Initial stages of heavy ion collisions: exercises

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1. Calculate (to lowest order, one tree diagram) the differential QED cross section $\mathrm{d}\sigma/\mathrm{d}t$ for $e^-\mu^- \to e^-\mu^-$ scattering neglecting all the masses. Express the result in terms of the Mandelstam invariants s, t and u. What is the high energy limit of the cross section, i.e. the limit t fixed, $s \sim -u \to \infty$?

2. In the previous problem, what happens if you replace the photon by a massless scalar particle? I.e. replace $g_{\mu\nu} \to 1$ in the photon propagator and $\gamma^\mu \to 1$ in the vertex. What is the high energy limit now? (This calculation is shorter than the previous one).

3. Derive the Green's function for the Helmholtz equation that was skipped in the lectures. Start from the definition $(\omega^2+\nabla^2)G(\mathbf{x},\mathbf{y}) = -\delta^3(\mathbf{x}-\mathbf{y})$. Defining the Fourier transform as
$$G(\mathbf{x},\mathbf{y}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} G(\mathbf{p})$$
calculate first $G(\mathbf{p})$ and then $G(\mathbf{x},\mathbf{y})$. You have to regularize the $\mathbf{p}$ integral; doing this with the substitution $\omega \to \omega + i\varepsilon$ will give the result we want here. What do you get if you replace $\omega \to \omega - i\varepsilon$ (we are assuming $\omega > 0$)?

4. (a) Show (this is easy) that if
$$\frac{\mathrm{d}\sigma_{\mathrm{el.}}}{\mathrm{d}^2\mathbf{q}_T} = \left|\frac{i}{2\pi}\int \mathrm{d}^2\mathbf{b}_T e^{-i\mathbf{q}_T\cdot\mathbf{b}_T}\Gamma(\mathbf{b}_T)\right|^2$$
then
$$\sigma_{\mathrm{el}} = \int \mathrm{d}^2\mathbf{b}_T|\Gamma(\mathbf{b}_T)|^2$$
(b) The total cross section is
$$\sigma_{\mathrm{tot}} = 2\int \mathrm{d}^2\mathbf{b}_T \mathrm{Re}[\Gamma(\mathbf{b}_T)],$$
and the partial wave unitarity bound is $|\Gamma(\mathbf{b}_T)|^2 \leq 2\mathrm{Re}[\Gamma(\mathbf{b}_T)]$, which leads to $\sigma_{\mathrm{el}} \leq \sigma_{\mathrm{tot}}$ (a pretty natural requirement). Where in the complex plane can $\Gamma(\mathbf{b}_T)$ be to satisfy this?
(c) Assuming that $\mathrm{Re}[\Gamma(\mathbf{b}_T)] = \Gamma_0 e^{-\mathbf{b}_T^2/(2B)}$ and $\mathrm{Im}[\Gamma(\mathbf{b}_T)] = 0.141\mathrm{Re}[\Gamma(\mathbf{b}_T)]$, $\sigma_{\mathrm{el}} =$ 25.4mb and $\sigma_{\mathrm{tot}} =$ 98.6mb, what are B and Γ_0 ? The cross section numbers for pp-scattering at $\sqrt{s} = 7\mathrm{TeV}$ come from TOTEM, Europhys.Lett. 101 (2013) 21002, https://cds.cern.ch/record/1472948/files/CERN-PH-EP-2012-239.pdf. Is the Gaussian $\mathbf{b}_T$-dependence consistent with Fig. 2 of the paper? ($t = -\mathbf{q}_T^2$)

5. Calculate
$$\int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ikx}}{k^2+m^2}.$$

This can be done the easy way using the theorem of residues; there is probably also a hard way without it. Then calculate the two dimensional transform needed in the virtual photon-dipole wave function

$$\int \mathrm{d}^2\mathbf{k}_T \frac{e^{i\mathbf{k}_T\cdot\mathbf{r}_T}}{\mathbf{k}_T^2+m^2}$$

and compare. This is perhaps easiest to do by integrating over the angle first, which leaves you with the Bessel $J_0(|\mathbf{k}_T||\mathbf{r}_T|) = J_0(kr)$. The integral over k then gives a modified Bessel K_0. Mathematica will do the k-integral for you, maybe even the two-dimensional one directly? Otherwise one probably needs integral representations of J_0 and K_0.

6. Fourier-transform the essential part of the LC wave function for emitting a soft gluon:

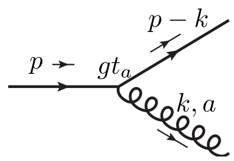
$$\int \mathrm{d}^2\mathbf{k}_T e^{i\mathbf{k}_T\cdot\mathbf{r}_T} \frac{\boldsymbol{\varepsilon}_T\cdot\mathbf{k}_T}{\mathbf{k}_T^2}.$$

This can be done analytically (even without mathematica!) by first integrating over the angle, which gives a Bessel function J_1 that is the derivative of J_0; thus the radial integral is easy. Note that there are two independent azimuthal angles, those of $\boldsymbol{\varepsilon}_T$ and $\mathbf{r}_T$. Surprisingly the integral is really convergent without any regularization.

7. (Kovchegov & Levin, 4.5 b) Solve the BK equation in zero transverse dimensions:

$$\partial_y N = \alpha_\mathrm{s} N - \alpha_\mathrm{s} N^2, \quad N(y=0) = N_0 \ll 1 \tag{1}$$

8. (Kovchegov & Levin, exercise 5.1 a,b) The gluon field radiated from a fast-moving quark can be written using usual covariant theory Feynman rules as



$$A_\mu^a(k) = -igt^a \frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \bar{u}_\sigma(p-k)\gamma^\nu u_\sigma(p)(2\pi)\delta((p-k)^2) \tag{2}$$

The incoming quark is on shell, with $p^\mu = (p^+, 0, \mathbf{0}_T)$.

(a) In covariant gauge we can use the eikonal vertex: assuming $p^+ \approx (p-k)^+ \gg k^+$ show or convince yourself (e.g. using the Gordon decomposition) that the leading high energy behavior is

$$\bar{u}_{\sigma'}(p-k)\gamma^\nu u_\sigma(p) \approx 2p^+\delta^{\nu+}\delta_{\sigma\sigma'} \tag{3}$$

(b) Then Fourier-transform

$$A_\mu^a(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ik\cdot x} A_\mu^a(k) \tag{4}$$

to get the field in coordinate space

$$A_{\mathrm{COV}}^{+a} = -\frac{g}{\pi} t^a \delta(x^-) \ln|\mathbf{x}_T|\Lambda \tag{5}$$

9. Consider two (independent of each other) transverse ($i, j \in \{1,2\}$) pure gauge fields that depend only on transverse coordinates $A_i^{(1,2)} = A_{i,a}^{(1,2)} t^a = \frac{-i}{g} U(\mathbf{x}_T)\partial_i U^\dagger(\mathbf{x}_T)$. Recall the expression for the field strength tensor $F_{\mu\nu}$ and show that these pure gauges have no longitudinal magnetic field $F_{ij}^{(1,2)} = 0$. Then consider a field that is the sum of the two: $A_i = A_i^{(1)} + A_i^{(2)}$: what is its magnetic field F_{ij}?