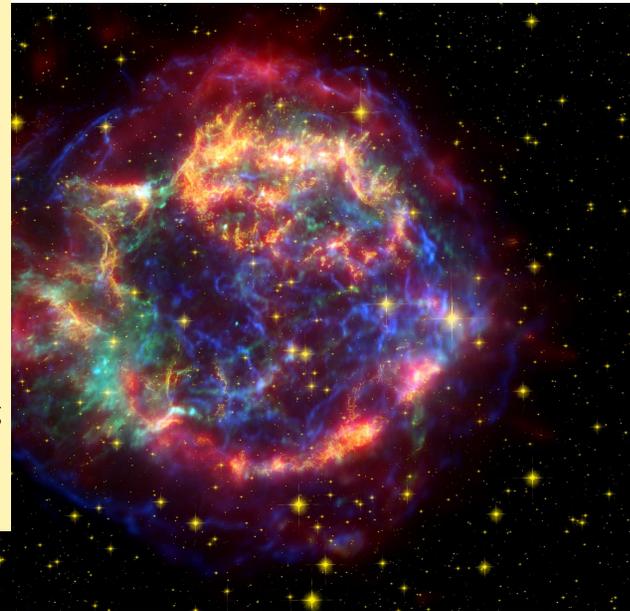
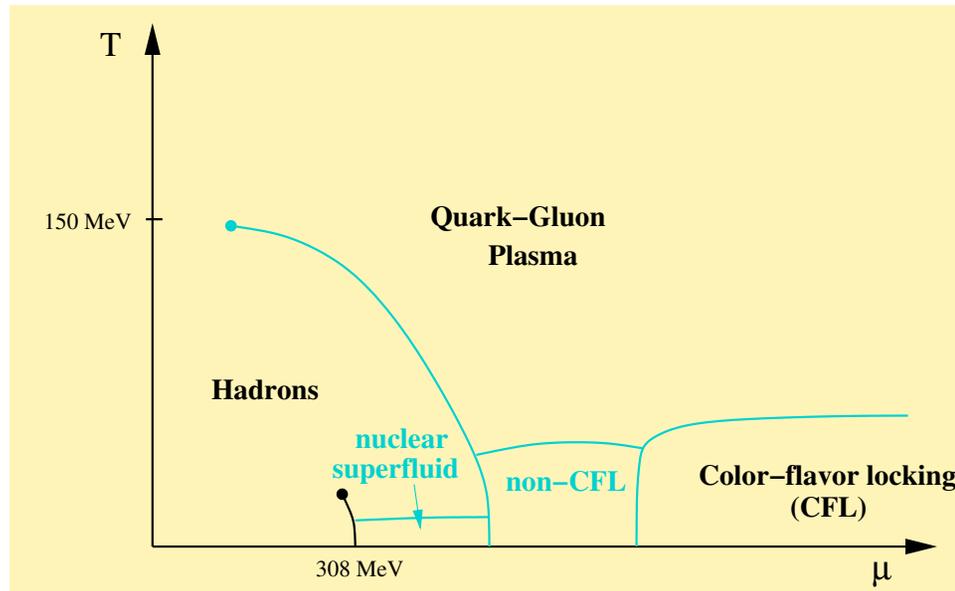


Dense matter in the QCD phase diagram and in compact stars



Basic questions

What happens to matter if you compress it more and more?

What is a compact star made of?

Outline

(1) Introduction and overview (pp 6 - 31)

- general remarks
- dense matter in the QCD phase diagram
- some selected astrophysical observations

(2) Dense quark matter (pp 33 - 50)

- basic thermodynamics → Problems I
- strange quark matter hypothesis
- equation of state

(3) Dense nuclear matter (pp 53 - 64)

- free nuclear matter → Problems II
- field-theoretical model
- saturation density and binding energy

(4) Cooper pairing in dense matter (pp 66 - 81)

- field-theoretical approach (sketch)
- fermionic excitations → Problems III
- solving the gap equation

(5) Color superconductivity (pp 83 - 106)

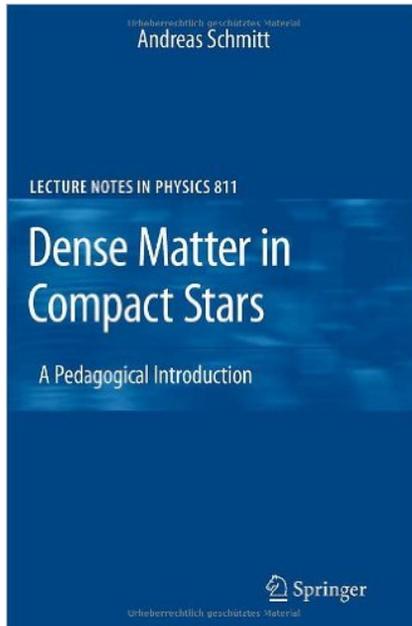
- color-flavor locked (CFL) quark matter
- stressed pairing and non-CFL color superconductors

(6) Transport in dense matter (pp 108 - 150)

- specific heat
- neutrino emissivity
- bulk viscosity

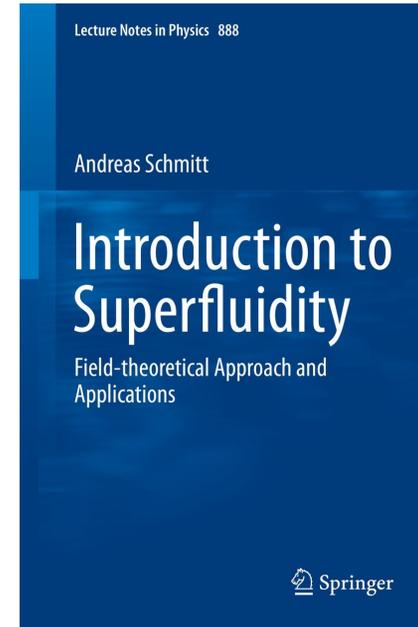
Literature

for (parts of) lectures (1) – (6), see



A. Schmitt, Lect. Notes Phys. 811, 1 (2010)
[arXiv:1001.3294 [astro-ph.SR]]

for more details about lectures (4), (5), see



A. Schmitt, Lect. Notes Phys. 888, 1-155 (2015)
[arXiv:1404.1284 [hep-ph]]

+ more specific reviews

Color superconductivity M. G. Alford, A. Schmitt, K. Rajagopal, T. Schäfer, RMP 80, 1455 (2008)

Transport in neutron stars A. Schmitt and P. Shternin, arXiv:1711.06520 [astro-ph.HE]

+ research papers quoted during the lectures

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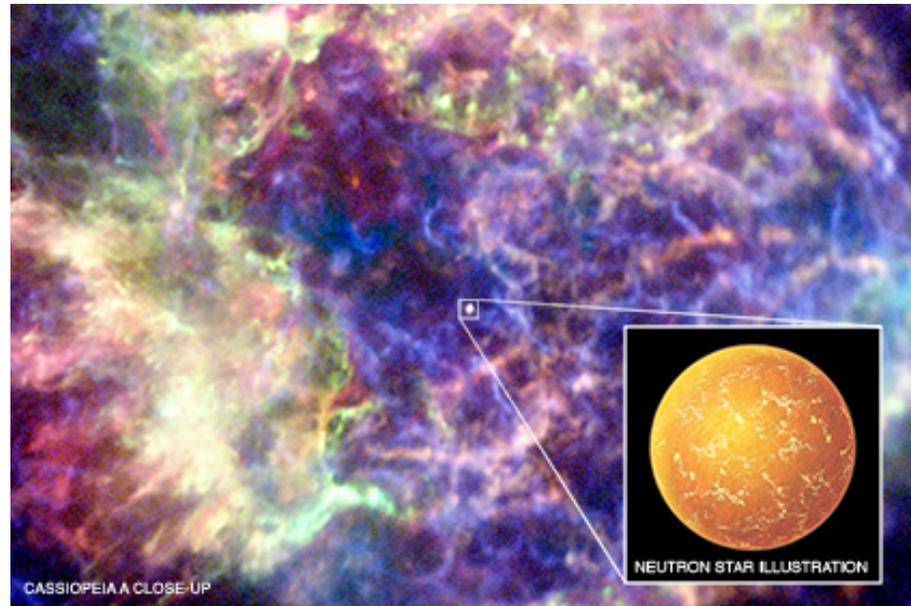
- specific heat
- neutrino emissivity
- bulk viscosity

Compact stars: densest matter in the universe

mass $\sim (1 - 2)M_{\odot}$

radius ~ 10 km

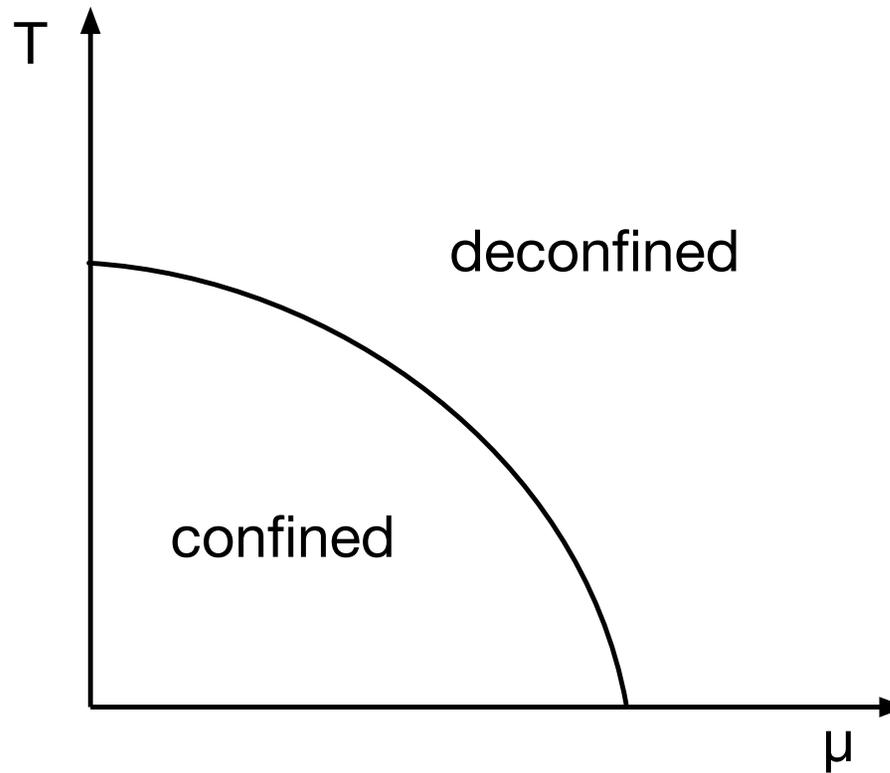
density $\lesssim 10 n_0$



→ at these extreme densities, fundamental physics becomes relevant

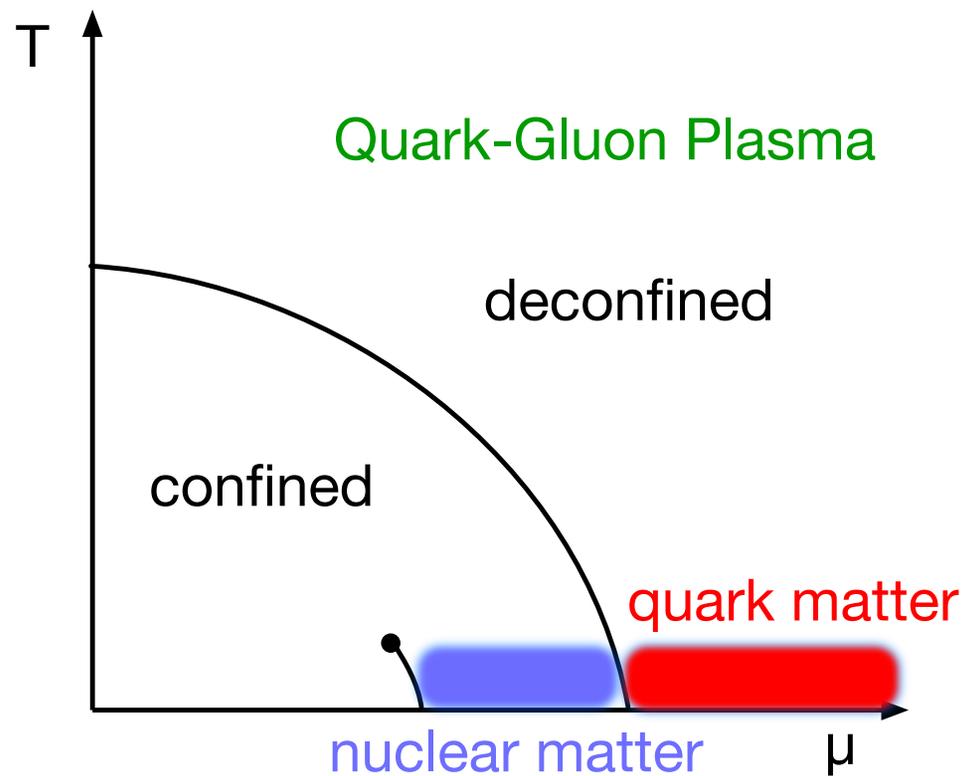
Schematically: nuclear matter and quark matter

- simplified QCD phase diagram

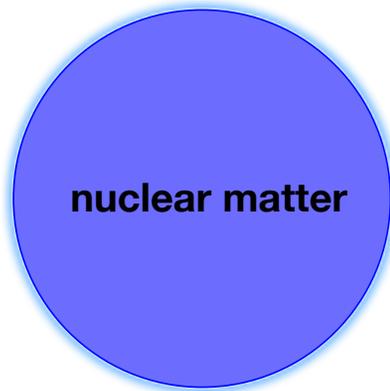


Schematically: nuclear matter and quark matter

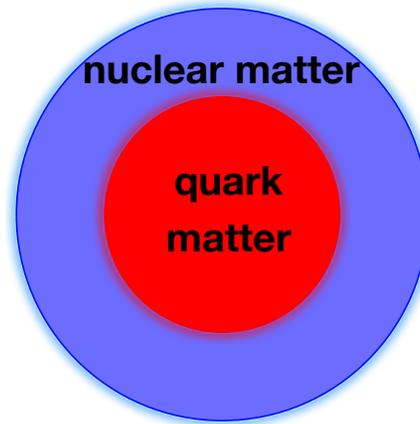
- simplified QCD phase diagram



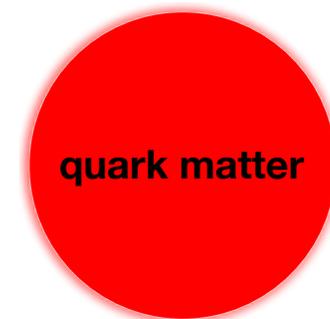
Compact star, simple view



Neutron star

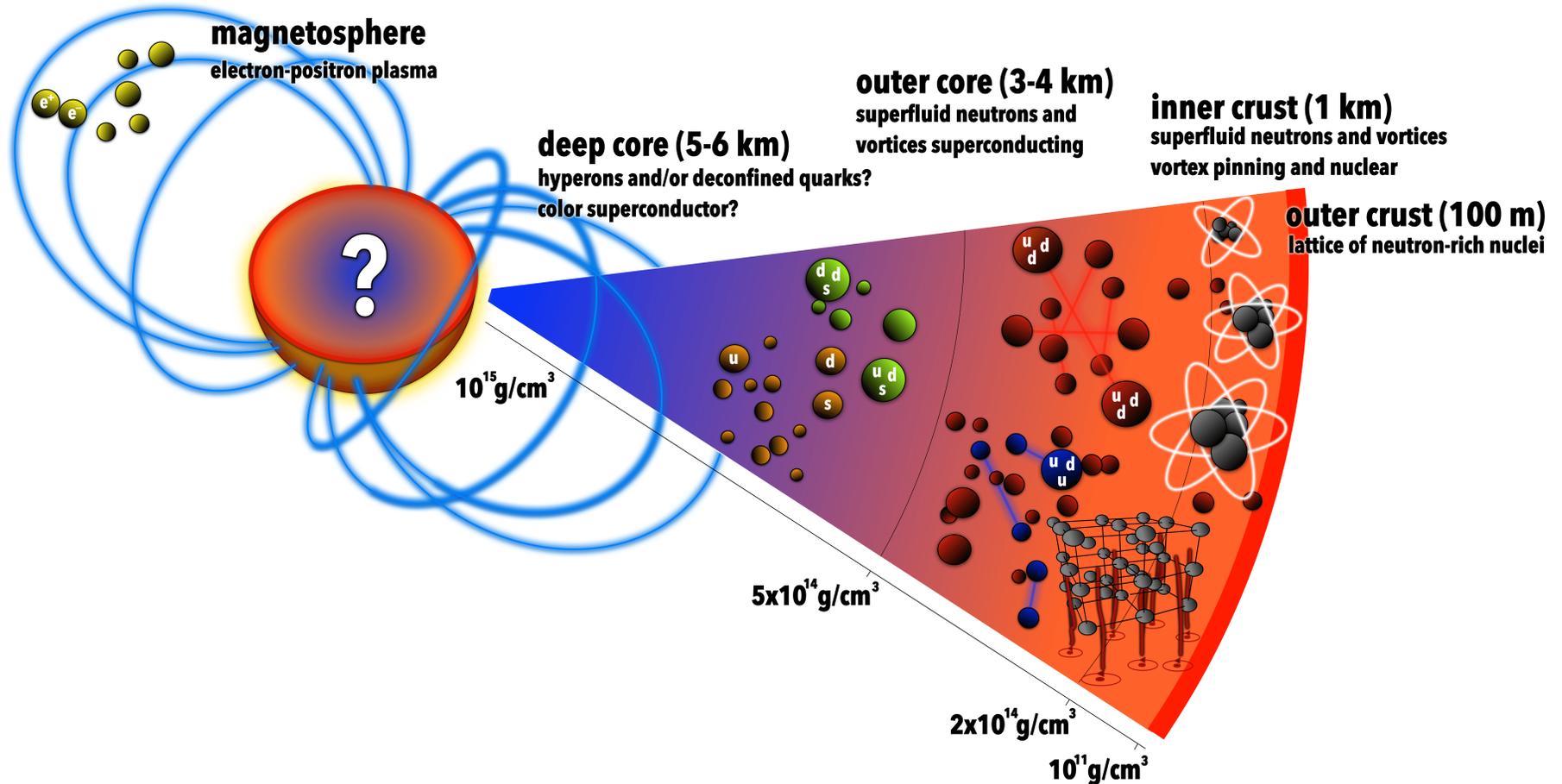


Hybrid star



Quark star
(Strange star)

Compact star, more detailed view

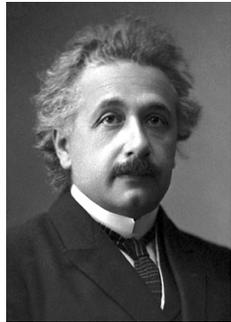
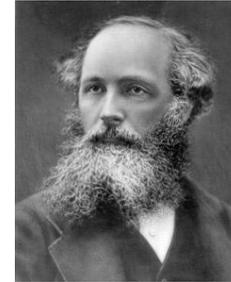


A. Watts *et al.*, arXiv:1501.00042 [astro-ph.SR]

Compact stars ...

... involve all fundamental forces

electromagnetism (magnetic field evolution, ...)



gravity (stability of the star, gravitational waves, ...)

weak interactions (neutrino emissivity, ...)



strong interactions (nuclear & quark matter, ...)

Compact stars ...

... relate to various fields in physics:

- **astrophysics/astronomy**

(birth of a compact star in supernova, neutron star mergers ...)

- **particle physics**

(quark matter, hyperons, meson condensation, ...)

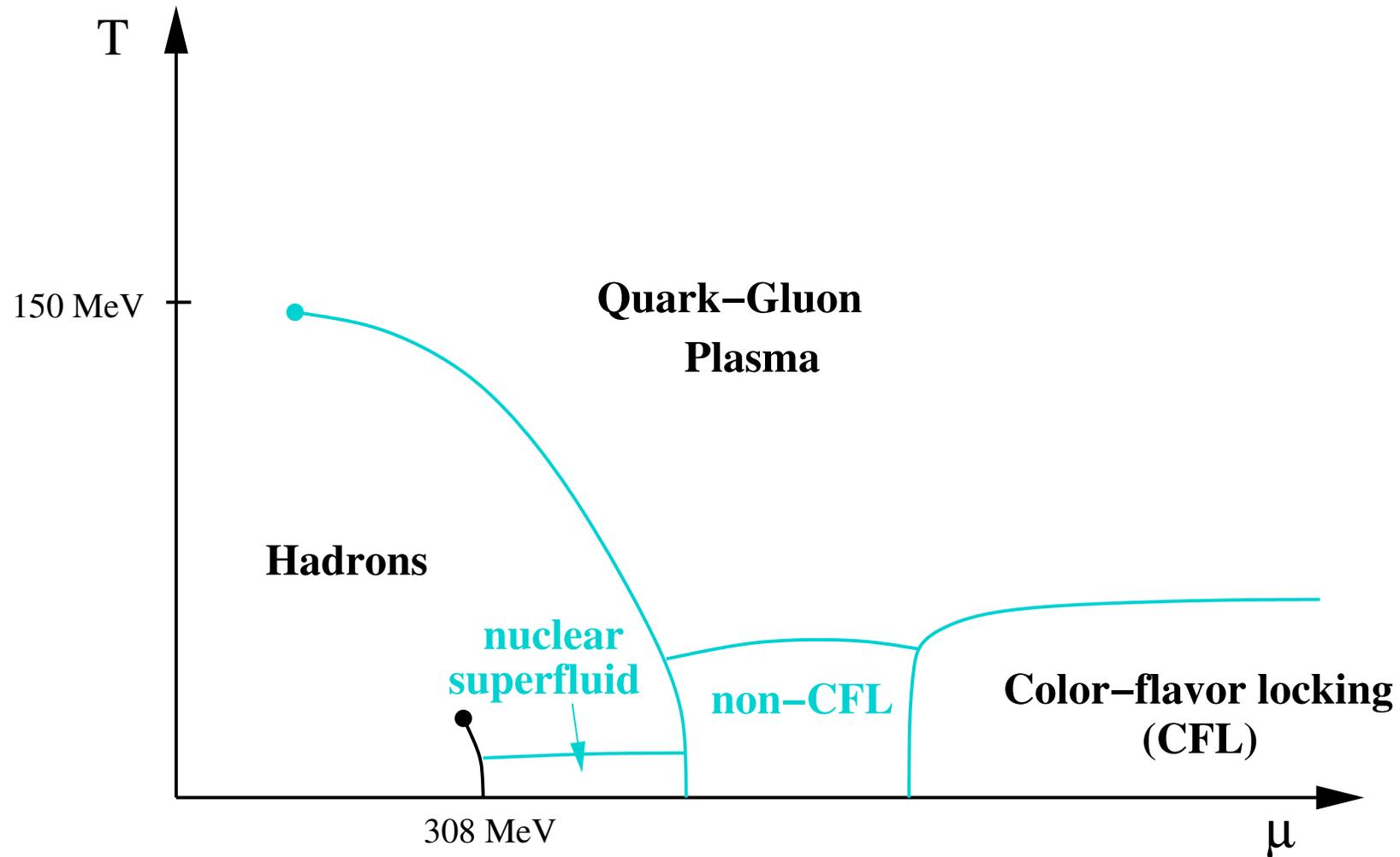
- **nuclear physics**

(dense nuclear matter, crust of the star, ...)

- **condensed matter physics**

(superfluidity/superconductivity, ion lattice in the crust, ...)

QCD at nonzero densities and temperatures



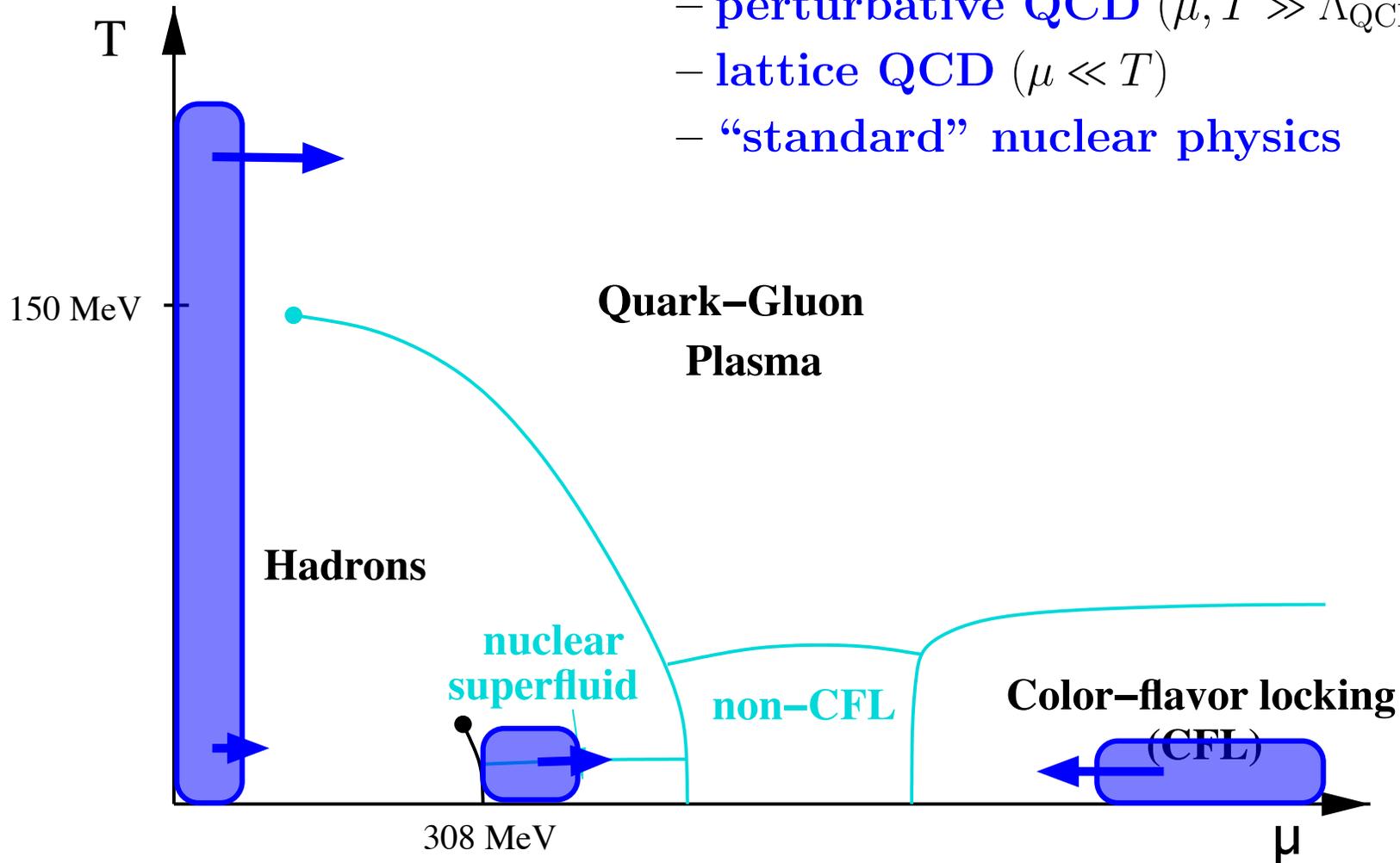
QCD at nonzero densities and temperatures

- rigorous methods

- perturbative QCD ($\mu, T \gg \Lambda_{\text{QCD}}$)

- lattice QCD ($\mu \ll T$)

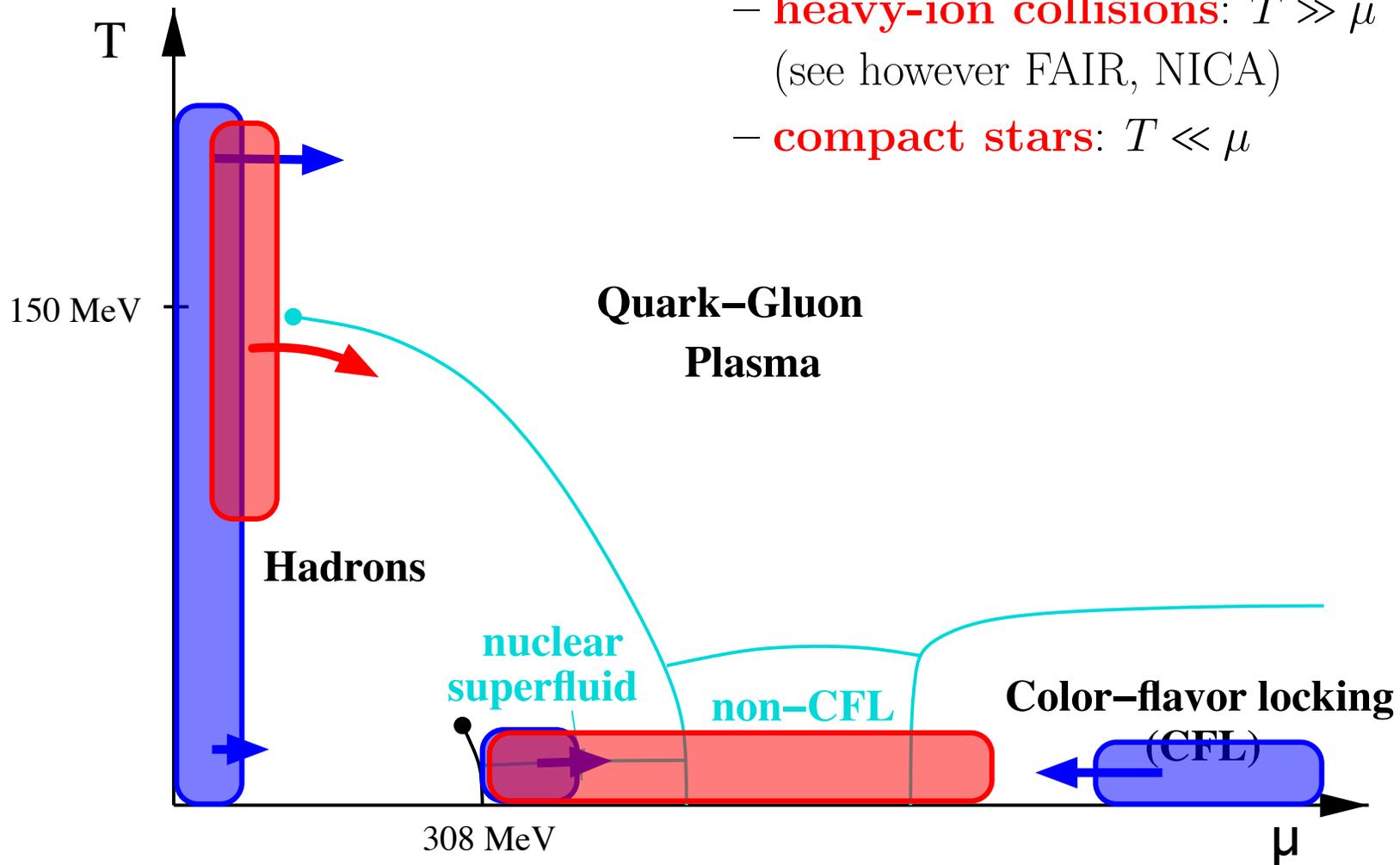
- “standard” nuclear physics



QCD at nonzero densities and temperatures

- **data**

- **heavy-ion collisions**: $T \gg \mu$
(see however FAIR, NICA)
- **compact stars**: $T \ll \mu$

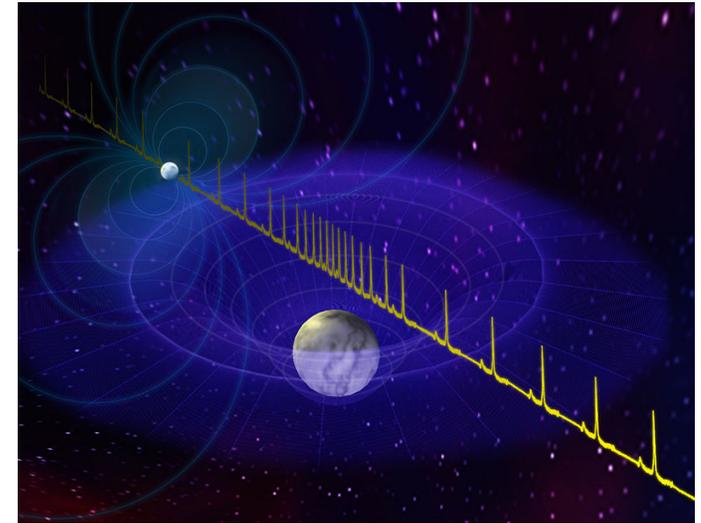
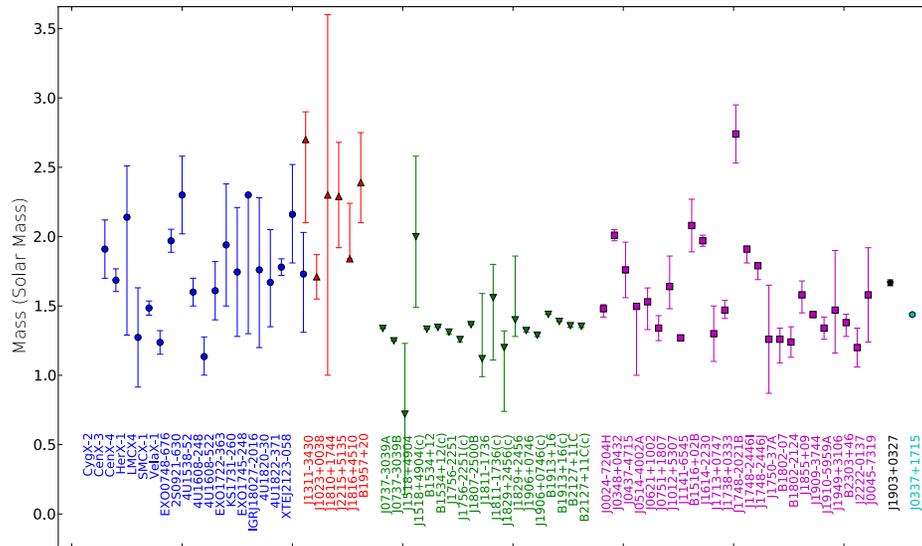


Theoretical tools to understand matter at compact star densities

- QCD
 - perturbative methods (at ultra-high densities) → see A. Vuorinen's lectures
 - lattice QCD (however, sign problem) → see G. Aarts' lectures
- effective theories
 - chiral perturbation theory in nuclear matter
 - effective theory of color-flavor locked quark matter
 - hydrodynamics
- phenomenological models
 - Nambu-Jona-Lasinio model
 - Ginzburg-Landau model
- non-perturbative methods/improvements
 - functional renormalization group → see J. Pawłowski's lectures
 - gauge-gravity duality

Some astrophysical observations and their relation to
fundamental physics

Neutron star masses (page 1/2): measurements



Neutron star masses [A. Watts *et al.*, arXiv:1501.00042]

Shapiro delay

- heaviest (accurately) known stars

$$M = 1.97 \pm 0.04 M_{\odot} \quad \text{P. Demorest et al., Nature 467, 1081 (2010)}$$

$$M = 2.01 \pm 0.04 M_{\odot} \quad \text{J. Antoniadis et al. Science 340, 6131 (2013)}$$

Neutron star masses (page 2/2): constraints on equation of state

equation of state $P(\epsilon)$ + TOV equation $\rightarrow M(R) \rightarrow$ maximal mass

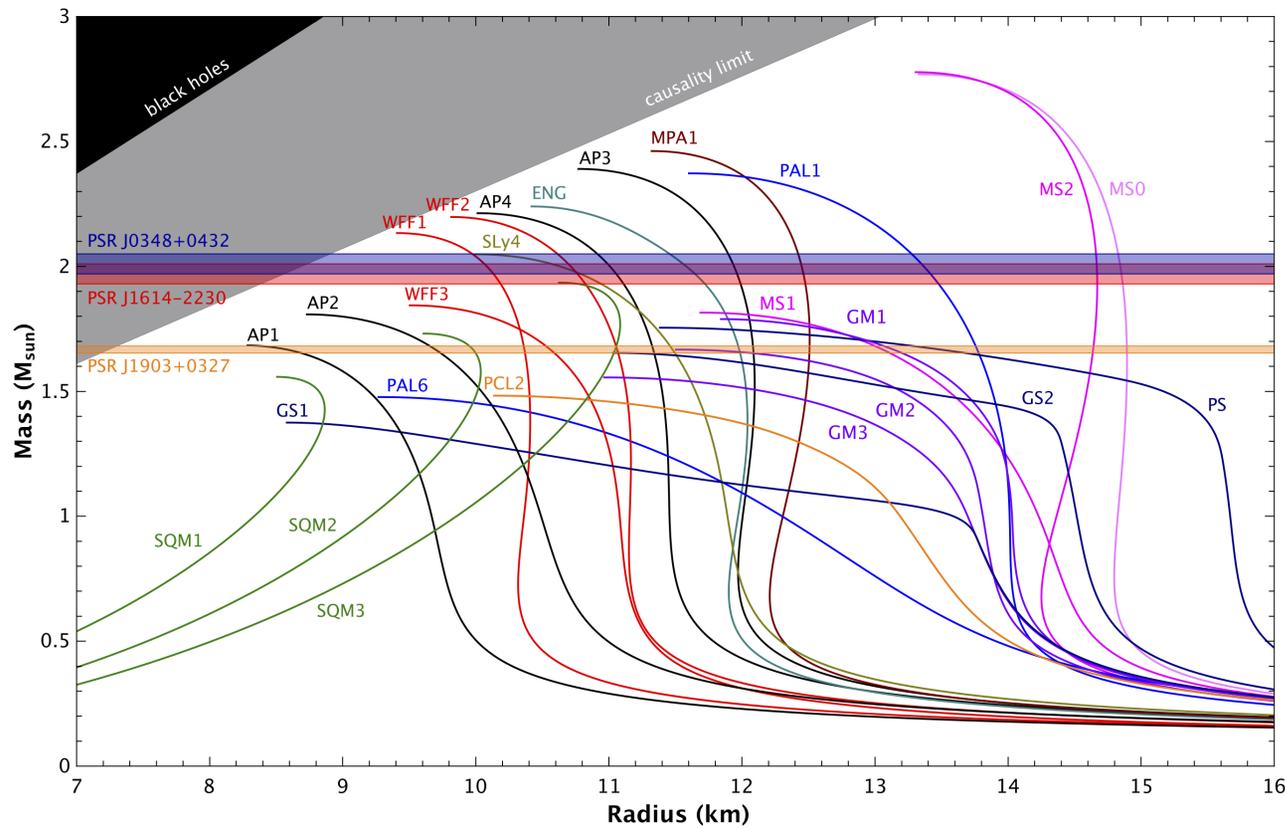
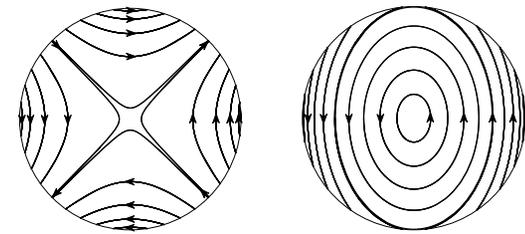


figure from <http://www3.mpifr-bonn.mpg.de>

r-mode instability (page 1/3): observational consequences

- **r-modes**: non-radial pulsation modes
 - **unstable** in a rotating star
 - star **spins down** by emitting **gravitational waves**

N. Andersson, *Astrophys. J.* 502, 708-713 (1998)



Polar View

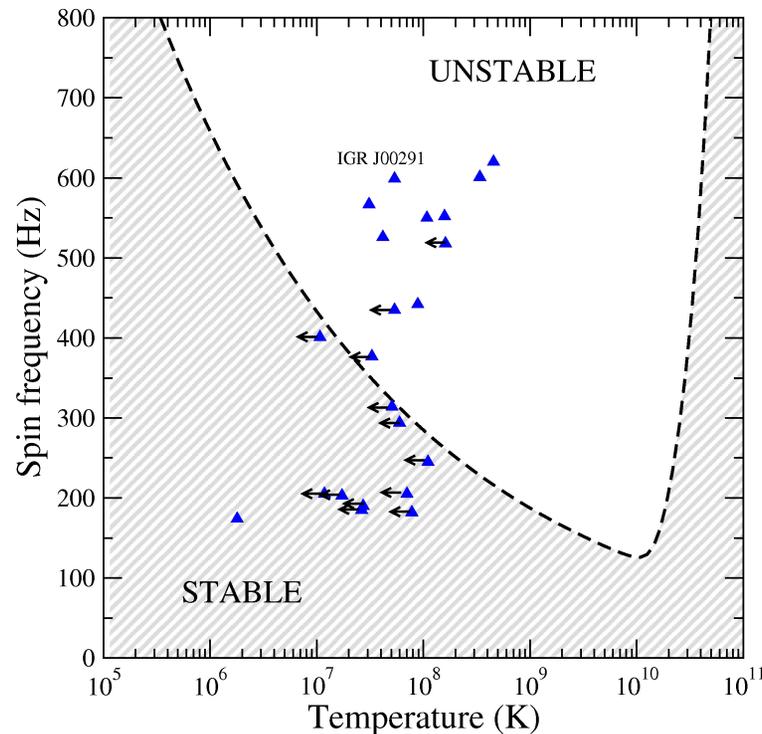
Equatorial View

L. Lindblom, *astro-ph/0101136*

- observables: (i) direct observation of **gravitational waves** → difficult
(grav. waves from **compact star mergers**, see below)
- (ii) stars should not be found in “**instability window**”

r-mode instability (page 2/3): puzzle

(ii) stars should not be found in *instability window*



- instability curve from *shear* (low T) and *bulk* (high T) *viscosity*
- probes *transport properties* of *nuclear* or *quark matter*

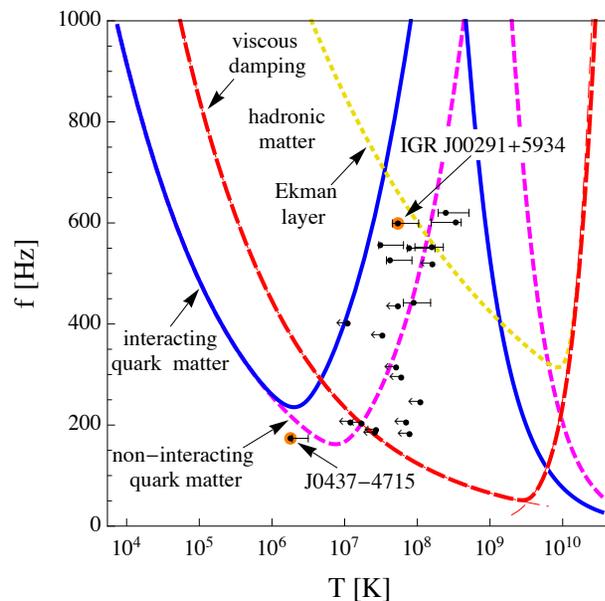
B. Haskell, et al., MNRAS 424, 93 (2012)

r-mode instability (page 3/3): possible solutions

- small saturation amplitude due to cutting of superfluid vortices through superconducting flux tubes

B. Haskell, K. Glampedakis and N. Andersson, MNRAS 441, 1662 (2014)

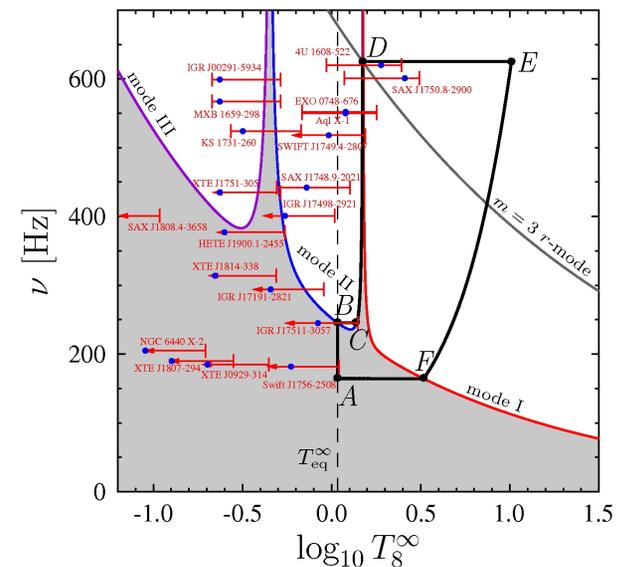
- quark matter (unpaired, non-Fermi liquid effects)



M. G. Alford, K. Schwenzer, PRL 113, 251102 (2014)

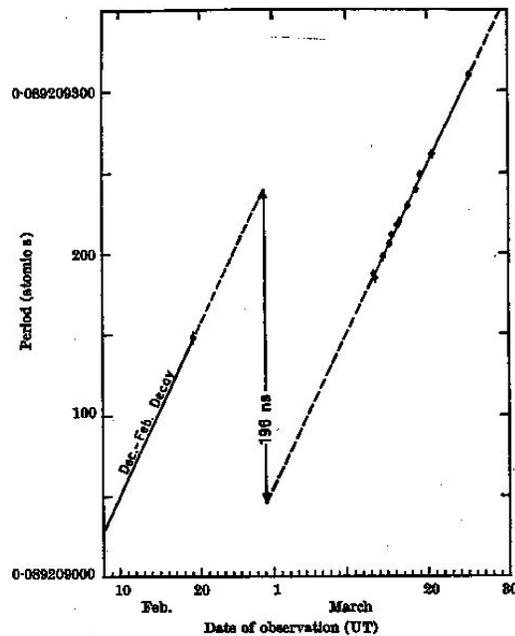
- coupling of “normal” r-mode to superfluid mode

M. E. Gusakov et al., PRL 112, 151101 (2014)

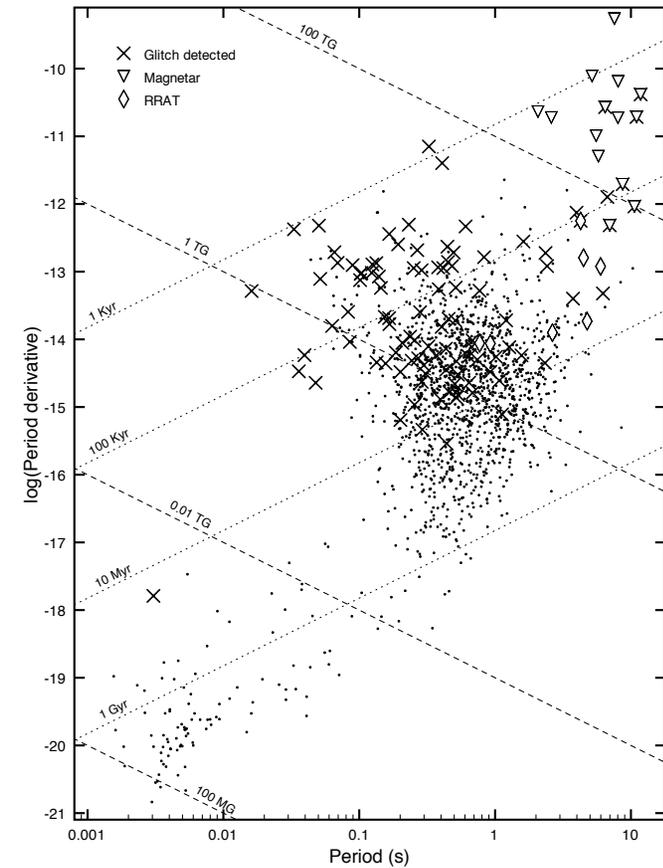


Pulsar glitches (page 1/3): observations

- pulsars usually spin-down steadily
- pulsar glitch = sudden spin-up
- first observed in Vela pulsar
V. Radhakrishnan, R.N. Manchester,
Nature 222, 228 (1969)



Espinoza *et al.*, *MNRAS* 414, 1679 (2011)

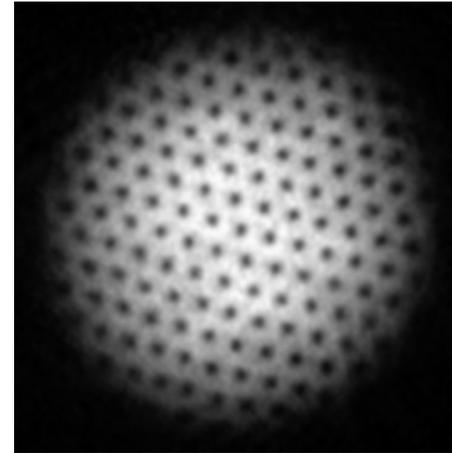


504 glitches observed in 187 pulsars (Jun 2018)
glitch table <http://www.jb.man.ac.uk/pulsar/glitches.html>

Pulsar glitches (page 2/3): explanation

- rotating superfluid \rightarrow vortex array

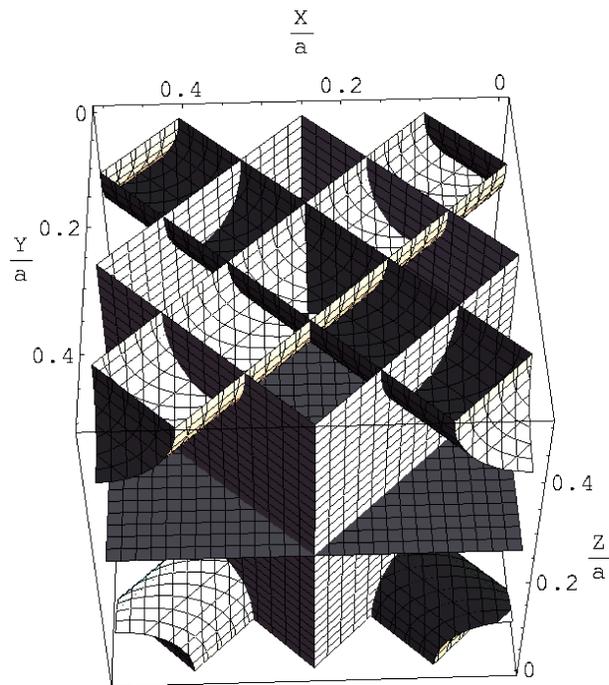
Vortices in rotating atomic superfluid
M. Zwierlein et al., Science 311, 492 (2006)



- **crust**: superfluid neutrons + ion lattice
- glitch mechanism:
vortex pinning and sudden (collective) **unpinning**
 \rightarrow sudden **transfer of angular momentum**
from superfluid to rest of star
P. W. Anderson, N. Itoh, Nature 256, 25 (1975)

Pulsar glitches (page 3/3): problems and alternatives

- huge glitches observed, $\Delta\Omega/\Omega \simeq 3 \times 10^{-5}$
R.N. Manchester, G. Hobbs, *Astrophys.J.* 736, L31 (2011)
- incompatible with superfluid entrainment in the crust?
“The crust is not enough” N. Andersson, et al., *PRL* 109, 241103 (2012)
“The crust may be enough” J. Piekarewicz, et al., *PRC* 90, 015803 (2014)



Crystalline CFL

- what triggers the collective unpinning?
superfluid two-stream instability?
N. Andersson, G.L. Comer, R. Prix, *PRL* 90, 091101 (2003)
A. Schmitt, *PRD* 89, 065024 (2014)
A. Haber, A. Schmitt, S. Stetina, *PRD* 93, 025011 (2016)
- alternative mechanism: crystalline CFL quark matter in the core?
K. Rajagopal and R. Sharma, *PRD* 74, 094019 (2006)
M. Mannarelli *et al.*, *PRD* 76, 074026 (2007)

Rapid cooling in Cas A (page 1/2)

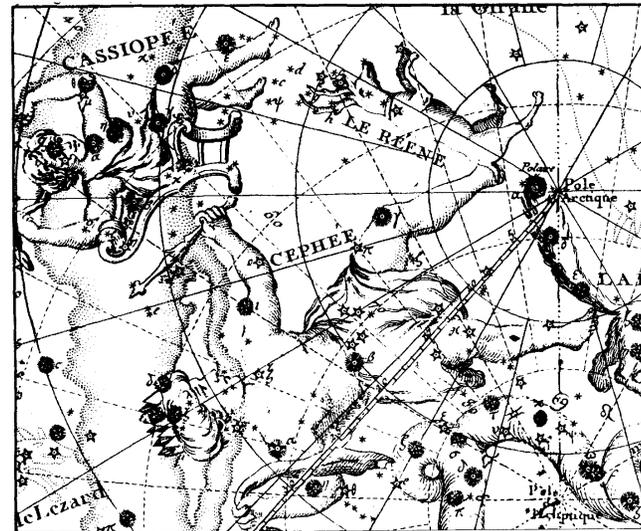
- young compact star (~ 340 yr)
at center of supernova remnant
Cassiopeia A (Cas A)

[supernova possibly observed historically

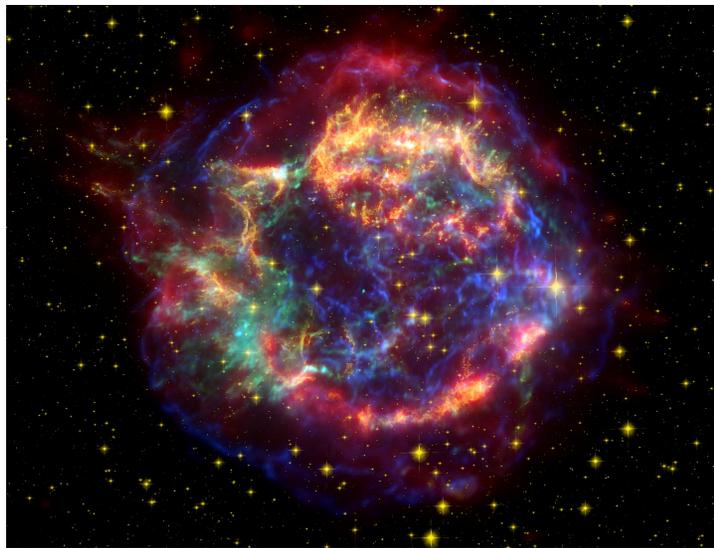
D.W. Hughes, *Nature* 285, 132 (1980)]

[compact star observed in 1999

H. Tananbaum, *IAUC* 7246, 1 (1999)]



From Atlas Céleste de Flamsteed,
l'Académie Royale de Science, Paris, 1776



Cas A, combined image from Spitzer and
Hubble Telescopes and Chandra X-ray

- rapid cooling observed:
temperature decrease of 1% - 3%
over 10 yr C. O. Heinke and W. C. G. Ho,
Astrophys. J. 719, L167 (2010); K.G. Elshamouty,
et al., *Astrophys. J.* 777, 22 (2013)

Rapid cooling in Cas A (page 2/2)

- superfluidity: neutrino emission suppressed at low T
- Cooper pair breaking and formation \rightarrow enhancement possible just below T_c

- rapid cooling due to transition to neutron superfluidity (in the presence of proton superc.)

D. Page, *et al.* PRL 106, 081101 (2011)

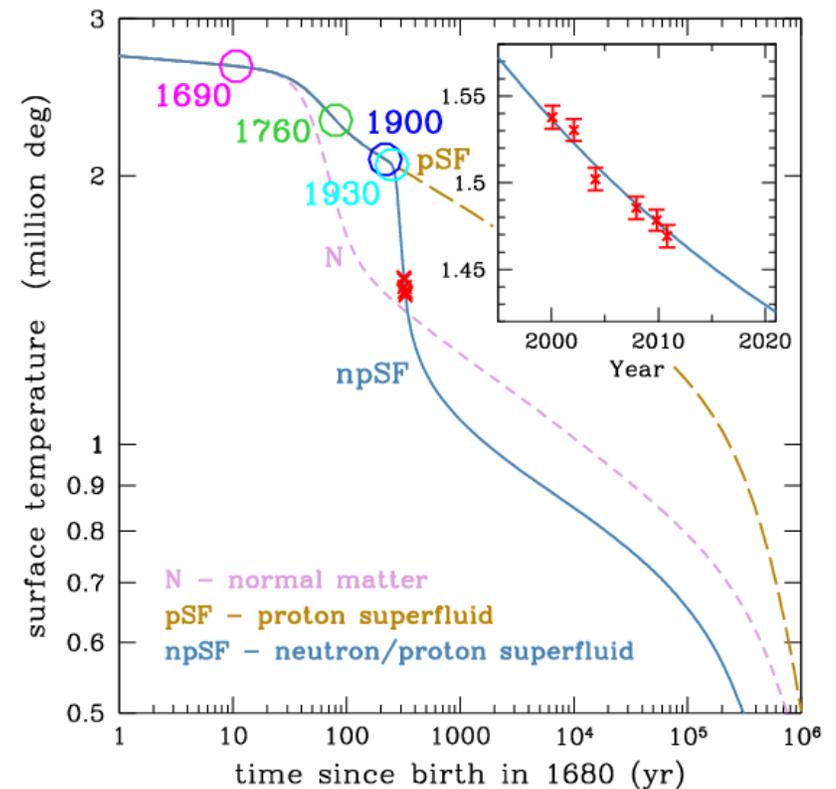
P. S. Shternin, *et al.* MNRAS 412, L108 (2011)

\rightarrow “measurement” of

$$T_c \simeq (5 - 8) \times 10^8 \text{ K}$$

- alternative explanation: 2SC \rightarrow LOFF transition in quark matter

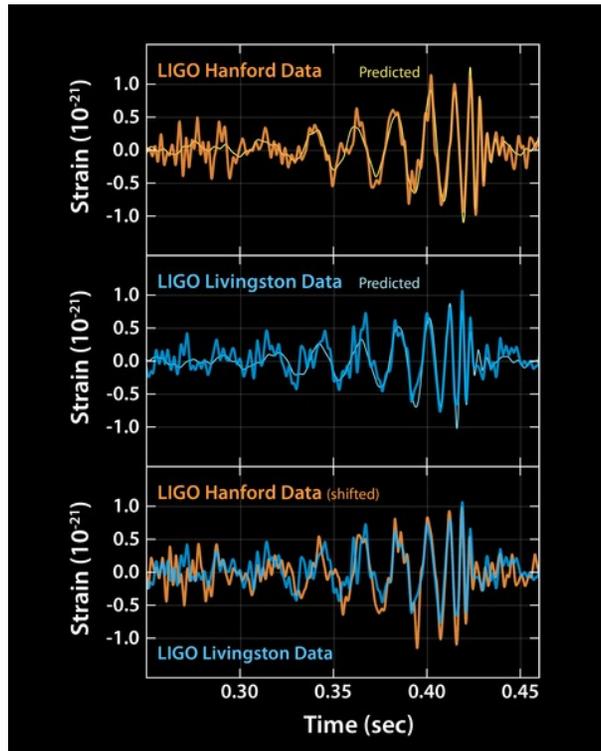
A. Sedrakian, A&A 555, L10 (2013)



W.C.G. Ho, *et al.*, PoS ConfinementX, 260 (2012)

Gravitational waves (page 1/3: detection)

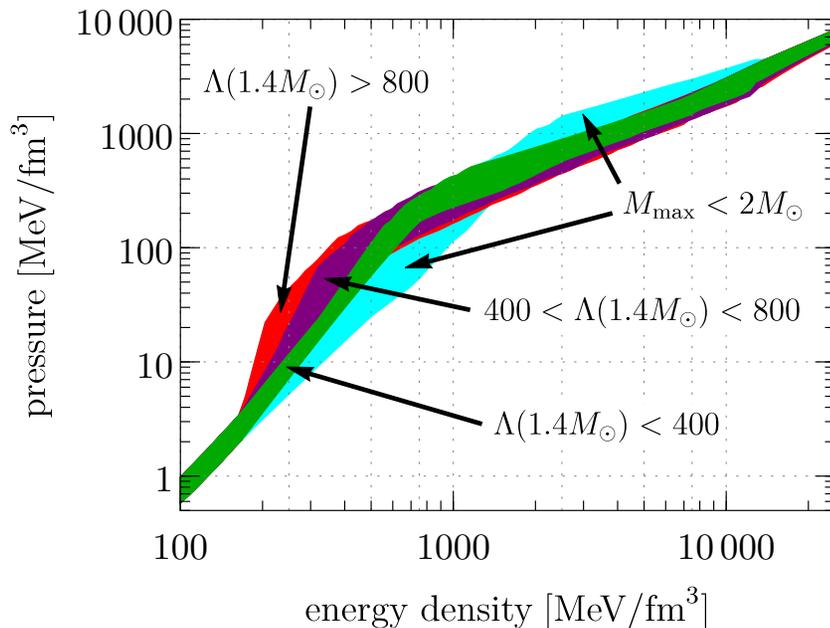
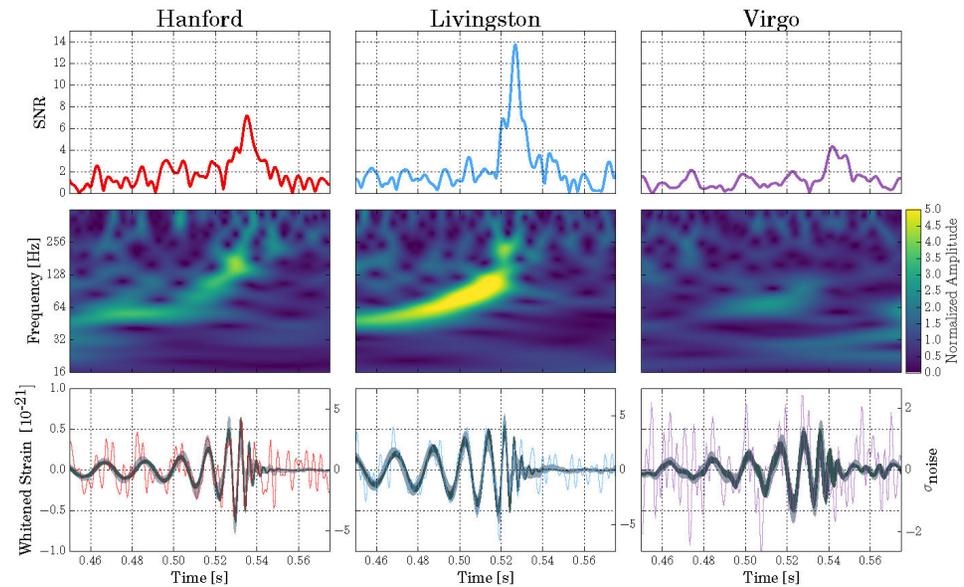
- gravitational waves: first detected by LIGO from **black hole merger** 2015 (Nobel Prize 2017)



- neutron stars as potential sources for gravitational waves:
 - neutron star mergers
 - ”mountains” (ellipticity + rotation)
 - oscillations (*r*-mode)

Gravitational waves (page 2/3: neutron star merger)

- gravitational waves detected from neutron star merger
LIGO and Virgo, PRL 119, 161101 (2017)
→ upper limit for tidal deformability Λ



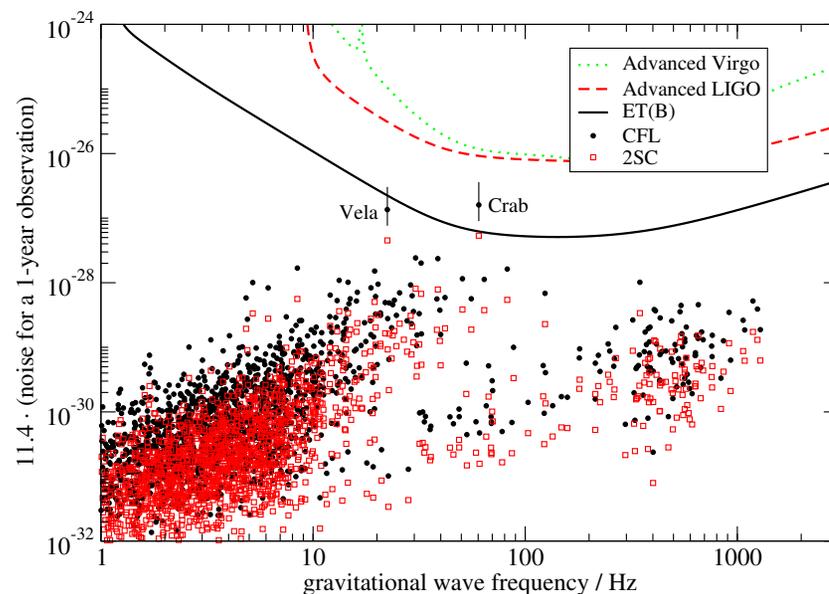
- 2-solar-mass stars:
EoS must be sufficiently stiff
- upper limit for Λ :
EoS must not be too stiff
(stiff EoS \rightarrow large stars \rightarrow large Λ)
- constrain family of EoSs
E. Annala *et al.*, arXiv:1711.02644 [astro-ph.HE]

Gravitational waves (page 3/3: mountains)

- ellipticity of star ("mountains"):
 - sustained by **crystalline structures** (e.g., crust of the star, mixed phases, LOFF phase, array of magnetic flux tubes, ...)
- misalignment of magnetic and rotational axis → **gravitational waves**

- for instance enhanced ellipticity of compact stars with flux tubes in quark matter core

K. Glampedakis, D. I. Jones and
L. Samuelsson, PRL 109, 081103 (2012)



Summary: compact stars are laboratories for fundamental physics

- matter inside compact stars is cold and dense ($\mu \gg T$) and very challenging to describe theoretically
- observations can be related to microscopic properties of dense matter
 - mass/radius \leftrightarrow equation of state
 - r-mode instability \leftrightarrow shear/bulk viscosity
 - pulsar glitches \leftrightarrow superfluidity
 - cooling \leftrightarrow neutrino emissivity
 - grav. waves (mergers) \leftrightarrow tidal deformability
 - grav. waves (mountains) \leftrightarrow crystalline structures

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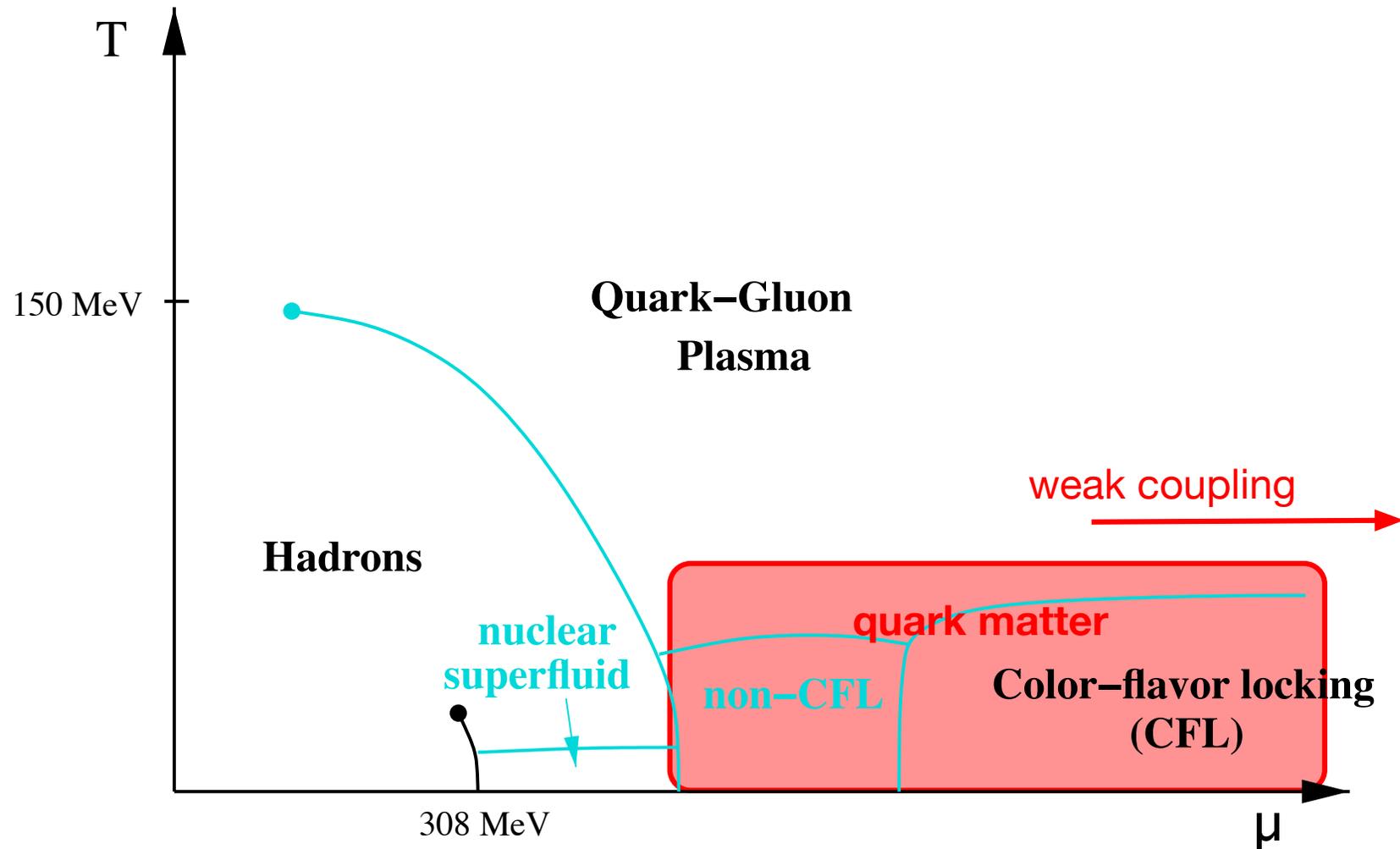
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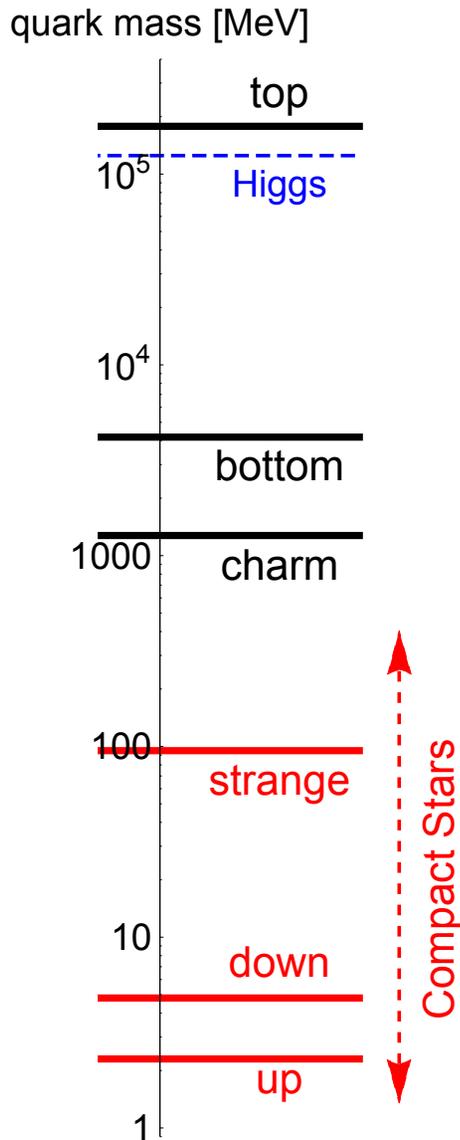
- specific heat
- neutrino emissivity
- bulk viscosity

Noninteracting quark matter

see Sec. 2.2 in [A. Schmitt, Lect. Notes Phys. 811, 1 \(2010\)](#)



Three-flavor quark matter



- quark chemical potential in compact stars
 $300 \text{ MeV} \lesssim \mu \lesssim 500 \text{ MeV}$

\Rightarrow three-flavor quark matter
 (ignore c,b,t)

- $0 \simeq m_u \simeq m_d \ll \mu$, but m_s not negligible
- remember electric charges:

$$q_u = \frac{2}{3}e, \quad q_d = q_s = -\frac{1}{3}e$$

Thermodynamics of free fermions

- pressure of fermions with mass m and spin $1/2$

$$P = -\epsilon + \mu n + Ts = 2T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left[1 + e^{-(E_k - \mu)/T} \right]$$

with chemical potential μ , temperature T , and

$$n = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} f_k \quad \text{number density}$$

$$\epsilon = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} E_k f_k \quad \text{energy density}$$

$$s = -2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} [(1 - f_k) \ln(1 - f_k) + f_k \ln f_k] \quad \text{entropy density}$$

and Fermi distribution and single-particle energy

$$f_k \equiv \frac{1}{e^{(E_k - \mu)/T} + 1}, \quad E_k = \sqrt{k^2 + m^2}$$

→ **Problems I**

Problems I: basic thermodynamic properties

Pressure for free fermions (upper sign) and bosons (lower sign):

$$P = \pm T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left[1 \pm e^{-(E_k - \mu)/T} \right], \quad E_k = \sqrt{k^2 + m^2}$$

1. Show that for fermions

$$s = \frac{\partial P}{\partial T} = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[(1 - f_k) \ln(1 - f_k) + f_k \ln f_k \right]$$

and derive the analogous expression for bosons

2. Derive expressions for the specific heat for bosons and fermions,

$$c_V = T \frac{\partial s}{\partial T}$$

and evaluate them

(a) for $T \gg m, \mu$ (fermions and bosons), using

$$\int_0^\infty dx \frac{x^4}{\cosh x + 1} = \frac{7\pi^4}{15}, \quad \int_0^\infty dx \frac{x^4}{\cosh x - 1} = \frac{8\pi^4}{15}$$

(b) for $T \ll \mu$ and $m = 0$ (only fermions), using

$$\int_0^\infty dx \frac{x^2}{\cosh x + 1} = \frac{\pi^2}{3}$$

Zero-temperature approximation

- compact stars: $T \ll \mu$
- $T = 0$:

$$f_k = \Theta(k_F - k)$$

with Fermi momentum

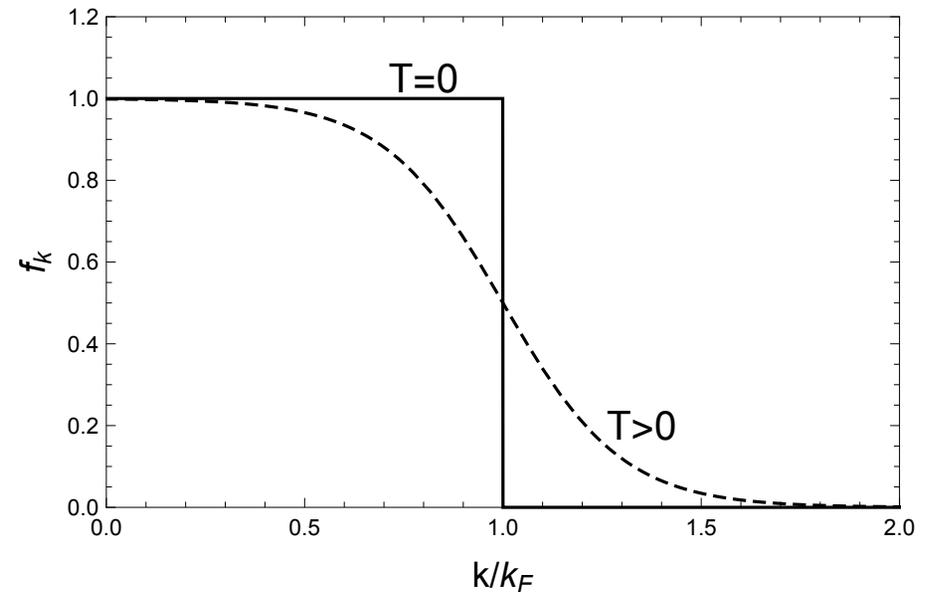
$$k_F = \sqrt{\mu^2 - m^2}$$

- analytic expressions for $T = 0$

$$n = \frac{1}{\pi^2} \int_0^{k_F} dk k^2 = \frac{k_F^3}{3\pi^2}$$

$$\epsilon = \frac{1}{\pi^2} \int_0^{k_F} dk k^2 \sqrt{k^2 + m^2} = \frac{1}{8\pi^2} \left[(2k_F^3 + m^2 k_F) \sqrt{k_F^2 + m^2} - m^4 \ln \frac{k_F + \sqrt{k_F^2 + m^2}}{m} \right]$$

$$P = \frac{1}{\pi^2} \int_0^{k_F} dk k^2 (\mu - \sqrt{k^2 + m^2}) = \frac{1}{24\pi^2} \left[(2k_F^3 - 3m^2 k_F) \sqrt{k_F^2 + m^2} + 3m^4 \ln \frac{k_F + \sqrt{k_F^2 + m^2}}{m} \right]$$



β -equilibrium and electric charge neutrality (page 1/2)

- **pure QCD**: quark chemical potentials μ_u, μ_d, μ_s independent
- **include weak interactions**: μ_u, μ_d, μ_s related through β -equilibrium

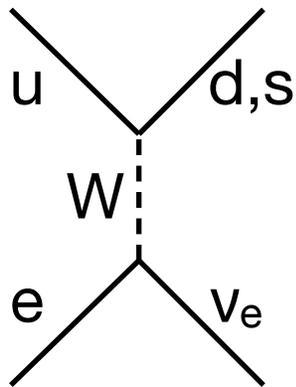
$$u + e \rightarrow d + \nu_e$$

$$u + e \rightarrow s + \nu_e$$

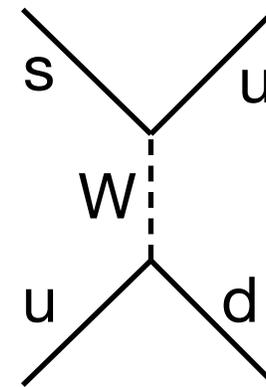
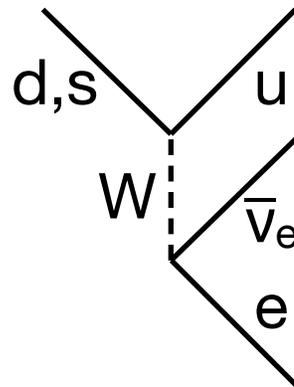
$$d \rightarrow u + e + \bar{\nu}_e$$

$$s \rightarrow u + e + \bar{\nu}_e$$

$$s+u \leftrightarrow d+u$$



leptonic



non-leptonic

β -equilibrium and electric charge neutrality (page 2/2)

- β -equilibrium

$$\mu_d = \mu_e + \mu_u, \quad \mu_s = \mu_e + \mu_u$$

(this automatically implies $\mu_d = \mu_s$)

- electric charge neutrality

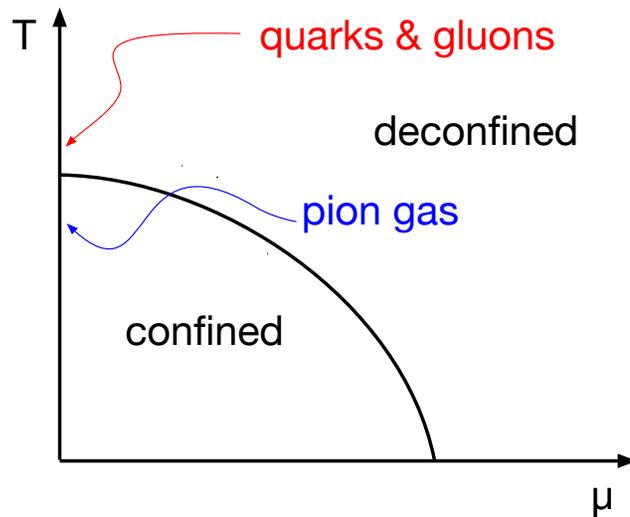
$$\sum_{f=u,d,s} q_f n_f - n_e = 0$$

(n_e electron density, q_f quark charges)

Bag model (page 1/2)

A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, PRD 9, 3471 (1974)

- for now consider $\mu = 0$ and nonzero T



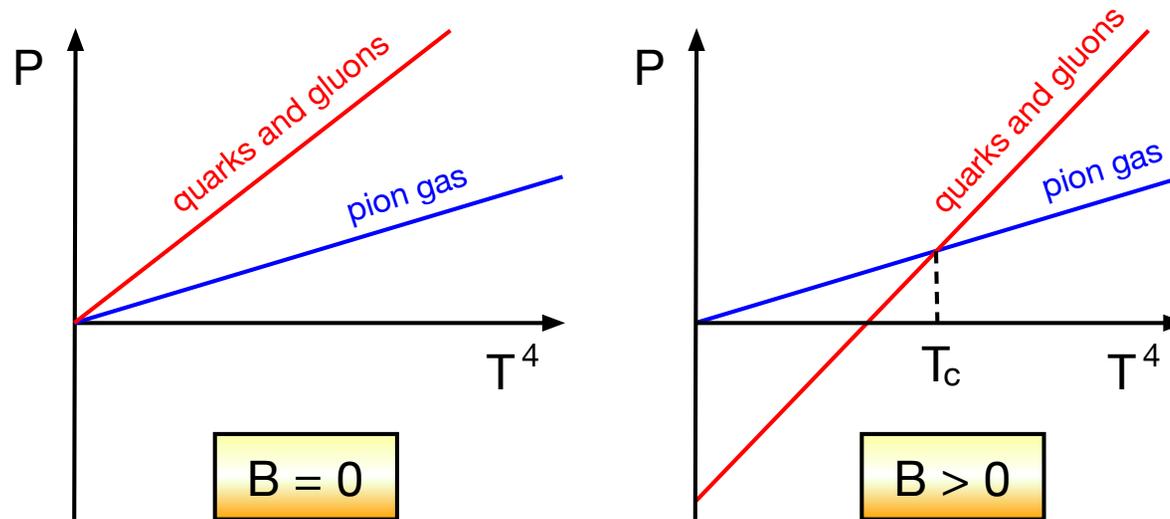
$$P_{\pi} = 3 \frac{\pi^2 T^4}{90}$$

$$P_{q,g} = 37 \frac{\pi^2 T^4}{90} - B$$

$$P_{\text{boson}} \simeq -T \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(1 - e^{-k/T}) = \frac{\pi^2 T^4}{90}$$

$$P_{\text{fermion}} \simeq T \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln(1 + e^{-k/T}) = \frac{7 \pi^2 T^4}{8 \cdot 90}$$

Bag model (page 2/2)



- without bag constant B : quarks and gluons “too favored”
- bag constant B is a (very crude!) model for confinement: pressure of the “bag” counterbalances microscopic pressure of quarks

$$P + B = \sum_f P_f, \quad \epsilon = \sum_f \epsilon_f + B$$

Strange quark matter hypothesis (page 1/4)

A. R. Bodmer, PRD 4, 1601 (1971); E. Witten, PRD 30, 272 (1984)

E. Farhi and R. L. Jaffe, PRD 30, 2379 (1984), PRD 30, 272 (1984)

- consider massless quarks (and $T = 0$, neglect electrons for now)

$$n_f = \frac{\mu_f^3}{\pi^2}, \quad \epsilon_f = \frac{3\mu_f^4}{4\pi^2}, \quad P_f = \frac{\mu_f^4}{4\pi^2} \quad \Rightarrow \quad P_f = \frac{\epsilon_f}{3}$$

and compute energy E per nucleon number A ,

$$\frac{E}{A} = \frac{\epsilon}{n_B}, \quad \left(\text{baryon density } n_B = \frac{1}{3} \sum_f n_f \right)$$

- at zero pressure, $P = 0$,

$$\frac{E}{A} = \frac{4B}{n_B}$$

Strange quark matter hypothesis (page 2/4)

	3-flavor quark matter (“strange quark matter”)	2-flavor quark matter
neutrality	$2n_u - n_d - n_s = 0$	$n_d = 2n_u$
chem. pot.	$\mu_u = \mu_d = \mu_s \equiv \mu$	$\mu_d = 2^{1/3} \mu_u$
E/A	$(4\pi^2)^{1/4} 3^{3/4} B^{1/4}$ $\simeq 829 \text{ MeV } B_{145}^{1/4}$	$(4\pi^2)^{1/4} (1 + 2^{4/3})^{3/4} B^{1/4}$ $\simeq 934 \text{ MeV } B_{145}^{1/4}$

$$B_{145}^{1/4} \equiv \frac{B^{1/4}}{145 \text{ MeV}}$$

Strange quark matter hypothesis (page 3/4)

- 3-flavor quark matter has lower energy than 2-flavor quark matter (additional Fermi sphere!)

$$\left. \frac{E}{A} \right|_{N_f=3} < \left. \frac{E}{A} \right|_{N_f=2}$$

- energy of 2-flavor quark matter must be larger than that of nuclear matter (since our world is made of nucleons, not quark matter)

$$\left. \frac{E}{A} \right|_{^{56}\text{Fe}} = \frac{56 m_N - 56 \cdot 8.8 \text{ MeV}}{56} = 930 \text{ MeV} < \left. \frac{E}{A} \right|_{N_f=2}$$

$$\Rightarrow B^{1/4} > 144.4 \text{ MeV}$$

Strange quark matter hypothesis (page 4/4)

- could 3-flavor quark matter be favored over nuclear matter?

$$\left. \frac{E}{A} \right|_{N_f=3} < \left. \frac{E}{A} \right|_{^{56}\text{Fe}} \quad \Rightarrow \quad B^{1/4} < 162.8 \text{ MeV}$$

→ if $145 \text{ MeV} < B^{1/4} < 162 \text{ MeV}$, 3-flavor quark matter is “absolutely stable” (at $P = 0$), while nuclear matter is metastable, (“strange quark matter hypothesis”)

- existence of ordinary nuclei does *not* rule out the hypothesis (need conversion of $\sim A$ up and down quarks into strange quarks)
- if the hypothesis is true:
 - strangelets could convert neutron stars into strange stars
 - if there are enough strangelets *every* neutron star should be converted, see however [A. Bauswein, et al., PRL 103, 011101 \(2009\)](#)

Equation of state (page 1/2)

- pressure (now taking into account m_s)

$$\sum_{i=u,d,s,e} P_i = \frac{\mu_u^4}{4\pi^2} + \frac{\mu_d^4}{4\pi^2} + \frac{3}{\pi^2} \int_0^{k_{F,s}} dk k^2 \left(\mu_s - \sqrt{k^2 + m_s^2} \right) + \frac{\mu_e^4}{12\pi^2}$$

with quark Fermi momenta $k_{F,u} \simeq \mu_u$, $k_{F,d} \simeq \mu_d$, $k_{F,s} = \sqrt{\mu_s^2 - m_s^2}$
and electron contribution $k_{F,e} \simeq \mu_e$

- write chemical potentials in terms of average quark chemical potential μ and μ_e (β -equilibrium)

$$\mu_u = \mu - \frac{2}{3}\mu_e, \quad \mu_d = \mu + \frac{1}{3}\mu_e, \quad \mu_s = \mu + \frac{1}{3}\mu_e$$

- solve charge neutrality

$$0 = \frac{\partial}{\partial \mu_e} \sum_{i=u,d,s,e} P_i = -\frac{2}{3}n_u + \frac{1}{3}n_d + \frac{1}{3}n_s + n_e$$

to lowest order in the strange quark mass

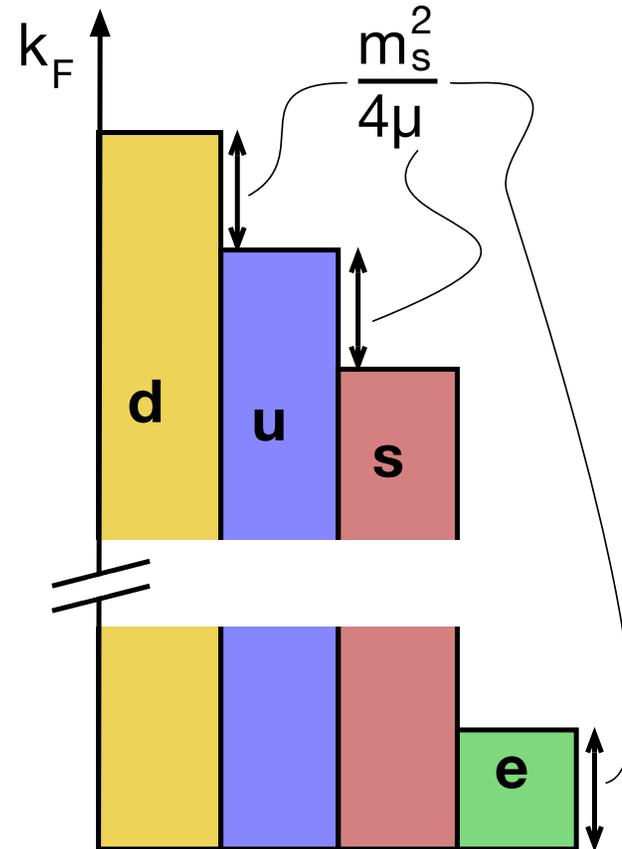
Equation of state (page 2/2)

$$\Rightarrow \mu_e \simeq \frac{m_s^2}{4\mu}$$

- equation of state

$$P \simeq \frac{\epsilon - 4B}{3} - \frac{\mu^2 m_s^2}{2\pi^2}$$

- splitting of Fermi surfaces
→ “stressed” Cooper pairing
[see lecture (5)]



Exercise: Show that the speed of sound c_s is, to lowest order in m_s ,

$$c_s^2 = \frac{\partial P}{\partial \epsilon} \simeq \frac{1}{3} \left(1 - \frac{m_s^2}{3\mu^2} \right)$$

Including interactions and Cooper pairing

- including interactions between (unpaired) quarks perturbatively
→ corrections in powers of α_s

G. Baym and S. A. Chin, PLB 62, 241 (1976)

B. A. Freedman and L. D. McLerran, PRD 16, 1169 (1977)

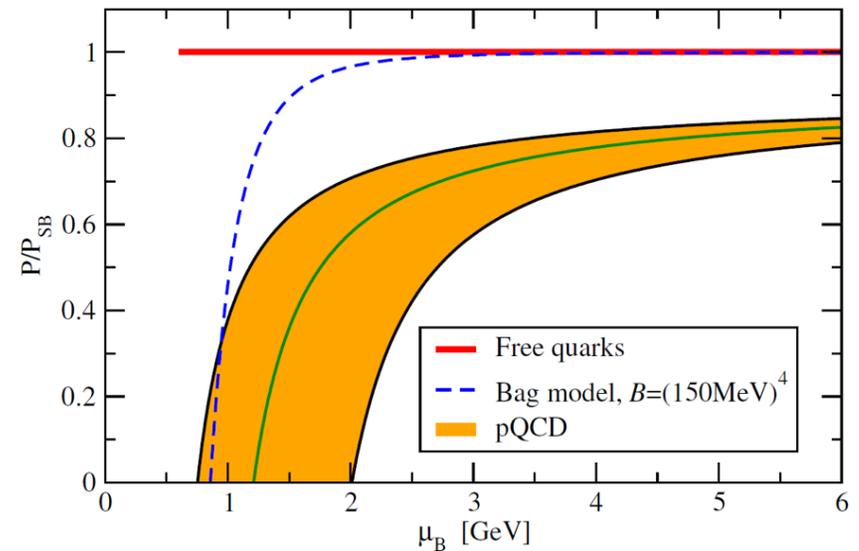
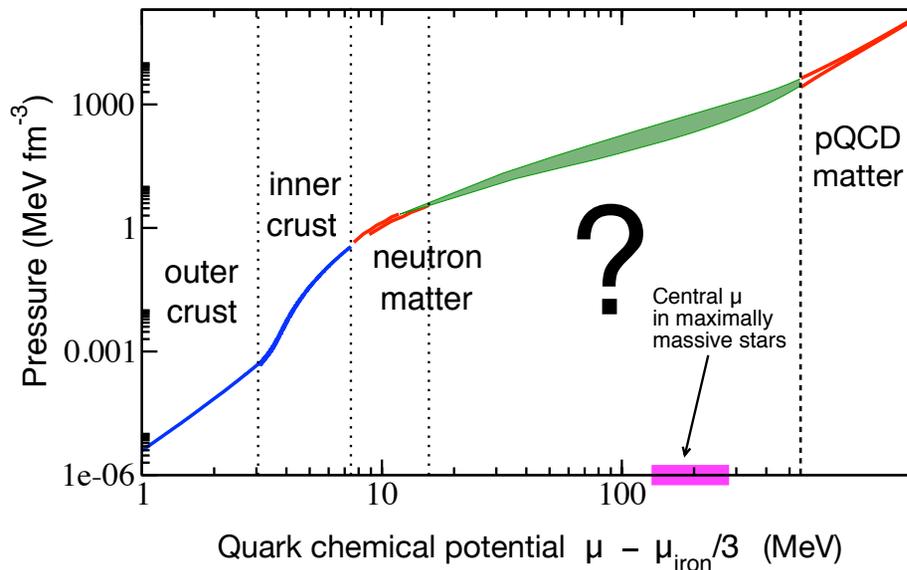
$$k_F = \mu \left(1 - \frac{2\alpha_s}{3\pi} \right)$$

- include energy gap Δ from Cooper pairing
to be discussed later

$$P \simeq \frac{3\mu^4}{4\pi^2} \left(1 - \frac{2\alpha_s}{\pi} \right) - \frac{3\mu^2}{4\pi^2} (m_s^2 - 4\Delta^2) - B$$

Recent studies of perturbative quark matter

- second-order corrections in α_s
A. Kurkela, P. Romatschke, A. Vuorinen
PRD 81, 105021 (2010)
- large corrections to bag model at all relevant densities!



- connect nuclear matter (low density) to perturbative QCD (high density)
A. Kurkela, E. S. Fraga, J. Schaffner-Bielich, A. Vuorinen, Astrophys. J. 789, 127 (2014)

Summary: unpaired quark matter

- strange quark matter hypothesis:

observations do not exclude the possibility that strange quark matter is the true ground state at zero pressure

- zero quark masses:

quark matter is particularly symmetric: $n_u = n_d = n_s$ (and no electrons)

- nonzero strange mass:

β -equilibrated, electrically neutral quark matter has $n_d > n_u > n_s$
(and nonzero n_e)

- perturbative results can be used to constrain equation of state at moderate densities

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- basic thermodynamics → Problems I
- strange quark matter hypothesis
- equation of state

(3) Dense nuclear matter (pp 53 - 64)

- free nuclear matter → Problems II
- field-theoretical model
- saturation density and binding energy

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- field-theoretical approach (sketch)
- fermionic excitations → Problems III
- solving the gap equation

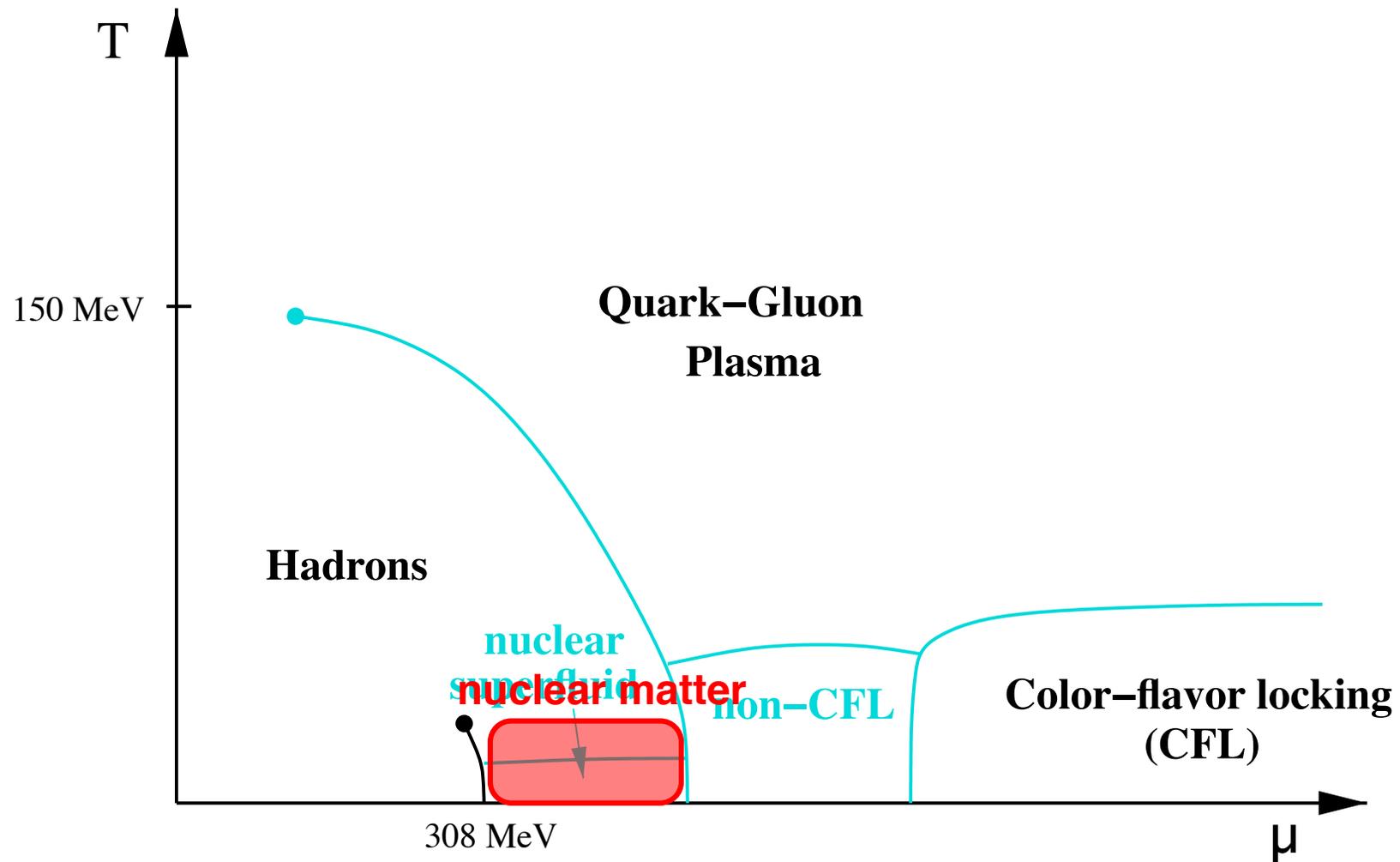
(5) Color superconductivity (pp 83 - 106)

- color-flavor locked (CFL) quark matter
- stressed pairing and non-CFL color superconductors

(6) Transport in dense matter (pp 108 - 150)

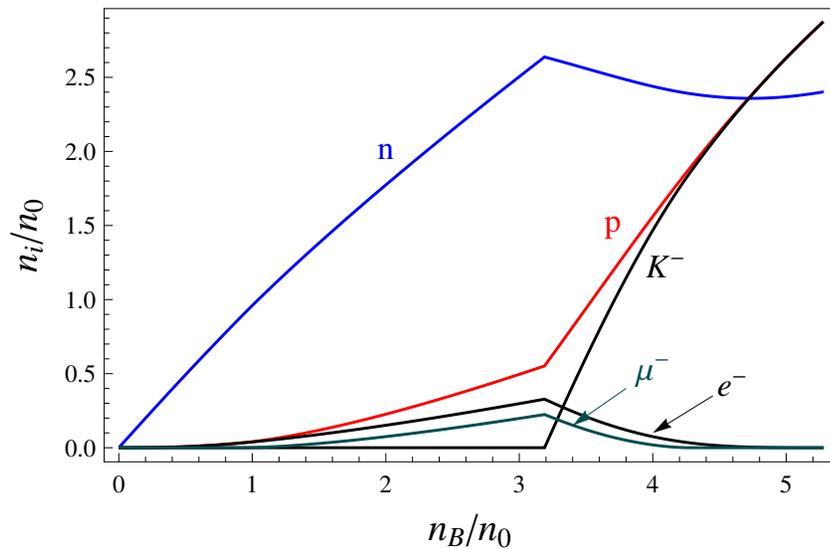
- specific heat
- neutrino emissivity
- bulk viscosity

Nuclear matter

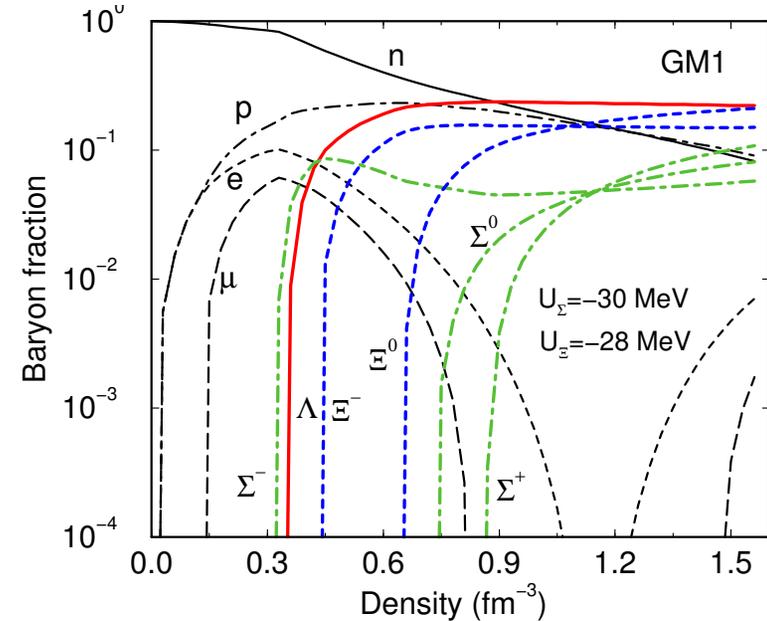


Nuclear matter

- "ordinary" nuclear matter: neutrons (n), protons (p), electrons (e)
non-interacting nuclear matter → **Problems II**
- more exotic phases possible at high density:
kaon condensation, hyperons, ...



A. Schmitt, Lect. Notes Phys. 811, 1 (2010)



J. Schaffner-Bielich, NPA 835, 279 (2010)

Problems II: non-interacting nuclear matter

1. Show that electrically neutral, non-interacting nuclear matter (n,p,e) at zero temperature and in β -equilibrium (assuming $\mu_\nu \simeq 0$)
 - (a) must contain protons in general, $n_p \neq 0$
 - (b) has a proton fraction $\frac{n_p}{n_B} = \frac{1}{9}$ in the ultra-relativistic limit
 - (c) obeys $\frac{n_p}{n_B} < \frac{1}{9}$ except for very small n_B (requires numerical evaluation)

2. Show that non-interacting, pure neutron matter in the non-relativistic limit has a "polytropic" equation of state,

$$P(\epsilon) = K\epsilon^p,$$

and compute K and p .

Basic properties of (interacting) nuclear matter

see Sec. 3.1 in A. Schmitt, Lect. Notes Phys. 811, 1 (2010)

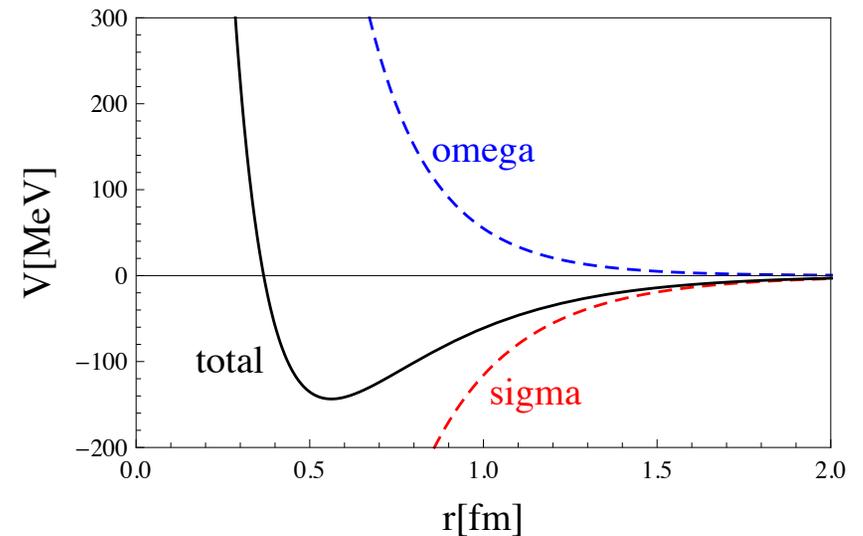
- relativistic, symmetric nuclear matter ("Walecka model")

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_N + \mu\gamma^0)\psi + g_\sigma\bar{\psi}\sigma\psi - g_\omega\bar{\psi}\gamma^\mu\omega_\mu\psi$$

$$+ \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu$$

(with μ introduced through $\mathcal{H} - \mu\mathcal{N}$)

- two parameters
(to be fitted later): g_σ , g_ω
- attractive and repulsive interaction through
sigma and omega exchange



Mean-field approximation

- replace meson fields by their vevs (space-time independent)

$$\sigma \rightarrow \langle \sigma \rangle, \quad \omega_\mu \rightarrow \langle \omega_0 \rangle \delta_{0\mu}$$

- mean-field Lagrangian

$$\mathcal{L}_{\text{mean-field}} = \bar{\psi} (i\gamma^\mu \partial_\mu - m_N^* + \mu^* \gamma_0) \psi - \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2} m_\omega^2 \langle \omega_0 \rangle^2$$

with

$$m_N^* \equiv m_N - g_\sigma \langle \sigma \rangle, \quad \mu^* \equiv \mu - g_\omega \langle \omega_0 \rangle$$

→ looks like non-interacting Lagrangian: interaction absorbed
in **effective mass** m_N^* and **effective chemical potential** μ^*

Pressure from partition function (page 1/2)

- partition function

$$\begin{aligned}
 Z &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\sigma \mathcal{D}\omega \exp \int_X \mathcal{L} \\
 &= e^{\frac{V}{T}(-\frac{1}{2}m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2}m_\omega^2 \langle \omega_0 \rangle^2)} \underbrace{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \int_X \bar{\psi} (i\gamma^\mu \partial_\mu - m_N^* + \mu^* \gamma_0) \psi}_{\det_{\text{Dirac}, K} \frac{-\gamma^\mu K_\mu - \gamma_0 \mu^* + m_N^*}{T}}
 \end{aligned}$$

with

$$\int_X \equiv \int_0^\beta d\tau \int d^3x, \quad X^\mu = (-i\tau, \mathbf{x}), \quad K^\mu = (-i\omega_n, \mathbf{k})$$

Thermal field theory: $Z = \text{Tr} e^{-\beta \hat{H}} = \int d\phi \langle \phi | e^{-\beta \hat{H}} | \phi \rangle \leftrightarrow \int d\phi \langle \phi | e^{-it_f \hat{H}} | \phi \rangle$

→ "imaginary time" τ and periodic boundary conditions for ϕ

(anti-periodic for fermions)

→ discrete energies → Matsubara frequencies $\omega_n = (2n + 1)\pi T$ (fermionic)

see A. Vuorinen's lecture

Pressure from partition function (page 2/2)

- pressure

$$P = \frac{T}{V} \ln Z$$

- 4-momentum sum = sum over Matsubara frequencies & 3-momentum integral

$$\frac{T}{V} \ln \det_K \rightarrow \frac{T}{V} \sum_K \ln \rightarrow T \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln$$

- determinant over Dirac space & summation over Matsubara sum & ignore "vacuum contribution" & neglect anti-baryons

$$P = -\frac{1}{2}m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2}m_\omega^2 \langle \omega_0 \rangle^2 + \underbrace{4T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln (1 + e^{-(E_k - \mu^*)/T})}_{P_N}$$

with $E_k = \sqrt{k^2 + (m_N^*)^2}$

Stationarity equations

- compute meson vevs from

$$0 = \frac{\partial P}{\partial \langle \sigma \rangle} = -m_\sigma^2 \langle \sigma \rangle - g_\sigma \frac{\partial P_N}{\partial m_N^*} \equiv -m_\sigma^2 \langle \sigma \rangle + g_\sigma n_s$$

$$0 = \frac{\partial P}{\partial \langle \omega_0 \rangle} = m_\omega^2 \langle \omega_0 \rangle - g_\omega \frac{\partial P_N}{\partial \mu^*} \equiv m_\omega^2 \langle \omega_0 \rangle - g_\omega n_B$$

- for given n_B the equations decouple and we need to solve

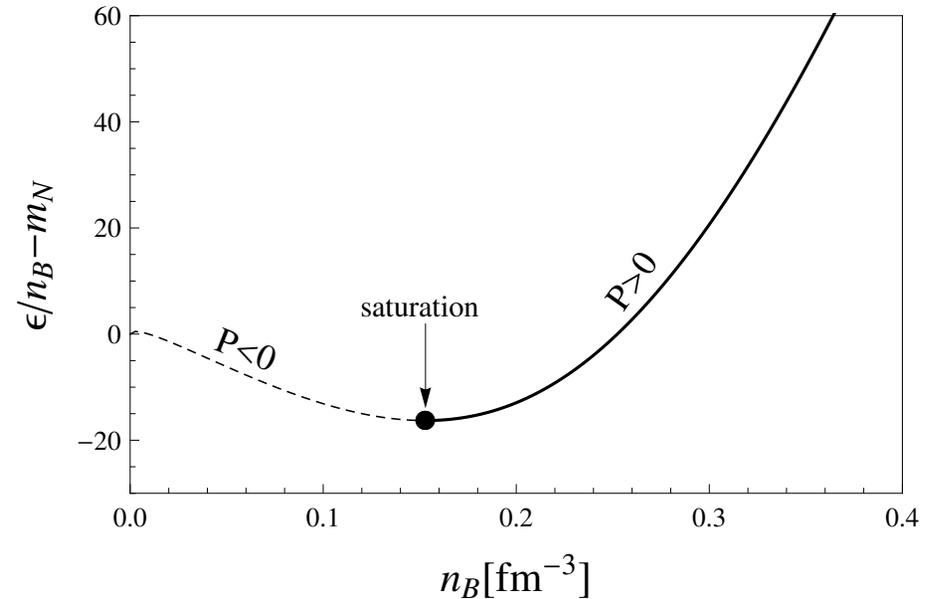
$$m_N^* = m_N - \frac{g_\sigma^2}{m_\sigma^2} n_s$$

for m_N^*

Saturation density and binding energy

- \exists minimum of $\epsilon/n_B = E/A$
at "saturation density"

$$n_0 \simeq 0.15 \text{ fm}^{-3}$$

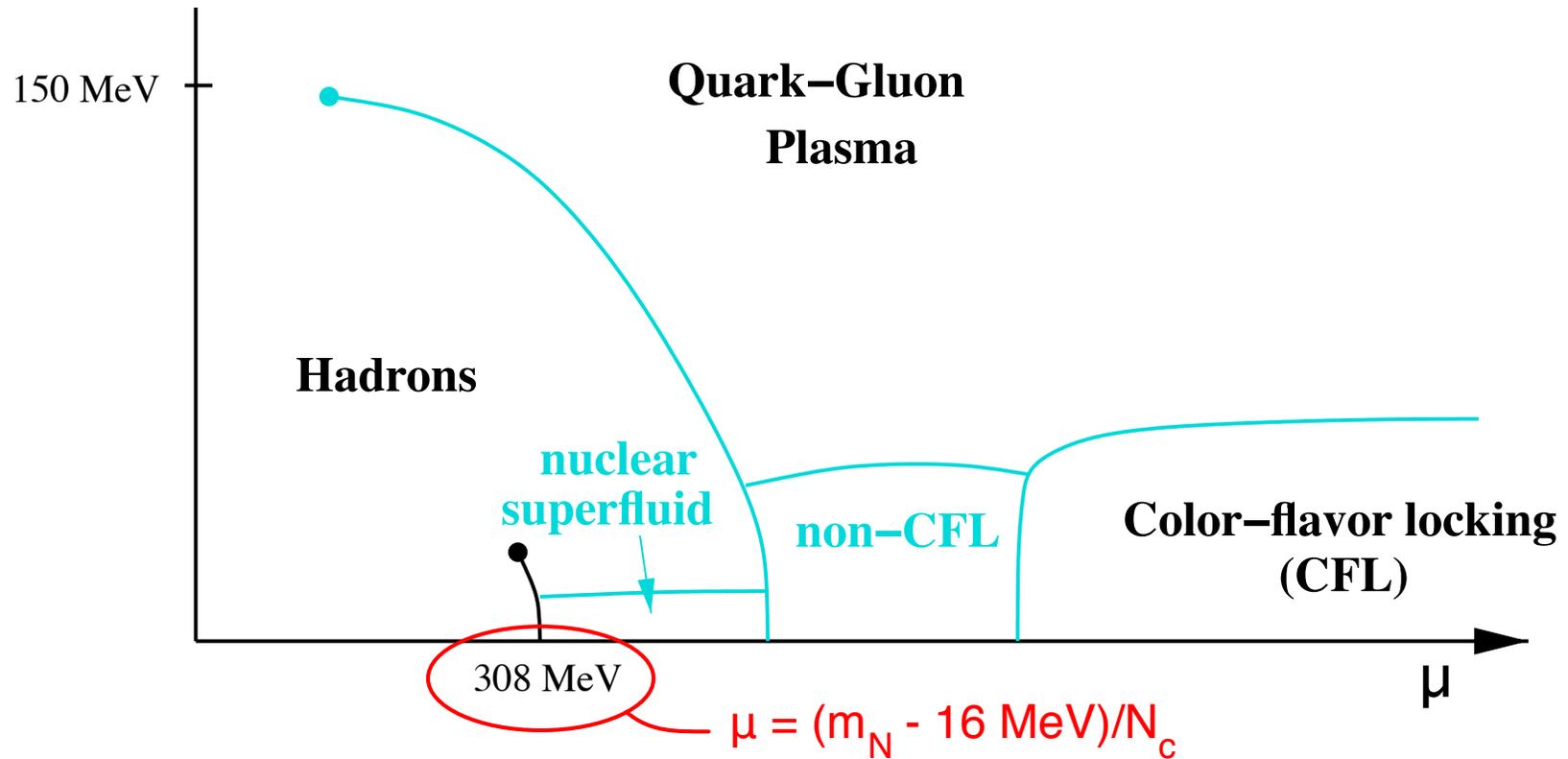


- semi-empirical energy

$$E = -a_1 A + \underbrace{a_2 A^{3/2}}_{\text{surface}} + \underbrace{a_3 \frac{Z^2}{A^{1/3}}}_{\text{Coulomb}} + \underbrace{a_4 \frac{(A - 2Z)^2}{A}}_{\text{(a)symmetry}}$$

- symmetric, infinite nuclear matter without EM has
binding energy $E_0 \equiv E/A = -a_1 = -16 \text{ MeV}$
- g_σ and g_ω fitted to reproduce n_0 and E_0

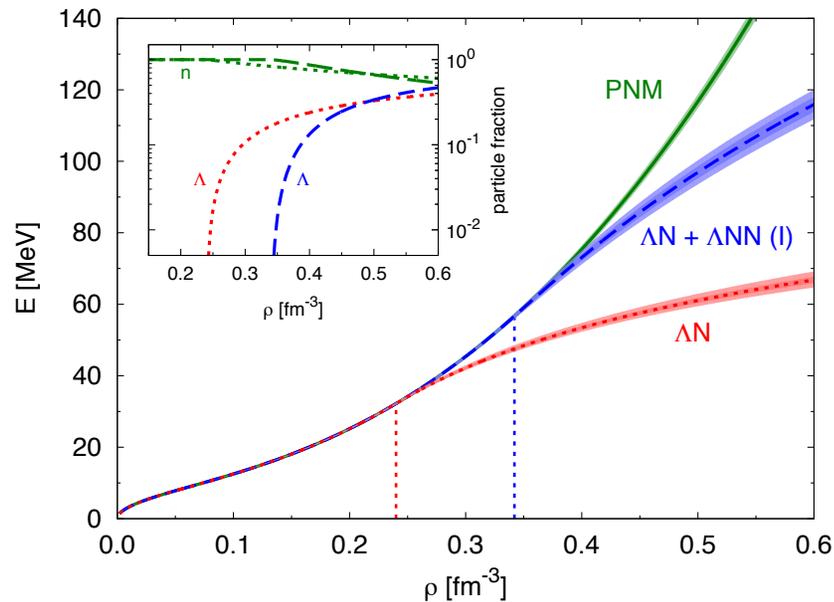
Saturation density in the QCD phase diagram



- $\mu_B < m_N - E_0$: vacuum with $P = 0$ and $n_B = 0$
- $\mu_B = m_N - E_0$: first-order phase transition to nuclear matter with $P = 0$ and $n_B = n_0$
- $\mu_B > m_N - E_0$: nuclear matter with $P > 0$ and $n_B > n_0$

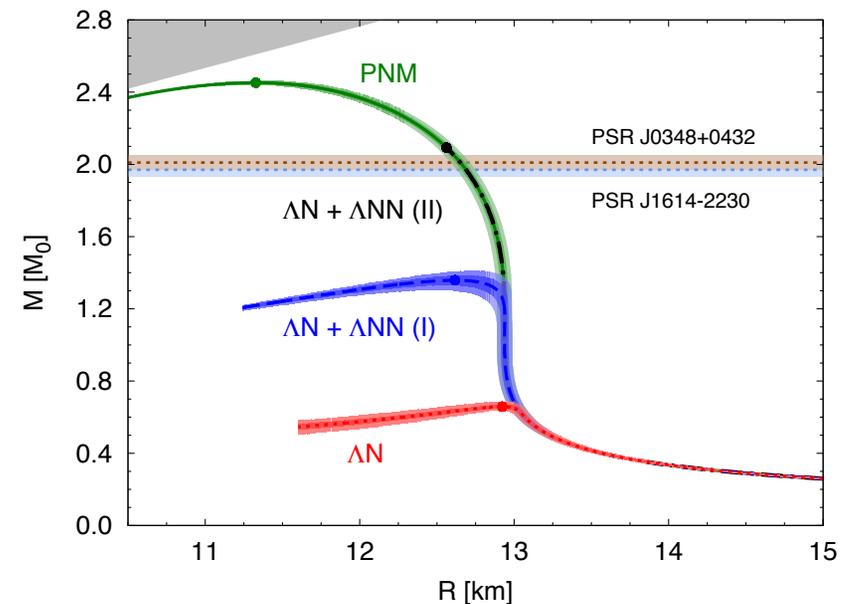
Current research (example 1/2): nuclear/hyperonic matter

D. Lonardonì, A. Lovato, S. Gandolfi and F. Pederiva, PRL 114, 092301 (2015)



- mass/radius relations
- no hyperons for " $\Lambda\text{N} + \Lambda\text{NN}$ (II)"

- PNM = pure neutron matter
- ΛN = two-body Λ -nucleon int.
- ΛNN = three-body Λ -nucleon int.
- different interactions lead to (very) different high-density EoSs

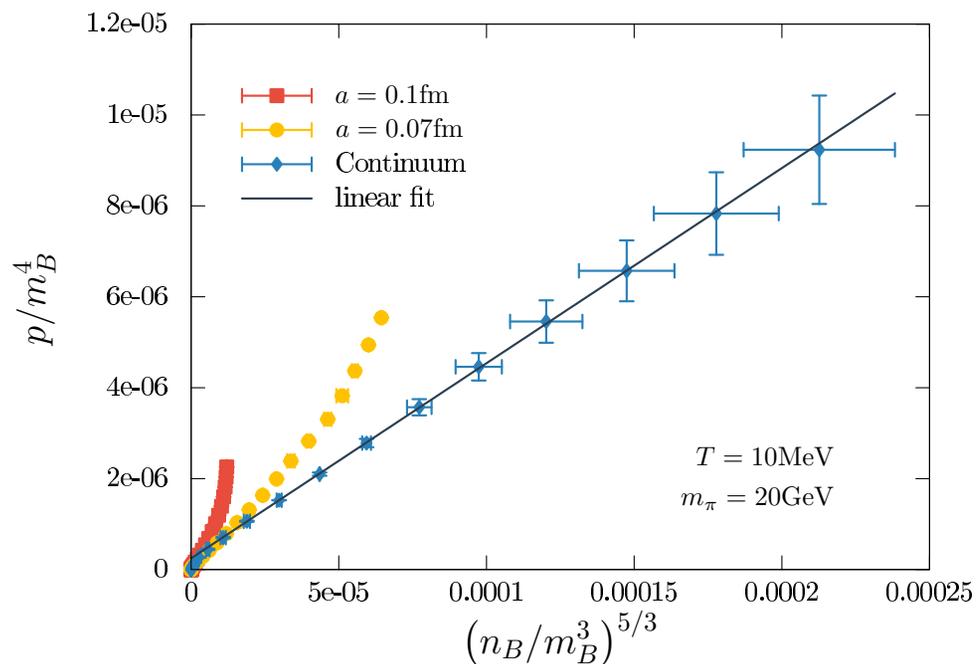


Current research (example 2/2): nuclear matter on the lattice

J. Glesaaen, M. Neuman and O. Philipsen, JHEP 1603, 100 (2016)

- lattice QCD: plagued by the "sign problem" at nonzero μ
- circumvent problem by strong-coupling expansion with (very!) heavy quarks (for other approaches see G. Aarts' lectures)

- baryon onset is seen
- nuclear matter seems to follow $P \propto \epsilon^{5/3}$ (see Problems II)



Summary: nuclear matter

- neutral nuclear matter in β -equilibrium is neutron-rich
→ "neutron star"
- symmetric nuclear matter has a "saturation density" n_0
and a "binding energy" E_0
- as a consequence, there is a first-order baryon onset
(liquid-gas transition) in the QCD phase diagram
- neutron star densities allow for "exotic" matter such as hyperons
- nuclear interactions at very high densities are poorly constrained by experiments
(hyperon-nucleon interaction even more so)
(and they are currently not possible to compute from first principles)

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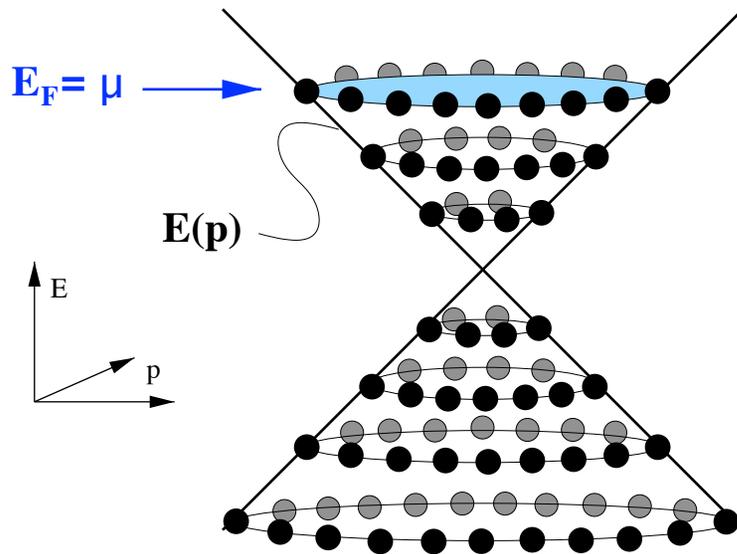
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- neutrino emissivity
- bulk viscosity

Cooper pairing of fermions



- free energy $\Omega = E - \mu N$
- no interactions: add fermion at $E = \mu$ without cost
- attractive interaction: add pair with gain
- pairs condense
→ “Cooper pairing”

This Bardeen-Cooper-Schrieffer (BCS) argument holds for electrons in a metal, ^3He atoms, [nucleons](#), [quarks](#), ...

Cooper pairing leads to superfluidity/superconductivity

superfluidity	superconductivity
frictionless “charge” transport through Cooper pair condensate (Bose-Einstein condensate in bosonic system); single fermions “gapped” <i>see below</i>	
spontaneous breaking of global symmetry (Cooper pairs neutral)	spontaneous breaking of local symmetry (Cooper pairs charged)
Goldstone mode (“phonon”)	Meissner effect (magnetic screening mass for gauge boson)

- Cooper pairing of quarks → **superfluidity** (baryon number charge) and/or **electromagnetic superconductivity** and/or **color superconductivity**

Cooper pairing in field theory (brief sketch, page 1/3)

for more details see chapter 5 of [A. Schmitt, Lect. Notes Phys. 888, 1 \(2015\)](#)

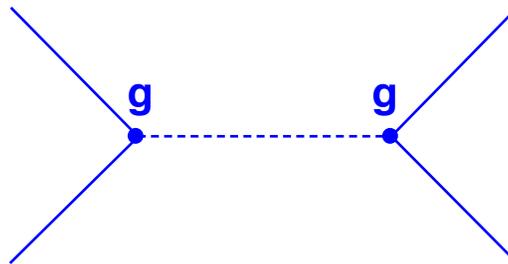
- fermions ψ plus attractive interaction mediated by bosons φ

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu + \gamma^0\mu - m)\psi - g\bar{\psi}\psi\varphi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2$$

(in QCD: $-g\bar{\psi}_\alpha\gamma^\mu T_a^{\alpha\beta}\psi_\beta A_\mu^a$)

- integrate out the bosonic fields to obtain fermion action

$$S = \int_{x,y} \left[\bar{\psi}(x)G_0^{-1}(x,y)\psi(y) + \frac{g^2}{2}\bar{\psi}(x)\psi(x)D(x,y)\bar{\psi}(y)\psi(y) \right]$$



Cooper pairing in field theory (brief sketch, page 2/3)

- mean field approximation (schematically)
 - write $\psi\psi = \langle\psi\psi\rangle + \text{fluctuations}$
 - drop terms quadratic in fluctuations: $g^2\psi\psi D\psi\psi \rightarrow \psi\Phi\psi$
with **gap matrix** $\Phi \rightarrow$ action becomes quadratic in the fields
- inverse propagator in "Nambu-Gorkov space"
(since $\langle\psi\psi\rangle$ really is $\langle\psi_C\bar{\psi}\rangle$)

$$\mathcal{S}^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Phi^- \\ \Phi^+ & [G_0^-]^{-1} \end{pmatrix}, \quad [G_0^\pm]^{-1} = \gamma^\mu K_\mu \pm \gamma^0 \mu - m$$

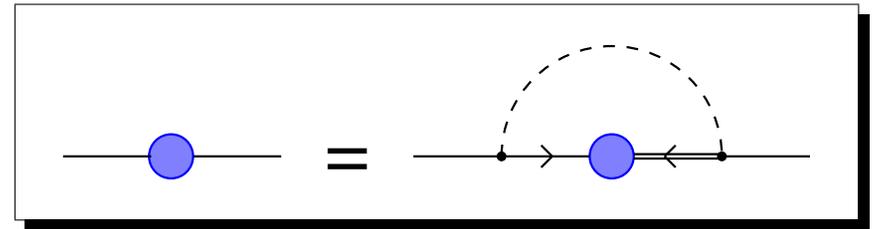
- mixing of particle and hole states through **gap matrix** Φ
- **anomalous propagators** F^\pm

$$\mathcal{S} = \begin{pmatrix} G^+ & F^- \\ F^+ & G^- \end{pmatrix}$$

Cooper pairing in field theory (brief sketch, page 3/3)

- gap equation (derived for instance by minimizing thermodynamic potential with respect to gap)

$$\Phi^+(P) = -g^2 \frac{T}{V} \sum_K D(P-K) F^+(K)$$



- ansatz for gap matrix

$$\Phi^\pm(P) = \pm \Delta(P) \gamma^5$$

→ even-parity, spin-singlet pairing, where fermions of the same chirality form Cooper pairs

for all possible Dirac structures, see D. Bailin and A. Love, Phys. Rept. 107, 325 (1984);
R. D. Pisarski and D. H. Rischke, PRD 60, 094013 (1999)

Quasi-particle excitations

- quasi-particle dispersion

→ **Problems III**

$$\epsilon_k^e \equiv \sqrt{(\mu - ek)^2 + \Delta^2}$$

- 4 poles of propagator $k_0 = \pm\epsilon_k^e$:

– upper sign:

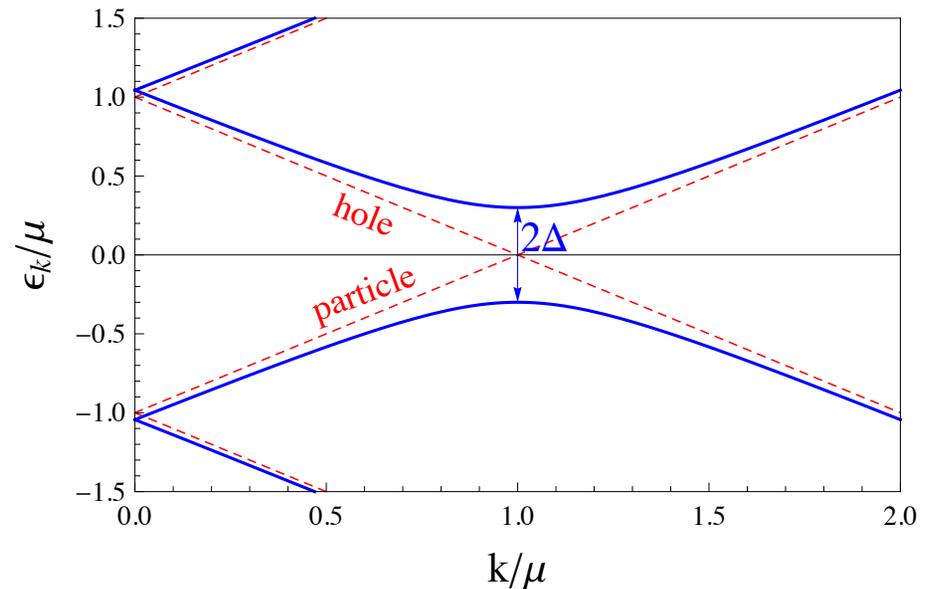
quasi-particles ($e = +$)

quasi-antiparticles ($e = -$)

– lower sign:

quasi-holes ($e = +$)

quasi-anti-holes ($e = -$)



- Δ energy gap in the quasi-particle spectrum
- quasi-particles are (k -dependent) mixtures of particles and holes

Problems III: fermionic excitations in a superfluid

1. Verify that $G^\pm = ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1}$ and $F^\pm = -G_0^\mp \Phi^\pm G^\pm$
2. Verify that in the ultra-relativistic limit ($m = 0$)

(a)

$$[G_0^\pm]^{-1} = \sum_{e=\pm} [k_0 \pm (\mu - ek)] \gamma^0 \Lambda_k^{\pm e}$$

(b)

$$G_0^\pm = \sum_e \frac{\gamma^0 \Lambda_k^{\mp e}}{k_0 \pm (\mu - ek)}$$

with the energy projectors $\Lambda_k^e \equiv \frac{1}{2} (1 + e\gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}})$. *Hint for (b): show that Λ_k^+ and Λ_k^- form a complete set of orthogonal projectors.*

3. With $\Phi^\pm = \pm\Delta\gamma^5$, show that the poles of G^\pm (for $m = 0$) are given by

$$k_0 = \pm\epsilon_k^e, \quad \epsilon_k^e = \sqrt{(\mu - ek)^2 + \Delta^2},$$

i.e., Δ is an "energy gap". Make use of the results of 1 and 2.

Gap equation with pointlike interaction (page 1/2)

- general gap equation (see above)

$$\Phi^+(P) = -g^2 \frac{T}{V} \sum_K D(P-K) F^+(K)$$

- gap equation with our ansatz for gap matrix

$$\Delta(P)\gamma^5 = -g^2 \frac{T}{V} \sum_K D(P-K) \frac{\Delta(K)\gamma^5 \Lambda_k^-}{k_0^2 - \epsilon_k^2}$$

(neglect antiparticle contribution, abbreviate $\epsilon_k \equiv \epsilon_k^+$)

- simplest case: pointlike interaction,

$$D^{-1}(Q) = -Q^2 + M^2 \simeq M^2$$

Gap equation with pointlike interaction (page 2/2)

- multiply by γ^5 , take trace, and perform Matsubara sum,

$$\Delta = G \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Delta}{2\epsilon_k} \tanh \frac{\epsilon_k}{2T}$$

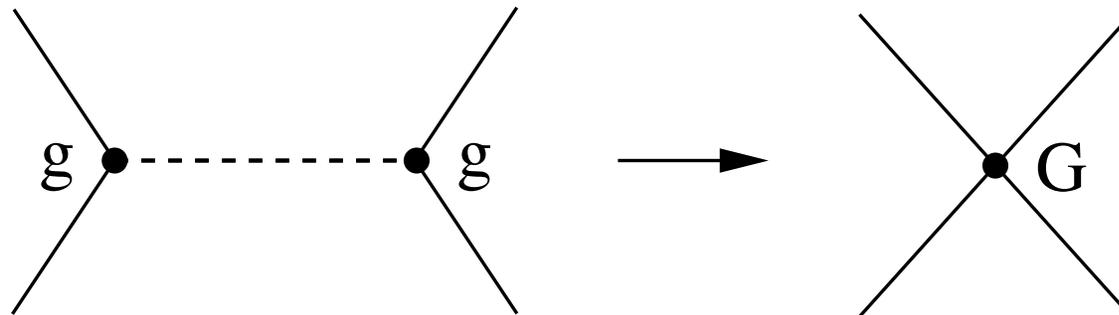
with (dimensionful) coupling constant

$$G = \frac{g^2}{2M^2}$$

→ Nambu-Jona-Lasinio(NJL) model

Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); Phys. Rev. 124, 246 (1961)

application to dense quark matter M. Buballa, Phys. Rept. 407, 205 (2005)



Zero-temperature solution (page 1/2)

- assume interaction is nonzero only in a small vicinity around the Fermi surface $[\mu - \delta, \mu + \delta]$, and $\Delta_0 \equiv \Delta(T = 0) \ll \delta \ll \mu$,

$$\Delta_0 \simeq \frac{\mu^2 G}{2\pi^2} \int_0^\delta d\xi \frac{\Delta_0}{\sqrt{\xi^2 + \Delta_0^2}}$$

$$(dk k^2 \simeq \mu^2 dk, \xi = k - \mu)$$

- after dividing by Δ_0 (trivial solution):
 Δ_0 nonzero for any coupling $G > 0$, no matter how small
(due to logarithmic divergence at Fermi surface)
→ instability towards Cooper pairing
- formal way of stating the instability: dimensional reduction of the dynamics of the system from 3+1 to 1+1 dimensions

Zero-temperature solution (page 2/2)

- compute Δ_0 with the help of

$$\int \frac{d\xi}{\sqrt{\xi^2 + \Delta_0^2}} = \ln \left[2 \left(\xi + \sqrt{\xi^2 + \Delta_0^2} \right) \right]$$

⇒ BCS gap

$$\Delta_0 \simeq 2\delta \exp\left(-\frac{2\pi^2}{G\mu^2}\right)$$

- weak coupling: gap is exponentially suppressed
- non-perturbative result (no Taylor expansion around $G = 0$)

Critical temperature (page 1/2)

- gap equation (see above)

$$\Delta = G \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Delta}{2\epsilon_k} \tanh \frac{\epsilon_k}{2T}$$

- divide by Δ , then take $\Delta \rightarrow 0$,

$$1 \simeq \frac{G\mu^2}{2\pi^2} \int_0^\delta \frac{d\xi}{\xi} \tanh \frac{\xi}{2T_c}$$

- with the new integration variable $z = \xi/(2T_c)$ and integration by parts

$$\frac{2\pi^2}{G\mu^2} = \ln z \tanh z \Big|_0^{\delta/(2T_c)} - \int_0^{\delta/(2T_c)} dz \frac{\ln z}{\cosh^2 z} \simeq \ln \frac{\delta}{2T_c} - \underbrace{\int_0^\infty dz \frac{\ln z}{\cosh^2 z}}_{-\gamma + \ln \frac{\pi}{4}}$$

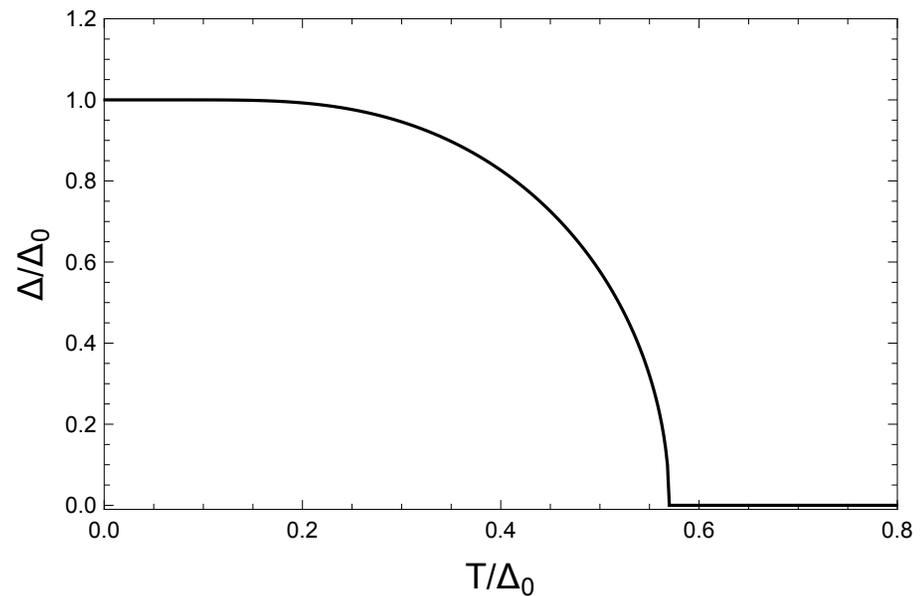
($\gamma \simeq 0.577$ Euler-Mascheroni constant)

Critical temperature (page 2/2)

- express T_c in terms of zero-temperature gap

$$T_c = \frac{e^\gamma}{\pi} \Delta_0 \simeq 0.57 \Delta_0$$

- general G and/or T :
numerical evaluation of gap
equation \rightarrow *exercise*



QCD gap equation (page 1/2)

for more details see chapter 4.3 in [A. Schmitt, Lect. Notes Phys. 811, 1 \(2010\)](#)

- gap equation

$$\Phi^+(P) = g^2 \frac{T}{V} \sum_K \gamma^\mu T_a^T F^+(K) \gamma^\nu T_b D_{\mu\nu}^{ab}(P-K)$$

- same structure, but difference in
 - Φ^+ matrix in color-flavor space
 - many different order parameters possible
 - gluon propagator $D_{\mu\nu}^{ab}(P-K)$
 - results in qualitative different result for the gap

QCD gap equation (page 2/2)

- zero-temperature gap at the Fermi surface

D. T. Son, PRD 59, 094019 (1999)

R. D. Pisarski and D. H. Rischke, PRD 61, 074017 (2000)

$$\Delta_0 \simeq 2b\mu \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

with $b \equiv 256\pi^4[2/(N_f g^2)]^{5/2}$

- notice g (not g^2) in exponential
- estimate magnitude of gap at compact star densities:
 $\mu \simeq 400 \text{ MeV} \rightarrow g \simeq 3.5 \rightarrow \Delta_0 \simeq 10 \text{ MeV} \gg T_{\text{compact star}}$
(including subleading effect not discussed here)
- compare to NJL result $\Delta_0 \simeq (10 - 100) \text{ MeV}$
 \rightarrow agreement despite completely different approach

Summary: Cooper pairing

- fermionic system at sufficiently low temperatures and (arbitrarily small!) attractive interaction shows Cooper pair condensation
- Cooper pair condensation → gap in the quasi-fermion spectrum
- Cooper pairing is expected to exist in compact stars in
 - nuclear matter: neutron superfluid, proton superconductor
(+ possibly hyperon pairing)
 - quark matter: color superconductor(phenomenological "evidence": glitches, cooling)

Outline

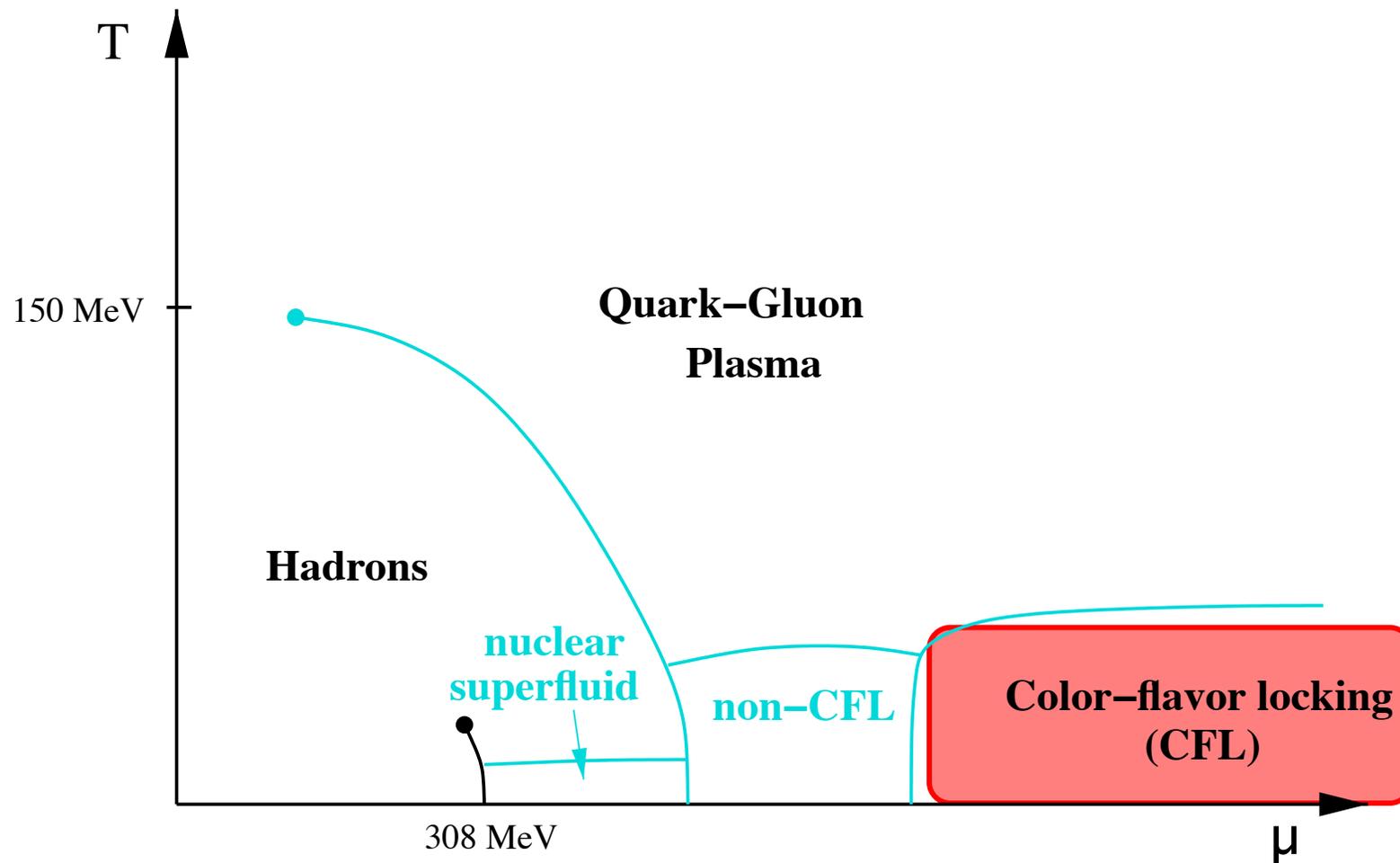
- (1) Introduction and overview (pp 6 - 31)
 - general remarks
 - dense matter in the QCD phase diagram
 - some selected astrophysical observations
- (2) Dense quark matter (pp 33 - 50)
 - basic thermodynamics → Problems I
 - strange quark matter hypothesis
 - equation of state
- (3) Dense nuclear matter (pp 53 - 64)
 - free nuclear matter → Problems II
 - field-theoretical model
 - saturation density and binding energy
- (4) Cooper pairing in dense matter (pp 66 - 81)
 - field-theoretical approach (sketch)
 - fermionic excitations → Problems III
 - solving the gap equation
- (5) Color superconductivity (pp 83 - 106)
 - color-flavor locked (CFL) quark matter
 - stressed pairing and non-CFL color superconductors
- (6) Transport in dense matter (pp 108 - 150)
 - specific heat
 - neutrino emissivity
 - bulk viscosity

Color-flavor locked (CFL) quark matter

M. Alford, K. Rajagopal, F. Wilczek, NPB 537, 443 (1999)

for a review of CFL and other color superconductors,

see M. Alford, K. Rajagopal, T. Schäfer, A. Schmitt, RMP 80, 1455 (2008)



CFL order parameter

- three-flavor quark matter: gap matrix in color-flavor space

$$\Phi^+ = \Delta \gamma_5 \mathcal{M}$$

- general (spin-singlet pairing)

$$\mathcal{M} \in [\bar{\mathbf{3}}]_c^a \otimes [\bar{\mathbf{3}}]_f^a, \quad \mathcal{M}_{ij}^{\alpha\beta} = \phi_A^B \epsilon^{\alpha\beta A} \epsilon_{ijB}$$

from $SU(3)_c$: $[\mathbf{3}]_c \otimes [\mathbf{3}]_c = [\bar{\mathbf{3}}]_c^a \oplus [\mathbf{6}]_c^s$ (attractive channel)

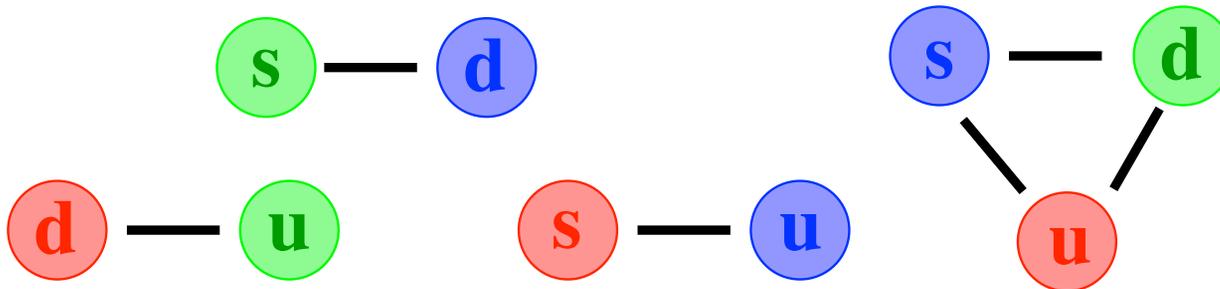
and $SU(3)_f$: $[\mathbf{3}]_f \otimes [\mathbf{3}]_f = [\bar{\mathbf{3}}]_f^a \oplus [\mathbf{6}]_f^s$ (overall antisymmetry)

- color-flavor locked order parameter

$$\phi_A^B = \delta_A^B$$

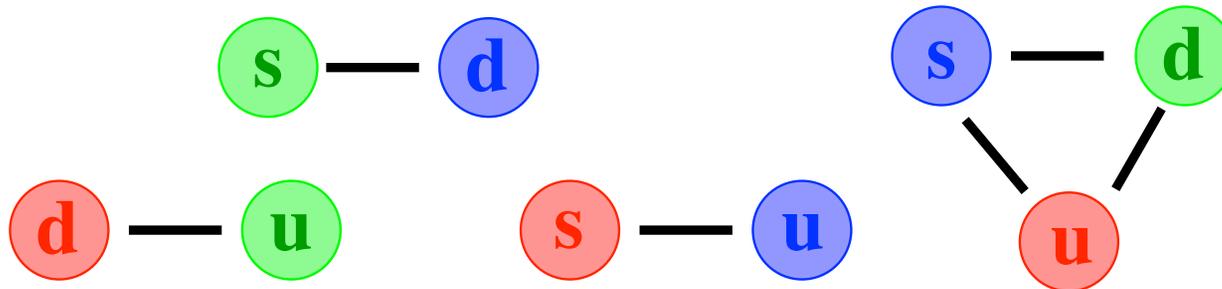
Pairing pattern of CFL

$$\mathcal{M} = \begin{array}{c} \begin{array}{cccccc} \mathbf{u} & \mathbf{d} & \mathbf{s} & \mathbf{u} & \mathbf{d} & \mathbf{s} & \mathbf{u} & \mathbf{d} & \mathbf{s} \end{array} \\ \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{c} \mathbf{u} \\ \mathbf{s} \\ \mathbf{p} \\ \mathbf{n} \end{array} \end{array}$$



Pairing pattern of CFL

$$\mathcal{M} = \begin{array}{c}
 \begin{array}{cccccccccc}
 \mathbf{u} & \mathbf{d} & \mathbf{s} & \mathbf{d} & \mathbf{u} & \mathbf{s} & \mathbf{u} & \mathbf{s} & \mathbf{d} & \\
 \hline
 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 \hline
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \\
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \\
 \hline
 \mathbf{p} & \mathbf{s} & \mathbf{n} & \mathbf{p} & \mathbf{s} & \mathbf{n} & \mathbf{p} & \mathbf{s} & \mathbf{n} & \\
 \hline
 \end{array}
 \end{array}$$



Understanding CFL in terms of symmetries

- "usual" fermionic superfluid: spontaneous breaking of $U(1) \rightarrow \mathbb{Z}_2$
(= Lagrangian invariant under $U(1)$ transformations, ground state $\langle \psi\psi \rangle$ only invariant under \mathbb{Z}_2)
- symmetry breaking pattern of CFL

$$\Rightarrow \underbrace{SU(3)_c \times SU(3)_L \times SU(3)_R}_{\supset U(1)_Q} \times U(1)_B \rightarrow \underbrace{SU(3)_{c+L+R}}_{\supset U(1)_{\tilde{Q}}} \times \mathbb{Z}_2$$

- blue: global symmetry
- red: local symmetry

- let's recall spontaneous symmetry breaking and QCD symmetries ...

Quick reminder: spontaneous symmetry breaking

- ϕ^4 theory

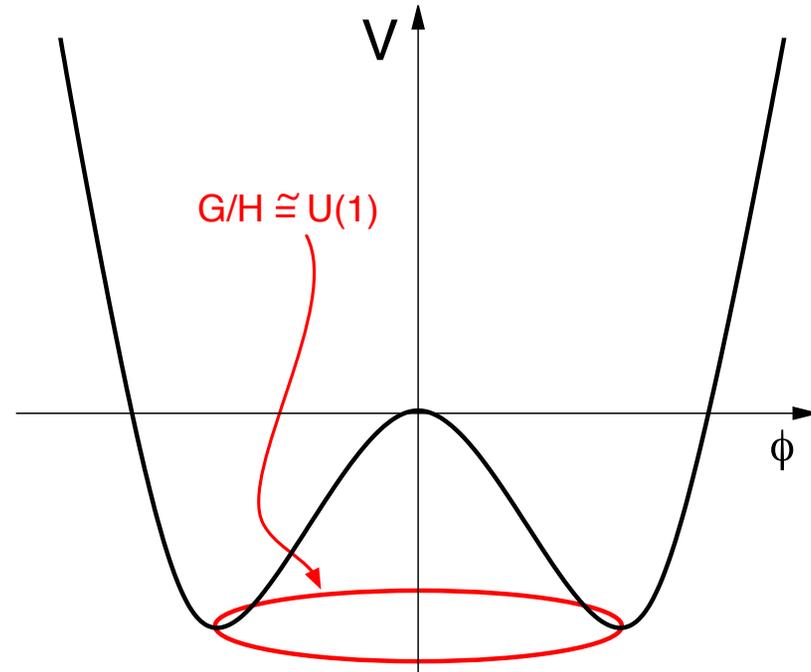
$$U(1) \rightarrow \mathbf{1}$$

- Goldstone theorem:

$$\exists \dim G - \dim H = 1$$

massless modes,

with $G = U(1)$, $H = \mathbf{1}$



- here: $G = SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B$
and $H = SU(3)_{c+L+R} \times \mathbb{Z}_2$

Symmetries of QCD

- QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\gamma^\mu D_\mu + \mu\gamma_0 - M)\psi - \frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a$$

with

$$M = \text{diag}(m_u, m_d, m_s)$$

$$D_\mu = \partial_\mu - igT_a A_\mu^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

- \mathcal{L}_{QCD} invariant under gauge group $SU(3)_c$,

$$\psi \rightarrow U\psi, \quad A_\mu \rightarrow UA_\mu U^{-1} + \frac{i}{g}U\partial_\mu U^{-1}, \quad U(x) \in SU(3)_c$$

Chiral symmetry (page 1/2)

- introduce left- and right-handed spinors

$$\psi_{R/L} \equiv P_{R/L}\psi, \quad P_R = \frac{1 + \gamma^5}{2}, \quad P_L = \frac{1 - \gamma^5}{2}$$

and write \mathcal{L}_{QCD} in terms of ψ_L and ψ_R

$$\begin{aligned} \mathcal{L}_{\text{QCD}} = & \bar{\psi}_R (i\gamma^\mu D_\mu + \mu\gamma_0)\psi_R + \bar{\psi}_L (i\gamma^\mu D_\mu + \mu\gamma_0)\psi_L \\ & - \bar{\psi}_R M\psi_L - \bar{\psi}_L M\psi_R + \mathcal{L}_{\text{gluons}} \end{aligned}$$

- consider $U(3)_R \times U(3)_L$ transformations

$$\psi_R \rightarrow e^{i\phi_R^a t_a} \psi_R, \quad \psi_L \rightarrow e^{i\phi_L^a t_a} \psi_L$$

$$(t_0 = \mathbf{1} \text{ and } t_a = T_a)$$

Chiral symmetry (page 2/2)

- for $M = 0$, \mathcal{L}_{QCD} is invariant under $U(3)_R \times U(3)_L$
(QCD at asymptotically large densities: $0 \simeq m_u \simeq m_d \simeq m_s \ll \mu$)
- in the real world, $U(3)_R \times U(3)_L$ “explicitly” broken
- write $U(3) \cong SU(3) \times U(1)$ and $U(1)_R \times U(1)_L \cong U(1)_B \times U(1)_A$ and take into account axial anomaly
→ QCD approximately (exact for $\mu \rightarrow \infty$) invariant under

$$\underbrace{SU(3)_R \times SU(3)_L}_{\text{“chiral symmetry”}} \times U(1)_B$$

- here $N_f = 3$, in general $SU(N_f)_R \times SU(N_f)_L$

Properties of CFL (page 1/2)

- CFL spontaneously breaks chiral symmetry

- “usual” chiral symmetry breaking: LR pairing $\langle \bar{\psi}_R \psi_L \rangle$

- CFL: LL, RR pairing $\langle \psi_R \psi_R \rangle, \langle \psi_L \psi_L \rangle$, however

$$SU(3)_c \times \underbrace{SU(3)_L \times SU(3)_R}_{\supset U(1)_Q} \times U(1)_B \rightarrow \underbrace{SU(3)_{c+L+R}}_{\supset U(1)_{\tilde{Q}}} \times \mathbb{Z}_2$$

- chiral symmetry broken through “locking” to color

- octet of pseudo-Goldstone modes K^0, K^\pm, π^0, \dots

D. T. Son and M. A. Stephanov, PRD 62, 059902 (2000)

→ effective theory for CFL just like usual chiral perturbation theory

P. F. Bedaque and T. Schäfer, NPA 697, 802 (2002)

Properties of CFL (page 2/2)

- CFL is a superfluid

$$SU(3)_c \times \underbrace{SU(3)_L \times SU(3)_R}_{\supset U(1)_Q} \times U(1)_B \rightarrow \underbrace{SU(3)_{c+L+R}}_{\supset U(1)_{\tilde{Q}}} \times \mathbb{Z}_2$$

- exactly massless Goldstone mode ϕ
- vortices in rotating CFL

- Is CFL an electromagnetic superconductor?

$$SU(3)_c \times \underbrace{SU(3)_L \times SU(3)_R}_{\supset U(1)_Q} \times U(1)_B \rightarrow \underbrace{SU(3)_{c+L+R}}_{\supset U(1)_{\tilde{Q}}}$$

- Cooper pairs neutral under $\tilde{Q} = Q + \frac{2}{\sqrt{3}}T_8$
- photon-gluon mixing with (small) mixing angle
 $\cos^2 \theta = 1 + \mathcal{O}(e^2/g^2)$
 (analogous to Weinberg angle in standard model)

Analogy to electroweak transition

Weinberg-Salam	CFL
$SU(2)_I \times U(1)_Y$ isospin, hypercharge	$SU(3)_c \times U(1)_Q$ color, electromagnetism
W_1, W_2, W_3, W_0	A_1, \dots, A_8, A
$SU(2)_I \times U(1)_Y \rightarrow U(1)_Q$	$SU(3)_c \times U(1)_Q \rightarrow U(1)_{\tilde{Q}}$
W^+, W^- $Z = \cos \theta_W W_3 + \sin \theta_W W_0$ $A = -\sin \theta_W W_3 + \cos \theta_W W_0$	A_1, \dots, A_7 $\tilde{A}_8 = \cos \theta A_8 + \sin \theta A$ $\tilde{A} = -\sin \theta A_8 + \cos \theta A$
W^+, W^-, Z (massive)	$A_1, \dots, A_7, \tilde{A}_8$ (massive)
A (massless)	\tilde{A} (massless)

Signature of CFL through "rotated electromagnetism"?

- magnetic field B of compact star has small \tilde{T}_8 component

→ color magnetic flux tube array

M. G. Alford and A. Sedrakian, *JPG* 37, 075202 (2010)

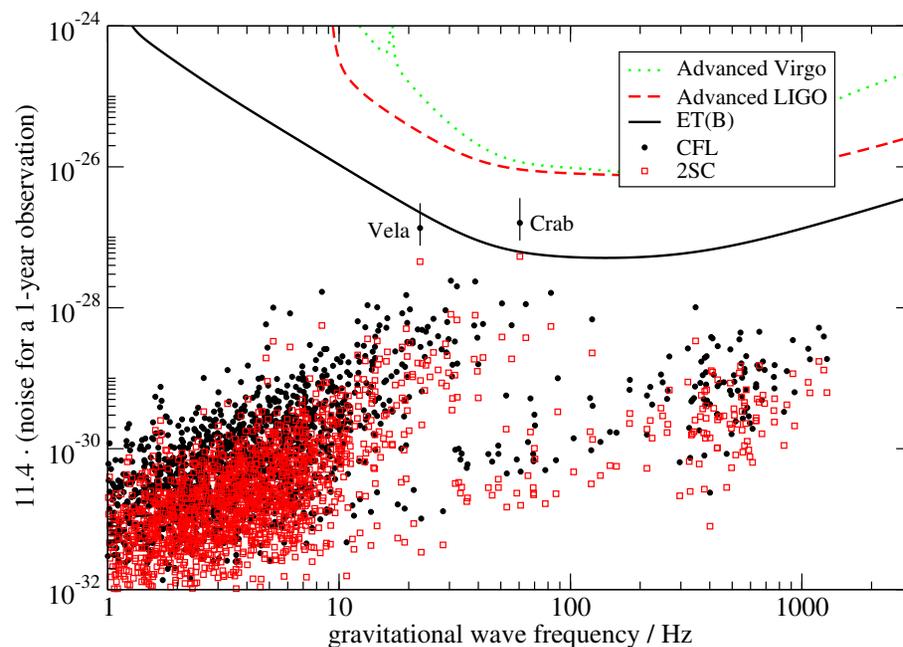
A. Haber and A. Schmitt, *JPG* 45, 065001 (2018)

→ ellipticity of compact stars
with CFL (or 2SC) quark
matter core

→ gravitational waves

K. Glampedakis, D. I. Jones and

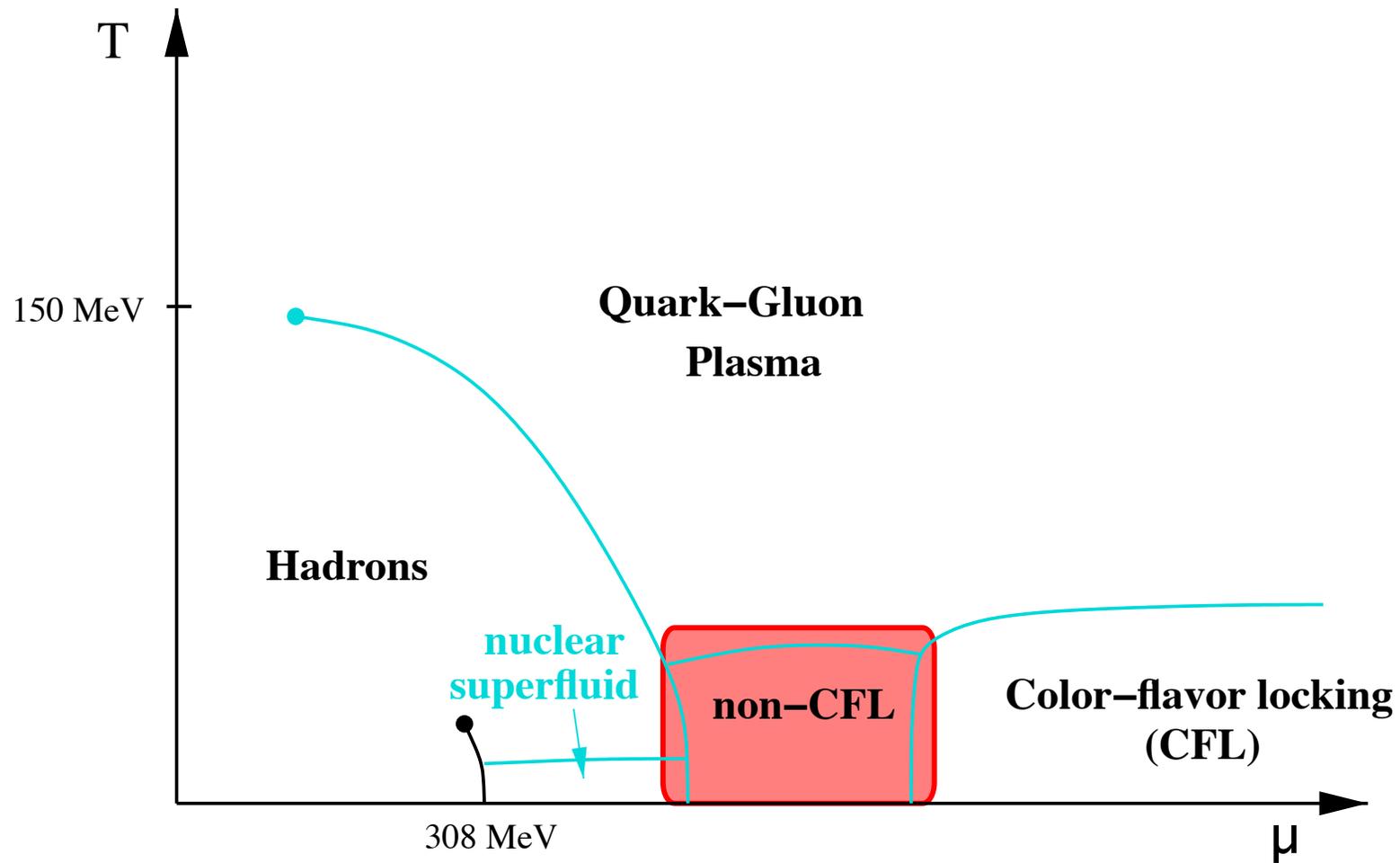
L. Samuelsson, *PRL* 109, 081103 (2012)



Summary: CFL quark matter

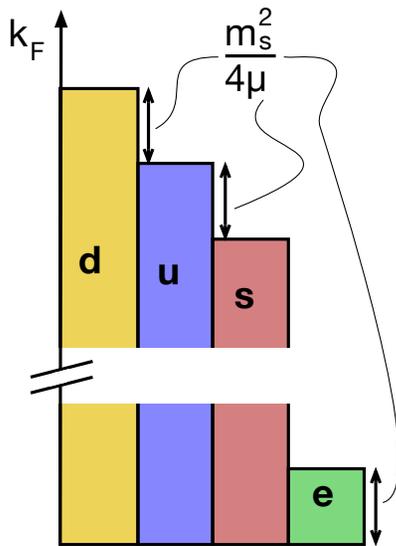
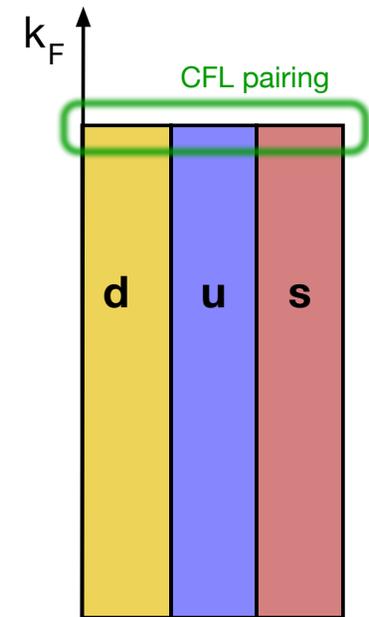
- the ground state of (three-flavor) QCD at asymptotically large densities (where the coupling is weak and the quark masses negligible) is CFL
- CFL involves Cooper pairing of all $N_c \times N_f = 9$ quarks in a particularly symmetric fashion [largest possible residual symmetry group $SU(3)_{c+L+R}$]
- from the symmetry breaking pattern we can read off various properties with potential relevance to compact star phenomenology (e.g., low-energy excitations of CFL: superfluid phonon, kaons, ...)

Dense, but not asymptotically dense, QCD



Stressed Cooper pairing (page 1/3)

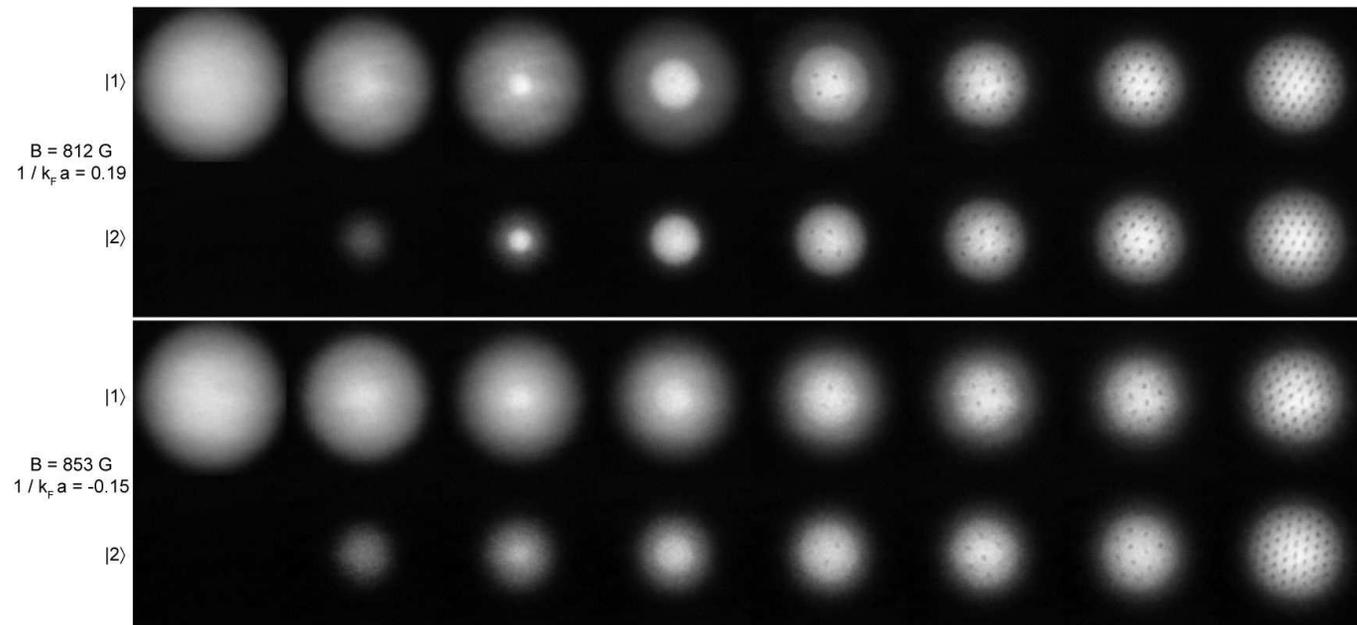
- asymptotically large densities
 - all 3 quark masses negligible \rightarrow very symmetric situation (e.g., chiral symmetry exact, same number of u , d , s)
 - particularly symmetric pairing pattern CFL is the ground state



- large, but not asymptotically large densities
 - “switch on” strange quark mass m_s
 - mismatched Fermi surfaces
 - “stressed” Cooper pairing?

Stressed Cooper pairing (page 2/3)

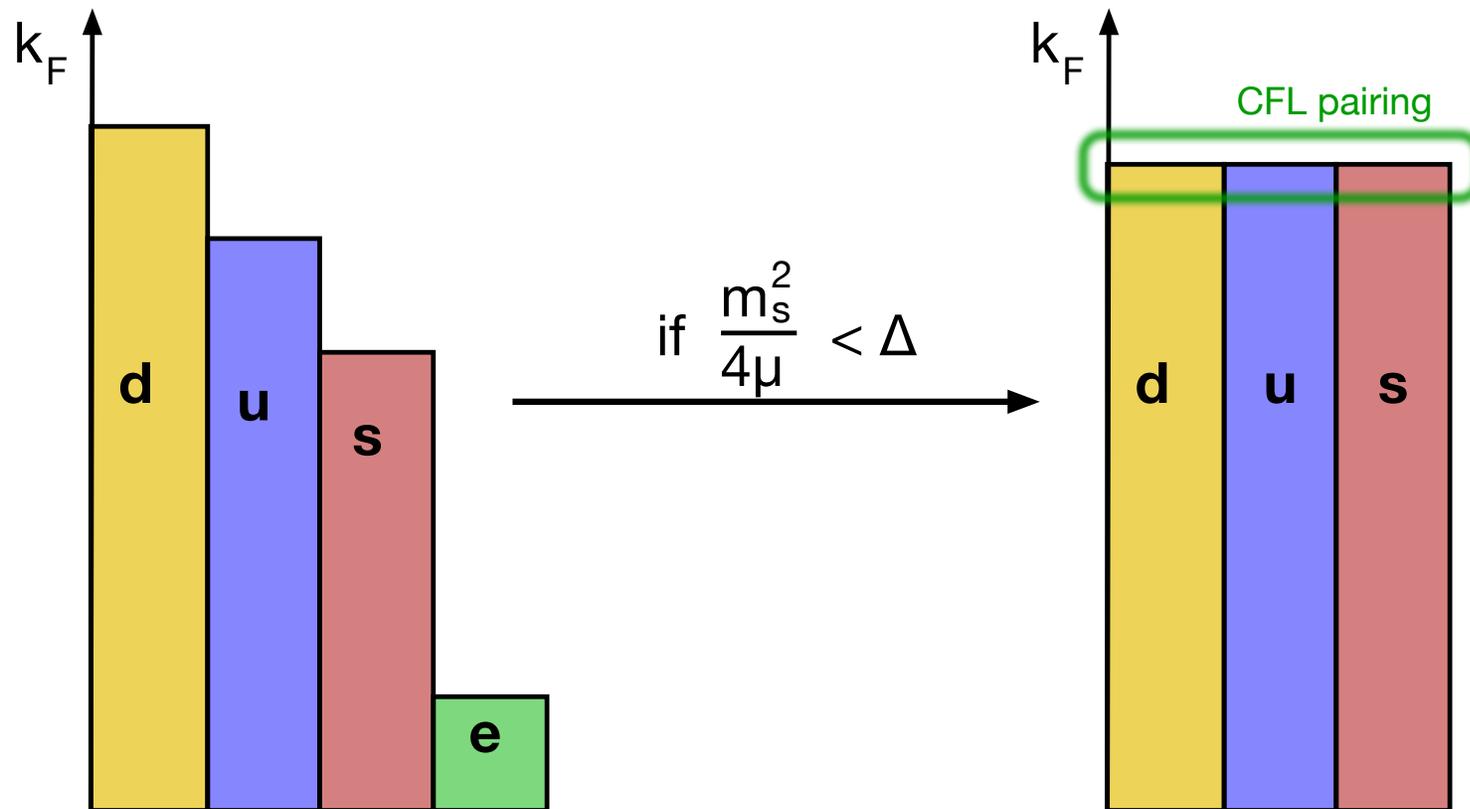
- stressed Cooper pairing is a general phenomenon
 - electronic superconductor in a magnetic field (Zeeman splitting)
B.S. Chandrasekhar, Appl. Phys. Lett. 1, 7 (1962); A.M. Clogston, PRL 9, 266 (1962)
LOFF phase in organic superconductor S. Tsuchiya, *et al.*, J. Phys. S. Jpn 84, 034703 (2015)
 - cold atomic gases: breakdown of pairing for large mismatch
M.W. Zwierlein, A. Schirotzek, C.H. Schunck, W. Ketterle, Science 311, 492 (2006)



general field-theoretical treatment, see chapter 9 of A. Schmitt, Lect. Notes Phys. 888, 1-155 (2015)

Stressed Cooper pairing (page 3/3)

- CFL favored if mismatch sufficiently small



Stressed pairing in quark matter: variants of CFL and non-CFL color superconductors (page 1/4)

- Kaon-condensed phases:
 $CFL-K^0$, $curCFL-K^0$
 P. Bedaque, T. Schäfer, NPA 697, 802 (2002)
 T. Schäfer, PRL 96, 012305 (2006)
 A. Schmitt, NPA 820, 49C (2009)

$curCFL-K^0$

counterpropagating currents:

K^0 -condensate

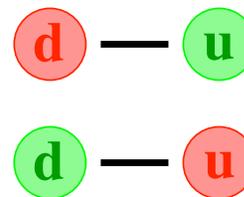
+ gapless fermions

- 2SC phase

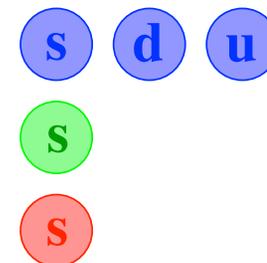
R. Rapp, T. Schäfer, E.V. Shuryak,
 M. Velkovsky, PRL 81, 53 (1998)

M.G. Alford, K. Rajagopal, F. Wilczek,
 PLB 422, 247 (1998)

paired:



unpaired:

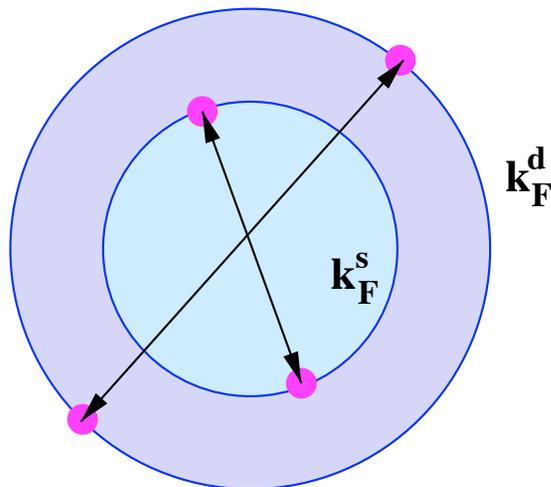
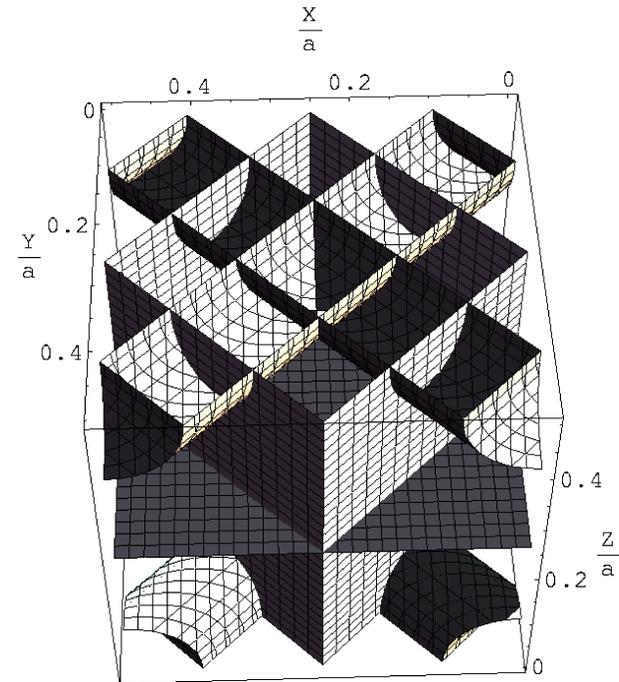


Stressed pairing in quark matter: variants of CFL and non-CFL color superconductors (page 2/4)

- Crystalline phases: LOFF

M. Alford, J. Bowers, K. Rajagopal, PRD 63, 074016 (2001)

M. Mannarelli, K. Rajagopal and R. Sharma, PRD 73, 114012 (2006)



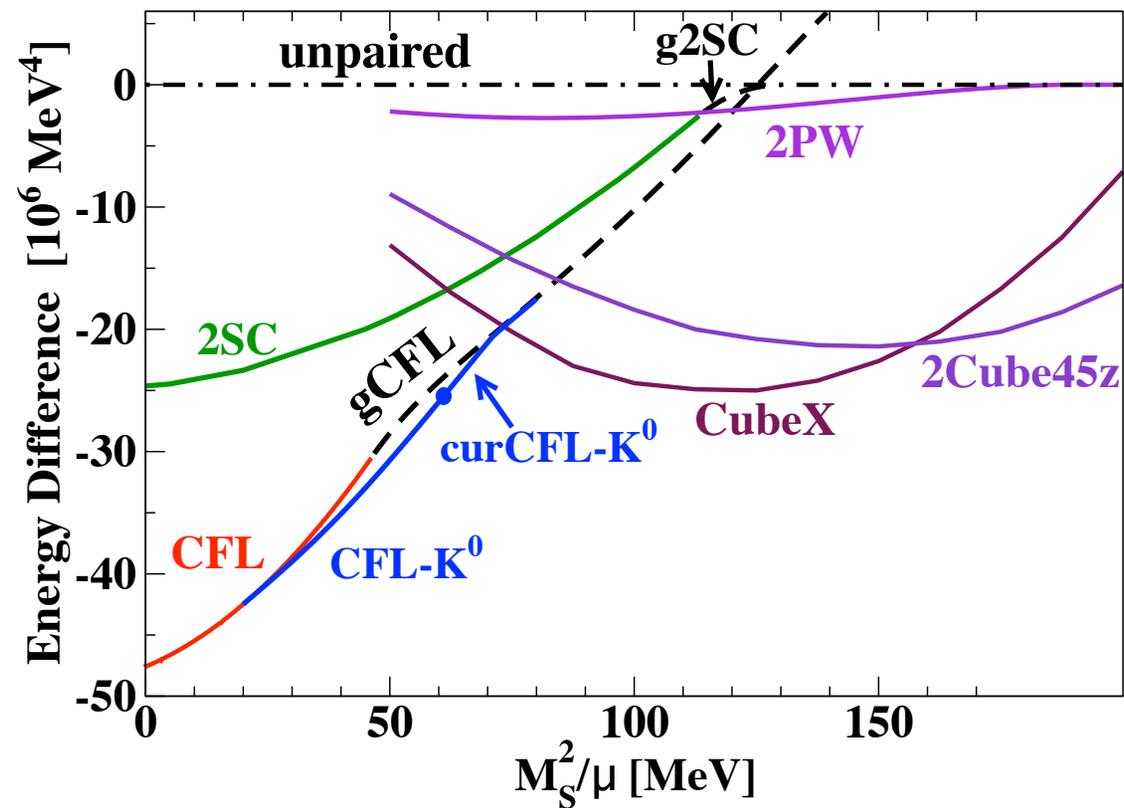
- Single-flavor pairing:
CSL, A-phase, polar phase ...

T. Schäfer, PRD 62, 094007 (2000)

A. Schmitt, PRD 71, 054016 (2005)

Stressed pairing in quark matter: variants of CFL and non-CFL color superconductors (page 3/4)

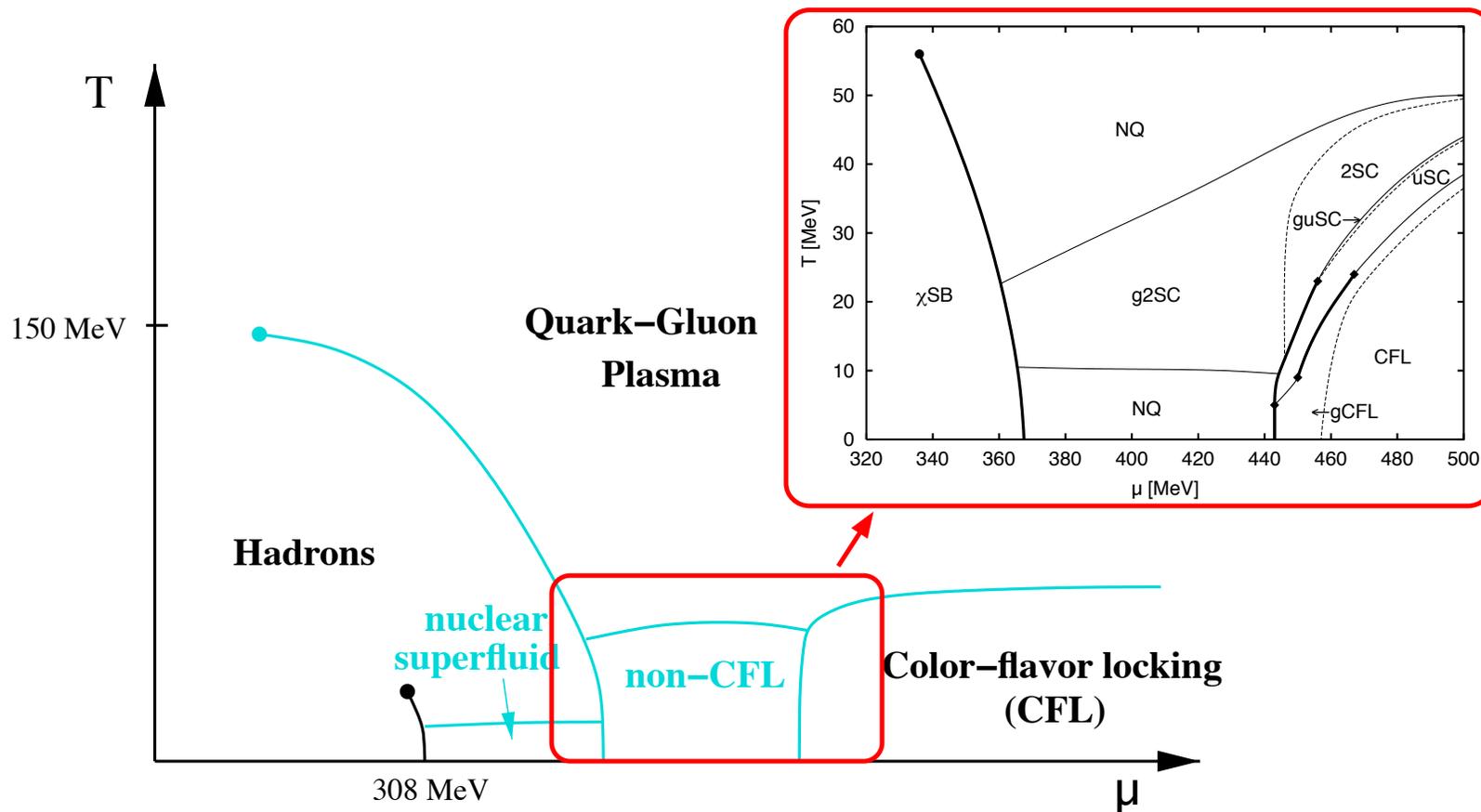
Free energy comparison of 3-flavor quark phases for $\Delta_{\text{CFL}} = 25$ MeV:
M. Alford, K. Rajagopal, T. Schäfer, A. Schmitt, RMP 80, 1455 (2008)



Stressed pairing in quark matter: variants of CFL and non-CFL color superconductors (page 4/4)

- for instance from NJL model

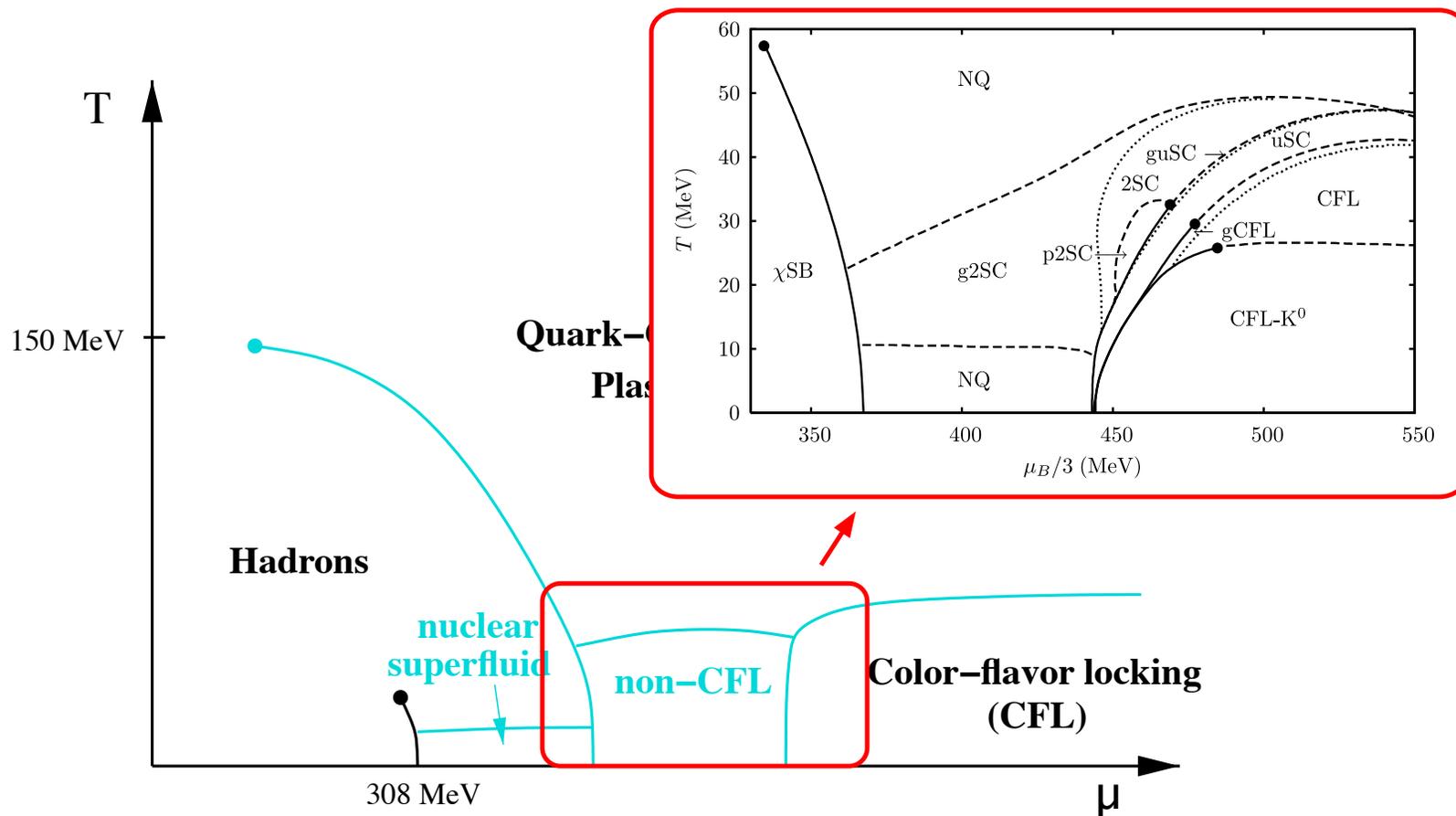
S. B. Ruester, V. Werth, M. Buballa, I. A. Shovkovy and D. H. Rischke, PRD 72, 034004 (2005)



Stressed pairing in quark matter: variants of CFL and non-CFL color superconductors (page 4/4)

- for instance from NJL model

H. Warringa, hep-ph/0606063



Summary: stressed Cooper pairing

- except for asymptotically large densities, quarks that “want” to pair have different Fermi surfaces
- Cooper pairing can sustain a small mismatch $\delta\mu$, but starts to break down when $\delta\mu \sim \Delta$
- CFL can “react” on a mismatch in various ways (kaon condensate, LOFF, ...)
- due to our poor knowledge of QCD at moderate densities, we do not know whether CFL (or a variant of it) persists down to nuclear matter densities

Outline

(1) Introduction and overview (pp 6 - 31)

- general remarks
- dense matter in the QCD phase diagram
- some selected astrophysical observations

(2) Dense quark matter (pp 33 - 50)

- basic thermodynamics → Problems I
- strange quark matter hypothesis
- equation of state

(3) Dense nuclear matter (pp 53 - 64)

- free nuclear matter → Problems II
- field-theoretical model
- saturation density and binding energy

(4) Cooper pairing in dense matter (pp 66 - 81)

- field-theoretical approach (sketch)
- fermionic excitations → Problems III
- solving the gap equation

(5) Color superconductivity (pp 83 - 106)

- color-flavor locked (CFL) quark matter
- stressed pairing and non-CFL color superconductors

(6) Transport in dense matter (pp 108 - 150)

- specific heat
- neutrino emissivity
- bulk viscosity

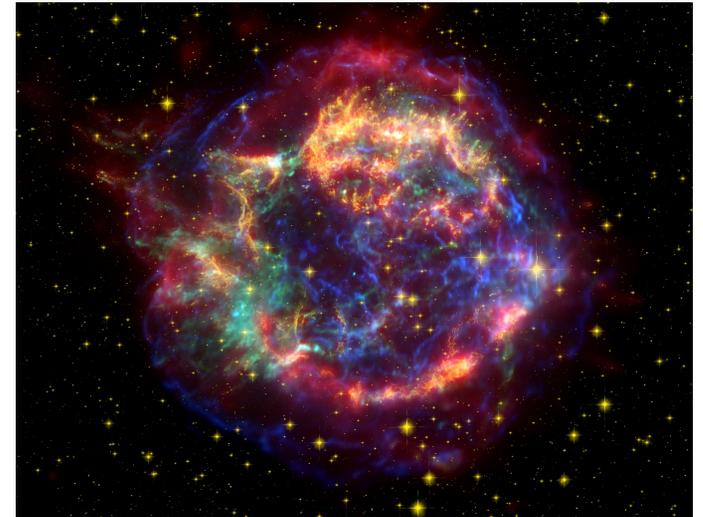
Transport in dense matter: overview (page 1/2)

- transport properties: out of (or near) equilibrium properties (as opposed to thermodynamics)
- need techniques such as kinetic theory, real-time thermal field theory, Boltzmann equation, hydrodynamics, ...
- calculate quantities such as shear and bulk viscosities, thermal and electric conductivities, neutrino emissivity, ...



Transport in dense matter: overview (page 2/2)

- highly relevant for compact stars: phases with similar thermodynamics can behave very different regarding transport (also important for heavy-ion collisions → lectures by T. Lappi and U. Heinz)
- for compact stars, need to consider many different scenarios for a review see [A. Schmitt and P. Shternin, 1711.06520 \[astro-ph.HE\]](#)
 - crust: electron transport in ion lattice, impurities, neutron (super)fluid, ...
 - core: baryonic and leptonic contribution, multi-fluid hydrodynamics, ...
 - core: transport in quark matter (and e^-), hyperons, mesons, ...



Transport in dense matter: 3 examples

- specific heat (\rightarrow cooling, thermodynamic property, but sensitive to low-energy excitations)
- neutrino emissivity (\rightarrow cooling)
- bulk viscosity (\rightarrow r -mode instability, merger dynamics (?))

Specific heat of a superfluid

see Sec. 4.1 in [A. Schmitt, Lect. Notes Phys. 811, 1 \(2010\)](#)

- entropy density

$$s = \frac{\partial P}{\partial T} = -2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} [(1 - f_k) \ln(1 - f_k) + f_k \ln f_k]$$

with (assume massless fermions)

$$f_k = \frac{1}{e^{\epsilon_k/T} + 1}, \quad \epsilon_k = \sqrt{(k - \mu)^2 + \Delta^2}$$

- specific heat (see [Problems I](#))

$$c_V \equiv T \frac{\partial s}{\partial T} = 2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \epsilon_k \frac{\partial f_k}{\partial T}$$

Specific heat: unpaired phase

- for $\Delta = 0$

$$c_V^0 \equiv \frac{1}{\pi^2} \int_0^\infty dk k^2 \frac{\epsilon_k^2}{T^2} \frac{e^{\epsilon_k/T}}{(e^{\epsilon_k/T} + 1)^2}$$

- for $T \ll \mu$ approximate

$$\int_0^\infty dk k^2 \simeq \mu^2 \int_0^\infty dk$$

$$\mu^2 \int_0^\infty dk F\left(\frac{k - \mu}{T}\right) = \mu^2 T \int_{-\mu/T}^\infty dx F(x) \simeq \mu^2 T \int_{-\infty}^\infty dx F(x)$$

with $x = (k - \mu)/T$

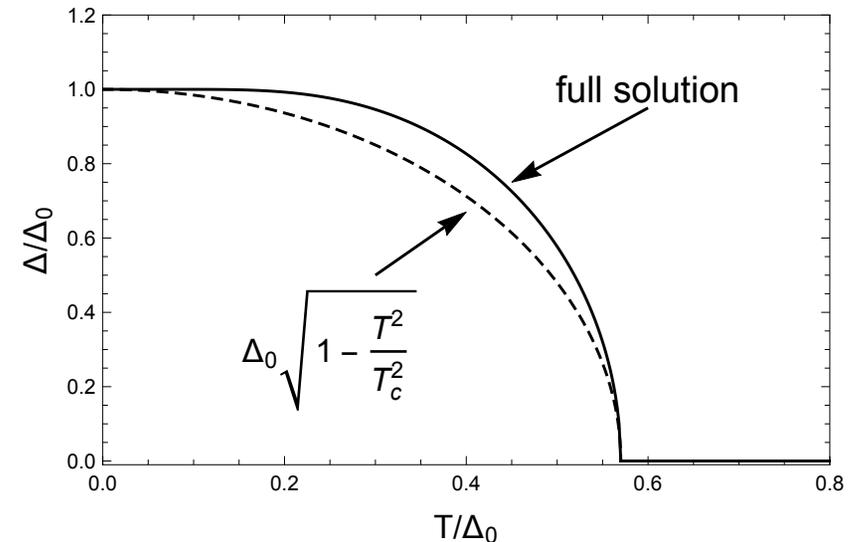
- this yields

$$c_V^0 \simeq \frac{\mu^2 T}{\pi^2} \int_0^\infty dx \frac{x^2}{1 + \cosh x} = \frac{\mu^2 T}{3}$$

Specific heat: paired phase (page 1/3)

- assume the following temperature-dependence of the gap

$$\Delta(T) = \Theta(T_c - T) \Delta_0 \sqrt{1 - \frac{T^2}{T_c^2}}$$



- with

$$\frac{\partial \Delta}{\partial T} = -\frac{\Delta_0^2 T}{T_c^2 \Delta} \Rightarrow \frac{\partial \epsilon_k}{\partial T} = -\frac{T \Delta_0^2}{\epsilon_k T_c^2} \Rightarrow \frac{\partial f_k}{\partial T} = \frac{1}{\epsilon_k} \frac{e^{\epsilon_k/T}}{(e^{\epsilon_k/T} + 1)^2} \left(\frac{\epsilon_k^2}{T^2} + \frac{\Delta_0^2}{T_c^2} \right)$$

we obtain

$$c_V = 2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^{\epsilon_k/T}}{(e^{\epsilon_k/T} + 1)^2} \left(\frac{\epsilon_k^2}{T^2} + \frac{\Delta_0^2}{T_c^2} \right)$$

Specific heat: paired phase (page 2/3)

- denote

$$\varphi \equiv \frac{\Delta}{T}$$

and apply above approximations for $T \ll \mu$,

$$c_V \simeq \frac{\mu^2 T}{\pi^2} \int_0^\infty dx \int_0^\pi d\theta \sin \theta \left(x^2 + \varphi^2 + \frac{\Delta_0^2}{T_c^2} \right) \frac{e^{\sqrt{x^2 + \varphi^2}}}{\left(e^{\sqrt{x^2 + \varphi^2}} + 1 \right)^2}$$

- small temperatures $T \ll \Delta$ and isotropic gap,

$$\frac{e^{\sqrt{x^2 + \varphi^2}}}{\left(e^{\sqrt{x^2 + \varphi^2}} + 1 \right)^2} \simeq e^{-\sqrt{x^2 + \varphi^2}} \simeq e^{-\varphi - \frac{x^2}{2\varphi}}$$

Specific heat: paired phase (page 3/3)

- this yields

$$c_V \simeq \frac{2\mu^2 T}{\pi^2} e^{-\varphi} \left[\int_0^\infty dx x^2 e^{-\frac{x^2}{2\varphi}} + \left(\varphi^2 + \frac{\Delta_0^2}{T_c^2} \right) \int_0^\infty dx e^{-\frac{x^2}{2\varphi}} \right]$$
$$\simeq \frac{\sqrt{2}\mu^2 T}{\pi^{3/2}} \varphi^{5/2} e^{-\varphi}$$

$$c_V \propto e^{-\Delta/T}$$

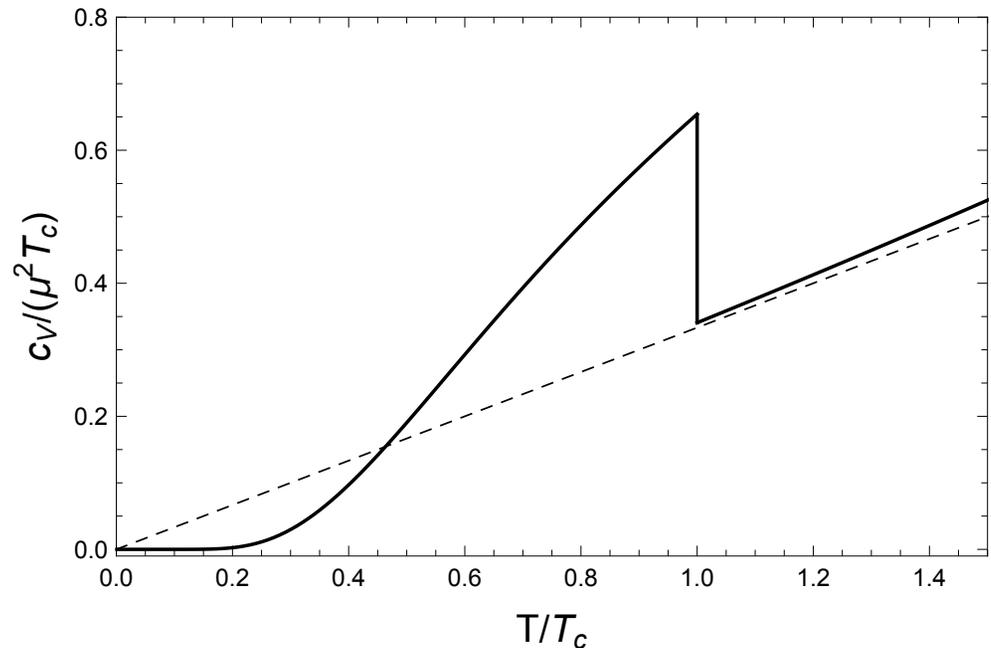
- exponential suppression of specific heat at small temperatures
- no states available to “store” heat

Jump at critical temperature

- c_V is discontinuous at T_c

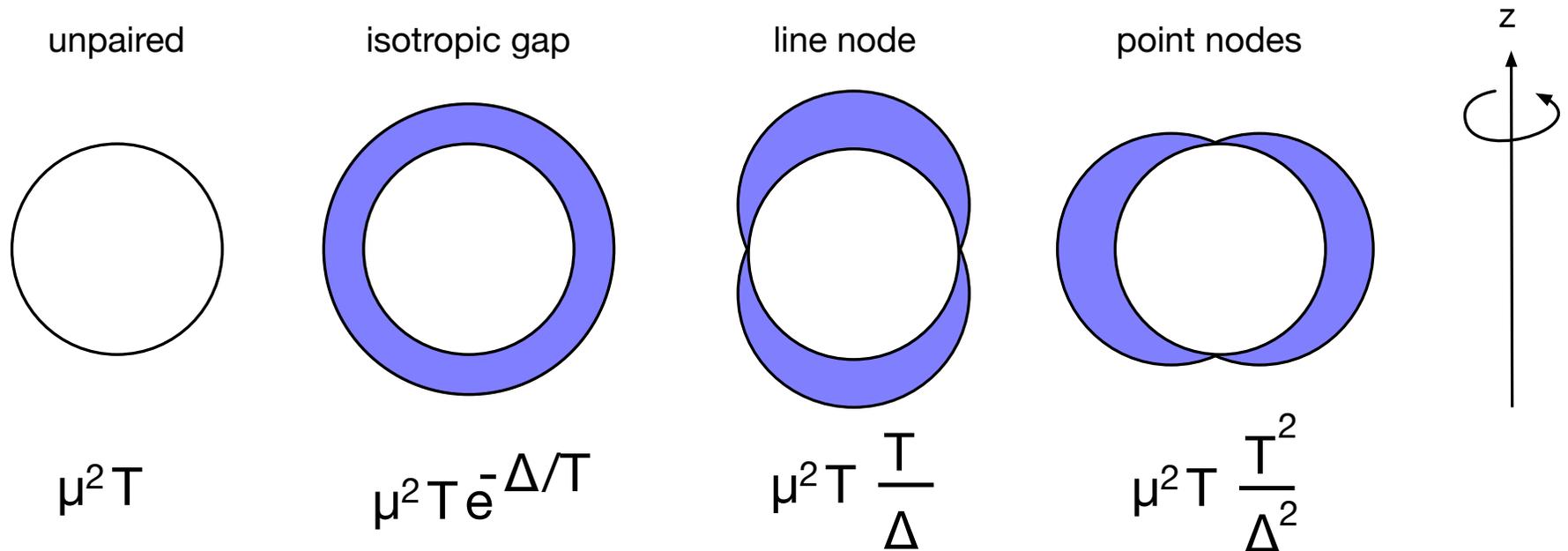
$$\begin{aligned} \Delta c_V &\equiv c_V(T \uparrow T_c) - c_V(T \downarrow T_c) \\ &= 2 \frac{\Delta_0^2}{T_c^2} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^{\epsilon_k/T}}{(e^{\epsilon_k/T} + 1)^2} \simeq \frac{\Delta_0^2 \mu^2}{\pi^2 T_c} \end{aligned}$$

- numerical evaluation for all $T \rightarrow$ *exercise*



Anisotropic gaps

- repeat calculation for $\Delta \rightarrow \Delta \sin \theta$, $\Delta \rightarrow \Delta \cos \theta$
- gaps with point or line nodes: superfluid ${}^3\text{He}$, single-flavor color superconductor, 3P_2 neutron superfluid, ...



- nodes: low-energy states available to store heat, but not “as many” as in fully unpaired case

Summary: specific heat

- specific heat is a very simple example to demonstrate effect of pairing gap
- at small temperatures, $T \ll \Delta$, the gap suppresses the specific heat exponentially (isotropic gap) or by a power law (anisotropic gap with nodes, depending on dimensionality of nodes)
- if fermions are gapped, other (low-energy) degrees of freedom dominate specific heat (e.g., phonon in a superfluid, kaons in CFL, ...)

Neutrino emissivity

see chapter 5 of [A. Schmitt, Lect. Notes Phys. 811, 1 \(2010\)](#)

- compact stars cool mainly via neutrino emission
(in their first $10^5 - 10^6$ yr)
- cooling is given through neutrino emissivity and specific heat

$$\epsilon_\nu(T) = -c_V(T) \frac{dT}{dt}$$

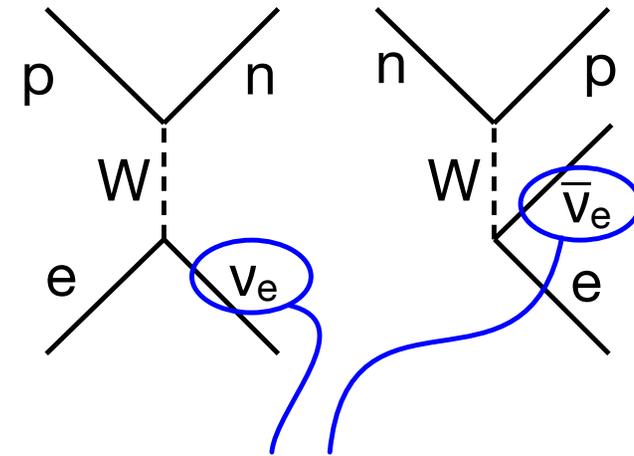
next: discuss “Urca processes” in nuclear matter (very briefly)
and quark matter (direct Urca, detailed calculation)

Urca processes^(*) in nuclear matter (page 1/2)

- “direct Urca processes”:
electron capture and β -decay

$$p + e \rightarrow n + \nu_e$$

$$n \rightarrow p + e + \bar{\nu}_e$$



(anti-) neutrinos leave the star

- three-momentum conservation (neglect neutrino momentum $\sim T$)

$$\mathbf{k}_{F,n} = \mathbf{k}_{F,p} + \mathbf{k}_{F,e}$$

^(*) named after the *Casino de Urca* in Rio de Janeiro

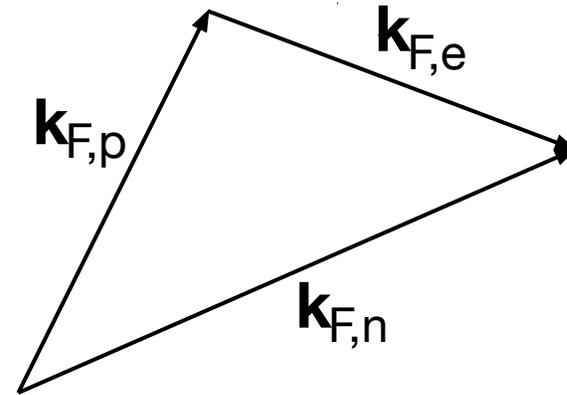
Urca processes in nuclear matter (page 2/2)

- triangle inequality

$$k_{F,n} < k_{F,p} + k_{F,e}$$

$$\Rightarrow k_{F,n} < 2k_{F,p}$$

($k_{F,p} = k_{F,e}$ in a neutral system)



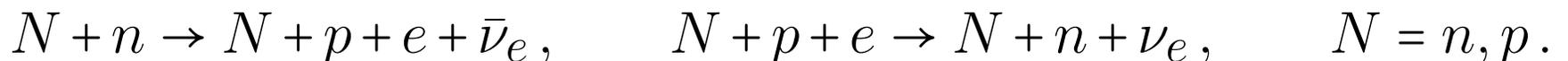
- with $n_i \propto k_{F,i}^3$ ($i = n, p$),

$$n_n < 8n_p \Rightarrow \frac{n_p}{n_B} > \frac{1}{9}$$

→ direct Urca in nuclear matter strongly suppressed

because proton fraction usually $\frac{n_p}{n_B} \lesssim 10\%$ (hence “neutron star”)

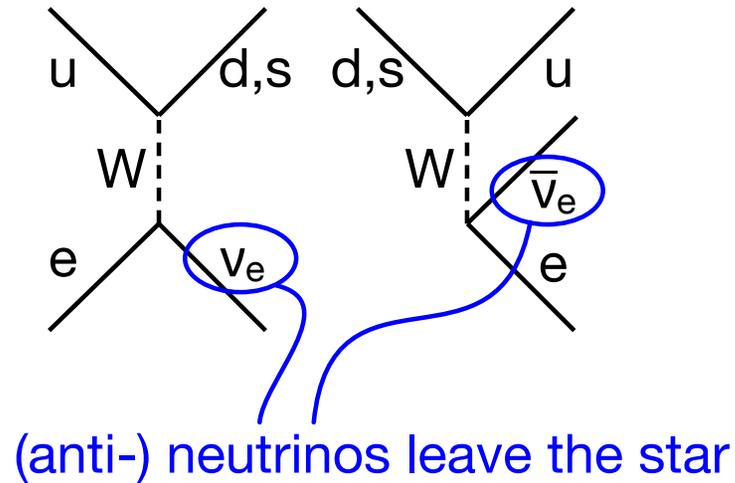
- instead: “modified Urca process”



Direct Urca processes in quark matter

$$d \rightarrow u + e + \bar{\nu}_e, \quad u + e \rightarrow d + \nu_e$$

$$s \rightarrow u + e + \bar{\nu}_e, \quad u + e \rightarrow s + \nu_e$$



- goals for the following:
 - understand the role of Cooper pairing for the emissivity
 - compute the emissivity of unpaired quark matter

How to compute the neutrino emissivity (page 1/4)

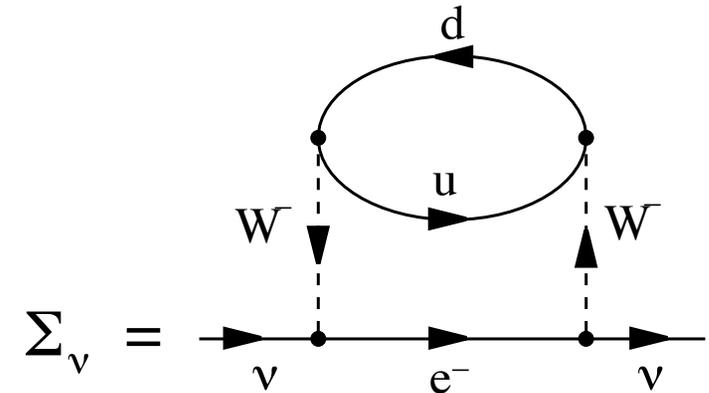
- starting point: kinetic equation

$$i\frac{\partial}{\partial t}\text{Tr}[\gamma_0 G_\nu^<(P_\nu)] = -\text{Tr}[G_\nu^>(P_\nu)\Sigma_\nu^<(P_\nu) - \Sigma_\nu^>(P_\nu)G_\nu^<(P_\nu)]$$

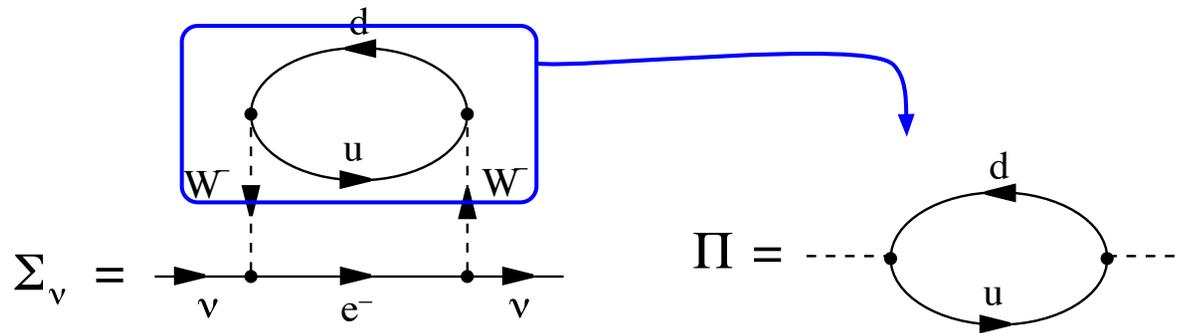
- “greater” and “lesser” neutrino propagators $G_\nu^>$ and $G_\nu^<$
and neutrino self-energies $\Sigma_\nu^>$ and $\Sigma_\nu^<$
→ real-time formalism, see for instance M. Le Bellac, *“Thermal Field Theory”* (2000)
- first term: neutrino gain from $d \rightarrow u + e + \bar{\nu}_e$, $u + e \rightarrow d + \nu_e$
second term: neutrino loss $u + e + \bar{\nu}_e \rightarrow d$, $d + \nu_e \rightarrow u + e$
(irrelevant since neutrinos, once created, leave the system)

How to compute the neutrino emissivity (page 2/4)

- neutrino self-energy from Urca process



- W -boson polarization tensor Π



$$\Pi^>(Q) = -2i[1 + f_B(q_0)]\text{Im} \Pi_R(Q), \quad \Pi^<(Q) = -2if_B(q_0)\text{Im} \Pi_R(Q)$$

How to compute the neutrino emissivity (page 3/4)

- the kinetic equation becomes

A. Schmitt, I. A. Shovkovy and Q. Wang, PRD 73, 034012 (2006)

$$\frac{\partial}{\partial t} f_\nu(t, \mathbf{p}_\nu) = \frac{G_F^2}{8} \int \frac{d^3 \mathbf{p}_e}{(2\pi)^3 p_\nu p_e} \underbrace{L_{\lambda\sigma} f_F(p_e - \mu_e)}_{\text{leptons}} \underbrace{f_B(p_\nu + \mu_e - p_e)}_{W\text{-boson}} \underbrace{\text{Im} \Pi_R^{\lambda\sigma}(Q)}_{\text{quarks}}$$

$$L^{\lambda\sigma} \equiv \text{Tr} [(\gamma \cdot P_e) \gamma^\sigma (1 - \gamma^5) (\gamma \cdot P_\nu) \gamma^\lambda (1 - \gamma^5)]$$

- no factor $1 - f_{\nu, \bar{\nu}}$ due to the absence of neutrino trapping ($f_{\nu, \bar{\nu}} \simeq 0$)
- vertex for the processes $d \leftrightarrow u + W^-$ and $e \leftrightarrow \nu + W^-$,

$$\Gamma^\mu = -\frac{e}{2\sqrt{2} \sin \theta_W} \gamma^\mu (1 - \gamma^5)$$

with the Weinberg angle θ_W ($V_{ud} \simeq 1$)

How to compute the neutrino emissivity (page 4/4)

- W -boson propagator approximated by M_W^{-2} ($M_W \simeq 80$ GeV), and

$$G_F = \frac{\sqrt{2}e^2}{8M_W^2 \sin^2 \theta_W} = 1.16637 \cdot 10^{-11} \text{ MeV}^{-2}$$

- neutrino emissivity = change in neutrino energy per unit time and volume,

$$\epsilon_\nu \equiv 2 \frac{\partial}{\partial t} \int \frac{d^3 \mathbf{p}_\nu}{(2\pi)^3} p_\nu f_\nu(t, \mathbf{p}_\nu)$$

(factor 2 accounts for antineutrinos)

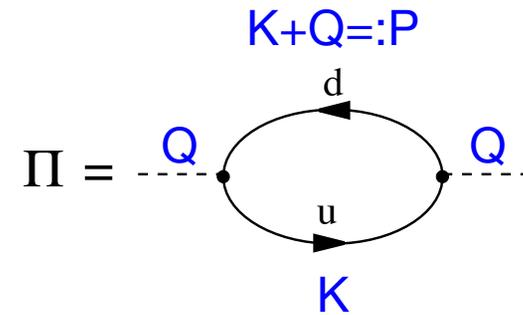
→ need to calculate $\frac{\partial}{\partial t} f_\nu(t, \mathbf{p}_\nu)$ from above, multiply by $2p_\nu$ and integrate over \mathbf{p}_ν

W -boson polarization tensor

$$\frac{\partial}{\partial t} f_\nu(t, \mathbf{p}_\nu) = \frac{G_F^2}{8} \int \frac{d^3 \mathbf{p}_e}{(2\pi)^3 p_\nu p_e} \underbrace{L_{\lambda\sigma} f_F(p_e - \mu_e)}_{\text{leptons}} \underbrace{f_B(p_\nu + \mu_e - p_e)}_{W\text{-boson}} \underbrace{\text{Im } \Pi_R^{\lambda\sigma}(Q)}_{\text{quarks}}$$

$$\Pi^{\lambda\sigma}(Q) = \frac{T}{V} \sum_K \text{Tr}[\Gamma_-^\lambda S(K) \Gamma_+^\sigma S(P)]$$

$S(K)$ propagator in Nambu-Gorkov space (for paired quark matter)



- Nambu-Gorkov vertices

$$\Gamma_\pm^\lambda = \begin{pmatrix} \gamma^\lambda(1 - \gamma^5) \tau_\pm & 0 \\ 0 & -\gamma^\lambda(1 + \gamma^5) \tau_\mp \end{pmatrix}$$

with $\tau_\pm \equiv (\tau_1 \pm i\tau_2)/2$ in flavor space (u and d interact)

Consider specific phase: 2SC

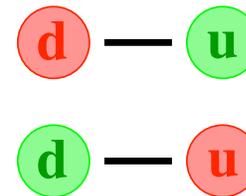
- propagators

$$G^\pm = \gamma^0 \Lambda_k^\mp \sum_{r=1,2} \mathcal{P}_r \frac{k_0 \mp (\mu - k)}{k_0^2 - \epsilon_{k,r}^2}$$

projectors in color space

$$\mathcal{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

paired:



unpaired:



- ignore strange quarks for simplicity [suppression due to Cabibbo angle, however larger phase space, Q. Wang, Z. g. Wang and J. Wu, PRD 74, 014021 (2006)]

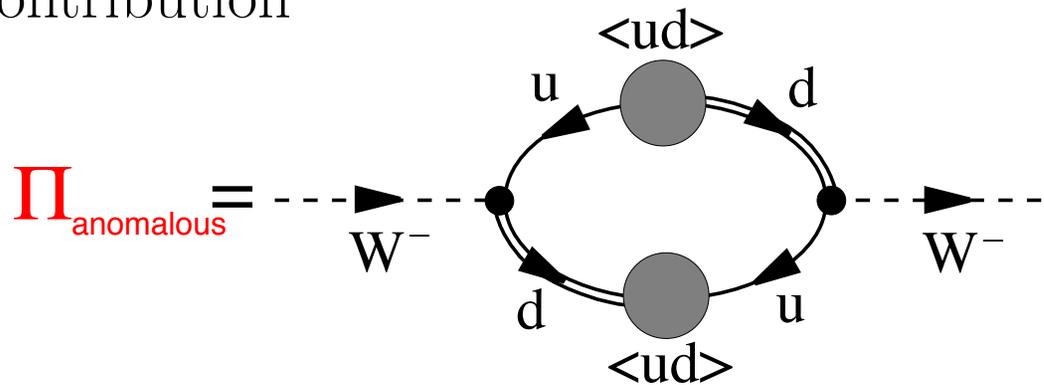
→ two contributions: gapped u , d and ungapped u , d
(due to color blindness of weak interaction: no mixed terms)

Anomalous contribution

- trace over Nambu-Gorkov space

$$\begin{aligned} \Pi^{\lambda\sigma}(Q) = & \frac{T}{V} \sum_K \left\{ \text{Tr} [\gamma^\lambda (1 - \gamma^5) \tau_- G^+(K) \gamma^\sigma (1 - \gamma^5) \tau_+ G^+(P)] \right. \\ & + \text{Tr} [\gamma^\lambda (1 + \gamma^5) \tau_+ G^-(K) \gamma^\sigma (1 + \gamma^5) \tau_- G^-(P)] \\ & - \text{Tr} [\gamma^\lambda (1 - \gamma^5) \tau_- F^-(K) \gamma^\sigma (1 + \gamma^5) \tau_- F^+(P)] \\ & \left. - \text{Tr} [\gamma^\lambda (1 + \gamma^5) \tau_+ F^+(K) \gamma^\sigma (1 - \gamma^5) \tau_+ F^-(P)] \right\} \end{aligned}$$

- anomalous contribution



- for simplicity, we ignore anomalous part in the following
however, it does give a (small) contribution P. Jaikumar, C. D. Roberts and A. Sedrakian,
PRC 73, 042801 (2006)

Sketch of calculation

- use the **Matsubara sum** \rightarrow *exercise, via contour integration*
see App. A of A. Schmitt, Lect. Notes Phys. 811, 1 (2010)

$$\begin{aligned}
 & T \sum_{k_0} \frac{k_0 - (\mu - k)}{k_0^2 - \epsilon_k^2} \frac{p_0 - (\mu - p)}{p_0^2 - \epsilon_p^2} \\
 &= -\frac{1}{4\epsilon_k \epsilon_p} \sum_{e_1, e_2} \frac{[\epsilon_k + e_1(\mu - k)][\epsilon_p + e_2(\mu - p)]}{q_0 - e_1\epsilon_k + e_2\epsilon_p} \frac{f_F(-e_1\epsilon_k) f_F(e_2\epsilon_p)}{f_B(-e_1\epsilon_k + e_2\epsilon_p)}
 \end{aligned}$$

- compute **imaginary part of retarded polarization tensor** with

$$q_0 \rightarrow q_0 - i\eta, \quad \lim_{\eta \rightarrow 0^+} \frac{1}{x \pm i\eta} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x)$$

- abbreviate the **Dirac trace** as

$$\mathcal{T}^{\lambda\sigma} \equiv \text{Tr} [\gamma^\lambda (1 - \gamma^5) \gamma_0 \Lambda_k^- \gamma^\sigma (1 - \gamma^5) \gamma_0 \Lambda_p^-]$$

- let's discuss gapped and ungapped contributions separately ...

Gapped contribution (page 1/4)

- gapped: putting everything together yields

$$\begin{aligned} \frac{\partial}{\partial t} f_\nu(t, \mathbf{p}_\nu) = & -\frac{\pi G_F^2}{4} \sum_{e_1 e_2} \int \frac{d^3 \mathbf{p}_e d^3 \mathbf{k}}{(2\pi)^3 (2\pi)^3 p_\nu p_e} L_{\lambda\sigma} \mathcal{T}^{\lambda\sigma} B_k^{e_1} B_p^{e_2} \\ & \times f_F(p_e - \mu_e) f_F(-e_1 \epsilon_k) f_F(e_2 \epsilon_p) \delta(q_0 - e_1 \epsilon_k + e_2 \epsilon_p) \end{aligned}$$

with the *Bogoliubov coefficients*

$$B_k^e \equiv \frac{1}{2} \left(1 + e^{\frac{\mu - k}{\epsilon_k}} \right)$$

- Bose distribution has canceled (W -boson does not appear in initial or final state)
- What does the **sum over $e_1, e_2 = \pm$** mean?

Gapped contribution (page 2/4)

- expected for $u + e \rightarrow d + \nu_e$: $f_e f_u (1 - f_d)$

- with Cooper pairing, all combinations appear!

$$f_e f_u f_d, \quad f_e f_u (1 - f_d), \quad f_e (1 - f_u) f_d, \quad f_e (1 - f_u) (1 - f_d)$$

→ recall that quasi-particles are mixtures of quasi-particles and holes and can appear on either side of the reaction process

- continue schematically,

$$\epsilon_\nu \sim \sum_{e_1, e_2 = \pm} \int_{v, x, y} \underbrace{\left(e^{v + e_1 \sqrt{y^2 + \varphi^2} - e_2 \sqrt{x^2 + \varphi^2}} + 1 \right)^{-1}}_{\text{electrons}} \underbrace{\left(e^{-e_1 \sqrt{y^2 + \varphi^2}} + 1 \right)^{-1}}_{u\text{-quarks}} \underbrace{\left(e^{e_2 \sqrt{x^2 + \varphi^2}} + 1 \right)^{-1}}_{d\text{-quarks}}$$

with

$$\varphi \equiv \frac{\Delta}{T}, \quad \int_{v, x, y} \equiv \int_0^\infty dv v^3 \int_0^\infty dx \int_0^\infty dy,$$

and

$$x = \frac{p - \mu_d}{T}, \quad y = \frac{k - \mu_u}{T}, \quad v = \frac{p_\nu}{T}$$

Gapped contribution (page 3/4)

- for small temperatures $\varphi \rightarrow \infty$ and

$$\epsilon_\nu \sim \int_{v,x,y} \left(\underbrace{\frac{1}{e^{\sqrt{x^2+\varphi^2}} + e^{v+\sqrt{y^2+\varphi^2}}}}_{(e_1,e_2)=(+,+)} + \underbrace{\frac{1}{e^{v+\sqrt{x^2+\varphi^2}} + e^{\sqrt{y^2+\varphi^2}}}}_{(e_1,e_2)=(-,-)} \right. \\ \left. + \underbrace{\frac{1}{e^v + e^{\sqrt{x^2+\varphi^2} + \sqrt{y^2+\varphi^2}}}}_{(e_1,e_2)=(-,+)} + \underbrace{\frac{1}{e^{v+\sqrt{x^2+\varphi^2} + \sqrt{y^2+\varphi^2}}}}_{(e_1,e_2)=(+,-)} \right)$$

- last two terms suppressed by $e^{-2\varphi}$, first two terms

$$\int_{v,x,y} \frac{1}{e^{\sqrt{x^2+\varphi^2}} + e^{v+\sqrt{y^2+\varphi^2}}} \simeq 2\varphi e^{-\varphi} \underbrace{\int_{v,x,y} \frac{1}{e^{x^2} + e^{v+y^2}}}_{\simeq 10.6}$$

Gapped contribution (page 4/4)

→ small T :

contribution of gapped quarks exponentially suppressed
and completely negligible if there are unpaired quarks in the system

- CFL: fermionic contribution negligible, Goldstone modes dominant
P. Jaikumar, M. Prakash and T. Schäfer, PRD 66, 063003 (2002)
- non-CFL quark matter (2SC, LOFF, ...):
unpaired quarks dominate neutrino emissivity

Unpaired quark matter (page 1/4)

$$\frac{\partial}{\partial t} f_\nu(t, \mathbf{p}_\nu) = -\frac{3\pi G_F^2}{4} \int \frac{d^3\mathbf{p}_e d^3\mathbf{k}}{(2\pi)^3 (2\pi)^3 p_\nu p_e} L_{\lambda\sigma} \mathcal{T}^{\lambda\sigma} \\ \times f_F(p_e - \mu_e) f_F(k - \mu_u) [1 - f_F(p - \mu_d)] \delta(p_e - p_\nu + k - p)$$

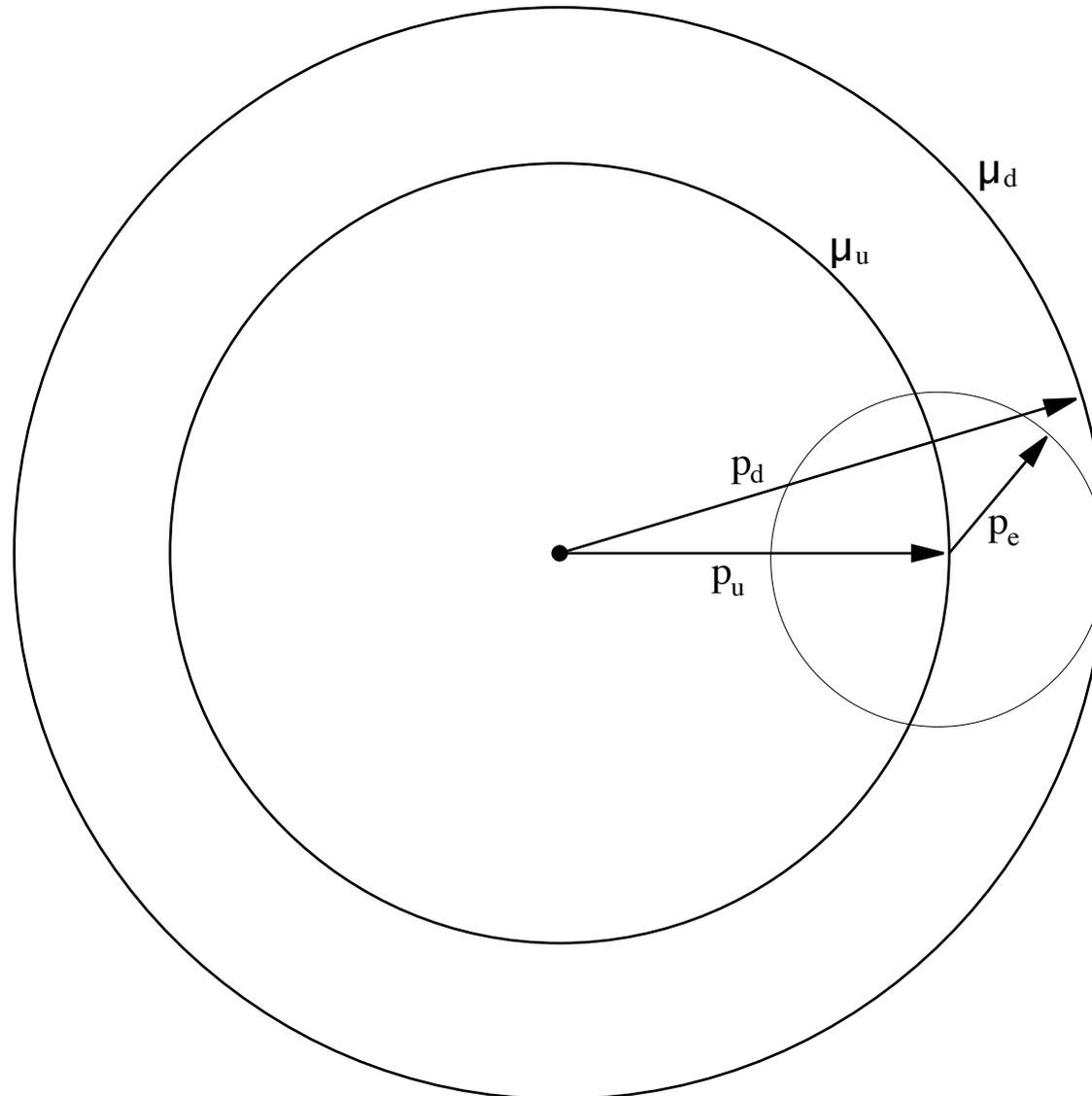
- $\mu_u + \mu_e = \mu_d$ (β -equilibrium)
- contribution would vanish without Fermi liquid corrections

$$p_{F,u/d} = \mu_{u/d}(1 - \kappa), \quad \kappa \equiv \frac{2\alpha_s}{3\pi}$$

(see earlier discussion of corrections to the equation of state)

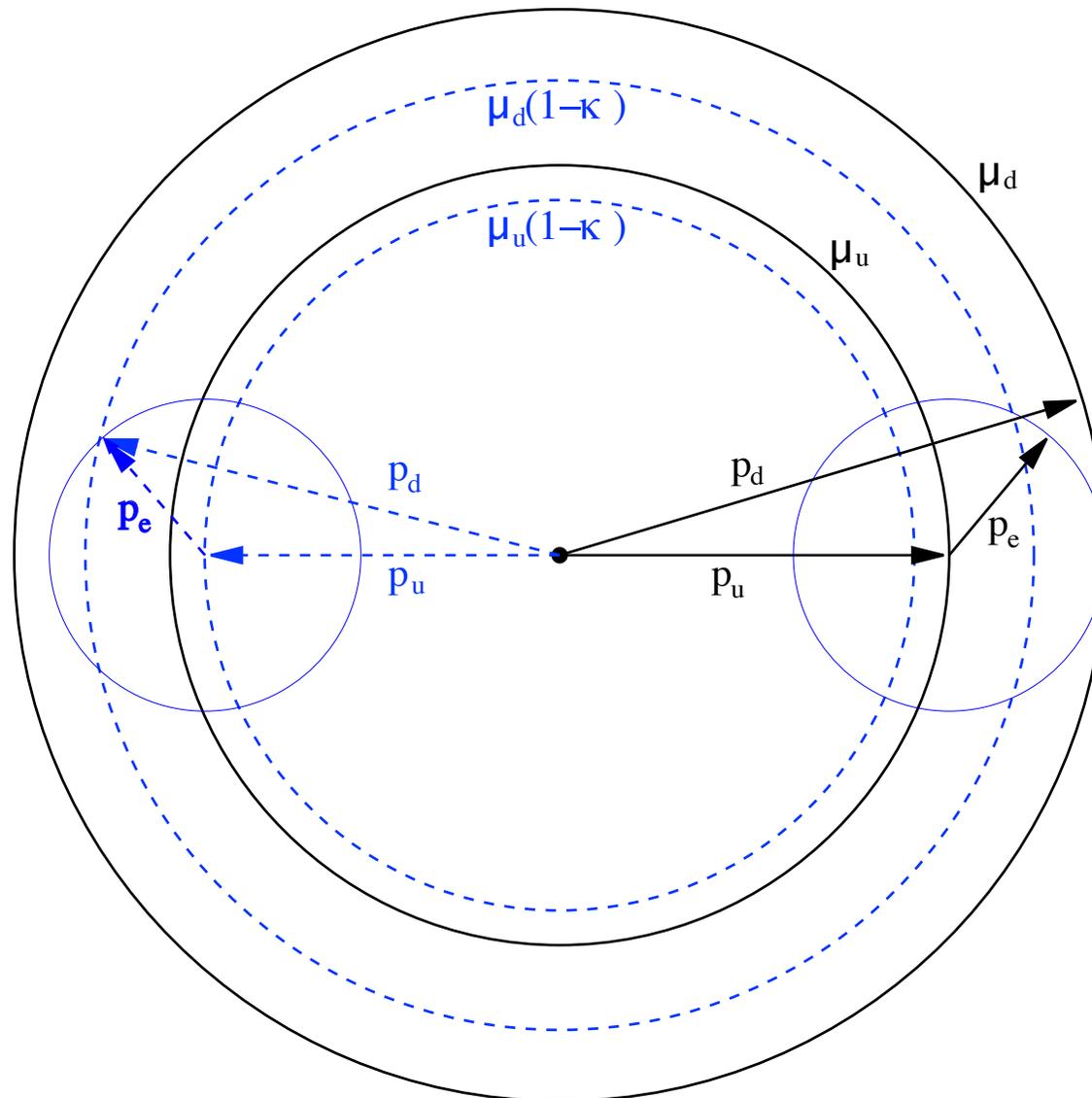
Unpaired quark matter (page 2/4)

- without Fermi liquid corrections



Unpaired quark matter (page 2/4)

- with Fermi liquid corrections



Unpaired quark matter (page 3/4)

- geometry fixes angle θ_{ud} between the u and d quarks,

$$\delta(p_e - p_\nu + k - p) \simeq \frac{\mu_e}{\mu_u \mu_d} \delta(\cos \theta_{ud} - \cos \theta_0), \quad \cos \theta_0 \equiv 1 - \kappa \frac{\mu_e^2}{\mu_u \mu_d}$$

- moreover \rightarrow *exercise*

$$L_{\lambda\sigma} \mathcal{T}^{\lambda\sigma} = 64(p_e - \mathbf{p}_e \cdot \hat{\mathbf{k}})(p_\nu - \mathbf{p}_\nu \cdot \hat{\mathbf{p}}) \simeq 128\mu_e p_\nu \kappa (1 - \cos \theta_{\nu d})$$

Unpaired quark matter (page 4/4)

- putting everything together

$$\begin{aligned} \epsilon_\nu &\simeq 128\alpha_s G_F^2 \mu_e \mu_u \mu_d T^6 \int \frac{d\Omega_{p_\nu}}{(2\pi)^3} \int \frac{d\Omega_p}{(2\pi)^3} \int \frac{d\Omega_k}{(2\pi)^3} \\ &\quad \times (1 - \cos \theta_{\nu d}) \delta(\cos \theta_{ud} - \cos \theta_0) \\ &\quad \times \int_0^\infty dv v^3 \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy f_F(v+x-y) f_F(y) [1 - f_F(x)] \end{aligned}$$

which gives

$$\epsilon_\nu = \frac{457}{630} \alpha_s G_F^2 \mu_e \mu_u \mu_d T^6$$

N. Iwamoto, PRL 44, 1637 (1980)

Cooling with quark direct Urca (page 1/2)

- relate change in temperature to emissivity and specific heat

$$\epsilon_\nu(T) = -c_V(T) \frac{dT}{dt}$$

(minus sign since ϵ_ν is energy *loss*)

- integrate:

$$t - t_0 = - \int_{T_0}^T dT' \frac{c_V(T')}{\epsilon_\nu(T')}.$$

- specific heat for 2-flavor quark matter (degeneracy factor $N_c = 3$)

$$c_V = (\mu_u^2 + \mu_d^2)T$$

Cooling with quark direct Urca (page 2/2)

- this yields

$$T(t) = \frac{T_0 \tau^{1/4}}{(t - t_0 + \tau)^{1/4}}, \quad \tau = \frac{315}{914} \frac{\mu_u^2 + \mu_d^2}{\alpha_s G_F^2 \mu_e \mu_u \mu_d} \frac{1}{T_0^4}$$

assume

$$\mu_d = 500 \text{ MeV}$$

$$\mu_u = 400 \text{ MeV}$$

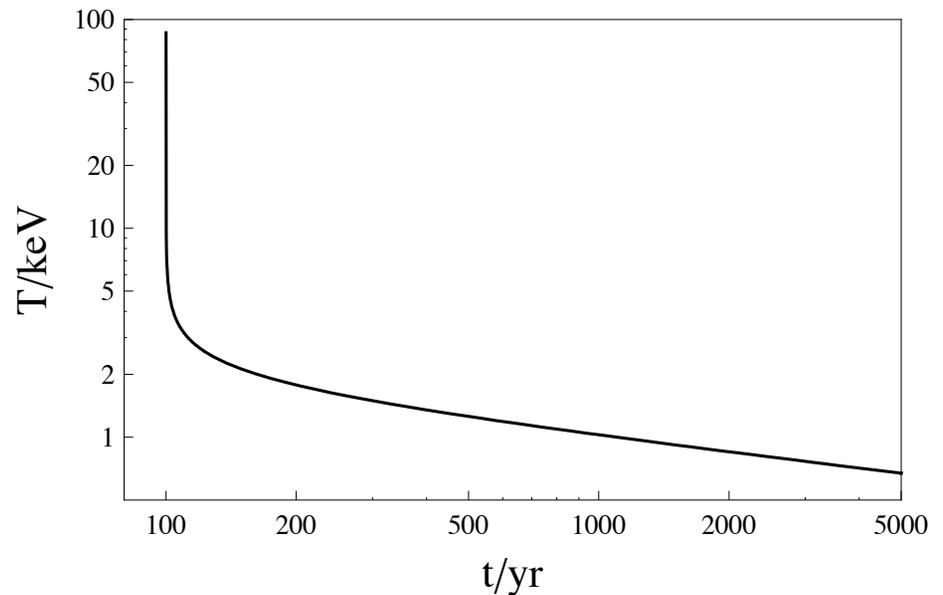
$$\mu_e = 100 \text{ MeV}$$

$$\alpha_s = 1$$

$$T_0 = 100 \text{ keV}$$

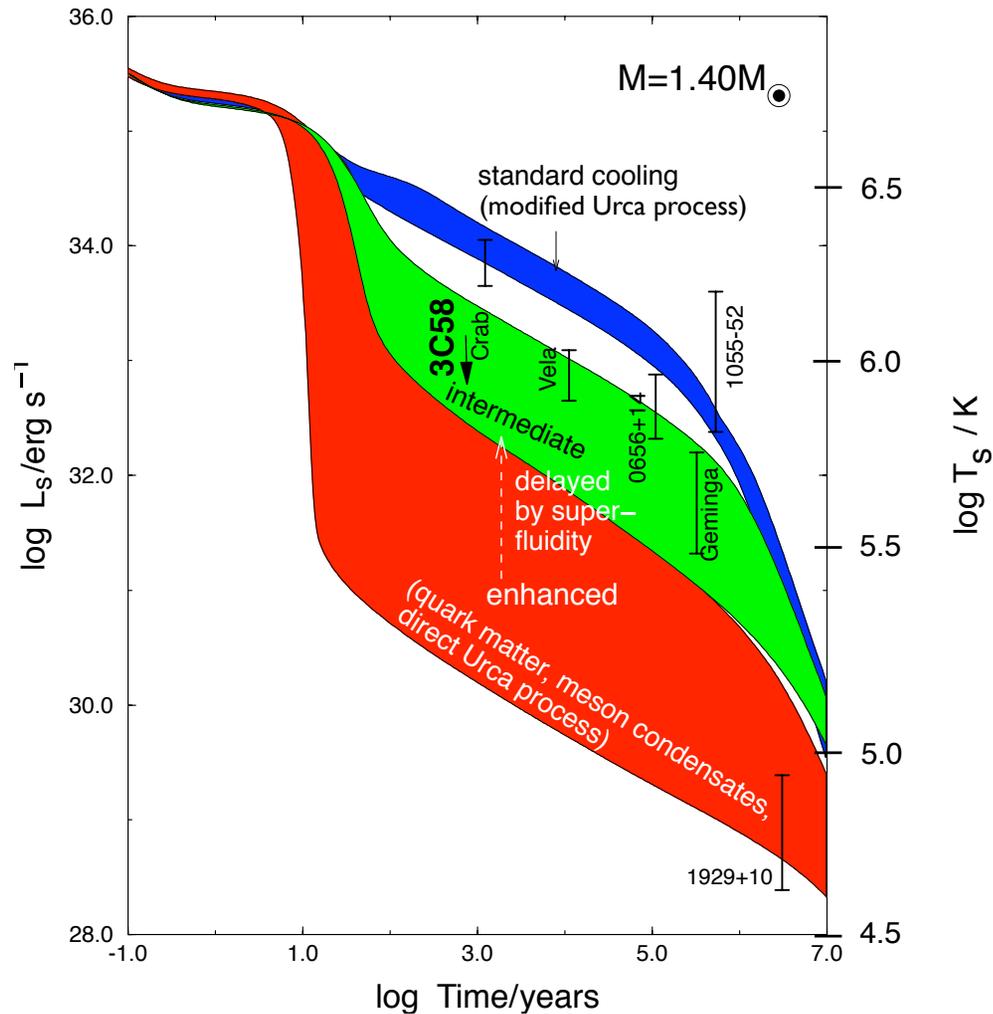
$$t_0 = 100 \text{ yr}$$

$$\Rightarrow \tau \simeq 10^{-5} \text{ yr} \simeq 5 \text{ min}$$

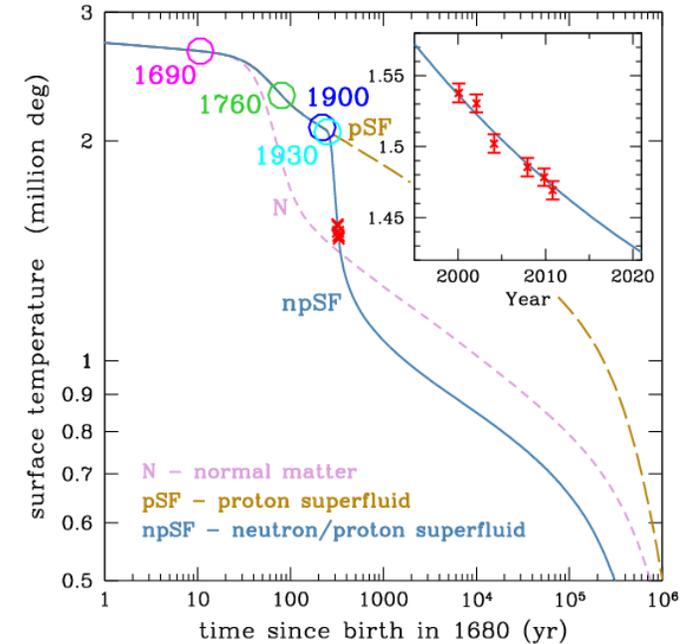


→ extremely fast cooling (too fast compared to observations)

Other cooling processes and observations



- superfluidity can *accelerate* cooling close to T_c see introduction



W.C.G. Ho, *et al.*, PoS ConfinementX, 260 (2012)

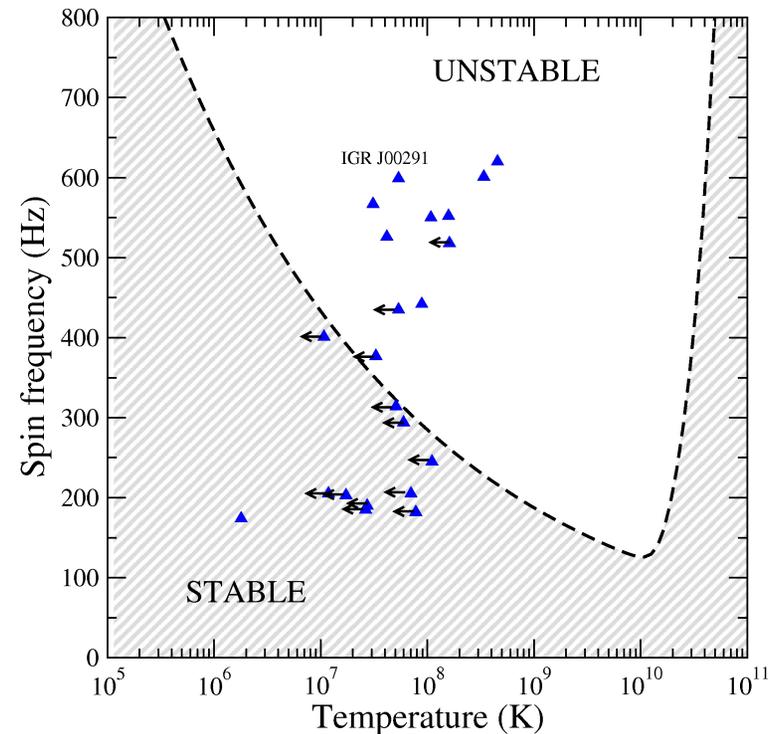
F. Weber, Prog. Part. Nucl. Phys. 54, 193 (2005)

Summary: neutrino emissivity and cooling of the star

- at low temperatures, $T \ll \Delta$, neutrino emissivity is exponentially suppressed by pairing gap
- unpaired quarks (if present, e.g., in 2SC, LOFF, ...) utterly dominate neutrino emissivity
- direct Urca leads to extremely efficient cooling (and is suppressed by small proton fraction in nuclear matter and/or by Cooper pairing)
- Cooper pairing allows for exotic processes (Cooper pair breaking and formation) that can even enhance the emissivity close to T_c (\rightarrow Cas A cooling)

Viscosity and r-mode instability

- instability window
B. Haskell *et al.*, MNRAS 424, 93 (2012)



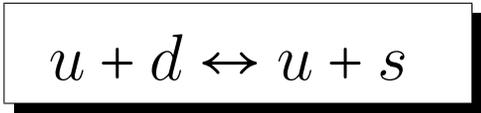
- instability curve given by shear (low T) and bulk (high T) viscosity
→ need to compute viscosities from microscopic physics
in the following: bulk viscosity

What is bulk viscosity?

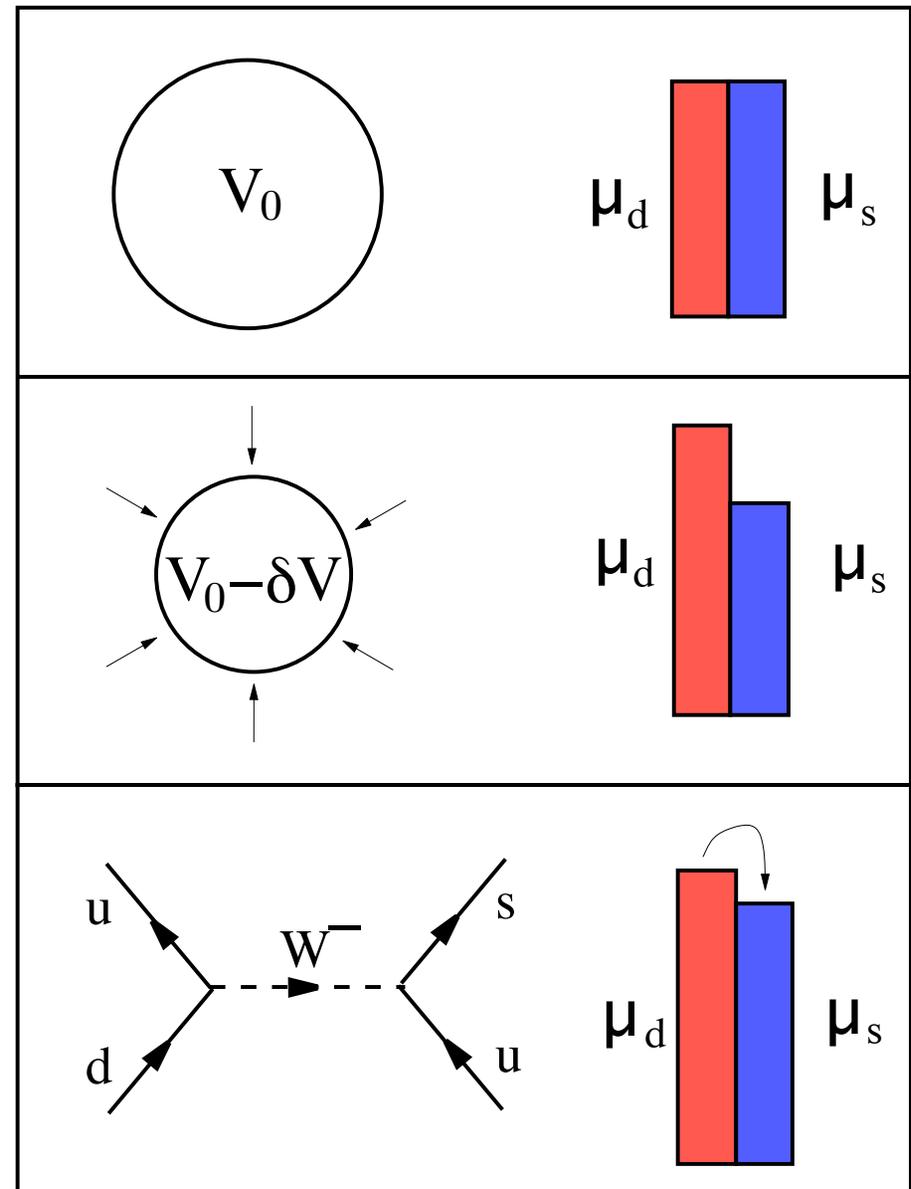
- volume oscillation
→ chemical
non-equilibrium

$$\mu_d - \mu_s \neq 0$$

- re-equilibration via



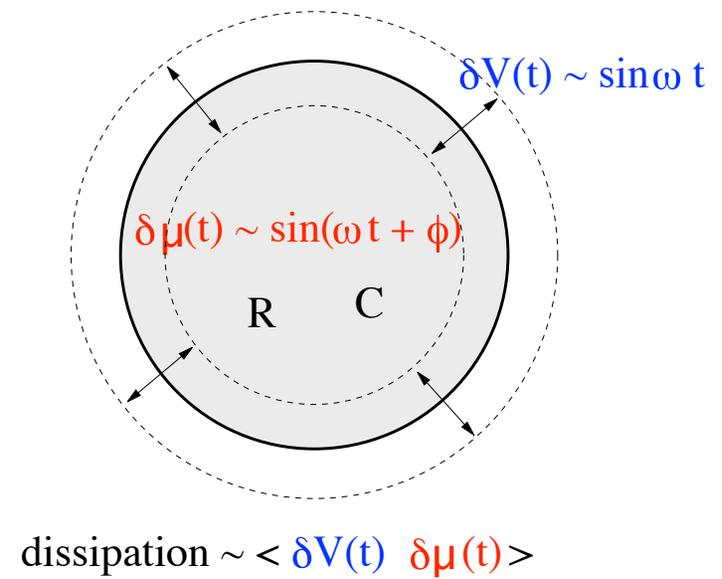
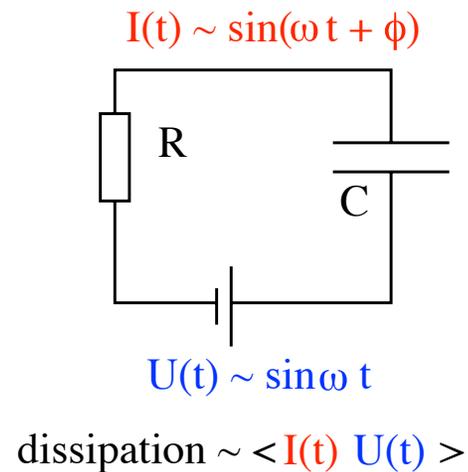
- **resonance phenomenon:**
external oscillation
vs. microscopic rate



Bulk viscosity is a resonance phenomenon

Just like an
electric circuit!

M. G. Alford, A. Schmitt,
JPG 34, 67 (2007)



“capacitance” $C \leftrightarrow$ inverse microscopic rate γ^{-1}
(slow process \rightarrow store large chemical energy)

“resistance” $R \leftrightarrow \left(n_u \frac{\partial \mu_d}{\partial n_u} + n_d \frac{\partial \mu_d}{\partial n_d} - n_s \frac{\partial \mu_s}{\partial n_s} \right)^{-1}$
(same dispersion for d and $s \rightarrow$ infinite “resistance” \rightarrow no
dissipation)

Bulk viscosity

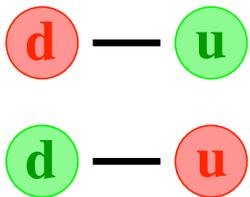
$$\zeta = \alpha \frac{\gamma}{\gamma^2 + \omega^2}$$

$$\alpha \equiv \frac{n_u \frac{\partial \mu_d}{\partial n_u} + n_d \frac{\partial \mu_d}{\partial n_d} - n_s \frac{\partial \mu_s}{\partial n_s}}{\frac{\partial \mu_d}{\partial n_d} + \frac{\partial \mu_s}{\partial n_s}}$$

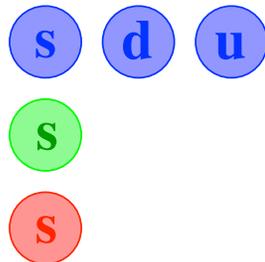
Compute rate for $u + d \leftrightarrow u + s$ in 2SC

$$\Gamma_{2SC} = 4 \left[\begin{array}{c} u \text{ (red)} \\ d \text{ (red)} \end{array} \Delta \right] W^- \left[\begin{array}{c} s \text{ (red)} \\ u \text{ (red)} \end{array} \Delta \right] + 2 \left[\begin{array}{c} u \text{ (red)} \\ d \text{ (red)} \end{array} \Delta \right] W^- \left[\begin{array}{c} s \text{ (blue)} \\ u \text{ (blue)} \end{array} \right] + 2 \left[\begin{array}{c} u \text{ (blue)} \\ d \text{ (blue)} \end{array} \right] W^- \left[\begin{array}{c} s \text{ (red)} \\ u \text{ (red)} \end{array} \Delta \right] + \left[\begin{array}{c} u \text{ (blue)} \\ d \text{ (blue)} \end{array} \right] W^- \left[\begin{array}{c} s \text{ (blue)} \\ u \text{ (blue)} \end{array} \right] \\ + 4 \left[\begin{array}{c} u \text{ (red)} \\ d \text{ (red)} \end{array} \Delta \right] W^- \left[\begin{array}{c} s \text{ (red)} \\ u \text{ (red)} \end{array} \Delta \right] + 2 \left[\begin{array}{c} u \text{ (red)} \\ d \text{ (red)} \end{array} \Delta \right] W^- \left[\begin{array}{c} s \text{ (blue)} \\ u \text{ (blue)} \end{array} \right]$$

paired:



unpaired:



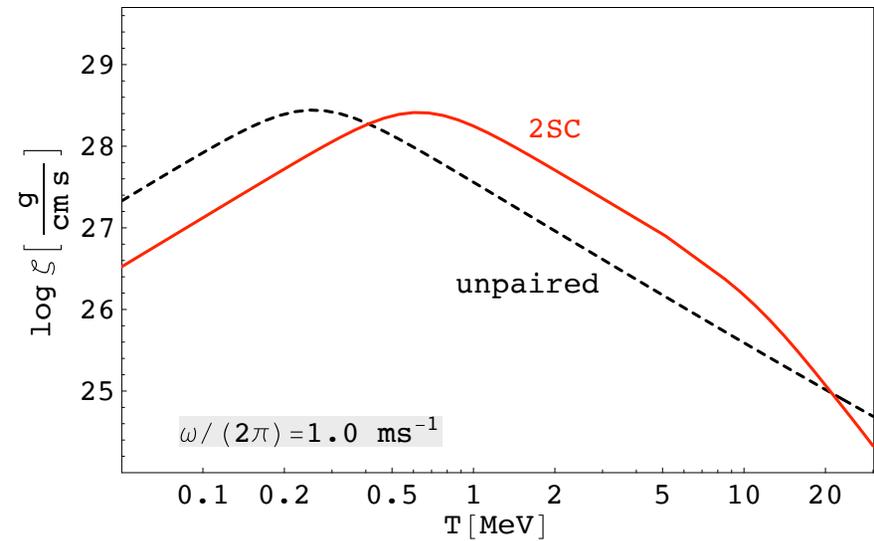
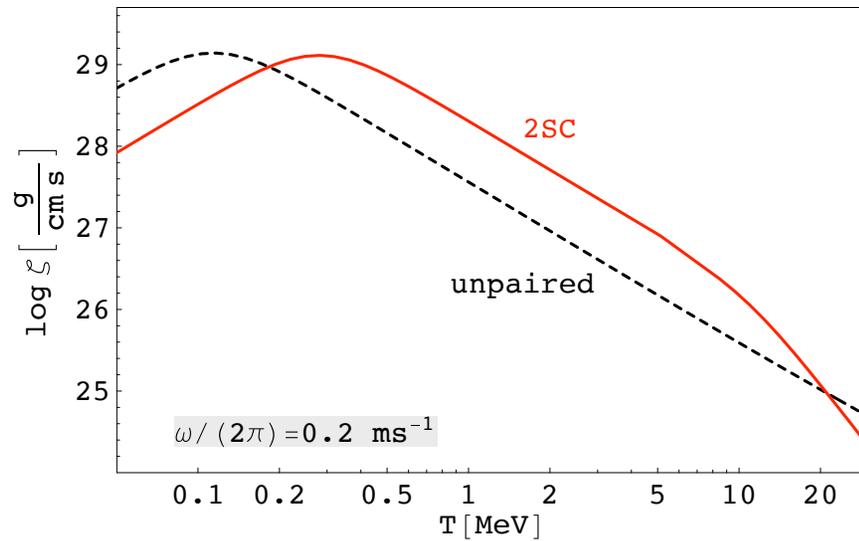
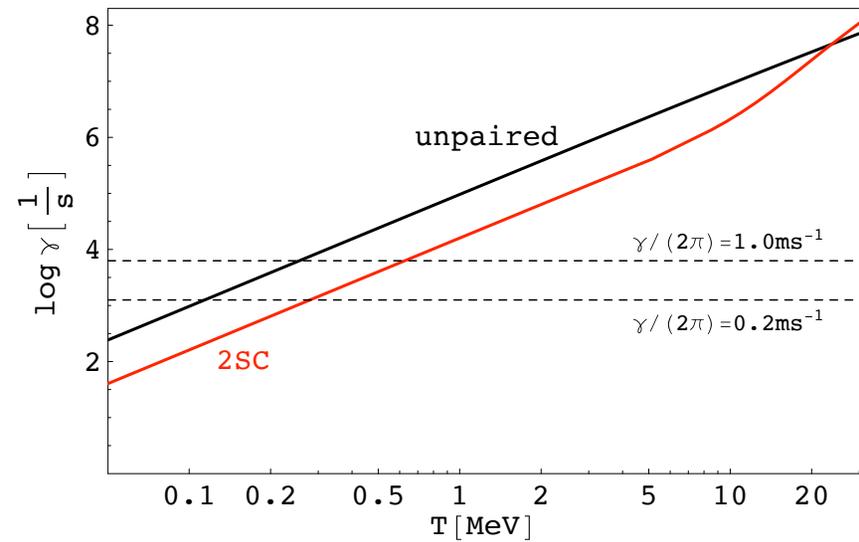
small temperatures,
 $T \ll T_c \simeq 30\text{MeV}$

$$\Gamma_{2SC} = \frac{1}{9} \Gamma_{\text{unpaired}}$$

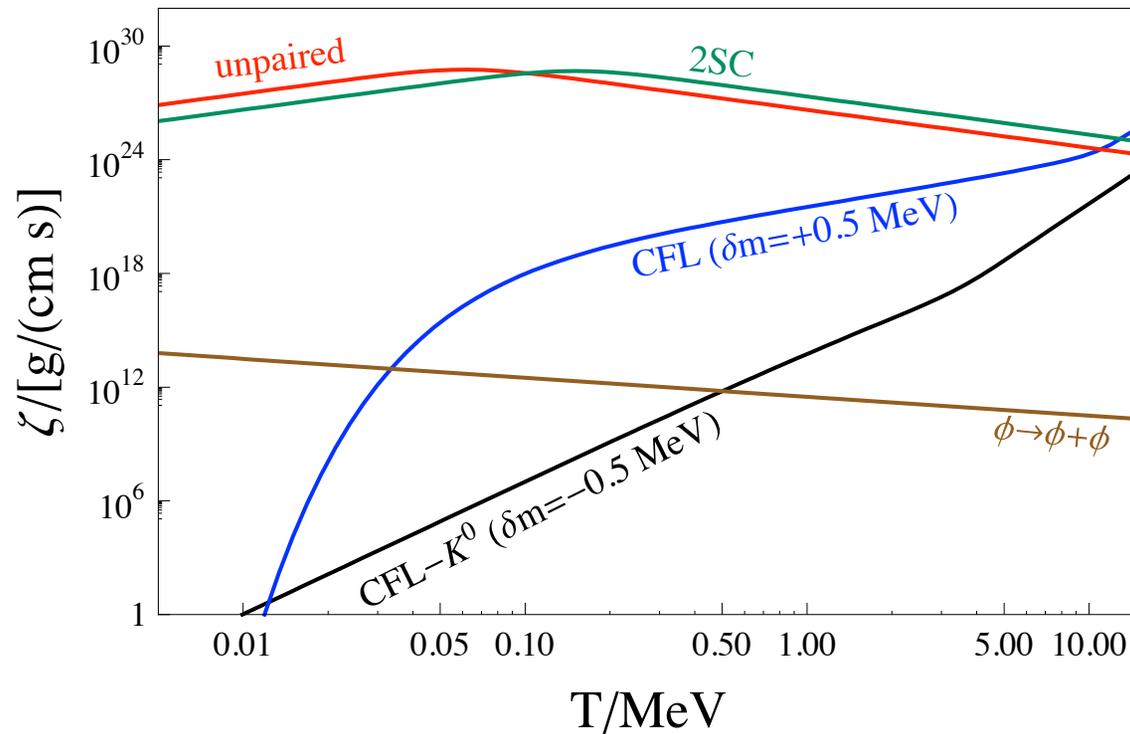
due to **exponential suppression**
 $\exp(-\Delta/T)$ of gapped modes

Results for bulk viscosity

$$\zeta = \alpha \frac{\gamma}{\gamma^2 + \omega^2}$$



Quark matter bulk viscosity: different phases



$$\omega/(2\pi) = 1 \text{ ms}^{-1}$$

$$\mu = 400 \text{ MeV}$$

$$\delta m \equiv m_{K^0} - \mu_{K^0}$$

unpaired from $u + d \leftrightarrow u + s$ J. Madsen, PRD 46, 3290 (1992)

unpaired from $u + e \leftrightarrow d + \nu_e$ B. A. Sa'd, I. A. Shovkoy and D. H. Rischke, PRD 75, 125004 (2007)

2SC from $u + d \leftrightarrow u + s$ M.G. Alford, A. Schmitt, JPG 34, 67-101 (2007)

CFL from $K^0 \leftrightarrow \phi + \phi$ M.G. Alford, M. Braby, S. Reddy, T. Schäfer, PRC 75, 055209 (2007)

CFL- K^0 from $K^0 \leftrightarrow \phi + \phi$ M.G. Alford, M. Braby, A. Schmitt, JPG 35, 115007 (2008)

CFL from $\phi \leftrightarrow \phi + \phi$ C. Manuel, F. Llanes-Estrada, JCAP 0708, 001 (2007)

Spin-one from $u + d \leftrightarrow u + s, u + e \leftrightarrow d + \nu_e$ X. Wang and I. A. Shovkoy, PRD 82, 085007 (2010)

Beyond the "standard" calculation

- a superfluid has three bulk viscosity coefficients
(due to relative motion of "superfluid" and "normal fluid")
I.M. Khalatnikov, *An introduction to the theory of superfluidity* (New York, 1989)
 - apply to CFL
 - CFL from $\phi \leftrightarrow \phi + \phi$ M. Mannarelli and C. Manuel, PRD 81, 043002 (2010)
 - CFL from $K^0 \leftrightarrow \phi + \phi$ R. Bierkandt and C. Manuel, PRD 84, 023004 (2011)
 - r -modes with CFL N. Andersson, B. Haskell and G. L. Comer, PRD 82, 023007 (2010)
- electroweak rate beyond linear order $\Gamma \propto \delta\mu$
 - J. Madsen, PRD 46, 3290 (1992)
 - M. G. Alford, S. Mahmoodifar and K. Schwenzer, JPG 37, 125202 (2010)
 - relevant for large r -mode amplitudes
- non-Fermi liquid effects in unpaired quark matter
 - K. Schwenzer, arXiv:1212.5242 [nucl-th]
 - seems to agree with astrophysical data in T - Ω plane
 - M. G. Alford and K. Schwenzer, PRL 113, 251102 (2014)

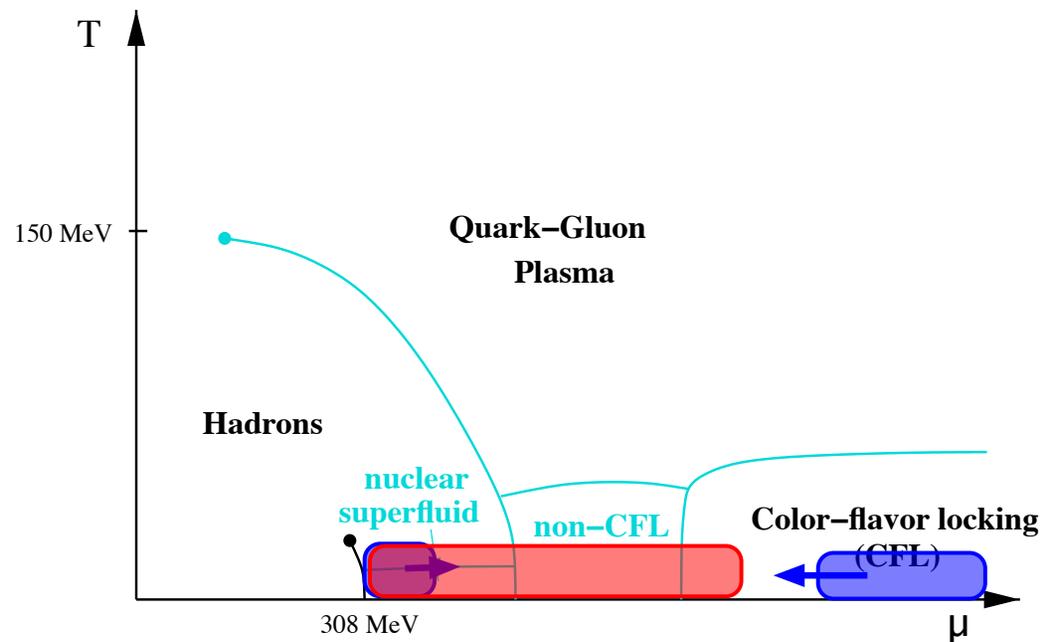
Conclusion (page 1/3)



- compact stars provide a laboratory for QCD, complementary to heavy-ion collisions ($\mu \gg T$ vs. $T \gg \mu$)
- even though we cannot perform “experiments” with compact stars, astrophysical data provides us with information about the star’s interior

Conclusion (page 2/3)

- we understand matter at the highest densities (CFL) and at densities inside nuclei, but compact stars live in between
- by relating observables to microscopic properties we are exploring this “in-between regime”



Conclusion (page 3/3)

- recent and coming progress through
 - more, and more precise, data (larger survey of stars, radius measurements, gravitational waves, ...)
 - pushing/expanding existing approaches (transport properties of quark/nuclear matter, model calculations, communication between nuclear/particle physicists and astrophysicists, ...)
 - novel methods (solutions to the sign problem, gauge-gravity duality, ...)



Solutions to problems

Contents

I. Basic thermodynamic properties	1
II. Non-interacting nuclear matter	3
III. Fermionic excitations in a superfluid	5

I. BASIC THERMODYNAMIC PROPERTIES

Problem:

The pressure for non-interacting fermions (upper sign) and bosons (lower sign) is given by

$$P = \pm T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \left[1 \pm e^{-(E_k - \mu)/T} \right], \quad E_k = \sqrt{k^2 + m^2}. \quad (1)$$

1. Show that for fermions

$$s = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} [(1 - f_k) \ln(1 - f_k) + f_k \ln f_k] \quad (2)$$

and derive the analogous expression for bosons

2. Derive expressions for the specific heat for bosons and fermions,

$$c_V = T \frac{\partial s}{\partial T} \quad (3)$$

and evaluate them

(a) for $T \gg m, \mu$ (fermions and bosons), using

$$\int_0^\infty dx \frac{x^4}{\cosh x + 1} = \frac{7\pi^4}{15}, \quad \int_0^\infty dx \frac{x^4}{\cosh x - 1} = \frac{8\pi^4}{15} \quad (4)$$

(b) for $T \ll \mu$ and $m = 0$ (only fermions), using

$$\int_0^\infty dx \frac{x^2}{\cosh x + 1} = \frac{\pi^2}{3} \quad (5)$$

Solution:

1. We abbreviate

$$x \equiv \frac{E_k - \mu}{T}, \quad \int_k \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}. \quad (6)$$

Then, for fermions, we have

$$f_k = \frac{1}{e^x + 1}, \quad (7)$$

from which we obtain the useful relation $e^x = (1 - f_k)/f_k$. Then we find

$$s_{\text{fermions}} = \frac{\partial P}{\partial T} = \int_k [\ln(1 + e^{-x}) + x f_k] = - \int_k [(1 - f_k) \ln(1 - f_k) + f_k \ln f_k]. \quad (8)$$

For bosons, we have

$$f_k = \frac{1}{e^x - 1}, \quad (9)$$

and thus $e^x = (1 + f_k)/f_k$. This yields

$$s_{\text{bosons}} = \frac{\partial P}{\partial T} = - \int_k [\ln(1 - e^{-x}) - x f_k] = \int_k [(1 + f_k) \ln(1 + f_k) - f_k \ln f_k]. \quad (10)$$

2. We compute for fermions (upper sign) and bosons (lower sign)

$$c_V = T \frac{\partial s}{\partial T} = T \int_k x \frac{\partial f_k}{\partial T} = \int_k \frac{x^2 e^x}{(e^x \pm 1)^2} = \int_k \frac{x^2}{e^x + e^{-x} \pm 2} = \frac{1}{2} \int_k \frac{x^2}{\cosh x \pm 1}, \quad (11)$$

(a) For sufficiently large temperatures, we can neglect m and μ , such that

$$c_V \simeq \frac{1}{4\pi^2} \int_0^\infty dk k^2 \frac{k^2}{T^2} \frac{1}{\cosh \frac{k}{T} \pm 1} = \frac{T^3}{4\pi^2} \int_0^\infty dy \frac{y^4}{\cosh y \pm 1} = \begin{cases} \frac{7\pi^2 T^3}{60} & (\text{fermions}) \\ \frac{2\pi^2 T^3}{15} & (\text{bosons}) \end{cases} \quad (12)$$

(b) For small temperatures we use the fact that the main contribution to the integral comes from the Fermi surface (the Fermi momentum for massless fermions simply is μ),

$$c_V = \frac{1}{4\pi^2} \int_0^\infty dk k^2 \frac{(k - \mu)^2}{T^2} \frac{1}{\cosh \frac{k - \mu}{T} + 1} \simeq \frac{\mu^2}{4\pi^2} \int_0^\infty dk \frac{(k - \mu)^2}{T^2} \frac{1}{\cosh \frac{k - \mu}{T} + 1}. \quad (13)$$

Introducing the new integration variable $y = (k - \mu)/T$ yields

$$c_V \simeq \frac{\mu^2 T}{4\pi^2} \int_{-\mu/T}^\infty dy \frac{y^2}{\cosh y + 1} \simeq \frac{\mu^2 T}{4\pi^2} \int_{-\infty}^\infty dy \frac{y^2}{\cosh y + 1} = \frac{\mu^2 T}{2\pi^2} \int_0^\infty dy \frac{y^2}{\cosh y + 1} = \frac{\mu^2 T}{6}. \quad (14)$$

II. NON-INTERACTING NUCLEAR MATTER

Problem:

1. Show that electrically neutral, non-interacting nuclear matter (n,p,e) at zero temperature and in β -equilibrium (assuming $\mu_\nu \simeq 0$)
 - (a) must contain protons in general, $n_p \neq 0$
 - (b) has a proton fraction $\frac{n_p}{n_B} = \frac{1}{9}$ in the ultra-relativistic limit
 - (c) obeys $\frac{n_p}{n_B} < \frac{1}{9}$ except for very small densities (requires numerical evaluation)
2. Show that non-interacting, pure neutron matter in the non-relativistic limit has a following "polytropic" equation of state,

$$P(\epsilon) = K\epsilon^{5/3}, \quad (15)$$

and compute K and p .

Solution:

1. (a) Neutrality requires $n_e = n_p$ and thus

$$k_{F,e} = k_{F,p} \quad (16)$$

With $\mu = \sqrt{k_F^2 + m^2}$ and the condition from β -equilibrium $\mu_e + \mu_p = \mu_n$ we have

$$\sqrt{k_{F,e}^2 + m_e^2} + \sqrt{k_{F,p}^2 + m_p^2} = \sqrt{k_{F,n}^2 + m_n^2}. \quad (17)$$

Suppose the system contains no protons (and then, because of neutrality, no electrons either), $k_{F,p} = 0$. Then, this equation becomes,

$$k_{F,n}^2 = (m_e + m_p)^2 - m_n^2. \quad (18)$$

The right-hand side is negative, because the neutron is slightly heavier than electron and proton together [that's why a neutron in vacuum decays into a proton and an electron (and an anti-neutrino)]. Hence there is no solution for $k_{F,n}$ and we conclude that protons must be present.

- (b) In the ultra-relativistic limit, $m_e \simeq m_n \simeq m_p \simeq 0$, Eq. (17) becomes

$$2k_{F,p} = k_{F,n} \quad (19)$$

Since $n \propto k_F^3$, this is equivalent to

$$8n_p = n_n, \quad (20)$$

and thus $\frac{n_p}{n_B} = \frac{1}{9}$ with $n_B = n_n + n_p$. That's why dense nuclear matter is neutron rich and hence the name neutron star.

- (c) For the numerical solution we replace $k_{F,e}$ and $k_{F,p}$ in Eq. (17) by $(3\pi^2 n_p)^{1/3}$ and $k_{F,n}$ by $[3\pi^2(n_B - n_p)]^{1/3}$, and solve the resulting equation numerically for n_p for given n_B . The result for a large range of n_B is shown in Fig. 1, and we see that $n_p \leq n_B/9$ with the upper limit approached asymptotically for $n_B \rightarrow \infty$. We also see that there is an onset density for neutrons below which the system only contains electrons and protons.

2. The non-relativistic limit is given by $m \gg k_F$. We can thus approximate the energy density as

$$\epsilon = \frac{1}{\pi^2} \int_0^{k_F} dk k^2 \sqrt{k^2 + m^2} \simeq \frac{m}{\pi^2} \int_0^{k_F} dk k^2 \left(1 + \frac{k^2}{2m}\right) = \frac{mk_F^3}{3\pi^2} + \mathcal{O}(k_F^5). \quad (21)$$

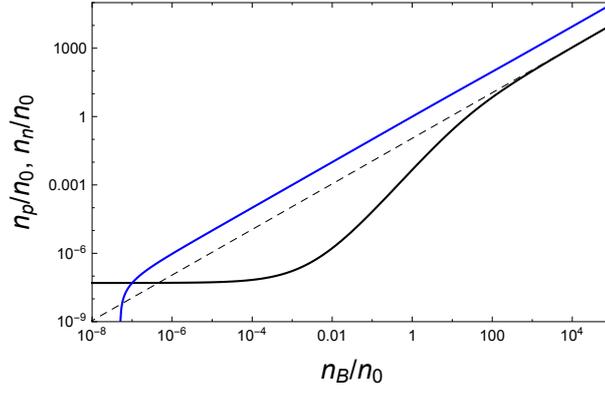


FIG. 1: Proton and neutron densities as a function of the total baryon number density, all given in units of nuclear saturation density, $n_0 \simeq 0.15 \text{ fm}^{-3} \simeq 1.15 \times 10^6 \text{ MeV}^3$. Dense matter in neutron stars only covers a small part of this logarithmic plot, $n_B \sim (1 - 10)n_0$. The dashed line is $n_B/9$.

The pressure becomes

$$\begin{aligned}
 P &= \frac{1}{\pi^2} \int_0^{k_F} dk k^2 (\mu - \sqrt{k^2 + m^2}) \simeq \frac{1}{\pi^2} \int_0^{k_F} dk k^2 \left[m \left(1 + \frac{k_F^2}{2m} \right) - m \left(1 + \frac{k^2}{2m} \right) \right] \\
 &= \frac{1}{2m\pi^2} \int_0^{k_F} dk k^2 (k_F^2 - k^2) = \frac{1}{2m\pi^2} \left(\frac{k_F^5}{3} - \frac{k_F^5}{5} \right) = \frac{k_F^5}{15m\pi^2},
 \end{aligned} \tag{22}$$

where $\mu = \sqrt{k_F^2 + m^2} \simeq m \left(1 + \frac{k_F^2}{2m} \right)$ has been used.

Putting these two results together yields the equation of state given in Eq. (15) with

$$p = \frac{5}{3}, \quad K = \left(\frac{3\pi^2}{m} \right)^{5/3} \frac{1}{15m\pi^2}. \tag{23}$$

III. FERMIONIC EXCITATIONS IN A SUPERFLUID

Problem:

1. Verify that $G^\pm = ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1}$ and $F^\pm = -G_0^\mp \Phi^\pm G^\pm$
2. Verify that in the ultra-relativistic limit ($m = 0$)

(a)

$$[G_0^\pm]^{-1} = \sum_{e=\pm} [k_0 \pm (\mu - ek)] \gamma^0 \Lambda_k^{\pm e} \quad (24)$$

(b)

$$G_0^\pm = \sum_e \frac{\gamma^0 \Lambda_k^{\mp e}}{k_0 \pm (\mu - ek)} \quad (25)$$

with the energy projectors $\Lambda_k^e \equiv \frac{1}{2} (1 + e\gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}})$. *Hint for (b): show that Λ_k^+ and Λ_k^- form a complete set of orthogonal projectors.*

3. With $\Phi^\pm = \pm \Delta \gamma^5$, show that the poles of G^\pm (for $m = 0$) are given by

$$k_0 = \pm \epsilon_k^e, \quad \epsilon_k^e = \sqrt{(\mu - ek)^2 + \Delta^2}, \quad (26)$$

i.e., Δ is an "energy gap". Make use of the results of 1 and 2.

Solution:

1. We have written the inverse propagator and the propagator in Nambu-Gorkov space as

$$\mathcal{S}^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Phi^- \\ \Phi^+ & [G_0^-]^{-1} \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} G^+ & F^- \\ F^+ & G^- \end{pmatrix} \quad (27)$$

It is easy to check that

$$\mathcal{S}^{-1} \mathcal{S} = 1. \quad (28)$$

With the expressions for G^\pm and F^\pm given in the problem, the 4 entries of this matrix are [in the order (1,1), (1,2), (2,1), (2,2)]

$$[G_0^+]^{-1} G^+ - \Phi^- G_0^- \Phi^+ G^+ = ([G_0^+]^{-1} - \Phi^- G_0^- \Phi^+) G^+ = [G^+]^{-1} G^+ = 1, \quad (29a)$$

$$-[G_0^+]^{-1} G_0^+ \Phi^- G^- + \Phi^- G^- = 0, \quad (29b)$$

$$\Phi^+ G^+ - [G_0^-]^{-1} G_0^- \Phi^+ G^+ = 0, \quad (29c)$$

$$-\Phi^+ G_0^+ \Phi^- G^- + [G_0^-]^{-1} G^- = ([G_0^-]^{-1} - \Phi^+ G_0^+ \Phi^-) G^- = [G^-]^{-1} G^- = 1. \quad (29d)$$

This problem is nothing but the formal inversion of a 2×2 matrix whose entries are (invertable) matrices themselves.

2. (a) All we need is the explicit form of the energy projectors and $(\gamma^0)^2 = 1$,

$$\begin{aligned} \sum_{e=\pm} [k_0 \pm (\mu - ek)] \gamma^0 \Lambda_k^{\pm e} &= [k_0 \pm (\mu - k)] \frac{1}{2} (\gamma^0 \pm \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) + [k_0 \pm (\mu + k)] \frac{1}{2} (\gamma^0 \mp \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) \\ &= \frac{1}{2} [k_0 \gamma^0 \pm k_0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} \pm (\mu - k) \gamma^0 + (\mu - k) \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} \\ &\quad + k_0 \gamma^0 \mp k_0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} \pm (\mu + k) \gamma^0 - (\mu + k) \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}] \\ &= k_0 \gamma^0 \pm \mu \gamma^0 - \boldsymbol{\gamma} \cdot \mathbf{k} = \gamma^\mu K_\mu \pm \mu \gamma^0. \end{aligned} \quad (30)$$

(b) We first show that Λ_k^+ and Λ_k^- form a complete set of orthogonal projectors. Completeness $\Lambda_k^+ + \Lambda_k^- = 1$ is obvious. Orthogonality is seen as follows,

$$\Lambda_k^+ \Lambda_k^- = \frac{1}{4}(1 + \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}})(1 - \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) = \frac{1}{4}(1 - \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) = \frac{1}{4}[1 + (\boldsymbol{\gamma} \cdot \hat{\mathbf{k}})^2] = 0. \quad (31)$$

Here we have used $\{\gamma^0, \gamma^i\} = 0$ and $(\boldsymbol{\gamma} \cdot \hat{\mathbf{k}})^2 = -1$:

$$\begin{aligned} (\boldsymbol{\gamma} \cdot \hat{\mathbf{k}})^2 &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{k}} \\ -\boldsymbol{\sigma} \cdot \hat{\mathbf{k}} & 0 \end{pmatrix}^2 = -(\boldsymbol{\sigma} \cdot \hat{\mathbf{k}})^2 = -\begin{pmatrix} \hat{k}_3 & \hat{k}_1 - i\hat{k}_2 \\ \hat{k}_1 + i\hat{k}_2 & -\hat{k}_3 \end{pmatrix}^2 \\ &= -\begin{pmatrix} \hat{k}_1^2 + \hat{k}_2^2 + \hat{k}_3^2 & 0 \\ 0 & \hat{k}_1^2 + \hat{k}_2^2 + \hat{k}_3^2 \end{pmatrix} = -1. \end{aligned} \quad (32)$$

Finally, we show that Λ_k^+ and Λ_k^- are projectors,

$$(\Lambda_k^\pm)^2 = \frac{1}{4}(1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}})(1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) = \frac{1}{4}[1 \pm 2\gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} - (\boldsymbol{\gamma} \cdot \hat{\mathbf{k}})^2] = \frac{1}{2}(1 \pm \gamma^0 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}}) = \Lambda_k^\pm. \quad (33)$$

Now we easily compute

$$[G_0^\pm]^{-1} G_0^\pm = \sum_{e_1 e_2} \frac{k_0 \pm (\mu - e_1 k)}{k_0 \pm (\mu - e_2 k)} \gamma_0 \Lambda_k^{\pm e_1} \gamma_0 \Lambda_k^{\mp e_2} = \sum_{e_1 e_2} \frac{k_0 \pm (\mu - e_1 k)}{k_0 \pm (\mu - e_2 k)} \Lambda_k^{\mp e_1} \Lambda_k^{\mp e_2} = \sum_e \Lambda_k^{\mp e} = 1, \quad (34)$$

where $\gamma^0 \Lambda_k^+ = \Lambda_k^- \gamma^0$ has been used. This problem shows that inversion of a matrix becomes very simple if the matrix is written in terms of orthogonal projectors.

3. We use the above results to compute the propagator

$$\begin{aligned} G^\pm &= ([G_0^\pm]^{-1} - \Phi^\mp G_0^\mp \Phi^\pm)^{-1} \\ &= \left(\sum_e \left\{ [k_0 \pm (\mu - ek)] \gamma^0 \Lambda_k^{\pm e} + \frac{\Delta^2 \gamma^5 \gamma^0 \Lambda_k^{\pm e} \gamma^5}{k_0 \mp (\mu - ek)} \right\} \right)^{-1} \\ &= \left\{ \sum_e \left[\underbrace{k_0 \pm (\mu - ek) - \frac{\Delta^2}{k_0 \mp (\mu - ek)}}_{\frac{k_0^2 - (\mu - ek)^2 - \Delta^2}{k_0 \mp (\mu - ek)}} \right] \gamma^0 \Lambda_k^{\pm e} \right\}^{-1} \\ &= \sum_e \frac{k_0 \mp (\mu - ek)}{k_0^2 - (\epsilon_k^e)^2} \gamma^0 \Lambda_k^{\mp e}, \end{aligned} \quad (35)$$

which shows that the poles of the propagator are indeed given by Eq. (26).