Exercise: Relativistic Bose gas at nonzero chemical potential

Consider a complex scalar field, with a global symmetry $\phi \to e^{i\alpha}\phi$. The (Euclidean) action is

$$S = \int d^4x \left(|\partial_\nu \phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4\right), \tag{1}$$

and the conserved charge reads

$$N = \int d^3x\, i \left[\phi^* \partial_4 \phi - (\partial_4 \phi^*)\phi\right]. \tag{2}$$

We take $m^2 > 0$, so that at vanishing and small μ the theory is in its symmetric phase.

In order to write down the euclidean path integral at nonzero μ, we have to revisit the derivation of the path integral with a bit more care (see e.g. Kapusta and Gale, *Finite-temperature field theory*). We start from the partition function,

$$Z = \mathrm{Tr}\, e^{-(H-\mu N)/T}, \tag{3}$$

and express the hamiltonian and conserved charge (densities) in terms of the canonical momenta $\pi_1 = \partial_4\phi_1, \pi_2 = \partial_4\phi_2$, where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. For example, the charge now takes the form

$$N = \int d^3x\, (\phi_2\pi_1 - \phi_1\pi_2). \tag{4}$$

Following the standard derivation of the path integral, one finds that the partition function then reads

$$Z = \int D\phi_1 D\phi_2 \int D\pi_1 D\pi_2 \exp \int d^4x \Big[i\pi_1\partial_4\phi_1 + i\pi_2\partial_4\phi_2 - \mathcal{H} + \mu(\phi_2\pi_1 - \phi_1\pi_2)\Big]. \tag{5}$$

Note that we use Euclidean time already at this stage.

i) Integrate out the momenta $\pi_{1,2}$ to arrive at the following expression for the euclidean action

$$\begin{aligned} S = & \int d^4x \left[(\partial_4 + \mu)\phi^*(\partial_4 - \mu)\phi + |\partial_i\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4\right] \\ = & \int d^4x \left[|\partial_\nu\phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\phi^*\partial_4\phi - \partial_4\phi^*\phi) + \lambda|\phi|^4\right]. \end{aligned} \tag{6}$$

ii) The corresponding lattice action, with lattice spacing $a_{\rm lat} \equiv 1$, is

$$S = \sum_x \Bigg[\left(2d + m^2\right) \phi_x^*\phi_x + \lambda \left(\phi_x^*\phi_x\right)^2 - \sum_{\nu=1}^4 \left(\phi_x^* e^{-\mu\delta_{\nu,4}}\phi_{x+\hat\nu} + \phi_{x+\hat\nu}^* e^{\mu\delta_{\nu,4}}\phi_x\right)\Bigg], \tag{7}$$

where the number of euclidean dimensions is $d = 4$. Show that this action reduces to Eq. (6) in the continuum limit.

iii) The complex field is written in terms of two real fields ϕ_a $(a = 1, 2)$ as $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Show that the lattice action then reads

$$\begin{aligned} S = \sum_x \Bigg[& \frac{1}{2}\left(2d + m^2\right)\phi_{a,x}^2 + \frac{\lambda}{4}\left(\phi_{a,x}^2\right)^2 - \sum_{i=1}^3 \phi_{a,x}\phi_{a,x+\hat i} \\ & - \cosh\mu\, \phi_{a,x}\phi_{a,x+\hat 4} + i \sinh\mu\, \varepsilon_{ab}\phi_{a,x}\phi_{b,x+\hat 4}\Bigg], \end{aligned} \tag{8}$$

where ε_{ab} is the antisymmetric tensor with $\epsilon_{12} = 1$, and summation over repeated indices is implied. Note that the '$\sinh\mu$' term is complex.

From now on the self-interaction is ignored and we take $\lambda = 0$. After going to momentum space, the action (8) reads

$$S = \sum_p \frac{1}{2}\phi_{a,-p}\left(\delta_{ab}A_p - \varepsilon_{ab}B_p\right)\phi_{b,p} = \sum_p \frac{1}{2}\phi_{a,-p}M_{ab,p}\phi_{b,p}, \tag{9}$$

where

$$M_p = \begin{pmatrix} A_p & -B_p \\ B_p & A_p \end{pmatrix}, \tag{10}$$

and

$$A_p = m^2 + 4\sum_{i=1}^{3}\sin^2\frac{p_i}{2} + 2\left(1-\cosh\mu\cos p_4\right), \qquad B_p = 2\sinh\mu\sin p_4. \tag{11}$$

iv) Show that the propagator corresponding to the action (9) is

$$G_{ab,p} = \frac{\delta_{ab}A_p + \varepsilon_{ab}B_p}{A_p^2+B_p^2}. \tag{12}$$

v) Demonstrate that the dispersion relation that follows from the poles of the propagator, taking $p_4 = iE_{\mathbf{p}}$, reads

$$\cosh E_{\mathbf{p}}(\mu) = \left(1+\frac{1}{2}\hat\omega_{\mathbf{p}}^2\right)\cosh\mu \pm \hat\omega_{\mathbf{p}}\sqrt{1+\frac{1}{4}\hat\omega_{\mathbf{p}}^2}\,\sinh\mu, \tag{13}$$

where

$$\hat\omega_{\mathbf{p}}^2 = m^2 + 4\sum_i \sin^2\frac{p_i}{2}. \tag{14}$$

vi) Show that this can be written as

$$\cosh E_{\mathbf{p}}(\mu) = \cosh\left[E_{\mathbf{p}}(0)\pm\mu\right], \tag{15}$$

such that the (positive energy) solutions are

$$E_{\mathbf{p}}(\mu) = E_{\mathbf{p}}(0)\pm\mu. \tag{16}$$

Sketch the spectrum. Note that the critical μ value for onset is $\mu_c = E_{\mathbf{0}}(0)$, so that one mode becomes exactly massless at the transition (Goldstone boson).
vii) The phase-quenched theory corresponds to $\sinh\mu = B_p = 0$. Show that the dispersion relation in the phase-quenched theory is

$$\cosh E_{\mathbf{p}}(\mu) = \frac{1}{\cosh\mu}\left(1+\frac{1}{2}\hat\omega_{\mathbf{p}}^2\right), \tag{17}$$

which corresponds to $E_{\mathbf{p}}^2(\mu) = m^2 - \mu^2 + \mathbf{p}^2$ in the continuum limit.
viii) Compare the spectrum of the full and the phase-quenched theory, when $\mu < \mu_c$. At larger μ, it is necessary to include the self-interaction to stabilize the theory. Based on what you know about symmetry breaking, sketch the spectrum in the full and the phase-quenched theory at larger μ as well.

Although the spectrum depends on μ, thermodynamic quantities do not. Up to an irrelevant constant, the logarithm of the partition function is

$$\ln Z = -\frac{1}{2}\sum_p \ln\det M = -\frac{1}{2}\sum_p \ln(A_p^2+B_p^2), \tag{18}$$

and some observables are given by

$$\langle|\phi|^2\rangle = -\frac{1}{\Omega}\frac{\partial\ln Z}{\partial m^2} = \frac{1}{\Omega}\sum_p\frac{A_p}{A_p^2+B_p^2}, \tag{19}$$

and

$$\langle n\rangle = \frac{1}{\Omega}\frac{\partial\ln Z}{\partial\mu} = -\frac{1}{\Omega}\sum_p\frac{A_pA_p'+B_pB_p'}{A_p^2+B_p^2}, \tag{20}$$

where $\Omega = N_\sigma^3 N_\tau$ and $A_p' = \partial A_p/\partial\mu$, $B_p' = \partial B_p/\partial\mu$.
ix) Evaluate the sums (e.g. numerically) to demonstrate that thermodynamic quantities are independent of μ in the thermodynamic limit at vanishing temperature.
This exercise is based on G. Aarts, JHEP **0905** (2009) 052 [arXiv:0902.4686 [hep-lat]].

Exercise: Fokker-Planck equation

Consider the Langevin process

$$\dot{x}(t) = K[x(t)] + \eta(t), \qquad K(x) = -S'(x), \qquad \langle \eta(t)\eta(t')\rangle_\eta = 2\lambda\delta(t-t'), \tag{21}$$

where λ normalises the noise and K is the drift term, which is derived from the action S. In order to study the equilibrium solution, i.e. the distribution to which the Langevin process (hopefully!) converges, we want to derive the associated Fokker-Planck equation

$$\partial_t \rho(x,t) = \partial_x \left(\lambda\partial_x - K\right)\rho(x,t), \tag{22}$$

for the distribution $\rho(x,t)$, defined via

$$\langle O[x(t)]\rangle_\eta = \int dx\, \rho(x,t) O(x), \tag{23}$$

with $O(x)$ a generic observable. Here the subscript η denotes noise averaging and will be dropped from now on.

To achieve this we consider the discretized process

$$\delta_n \equiv x_{n+1} - x_n = \epsilon K_n + \sqrt{\epsilon}\eta_n, \qquad \langle \eta_n \eta_{n'}\rangle = 2\lambda\delta_{nn'}. \tag{24}$$

i) Show that

$$\begin{aligned}\langle O(x_{n+1})\rangle - \langle O(x_n)\rangle &= \langle O'(x_n)\delta_n + \frac{1}{2}O''(x_n)\delta_n^2 + \ldots\rangle \\ &= \epsilon\langle O'(x_n)K_n + \lambda O''(x_n)\rangle + \mathcal{O}(\epsilon^{3/2}).\end{aligned} \tag{25}$$

In the $\epsilon \to 0$ limit, this gives

$$\partial_t\langle O(x)\rangle = \langle O'(x)K(x) + \lambda O''(x)\rangle. \tag{26}$$

ii) Use Eq. (23) to demonstrate that this yields the Fokker-Planck equation (22) for $\rho(x,t)$. What should λ be in order to obtain the desired equilibrium distribution, $\rho(x) \sim \exp(-S(x))$?
iii) We now repeat the analysis for the Langevin process with a complex drift, $K(z) = -S'(z) \in \mathbb{C}$. We write this as a real Langevin process in the complex plane, i.e. with the complex Langevin equations,

$$\begin{aligned}\dot{x} &= K_x + \eta_x, \qquad K_x = -\mathrm{Re}\, S'(z), \qquad \langle \eta_x(t)\eta_x(t')\rangle = 2\lambda_x\delta(t-t'), \\ \dot{y} &= K_y + \eta_y, \qquad K_y = -\mathrm{Im}\, S'(z), \qquad \langle \eta_y(t)\eta_y(t')\rangle = 2\lambda_y\delta(t-t').\end{aligned} \tag{27}$$

By writing $z = x + iy$, show that these Langevin equations are indeed equivalent to

$$\dot{z} = -S'(z) + \eta, \qquad \langle \eta(t)\eta(t')\rangle = 2\delta(t-t'). \tag{28}$$

Express η in terms of $\eta_{x,y}$ and derive the necessary restrictions on $\lambda_{x,y}$ (answer: $\lambda_x - \lambda_y = 1$). The case $\lambda_y > 0$ is referred to as complex noise.
iv) The distribution $P(x,y;t)$ is now defined via

$$\langle O[x(t) + iy(t)]\rangle_\eta = \int dxdy\, P(x,y;t) O(x+iy). \tag{29}$$

Show that $P(x,y;t)$ satisfies

$$\partial_t P(x,y;t) = \left[\partial_x\left(\lambda_x\partial_x - K_x\right) + \partial_y\left(\lambda_y\partial_y - K_y\right)\right] P(x,y;t). \tag{30}$$

This is reviewed e.g. in Damgaard and Hüffel, Phys. Rept. **152** (1987) 227. For real Langevin dynamics, one can prove that the process converges, i.e. that the Fokker-Planck equation (22) converges to the equilibrium solution exponentially fast. For a complex action, this general proof breaks down. In fact, Eq. (30) has no generic solutions! Complex noise and especially its problems are discussed in G. Aarts, E. Seiler and I. O. Stamatescu, Phys. Rev. D **81** (2010) 054508 [arXiv:0912.3360 [hep-lat]].

Exercise: Complex Gaussian model

For Gaussian models with a complex action, the Fokker-Planck equation (30) can be solved and one can show that the expected results are obtained. Consider the complex integral

$$Z = \int_{-\infty}^{\infty} dx\, \rho(x), \qquad \rho(x) = e^{-S}, \qquad S = \frac{1}{2}\sigma x^2, \qquad \sigma = a + ib. \tag{31}$$

i) Show that the corresponding complex Langevin equations are given by

$$\dot{x} = K_x + \eta, \qquad K_x = -ax + by, \tag{32}$$

$$\dot{y} = K_y, \qquad K_y = -ay - bx, \tag{33}$$

where $\langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$. We consider real noise only.
ii) Demonstrate that these Langevin equations are solved by

$$x(t) = e^{-at}\left[\cos(bt)x(0) + \sin(bt)y(0)\right] + \int_0^t ds\, e^{-a(t-s)}\cos[b(t-s))]\eta(s), \tag{34}$$

$$y(t) = e^{-at}\left[\cos(bt)y(0) - \sin(bt)x(0)\right] - \int_0^t ds\, e^{-a(t-s)}\sin[b(t-s)]\eta(s). \tag{35}$$

iii) Show that the expectation values in the infinite time limit are given by

$$\langle x^2\rangle = \frac{1}{2a}\frac{2a^2+b^2}{a^2+b^2}, \qquad \langle y^2\rangle = \frac{1}{2a}\frac{b^2}{a^2+b^2}, \qquad \langle xy\rangle = -\frac{1}{2}\frac{b}{a^2+b^2}. \tag{36}$$

iv) Demonstrate that this yields the desired result

$$\langle x^2\rangle \to \langle (x+iy)^2\rangle = \frac{a-ib}{a^2+b^2} = \frac{1}{a+ib} = \frac{1}{\sigma}. \tag{37}$$

v) The Fokker-Planck equation for the (real and positive) weight $P(x,y;t)$, defined via

$$\langle O(x(t)+iy(t))\rangle = \int dxdy\, P(x,y;t)O(x+iy), \tag{38}$$

is given by

$$\partial_t P(x,y;t) = \left[\partial_x\left(\partial_x - K_x\right) - \partial_y K_y\right] P(x,y;t) \tag{39}$$

Since the original integral is Gaussian, the equilibrium distribution $P(x,y)$ is also Gaussian and can be written as

$$P(x,y) = N\exp\left[-\alpha x^2 - \beta y^2 - 2\gamma xy\right], \tag{40}$$

where N is a normalization constant.
Using the Fokker-Planck equation, show that the coefficients are given by

$$\alpha = a, \qquad \beta = a\left(1 + \frac{2a^2}{b^2}\right), \qquad \gamma = \frac{a^2}{b}, \tag{41}$$

and demonstrate that this gives the previously computed expectation values

$$\langle x^2\rangle = \frac{\int dxdy\, P(x,y)x^2}{\int dxdy\, P(x,y)}, \tag{42}$$

etc.
vi) From the equivalence

$$\int dx\, \rho(x)O(x) = \int dxdy\, P(x,y)O(x+iy), \tag{43}$$

it follows that the real distribution is related to the original complex one via

$$\rho(x) = \int dy\, P(x-iy,y). \tag{44}$$

Verify this explicitly (up to the undetermined normalization).
This is simple version of the problem treated in G. Aarts, JHEP **0905** (2009) 052 [arXiv:0902.4686 [hep-lat]].