\left(A_{i}^{(2)}\left(\mathbf{x}_{\perp}\right)-A_{i}^{(3)}\left(\mathbf{x}_{\perp}\right)\right)  \tag{143}\\
& =-\delta\left(x^{-}\right) \theta\left(x^{+}\right) \partial_{i}\left(A_{i}^{(1)}\left(\mathbf{x}_{\perp}\right)\right)=\left.J^{+}\right|_{x^{+}>0}  \tag{144}\\
i g\left[A_{i,-} \partial_{-} A_{i}\right] & =i g[A_{i}, \overbrace{A_{i}^{(2)}\left(\mathbf{x}_{\perp}\right)-A_{i}^{(3)}\left(\mathbf{x}_{\perp}\right)}^{-A_{i}^{(1)}\left(\mathbf{x}_{\perp}\right)}] \delta\left(x^{-}\right) \theta\left(x^{+}\right)  \tag{145}\\
& =i g\left[A_{i}^{(2)}, A_{i}^{(1)}\left(\mathbf{x}_{\perp}\right)\right] \delta\left(x^{-}\right) \theta\left(x^{+}\right) \tag{146}
\end{align*}
$$

$\Sigma=0 \Longrightarrow$

$$
\begin{align*}
\left.A_{i}^{(3)}\right|_{\tau=0} & =A_{i}^{(1)}+A_{i}^{(2)}  \tag{147}\\
\left.A^{\eta}\right|_{\tau=0} & =\frac{i g}{2}\left[A_{i}^{(1)}, A_{i}^{(2)}\right] \tag{148}
\end{align*}
$$



- $\tau=0$ longitudinal $E^{z} \sim\left[A_{i}^{(1)}, A_{i}^{(2)}\right]$ and $B^{z} \sim \varepsilon^{i j}\left[A_{i}^{(1)}, A_{j}^{(2)}\right]$
- $\perp$ correlation length $1 / Q_{s}$.

One way to think of the longitudinal initial electric and magnetic fields is to note that in the absence of color charges the fields satisfy the nonabelian Gauss law and Bianchi identities, stating that the covariant divergences of the chromoelectric and -magnetic fields vanish. It is interesting to separate the commutator terms from the linear ones. Then one can interpret the nonlinear terms as kind of effective electric and magnetic field densities generated by the interaction of the electric and magnetic fields of one nucleus with the pure gauge potential of the other one.

- Gauss

$$
\begin{equation*}
\left[D_{i}, E^{i}\right]=0 \quad \Longrightarrow \quad \partial_{i} E^{i}=i g\left[A^{i}, E^{i}\right] \tag{149}
\end{equation*}
$$

- Bianchi

$$
\begin{equation*}
\left[D_{i}, B^{i}\right]=0 \quad \Longrightarrow \quad \partial_{i} B^{i}=i g\left[A^{i}, B^{i}\right] \tag{150}
\end{equation*}
$$

Inside the future light cone the field equations have to be solved numerically. This can be done with relatively standard real time lattice gauge field methods.


In this plot the transverse field notation means

$$
\begin{align*}
& E_{T}^{2}=E_{x}^{2}+E_{y}^{2}  \tag{151}\\
& B_{T}^{2}=B_{x}^{2}+B_{y}^{2} \tag{152}
\end{align*}
$$

It is interesting to see what this means in terms of the energy momentum tensor. For pure gauge Yang-Mills theory the diagonal components are given by

$$
\begin{align*}
\varepsilon & ==\frac{1}{2}\left[E_{x}^{2}+E_{y}^{2}+E_{z}^{2}+B_{x}^{2}+B_{y}^{2}+B_{z}^{2}\right]  \tag{153}\\
p_{x} & ==\frac{1}{2}\left[-E_{x}^{2}+E_{y}^{2}+E_{z}^{2}-B_{x}^{2}+B_{y}^{2}+B_{z}^{2}\right]  \tag{154}\\
p_{y} & ==\frac{1}{2}\left[E_{x}^{2}-E_{y}^{2}+E_{z}^{2}+B_{x}^{2}-B_{y}^{2}+B_{z}^{2}\right]  \tag{155}\\
p_{z} & ==\frac{1}{2}\left[E_{x}^{2}+E_{y}^{2}-E_{z}^{2}+B_{x}^{2}+B_{y}^{2}-B_{z}^{2}\right] \tag{156}
\end{align*}
$$

At the initial condition one starts with negative longitudinal pressure $p_{z}=-p_{x}=-p_{z}=-\varepsilon$. This is similar to the longitudinal chromoelectric field in a QCD string, which has a string tension, i.e. negative longitudinal pressure. here is a plot from [26]:


After a time $\tau \sim 1 / Q_{\mathrm{s}}$ the negative pressure approaches zero, and the transverse pressure becomes half of the energy density. This is the starting point for the isotropization of the system, which requires going beyond the classical approximation.

In order to interpret these fields in terms of a number density of gluons (which is possible only after a time $\tau \gtrsim 1 / Q_{\mathrm{s}}$ ), one needs to decompose the solution in Fourier $\mathbf{k}_{\perp}$-modes. Knowing that this is a one scale problem, with the only dimensionful scale, one knows already beforehand that the gluon number distribution has to have the parametric form

$$
\begin{equation*}
\frac{\mathrm{d} N_{\mathrm{g}}}{\mathrm{dyd} \mathrm{~d}^{2} \mathbf{x}_{\perp} \mathrm{d}^{2} \mathbf{p}_{\perp}}=\frac{1}{\alpha_{\mathrm{s}}} f\left(\frac{p_{T}}{Q_{\mathrm{s}}}\right) \tag{157}
\end{equation*}
$$

In the dilute limit one can recover a perturbative $k_{T}$-factorization result for the gluon density. One first has to transform the fields to transverse Coulomb gauge $\partial_{i} A_{i}=0$; this is taking use of an additional gauge freedom in the Fock-Schwinger gauge to perform $\tau$-independent gauge transformations. Assuming that the fields are small, one neglects the nonlinear terms in the equations of motion for $\tau>0$. The linearized equations are then wave equations in an expanding geometry, separately for each transverse momentum mode.

$$
\begin{align*}
& \left(\tau^{2} \partial_{\tau}^{2}+\tau \partial_{\tau}+\tau^{2} \mathbf{k}_{\perp}^{2}\right) A_{i}\left(\tau, \mathbf{k}_{\perp}\right)=0  \tag{158}\\
& \left(\tau^{2} \partial_{\tau}^{2}-\tau \partial_{\tau}+\tau^{2} \mathbf{k}_{\perp}^{2}\right) A_{\eta}\left(\tau, \mathbf{k}_{\perp}\right)=0 \tag{159}
\end{align*}
$$

$$
\begin{equation*}
\Longrightarrow A_{i}\left(\tau, \mathbf{k}_{\perp}\right)=A_{i}\left(\tau=0, \mathbf{k}_{\perp}\right) J_{0}\left(\left|\mathbf{k}_{\perp}\right| \tau\right) \quad A^{\eta}\left(\tau, \mathbf{k}_{\perp}\right)=-\frac{1}{\tau\left|\mathbf{k}_{\perp}\right|} A^{\eta}\left(\tau=0, \mathbf{k}_{\perp}\right) J_{1}\left(\left|\mathbf{k}_{\perp}\right| \tau\right) \tag{160}
\end{equation*}
$$

- These modes are (boost invariant) plane waves. Thus they can be interpreted as particles, gluons.
- In this approximation (interactions only at $\tau=0$, free propagation after that), there is no contradiction between a classical field and particle description of the fields. The negative longitudinal pressure at $\tau=0$ is due to the boost invariance of the situation. This imposes a specific restriction on the two possible linear polarization states for gluons with only transverse momentum. The polarization state with a electric field in the $z$-direction and a magnetic field in a transverse direction starts in a phase where there is only an electric field, whereas the other polarization state with a magnetic field in the $z$-direction and an elecric field in a transverse direction has to start in a phase where there is only a magnetic field. The different modes present in these two polarization states oscillate in time with slightly different frequencies, and later get out of sync. Thus after a time $\tau \sim 1 / Q_{\mathrm{s}}$ the phases of these oscillations are more randomly distributed, and the energy momentum tensor is just dictated by the boost invariance, i.e. absence of longitudinal momentum, i.e. zero longitudinal pressure.
- The initial fields related to Wilson lines, and via that to the dipole amplitude measured in DIS.

In this dilute limit, when we have turned off all the interactions between the gluons after $\tau=0$, the number distribution is given by a $k_{T}$-factorization formula.

$$
\begin{equation*}
\frac{\mathrm{d} N}{\mathrm{dy} \mathrm{~d}^{2} \mathbf{k}_{\perp}}=\frac{\alpha_{\mathrm{s}}}{S_{\perp}} \frac{2}{C_{\mathrm{F}}} \frac{1}{k_{T}^{2}} \int \mathrm{~d}^{2} \mathbf{q}_{\perp} \varphi^{\mathrm{dip}}\left(\mathbf{q}_{\perp}\right) \varphi^{\mathrm{dip}}\left(\left|\mathbf{k}_{\perp}-\mathbf{q}_{\perp}\right|\right) . \tag{161}
\end{equation*}
$$

This calculation can also be repeated by assuming that one of the two colliding objects is dilute (a theorist's " pA " collision).

However, no such analytic calculation exists for a dense-dense system, an "AA" collision. Also in the "AA" case, one can numerically verify that the $k_{T}$-factorization formula works for high $k_{T}$ gluons. However, beacuse of the explicit $1 / k_{T}{ }^{2}$ in front, it cannot be integrated to give a finite total gluon multiplicity. The full result of the numerical calculation, on the other hand, can. Sometimes this is fixed by an ad hoc cutoff

$$
\begin{equation*}
\frac{\mathrm{d} N}{\mathrm{~d}^{2} \mathbf{p}_{\perp} \mathrm{d} y}=\frac{1}{\alpha_{\mathrm{s}}} \frac{1}{\mathbf{p}_{\perp}^{2}} \int_{\mathbf{k}_{\perp}}\left[\theta\left(p_{T}-k_{T}\right)\right] \phi_{y}\left(\mathbf{k}_{\perp}\right) \phi_{y}\left(\mathbf{p}_{\perp}-\mathbf{k}_{\perp}\right) \tag{162}
\end{equation*}
$$

Here is a comparison of the $k_{T}$-factorized formula (161) to the result of the full CYM calculation, and also to the version with the cutoffs (162), showing that the formula without these cutoffs is correct:


Here the calculation is done in the MV model with two very different values for $g^{2} \mu$ in the two colliding nuclei: in this case the result for ad hoc cutoff depends very much on whether one cuts off the spectrum at $k_{T}<p_{T}$ or $\left|\mathbf{p}_{\perp}-\mathbf{k}_{\perp}\right|<p_{T}$.

Here is the same comparison in the case of a dense-dense collision


These plots are from [27].
Here is a comparison with experimental data from pA collisions; the theory predictions here are calculated with the $k_{T}$-factorization formula (161), including a dipole gluon distribution satisfying the BK evolution equation.


The plot is from ALICE [28], the CGC calculations from [29].

## 9 Bottom-up thermalization

### 9.1 Starting point

We have now seen what the "glasma" color fields at the initial stage of a heavy io collision look like. In particular we have seen that the initial fields are boost invariant. In terms of momenta of gluons this corresponds to an extremely anisotropic momentum space distribution of gluons. This is still very far from a thermal equilibrium that could be matched to hydrodynamics. In particular the important problem is that of
isotropization. The longitudinal expansion of the system constantly tries to make the momentum distribution more anisotropic, fighting against interactions that try to isotropize the system.

The "extreme" weak coupling parametric analysis of how the interactions eventually isotropize the system is known as the "bottom-up" thermalization scenario, and was developed in Ref. [30]. The qualitative features of this parametric scenario have later been confirmed in explicit numerical studies both in with Classical Yang-Mills simulations (valid for the first part of the scenario) and in effective kinetic theory. For simplicity this analysis concerns a system that is boos in variant in the longitudinal direction, and inifinitely large and homogenous in the transverse ones. Physically this is justified when the collision energy $\sqrt{s}$ is large enough, and when we are interested in times that are early

Assumptions:

- $\alpha_{\mathrm{s}} \ll 1$
- Hard scale $Q_{\mathrm{s}}$, soft $m_{D} \ll Q_{\mathrm{s}}$
- At $Q_{\mathrm{s}} \tau \sim 1$ overoccupied $f \sim 1 / \alpha_{\mathrm{s}}$
- Boost invariant expansion: $p_{z} \ll p_{T} \sim Q_{\mathrm{S}}$

At what $Q_{\mathrm{s}} \tau$ is $p_{z} \sim p_{T} \quad \& \quad f \sim 1$ ?

### 9.2 First stage: classical fields

Let us first see what happens completely in the absence of interactions. The important feature of this system is that longitudinal momenta are redshifted by the expansion in the longitudinal direction, we have:

$$
\begin{equation*}
p_{T} \sim \text { const. } \quad \boldsymbol{p}_{z} \sim \frac{1}{\tau} \tag{163}
\end{equation*}
$$

To see this formally in kinetic theory we can take the Boltzmann equation for a general Lorentz frame

$$
\begin{equation*}
p^{\mu} \partial_{\mu} f(p, x)=C[f] \tag{164}
\end{equation*}
$$

In the absence of interactions we take the collision term to be zero; $C=0$. Now we are looking for boost invariant, transversally homogenous solutions for massless particles. In this case we can parametrize the coordinates and momenta as

$$
\begin{align*}
p^{0} & =p_{T} \cosh y  \tag{165}\\
p^{z} & =p_{T} \sinh y  \tag{166}\\
t & =\tau \cosh \eta  \tag{167}\\
z & =\tau \sinh \eta \tag{168}
\end{align*}
$$

In terms of these variables the equation (164) is

$$
\begin{equation*}
\left(\partial_{\tau}+\tanh (\eta-y) \frac{1}{\tau} \partial_{\eta}\right) f=0 \tag{169}
\end{equation*}
$$

We must look for a solution of (164) that only depends on $p_{T}, \tau$ and $\xi \equiv \eta-y$. For this we can replace $\partial_{\eta}=\partial_{\xi}$. The equation is now

$$
\begin{equation*}
\left(\partial_{\tau}+\tanh (\xi) \frac{1}{\tau} \partial_{\xi}\right) f=0 \tag{170}
\end{equation*}
$$

It is easy to check that this equation is satisfied by any function that only depends on $\tau$ and $\xi$ through the combination $\tau \sinh \xi$ :

$$
\begin{equation*}
f\left(p_{T}, \tau \sinh \xi\right) \tag{171}
\end{equation*}
$$

From this it is easy to deduce that the (r.m.s.) average value of the longitudinal momentum $p^{z}=p_{T} \sinh \xi$ behaves as $\sqrt{\left\langle p_{z}^{2}\right\rangle} \sim 1 / \tau$. In the following we simply denote $\sqrt{\left\langle p_{z}^{2}\right\rangle}$ by $p_{z}$.

A similar behavior is also seen in fully boost invariant classical Yang-Mills simulations.

## No interactions

- $\operatorname{cst} f \sim 1 / \alpha_{\mathrm{s}}$
- $p_{T} \sim Q_{\mathrm{s}} \sim$ cst.
- $p_{z} \sim 1 / \tau$
- Hard gluon number density

$$
\begin{equation*}
n_{h} \sim \int_{\mathbf{p}} f \sim p_{z} p_{T}{ }^{2} \frac{1}{\alpha_{\mathrm{s}}} \sim \frac{1}{\alpha_{\mathrm{s}}} \frac{Q_{\mathrm{s}}^{2}}{\tau} \tag{172}
\end{equation*}
$$

## $\Longrightarrow$ same with elastic collisions

Now let us introduce also elastic collisions, which keep the hard particle number density constant, but start to fight against the redshift from the longitudinal expansion. Note that since we now have collisions that can modify the longitudinal momentum of gluons, the typical longitudinal momentum does not behave like $p_{z} \sim 1 / \tau$ any more. Consequently also the phase space density of the hard gluons, which can be estimated as the total (so far conserved) number density divided by the phase space volume $p_{z} p_{T}{ }^{2}$ occupied by the gluons. Energy and momentum conservation dictates that also the typical transverse momentum of the gluons changes, but this change is of order $p_{z}$ or $m_{D}$, which is so small that it does not change our estimate $p_{T} \sim Q_{\mathrm{s}}$. An important quantity here is the Debye mass, which regulates the otherwise infrared divergent total cross section, and gives the size of the transverse momentum kick received by a particle in a typical collision. The Debye mass is the only dimensionful scale in the scattering cross section in the highenergy (small angle) scattering limit, which dominates the total cross section in QCD (or any gauge theory). To estimate the total scattering cross section between hard particles we count the appropriate power of the coupling constant from the Feynman diagram. This is then multiplied by the appropriate power of the Debye mass using dimensional analysis. Any corrections to this estimate are suppressed by powers of $m_{D} / \sqrt{s}$, where for a hard-hard scattering the total energy is of the order of the hard momentum scale $\sqrt{s} \sim Q_{\mathrm{s}}$.

Elastic collisions:

$$
\begin{align*}
& \text { hard'0000000500000000 hard } \\
& \text { o } \sim m_{D}  \tag{173}\\
& \text { hard } 000000000000000 \text { hard }  \tag{174}\\
& \sigma \sim \frac{\alpha_{\mathrm{s}}^{2}}{m_{D}^{2}} \\
& m_{D}^{2} \sim \alpha_{\mathrm{s}} \int_{\mathbf{p}} \frac{f(\mathbf{p})}{p} \sim \frac{\alpha_{\mathrm{s}}}{Q_{\mathrm{s}}} n_{h} \sim \frac{Q_{\mathrm{s}}}{\tau}
\end{align*}
$$

Collision rate

$$
\begin{gather*}
\frac{\mathrm{d} N_{\text {coll }}}{\mathrm{d} \tau} \sim \sigma n_{h} \overbrace{(1+f)}^{\text {Bose enhancement }} \\
f \sim \frac{n_{h}}{p_{z} Q_{\mathrm{s}}^{2}} \gg 1 \tag{175}
\end{gather*}
$$

In a steady state, the longitudinal momentum $p_{z}$ gained by the particles per unit time must be the same as the typical $p_{z}$ divided by the lifetime of the system $\tau$. To see this more formally let us assume quite generally that the typical momentum behaves with a power law in time $p_{z}^{2} \sim \tau^{\alpha}$, for which $\mathrm{d} p_{z}^{2} / \mathrm{d} \tau \sim p_{z}^{2} / \tau$; in a steady state this must be true for the competing contributions of the redshifting that decreases the longitudinal momenta and the collisions that enhance it. The collisional enhancement per unit time is given by the rate of collisions times the typical momentum kick gained in each collision, $m_{D}^{2}$ :

$$
\begin{equation*}
\frac{p_{z}^{2}}{\tau} \sim \frac{\mathrm{~d} p_{z}^{2}}{\mathrm{~d} \tau} \sim m_{D}^{2} \frac{\mathrm{~d} N_{\text {coll }}}{\mathrm{d} \tau} \sim m_{D}^{2} \frac{\alpha_{\mathrm{s}}^{2}}{m_{D}^{2}} \frac{Q_{\mathrm{s}}^{2}}{\alpha_{\mathrm{s}} \tau} \frac{Q_{\mathrm{s}}^{2}}{\alpha_{\mathrm{s}} \tau Q_{\mathrm{s}}^{2} p_{z}} \sim \frac{Q_{\mathrm{s}}^{2}}{p_{z} \tau} \Longrightarrow p_{z} \sim \frac{Q_{\mathrm{s}}}{\left(Q_{\mathrm{s}} \tau\right)^{1 / 3}} \tag{177}
\end{equation*}
$$

Note that $p_{z} \gg m_{D}$, which is a necessary consistency requirement, since we estimated the scattering cross section with a 3-dimensional formula, assuming that all the components of the hard momenta are indeed hard compared to the screening scale $m_{D}$. Also note that this is a regime when the occumation number is large, and therefore can be described by a classical field approximation. This can be seen also in the fact that the values of the relevant momentum scales $p_{T} \sim Q_{\mathrm{s}}, p_{z}$ and $m_{D}$ only depend on the initial hard scale $Q_{\mathrm{s}}$ and $\tau$, but not at all on the coupling. The fact that the coupling scales out of the dynamics completely like this is a feature of the classical field evolution. We can see when this regime ends by checking when the classical high occupation number regime ceases to be valid:

$$
\begin{equation*}
f \sim \frac{1}{\alpha_{\mathrm{s}}} \frac{1}{\left(Q_{\mathrm{s}} \tau\right)^{2 / 3}} \sim 1 \Longrightarrow Q_{\mathrm{s}} \tau \sim \alpha_{\mathrm{s}}^{-3 / 2} \tag{178}
\end{equation*}
$$

### 9.3 Second stage: soft particles start to play a role

As the hard particles at a scale $Q_{\mathrm{s}}$ are happily interacting during the first stage, one starts to create a bath of soft particles at a lower scale, with momenta suppressed by a power of the coupling compared to $Q_{\mathrm{s}}$. The arguments in the previous section are not fully valid any more, since now it is the 1 , and not the $f$, that dominates in the Bose enhancement factor $1+f$ in the scattering rates. In this next stage of the processm, the soft particles will dominate Debye screening, but the hard particles will still carry the dominant part of the gluon number, and energy, of the system. To make the derivation a little bit shorter we will first assume that this is the case, and self-consistently estimate $m_{D}$ and the number density of soft gluons $n_{s}$. One can then, a posteriori, check that indeed the soft gluons dominate $m_{D}$ byt not the total multiplicity.

The soft momentum scale (it is isotropic, because the soft sector interacts with itself very strongly) is the product of the number of elastic collisions between soft particles and other (i.e. hard at this stage) particles and the typical momentum obtained by the soft particle in these collisions, which is of the order $m_{D}$. The number of collisions is given by the time times the collision rate, and the collision rate is the number density of scattering centers times the cross section. At this stage the scattering centers are still the hard particles, for which the number density (172) estimate is still valid. In principle there should be a Bose enhancement factor in (179), but now the number density of the hard particles is $f \ll 1$ so it does not matter.

Soft momentum scale

$$
\begin{equation*}
k_{\mathrm{s}}^{2} \sim N_{\mathrm{coll}} m_{D}^{2} \sim \tau \frac{\alpha_{\mathrm{s}}^{2}}{m_{D}^{2}} \frac{Q_{\mathrm{s}}^{3}}{\alpha_{\mathrm{s}} Q_{\mathrm{s}} \tau} m_{D}^{2} \sim \alpha_{\mathrm{s}} Q_{\mathrm{s}}^{2} \quad \Longrightarrow \quad \text { indep. of } \tau \tag{179}
\end{equation*}
$$

The number density of soft gluons and the Debye scale both depend on each other, so they have to be solved self-consistently:

Producing soft particles (Bethe-Heitler):


Soft gluon number density

$$
\begin{equation*}
n_{s} \sim \tau \frac{\mathrm{~d} N^{\mathrm{BH}}}{\mathrm{~d} \tau} \sim \tau \overbrace{n_{h}^{2}}^{\left(\frac{Q_{\mathrm{s}}^{2}}{\alpha_{\mathrm{s}} \tau}\right)^{2}}(1+\overbrace{f_{h}}^{\ll 1})^{2} \frac{\alpha_{\mathrm{s}}^{3}}{m_{D}^{2}} \sim \frac{\alpha_{\mathrm{s}} Q_{\mathrm{s}}^{4}}{m_{D}^{2} \tau} \tag{181}
\end{equation*}
$$

Soft-dominated Debye scale

$$
\begin{equation*}
m_{D}^{2} \sim \frac{\alpha_{\mathrm{s}} n_{s}}{k_{\mathrm{s}}} \tag{182}
\end{equation*}
$$

$$
\begin{align*}
n_{\mathrm{s}} & \sim \frac{\alpha_{\mathrm{s}}^{1 / 4} Q_{\mathrm{s}}^{3}}{\left(Q_{\mathrm{s}} \tau\right)^{1 / 2}}  \tag{183}\\
m_{D}^{2} & \sim \frac{\alpha_{\mathrm{s}}^{3 / 4} Q_{\mathrm{s}}^{2}}{\left(Q_{\mathrm{s}} \tau\right)^{1 / 2}} \tag{184}
\end{align*}
$$

Comparing these results to the contribution of the hard modes to the Debye mass, eq. (174), we can indeed see that now the soft particle contribution to the Debye mass is larger than that of the hard particles, as we anticipated when using Eq. (182) to estimate the Debye scale. Comparing the soft momentum scale (179) to the Debye scale, remembering that now $Q_{\mathrm{s}} \tau \gg \alpha_{\mathrm{s}}^{-3 / 2}$, we also see that $k_{s} \gg m_{D}$. This means that indeed, as assumed in the estimate (179), the soft particles need to accumulate their momentum through several collisions with the hard particles.

On the other hand, we have assumed that the although the relative share of hard particles to the total gluon number density is decreasing, they still dominate the particle number density, i.e. $n_{h} \gg n_{s}$ (this was assumed e.g. when calculating the production rate for soft particles in eq. (181)). Let us now see when this condition ceases to be valid.

Stage ends:

$$
\begin{equation*}
n_{h} \sim \frac{Q_{\mathrm{s}}^{2}}{\alpha_{\mathrm{s}} \tau} \sim n_{s} \sim \frac{\alpha_{\mathrm{s}}^{1 / 4} Q_{\mathrm{s}}^{3}}{\left(Q_{\mathrm{s}} \tau\right)^{1 / 2}} \Longrightarrow Q_{\mathrm{s}} \tau \sim \alpha_{\mathrm{s}}^{-5 / 2} \tag{185}
\end{equation*}
$$

Note that during this stage $\alpha_{\mathrm{s}}^{-3 / 2} \ll Q_{\mathrm{s}} \tau \ll \alpha_{\mathrm{s}}^{-5 / 2}$ the occupation number of the hard modes has been $f_{h} \ll 1$ (Exercise: what is the time dependence? Note that this depends on the estimate (177) which is now modified both because there is no Bose enhancement and because the Debye mass is determined by the soft modes). The occupation number of the soft modes $f_{s} \sim n_{s} / k_{s}^{3}$, on the other hand, has been $\gg 1$ during this stage. They are, however, only a small part of the total number density, so overall the system has been underoccupied.

### 9.4 Last stage: energy loss

In the last stage of the development, most of the particle number is in the soft particles, whose momentum distribution is already isotropic. However, there is still a remaining collection of hard particles, which control the energy density. In order to isotropize the energy momentum tensor, they have to lose their energy. This happens when collisions with the soft particles leads to the hard particles losing their energy. In fact, we are diong a calculation of jet energy loss. The physical description of this last stage depends on the destructive interference between gluon emissions, i.e. the Landau-Pomeranchuk-Migdal (LPM) effect.

To get an estimate for how this works we look at a hard gluon, with energy $\sim Q_{\mathrm{s}}$. It goes through a plasma of softer gluons, colliding with scattering centers in this plasma with a cross section $\sigma \sim \alpha_{\mathrm{s}}^{2} / m_{D}^{2}$. The inverse mean free path of the particle in the plasma is the density of scattering centers times the cross section: $\lambda^{-1} \sim n_{s} \sigma$, this is how often the particle encounters a scattering target per unit time/length. A high energy particle will coherently see many scattering centers. If it propagates a time/distance $t_{\mathrm{f}}$ it will see $t_{\mathrm{f}} / \lambda$ of them, each time picking up a transverse momentum $\sim m_{D}^{2}$. Thus it will pick up a total transverse momentum $k_{T}^{2}=m_{D}^{2} t_{\mathrm{f}} / \lambda$. What is then this formation time $t_{\mathrm{f}}$ ? The emission of a gluon with transverse momentum $k_{T}$ and longitudinal momentum $k_{\mathrm{br}}$ means that one is emitting the gluon at an angle $\theta \approx k_{T} / k_{\mathrm{br}}$. In order for the emission to really take place so that the emitted gluon and its parent are separate particles, the two-particle system has to propagate far enough that the coordinate separation between emitted gluon and parent, $t_{\mathrm{f}} \theta$ is larger than the transverse size or wavelength of the emitted gluon $k_{T}$, i.e.

$$
\begin{equation*}
t_{\mathrm{f}} \frac{k_{T}}{k_{\mathrm{br}}} \sim k_{T} \quad \Longrightarrow \quad t_{\mathrm{f}} \sim \frac{k_{\mathrm{br}}}{k_{T}{ }^{2}} \tag{186}
\end{equation*}
$$

Emitting a very high energy gluon takes a long time, this is what limits how fast you can lose energy by gluon emissions. Within this formation time $t_{\mathrm{f}}$ all the interactions with the target are coherrent: the emitted gluon accumulates tranverse momentum from all the scattering centers it sees on its way. Within this formation time, one can therefore emit only up to one gluon, in fact one emits one gluon with a probability $\alpha_{\mathrm{s}}$ and zero gluons with probability $1-\alpha_{\mathrm{s}}$. Therefore the rate for emitting gluons with momentum $k_{b r}$ is $1 / t_{\mathrm{br}} \sim \alpha_{\mathrm{s}} / t_{\mathrm{f}}$, which depends on $k_{t}$ extbr. Since the formation time both determines how much transverse momentum one
can accumulate from the medium, and depends on the transverse momentum itself, it has to be determined self-consistently in the following way:


LPM interference: max 1 emission during formation time

$$
\begin{equation*}
t_{\mathrm{f}} \sim \frac{k_{\mathrm{br}}}{k_{T}{ }^{2}} \tag{187}
\end{equation*}
$$

Mean free path

$$
\begin{gather*}
\frac{1}{\lambda}=n_{s} \sigma=n_{s} \frac{\alpha_{\mathrm{s}}^{2}}{m_{D}^{2}}  \tag{188}\\
k_{T}^{2}=m_{D}^{2} \frac{1}{\lambda} t_{\mathrm{f}} \Longrightarrow \frac{1}{t_{\mathrm{f}}} \sim \frac{\alpha_{\mathrm{s}} \sqrt{n_{s}}}{\sqrt{k_{\mathrm{br}}}} \tag{189}
\end{gather*}
$$

## Emission rate

$$
\begin{equation*}
\frac{1}{t_{\mathrm{br}}}=\frac{\alpha_{\mathrm{s}}}{t_{\mathrm{f}}}=\frac{\alpha_{\mathrm{s}}^{2} \sqrt{n_{s}}}{\sqrt{k_{\mathrm{b} r}}} \tag{190}
\end{equation*}
$$

Now we have to determine how fast these kind of emissions lead to the hard particles losing all of their energy to the bath of soft particles, which already contains the particle number of the system. Again we have to self-consistently solve for two uknown quantities, the temperature of the bath $T$ and the typical energy loss in a branching, $k_{s}$. To relate these we need two conditions. One is an eqation that, through the medium density in eq. (190), relates the branching momentum to the temperature $T$. For this comparison we agein need to equate, similarly to before, the rate of emissions to the lifetime of the system. The other one is obtained from a comparison of the energy transfer from the hard particle to the medium to the rate of change of the temperature of the medium. In equations:

$$
\begin{equation*}
\frac{1}{t_{\mathrm{br}}} \sim \frac{1}{\tau} \sim \frac{\alpha_{\mathrm{s}}^{2} T^{3}}{\sqrt{k_{\mathrm{br}}}} \Longrightarrow \quad k_{\mathrm{br}} \sim \alpha_{\mathrm{s}}^{4} T^{3} \tau^{2} \tag{191}
\end{equation*}
$$

Energy flow (per unit volume)

$$
\begin{equation*}
k_{\mathrm{br}} \frac{\mathrm{~d} N_{\mathrm{br}}}{\mathrm{~d} \tau}=k_{\mathrm{br}} \frac{n_{h}}{t_{\mathrm{br}}} \sim k_{\mathrm{b} r} \frac{1}{\tau} \frac{Q_{\mathrm{s}}^{2}}{\alpha_{\mathrm{s}} \tau}=\frac{\mathrm{d}\left(T^{4}\right)}{\mathrm{d} \tau} \sim T^{3} \frac{\mathrm{~d} T}{\mathrm{~d} \tau} \Longrightarrow T \sim \alpha_{\mathrm{s}}^{3} Q_{\mathrm{s}}^{2} \tau \tag{192}
\end{equation*}
$$

So the temperature of the soft gluon bath rises very rapidly! The hard gluons have lost all their energy when the temperature of the bath, and thus the energy lost in one branching, is of the same order as the original energy of the hard gluons:

$$
\begin{gather*}
k_{\mathrm{br}} \sim \alpha_{\mathrm{s}}^{4} T^{3} \tau^{2} \sim Q_{\mathrm{s}} \Longrightarrow \quad Q_{\mathrm{s}} \tau \sim \alpha_{\mathrm{s}}^{-13 / 5}  \tag{193}\\
T_{\text {final }} \sim \alpha_{\mathrm{s}}^{2 / 5} Q_{\mathrm{s}}, \quad n_{\mathrm{s}}^{\text {final }} \sim \alpha_{\mathrm{s}}^{6 / 5} Q_{\mathrm{s}}^{3} \tag{194}
\end{gather*}
$$

This is a fascinating result: the formal weak coupling result for the isotropization time of the system is given by the fractional power $\alpha_{\mathrm{s}}^{-13 / 5}$ of the coupling. The nontrivial result is a consequence of many physical processes in gauge theory contributing to different stages of the process. Let us recapitulate some of the necessary ingredients:

- Overoccupied $f \sim 1 / \alpha_{\mathrm{s}}$ initial condition dominated by perturbative scale $Q_{\mathrm{s}}$
- Longitudinal expansion that tends to make the momentum distribution anisotropic
- Development of a softer scale $m_{D}$ that is initially independent of $\alpha_{\mathrm{s}}$ but later proportional to it, characterizing the collisions.
- Elastic scattering that tries to counteract the expansion and isotropize the momenta
- Inelastic scattering that creates a bath of softer gluons, increasing the particle number
- Interference, i.e. LPM effect, in the scattering of the remainign hard particles with this soft medium

It is nontrivial to build a quantitative calculation that includes all of these effects in a reasonable way. In fact, no one single calculation does so at the moment. Some state of the art methods used in these different stages are

1. Boost invariant CYM up to $Q_{\mathrm{s}} \tau \sim 1$; matched in some way to information from nuclear wavefunctions that can be obtained from DIS.
2. 3-dimensional classical Yang-Mills can go from $Q_{\mathrm{s}} \tau \sim 1$ to $Q_{\mathrm{s}} \tau \sim \alpha_{\mathrm{s}}^{-3 / 2}$ and has been seen to reproduce the first stage of the evolution process, demonstrating that the "bottom-up" scenario was the right one of several proposed weak coupling scaling solutions see e.g. [31]
3. After $Q_{\mathrm{s}} \tau \gg 1$, when the occupation number has fallen below $1 \alpha_{\mathrm{s}}$ due to some elastic scatterings taking place, one can use en effective kinetic theory description [32] that includes both elastic and inelastic interactions, Debye screening, and LPM interference, to evolve the system towards hydrodynamics. This was done e.g. in the calclation of [33].
4. The kinetic theory takes one close to isotropy so that the deviations from it are small; one can then match to viscous hydrodynamics which is a systematical way to treat systems close to local thermal equilibrium; again there is a common regime of validity for both theories.
5. In the earliest stages of this process, this is not actually the whoel story. Anisotropic systems of gauge fields exhibit plasma instabilities that were not included in this picture. They affect the transition from stage 1 to 2 , which in any case has been by far the sketchiest (i.e. no quantitative description at all) part in these lectures
6. In addition, there have been many studies of thermalization in the limit of strong coupling, but this would take us too far afield from the topic of these lectures.


Figure 2: Comparison of CYM simulation from [31], compared to several proposed weak coupling scaling laws for the development of the occupation number and anisotropy in the 3-dimensional classical field phase.



These plots are from the effective kinetic theory simulation of [33], tracing the steps of the bottom-up thermalization scenario in an actual calculation.

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[^0]:    ${ }^{1}$ More precisely, a general solution of the wave equation can always be expressed as a linear superposition of these oscillations, and because the wave equation is linear these oscillations are not coupled to each other

[^1]:    ${ }^{2}$ Of course, if you are building a detector for DIS experiments at the Electron-Ion Collider, $y$ is the most important variable for you: it determines what direction the electron flies in, and you really have to measure very precisely this electron.

